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On the Foiaş and Strătilă theorem

by

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Abstract. We extend the Foiaş and Strătilă theorem to the case of L^2-functions whose spectral measures are continuous and concentrated on an independent Helson set, and to ergodic actions of locally compact second countable abelian groups. We first prove it for functions satisfying Carleman's condition for the Hamburger moment problem, without the assumption that the spectral measure is supported by a Helson set. Then we show independently that the spectral projector associated with a Helson set preserves each L^p-space, with an appropriate bound of the corresponding norm.

1. Introduction

1.1. Main results. The Foiaş and Strătilă theorem [9] asserts that, given an ergodic measure-preserving automorphism T on a standard probability space, if the spectral measure σ of a non-zero square-integrable complex function f is continuous and supported on a *Kronecker set* [21], then the process $(f \circ T^n)$ is Gaussian. The dynamical system generated by the process $(f \circ T^n)$ is then determined up to a metric isomorphism by σ, the spectral measure of the process. Except for the case of discrete spectrum and non-deep extensions (see [14]), this is the only result of spectral determination in ergodic theory.

It implies strong ergodic properties for these *Gaussian–Kronecker* automorphisms. Factors and self-joinings of such systems can be completely described [23]. In [15], we extend these properties and prove disjointness results for a wider class of Gaussian automorphisms, the "*GAG*" automorphisms, which include all Gaussian automorphisms with simple spectrum and thus mixing cases, but all the results there eventually rely on the Foiaş and Strătilă theorem.

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Our main result is its extension to the larger class of *algebraically independent Helson sets*. We also extend it to ergodic actions of locally compact second countable (*lcsc*) abelian groups.

Let $(X,\mathcal{B},\mu)$ denote a standard probability space and let $T=(T_g)_{g\in G}$ be an action of an lcsc abelian group G on X by measure-preserving automorphisms (for sake of simplicity, we use the same notation as for a single measure-preserving automorphism).

The Fourier transform of a complex Borel measure σ on the dual group $\Gamma=\widehat{G}$ is defined by $\widehat{\sigma}(g)=\int_\Gamma \gamma(g)\,d\sigma(\gamma)$ for $g\in G$, and the spectral measure σ_f of $f\in L^2(\mu)$ is the finite positive Borel measure on Γ given by

$$\widehat{\sigma}_f(g)=(f\circ T_g\mid f)\qquad (g\in G).$$

A closed set $K\subset\Gamma$ is a *Helson* set with constant α ($0<\alpha\le 1$), or a *Helson-*α set if, for every complex Borel measure σ on K,

$$\sup_{g\in G}|\widehat{\sigma}(g)|\ge\alpha\|\sigma\|,$$

where $\|\cdot\|$ denotes the total variation norm. An equivalent, more usual definition is that K is a Helson set if each continuous function on K, vanishing at infinity if Γ is not compact, is the restriction to K of the Fourier transform of an integrable function.

THEOREM 1. *Assume that $(T_g)_{g\in G}$ is ergodic and let f be a non-zero function in $L^2(\mu)$. If the spectral measure of f is continuous and concentrated on an algebraically independent Helson set, then f has a Gaussian distribution.*

Closed Kronecker sets are Helson-1 sets and are also algebraically independent [21, 16]. Whereas the Kronecker assumption implies that the process under consideration is *rigid*, i.e. such that there exists a subsequence (T^{n_j}) converging in measure to the identity, Theorem 1 allows us to get *mildly mixing* examples, but not yet strongly mixing examples.

We shall first prove a result where the assumption that σ_f is supported on a Helson set is replaced by an additional hypothesis on the moments of f.

DEFINITION. Given a positive measure μ, we denote by $\mathcal{C}(\mu)$ the class of all $f\in\bigcap_{2\le p<\infty}L^p(\mu)$ such that

$$\sum_{p=2}^{\infty}1/\|f\|_p=\infty.$$

For real random variables, this condition is known as Carleman's condition for the Hamburger moment problem.

THEOREM 2. *Assume that $(T_g)_{g\in G}$ is ergodic and let f be a non-zero function in $\mathcal{C}(\mu)$. If the spectral measure of f is continuous and concentrated on an algebraically independent compact set, then f has a Gaussian distribution.*

We recall that a subspace H of $L^2(\mu)$ is a *Gaussian subspace* if every non-zero function in H has a Gaussian distribution, and then the joint distribution of any finite family of functions in H is Gaussian.

COROLLARY 1. *Under the assumptions of Theorem* 1 *or of Theorem* 2, *the functions* $f \circ T_g$ $(g \in G)$ *span a Gaussian space. In case of a single measure-preserving automorphism* T, *the process* $(f \circ T^n)_{n\in\mathbb{Z}}$ *is Gaussian.*

Theorem 1 follows from Theorem 2 and from a result that may be of independent interest, which applies to any action of an lcsc abelian group acting by measure-preserving automorphisms:

THEOREM 3. *Let* K *be a compact Helson subset of* Γ. *Then, for* $2 \le p < \infty$, *the spectral projector* π_K *corresponding to* K *maps* $L^p(\mu)$ *into itself with*

$$\|\pi_K f\|_p \le Cp\|f\|_p \quad \text{for every } f \in L^p(\mu),$$

where C *depends only on the Helson constant of* K.

In particular, $\pi_K(L^\infty(\mu)) \subset \mathcal{C}(\mu)$.

The last theorems highlight the different roles of algebraic independence and harmonic properties of the support of the spectral measure. Under only the assumption that the spectral measure is continuous and supported by an independent compact set, which would yield mixing examples, the problem remains open and there is little hope to get a positive answer. However, under the weaker assumption that the Gaussian automorphism corresponding to the spectral measure has a simple spectrum, there are examples where the spectral determination property does not hold.

In Section 2, we show how to extend the proofs to the case of group actions. The proof of Theorem 2 is given in Section 3. Those of Theorems 3 and 1 follow in Section 4. Section 5 contains the examples mentioned above and a few additions.

1.2. Notation, definitions and preliminaries. We refer to [2, 18, 19] for basic definitions and results in ergodic theory and spectral theory of dynamical systems. For harmonic analysis, we refer to [21], and to [16] for definitions and properties of thin sets.

Here, the abelian group G acting on $(X, \mathcal{B}, \mu)$ is locally compact, second countable, and non-compact. For $g \in G$ we also denote by T_g the unitary operator $f \mapsto f \circ T_g$ on $L^2(\mu)$. We shall denote its dual group Γ multiplicatively and, for $\gamma \in \Gamma$ and $g \in G$, $\gamma(g)$ is also denoted $\langle \gamma, g\rangle$. In the case $G = \mathbb{Z}$ we identify $\Gamma = \mathbb{T}$ with $\mathbb{S}^1$.

The group generated by $K \subset \Gamma$ is denoted $\mathrm{Gp}(K)$. We shall need algebraic independence in the sense that if K_1, K_2 are two disjoint subsets of K, then $\mathrm{Gp}(K_1)$ and $\mathrm{Gp}(K_2) \setminus \{1\}$ are still disjoint; thus the definition of independence can be taken here in the weaker sense: when $\gamma_1, \dots, \gamma_k \in K$

and $n_1, \ldots, n_k \in \mathbb{Z}$, then $\gamma_1^{n_1} \times \cdots \times \gamma_k^{n_k} = 1$ only if $\gamma_1^{n_1} = \cdots = \gamma_k^{n_k} = 1$ ($n_1 = \cdots = n_k = 0$ is not required, and this allows finite order elements).

Let $\mathcal{B}(f)$ (resp. $\mathcal{B}(H)$) denote the sub-σ-algebra generated by a measurable function f (resp. a subset H of $L^2(\mu)$). If $f \in L^2(\mu)$, the closed invariant subspace of $L^2(\mu)$ generated by f is denoted $Z(f)$, so the factor generated by f is $\mathcal{B}(Z(f))$ (factors will be taken as invariant sub-σ-algebras).

By a *measure* σ on Γ we always mean a complex Borel measure on Γ. The support of σ, $\operatorname{supp}(\sigma)$, will always mean its *topological support*, i.e. the smallest closed set of Γ on which $|\sigma|$ is concentrated. We denote by $\widetilde{\sigma}$ the measure defined by $\widetilde{\sigma}(B) = \bar{\sigma}(B^{-1})$ for every Borel subset B of Γ; given $f \in L^2(\mu)$, the spectral measure of $\bar{f}$ is $\widetilde{\sigma}_f$ and it is concentrated on $\operatorname{supp}(\sigma_f)^{-1}$. If σ is the spectral type of T, defined up to equivalence of measures, the spectral representation yields an isometry $\varphi \mapsto \varphi(T)$ from $L^\infty(\sigma)$ onto a subalgebra of $\mathcal{L}(L^2(\mu))$ such that $\varphi(T)$ corresponds to multiplication by φ, and in particular each $g \in G$, taken as a character of Γ, corresponds to T_g.

Given a Borel subset K of Γ, $\mathbf{1}_K(T)$ is the spectral projector of $L^2(\mu)$ corresponding to K, which we denote by π_K. For $f \in L^2(\mu)$, we have $\overline{\pi_K f} = \pi_{K^{-1}} \bar{f}$.

In our proofs, the operation of $A(\Gamma)$ on the spaces $L^p(\mu)$ plays a major role. Recall that $A(\Gamma)$ denotes the Banach algebra of Fourier transforms of integrable functions, equipped with the norm inherited from the $L^1(G)$-norm. Given $\varphi \in A(\Gamma)$, the operator $\varphi(T)$ sends each $L^p(\mu)$ ($1 \le p \le \infty$) into itself, with norm $\|\varphi(T)\|_{\mathcal{L}(L^p(\mu))} \le \|\varphi\|_{A(\Gamma)}$, and thus it also preserves $\mathcal{C}(\mu)$.

Concerning Gaussian automorphisms, we shall use the definitions and notation of [15]. In particular, if σ is a continuous symmetric measure on $\mathbb{T}$, we denote by T_σ the Gaussian automorphism defined by the real Gaussian process of spectral measure σ. However, like Foiaş and Strătilă [9], we consider complex-valued Gaussian processes $(f \circ T^n)$, which will be more convenient.

Since an L^2-limit of Gaussian functions is still Gaussian, we have the following elementary lemma, which shows that it is sufficient to prove Theorem 1 in the case when σ_f has compact support:

Lemma 1. *Let $f \in L^2(\mu)$. If (K_n) is a sequence of Borel subsets of Γ such that $\sigma_f(\Gamma \setminus K_n) \to 0$ and each $\pi_{K_n} f$ is Gaussian, then f is Gaussian.*

Furthermore, for the proofs of Theorems 1 and 2, we can restrict ourselves to countable group actions. Indeed, let G_0 be a countable dense subgroup of G, endowed with the discrete topology. Then Γ is continuously embedded in the compact group $\Gamma_0 = \widehat{G}_0$, the spectral measures of f for the actions of G and G_0 being identified in this embedding. Clearly, if a set K in Γ is independent and compact in Γ, these properties still hold in Γ_0. The Fourier transform of a measure concentrated on K is continuous in the topology of G, hence by density of G_0 in G, directly from the definition, if K is Helson in Γ, the same holds in Γ_0.

So, *we henceforth assume that the abelian group G is countable and discrete, so that Γ is compact and metrizable.*

Note also that, given $\varepsilon > 0$, we can choose a Cantor subset K with $\sigma_f(\Gamma \setminus K) < \varepsilon$; indeed, this is a standard fact for finite positive measures on $\mathbb{T}^{\mathbb{N}}$ and the dual group of a countable abelian group is naturally embedded as a compact subgroup of $\mathbb{T}^n$. By Lemma 1, if we find that every such $\pi_K f$ is Gaussian, then f itself is Gaussian.

2. Zsido's theorem and abelian group actions

2.1. Spectral process. Let f be a non-zero function in $L^2(\mu)$ with a continuous spectral measure.

We consider first the case of a single measure-preserving automorphism T. Let $\gamma(t) = \exp(2\pi i t)$ for $t \in [0,1]$. The spectral process corresponding to f is defined by

$$f_t = \pi_{\gamma([0,t))} f \qquad (t \in [0,1]).$$

As in the proof of C. Foiaş and S. Strătilă, the main argument in the proofs of Theorems 1 and 2 will be to prove that this process has independent increments. As σ_f is continuous, $Z(f)$ is orthogonal to all eigenfunctions and in particular every f_t has zero mean. In order to deduce that (f_t) is a Gaussian process, it is then sufficient to know that it admits a version with a.s. continuous sample functions (see e.g. [5, Chap. VIII, Theorem 7.1]). The latter fact has been proved independently by L. Zsido [26]:

THEOREM (Zsido). *Assume that T is ergodic and that σ_f is continuous. If the spectral process (f_t) has independent increments, then it is Gaussian.*

For a countable abelian group action $(T_g)_{g\in G}$ we want to construct a process with similar properties.

The following lemma will apply when $K = \operatorname{supp} \sigma_f$ is a Cantor set. Then K is homeomorphic to a compact subset L of $[0,1]$. We can furthermore assume that L contains the endpoints 0 and 1. Choose a homeomorphism γ from L onto K and let

$$f_t = \pi_{\gamma([0,t)\cap L)} f \qquad (t \in [0,1]). \tag{1}$$

LEMMA 2. *Assume that the action (T_g) is ergodic, that σ_f is continuous with compact support K, and that γ is a homeomorphism of a compact set $L \subset [0,1]$ containing 0 and 1 onto K, and let (f_t) be defined by (1).*

Then, for every $t \in (0,1)$, the spectral measures σ_{f_t} and $\sigma_{f_1 - f_t}$ are concentrated on disjoint open sets of K. Moreover, if this process has independent increments, it admits a version with almost surely continuous sample functions and thus it is Gaussian.

Proof. The first assertion is immediate from the facts that

$$f_1 - f_t = \pi_{\gamma([t,1]\cap L)} f = \pi_{\gamma((t,1]\cap L)} f$$

since σ_f is continuous, and that $\gamma([0,t)\cap L)$ and $\gamma((t,1]\cap L)$ are disjoint open subsets of K.

For the second assertion, we proceed by slight modifications of Zsido's proof [26]. By the same classical argument of probability theory (see [5]), (f_t) has a version where for almost every x the sample function $t \mapsto (f_t(x))$ is right continuous with left limits (*càdlàg*) and thus has a jump $\Delta f_t(x)$ at every $t \in [0,1]$. Since the complement of L consists of countably many open intervals (t'_n, t''_n), we can moreover let $f_t(x) = f_{t'_n}(x)$ everywhere on each of these intervals, so that almost all sample functions are continuous on $[0,1] \setminus L$.

Hence, it remains to show that almost all sample functions are continuous on L. To this end we will show that, given $g \in G$, there is a set F of full measure in X such that for every $x \in F$,

$$\Delta f_t(T_g x) = \langle \gamma(t), g\rangle\, \Delta f_t(x) \quad \text{for every } t \in L. \tag{2}$$

Let $\eta > 0$. Since the map $t \mapsto \langle \gamma(t), g\rangle$ is continuous on L, we can find a finite subdivision $t_0 = 0 < t_1 < \cdots < t_k = 1$ of points in L such that $|\langle \gamma(t), g\rangle - \langle \gamma(t_j), g\rangle| < \eta$ on each $[t_{j-1}, t_j] \cap L$. Let then $f_j = f_{t_j} - f_{t_{j-1}} = \pi_{\gamma([t_{j-1},t_j)\cap L)} f$, $(1 \le j \le k)$. Since T_g corresponds in the spectral representation to the multiplication by $\langle \cdot, g\rangle$, it follows that

$$\|f_j \circ T_g - \langle \gamma(t_j), g\rangle f_j\|_2 \le \eta \|f_j\|_2,$$

hence the set N'_j of all $x \in X$ where

$$|f_j(T_g x) - \langle \gamma(t_j), g\rangle f_j(x)| \ge \eta^{1/2}$$

has measure $\le \eta \|f_j\|^2$.

Moreover, the set N''_j of all x where $|f_j(x)| \ge \eta^{-1/2}$ also has measure $\le \eta \|f_j\|^2$ and outside N''_j we have, for all $t \in [t_{j-1}, t_j]$,

$$|\langle \gamma(t_j), g\rangle f_j(x) - \langle \gamma(t), g\rangle f_j(x)| < \eta^{1/2}.$$

Let N_η be the union of all N'_j and N''_j $(1 \le j \le k)$. Since (f_t) has orthogonal increments,

$$\mu(N_\eta) \le 2\eta \sum_{j=1}^{k} \|f_j\|_2^2 = 2\eta \|f\|_2^2,$$

and, for each $x \notin N_\eta$ and every $t \in L$, if $t \in [t_{j-1}, t_j] \cap L$ then

$$|f_j(T_g x) - \langle \gamma(t), g\rangle f_j(x)| < 2\eta^{1/2}. \tag{3}$$

Let (η_n) be a sequence of positive reals converging to 0 and let F_g be the set of $x \in X$ where the sample functions are càdlàg and which belong to infinitely many sets $[0,1] \setminus N_{\eta_n}$. Then F_g has full measure, and if $x \in F_g$, then (3) holds for an infinite subsequence of (η_n) for every $t \in L$, with the corresponding $j = j_n(t)$. As $f_{j_n(t)}(x)$ converges pointwise to the jump $\Delta f_t(x)$, it follows that (2) holds for every $x \in F_g$.

Since G is countable, the invariant set $F = \bigcap_{g,g' \in G} T_g F_{g'}$ has full measure and if $x \in F$ then (2) holds for all g in G. It follows that, on F, the modulus $|\Delta f_t(x)|$ of the jump is invariant under the action of G for all $t \in L$. The rest of the proof is as in [26]: by ergodicity, in any given subinterval of $[0,1]$ the number of jumps of $t \mapsto f_t(x)$ with modulus $\ge \delta > 0$, as a function of x, must be a.e. constant on F, whence non-zero jumps can only occur at points $t \in [0,1]$ not depending on x, and thus correspond to eigenfunctions in the closed invariant subspace generated by f, which would contradict the hypothesis that σ_f is continuous. ■

2.2. A reduction. The following lemma expresses a criterion for f to be Gaussian, independently of the construction of a spectral process.

LEMMA 3. *Assume that (T_g) is ergodic and σ_f-continuous. In order for f to be Gaussian, it is sufficient that, for any disjoint open sets U and V in Γ, the factors generated by $\pi_U f$ and $\pi_V f$ are independent.*

Proof. Assume that this condition holds. By Lemmas 1 and 2, it is enough to show that, for each Cantor set K in Γ, the corresponding process (f_t) has independent increments. Now, for every $t \in (0,1)$, by the hypothesis and the first assertion of Lemma 2, the factors generated by f_t and $f_1 - f_t$ are independent. A priori, we have to show that for every finite sequence $t_0 = 0 < t_1 < \cdots < t_k = 1$ the variables $(f_{t_j} - f_{t_{j-1}})_{1 \le j \le k}$ are independent; but, given $0 < t' < t'' < 1$, we have $f_{t''} - f_{t'} \in Z(f_{t''}) \cap Z(f_1 - f_{t'})$, so the conclusion follows by induction. ■

Of course, by regularity of σ_f, the same condition with disjoint compact sets instead of open sets is also sufficient, and in fact it will hold for any pair of disjoint Borel sets. Notice that the condition for symmetric Borel sets is also necessary in an invariant Gaussian space, since two orthogonal real functions in a Gaussian space are independent.

3. Group property and Carleman's condition. We will now prove Theorem 2. We need the following result of Foiaş [7] (see also [8]), which states that the *topological supports* of spectral measures of locally compact group actions in ergodic theory satisfy a "group property":

THEOREM (Foiaş). *Let f_1, $f_2 \in L^2(\mu)$. If $f_1 f_2 \in L^2(\mu)$ then*

$$\operatorname{supp}(\sigma_{f_1 f_2}) \subset \operatorname{supp}(\sigma_{f_1}) \cdot \operatorname{supp}(\sigma_{f_2}).$$

Lemma 4. *Let $f \in \mathcal{C}(\mu)$ and let U be an open set in Γ. The spectral type of T restricted to the factor generated by $\pi_U f$ is concentrated on* $\mathrm{Gp}(U \cap \mathrm{supp}(\sigma_f))$.

Proof. By the Stone–Weierstrass theorem, the space of functions in $A(\Gamma)$ with compact support contained in U is dense in the space of all continuous functions on Γ vanishing outside U, so we can choose a sequence (φ_n) of functions in $A(\Gamma)$ with $\|\varphi_n\|_{A(\Gamma)} \le 1$ and $\mathrm{supp}(\varphi_n) \subset U$ which span a dense subspace of $L^2(\sigma_f|_U)$. Then $\mathrm{supp}(\sigma_{\varphi_n(T)f}) \subset U \cap \mathrm{supp}(\sigma_f)$ and, by the spectral isomorphism, the functions $(\varphi_n(T)f)$ span a dense subspace of $Z(\pi_U f) = \pi_U Z(f)$.

Let $(f_n)_{n\ge1}$ be the sequence of all functions $\mathrm{Re}\,\varphi_n(T)f$ and $\mathrm{Im}\,\varphi_n(T)f$, reordered. It is a sequence of real functions generating a dense subspace of $Z(\pi_U f) + Z(\overline{\pi_U f})$, so that $\mathcal{B}(Z(\pi_U f)) = \mathcal{B}(\{f_n\}_{n\ge1})$. For every $n \ge 1$, we have $\|f_n\|_p \le \|f\|_p$ for all $p \ge 2$, f_n belongs to $\mathcal{C}(\mu)$, and $\sigma_{f_n} \ll \sigma_f|_U + \widetilde{\sigma}_f|_{U^{-1}}$ so that $\mathrm{supp}(\sigma_{f_n}) \subset \mathrm{Gp}(U \cap \mathrm{supp}(\sigma_f))$.

Foiaş' theorem applies to powers of f_n and their finite products, which all belong to $L^2(\mu)$, and thus their spectral measures are all concentrated on $\mathrm{Gp}(U \cap \mathrm{supp}(\sigma_f))$; since the spectral measure of a sum is absolutely continuous with respect to the sum of the spectral measures of its terms, this remains true by linearity for all polynomials in the functions f_n.

The result will follow if we show that these polynomials are dense in $L^2(\mathcal{B}(\{f_n\}_{n\ge1}))$, and to prove this it is enough to show that for any $n \ge 1$ the polynomials in $f_1, \ldots, f_n$ are dense in $L^2(\mathcal{B}(\{f_j\}_{1\le j\le n}))$.

This is a classic result in the case of a single function satisfying Carleman's condition, and we just need to extend it. As in that case, we shall use quasi-analytic classes, for which we refer to W. Rudin's book [22, Chap. 19].

Fix $n \ge 1$ and let ν be the joint distribution of $f_1, \ldots, f_n$, so that, for any positive measurable function h on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} h(t_1, \ldots, t_n)\, d\nu(t_1, \ldots, t_n) = \int_X h(f_1(x), \ldots, f_n(x))\, d\mu(x).$$

Under the map $h \mapsto h \circ (f_1, \ldots, f_n)$, $L^2(\mathcal{B}(\{f_j\}_{1\le j\le n}))$ is isomorphic to $L^2(\nu)$, the f_j corresponding to the coordinate functions, so we have to show that the polynomials are dense in $L^2(\nu)$.

Let h be a function in $L^2(\nu)$ orthogonal to all polynomials, and consider the Fourier transform given on $\mathbb{R}^n$ by

$$\Phi(s_1, \ldots, s_n) = \int_{\mathbb{R}^n} \exp\Big(i \sum_{j=1}^n s_j t_j\Big) h(t_1, \ldots, t_n)\, d\nu(t_1, \ldots, t_n).$$

For every sequence $k_1, \dots, k_n$ of natural numbers,

$$\int_{\mathbb{R}^n} |t_1^{k_1} \cdots t_n^{k_n}|^2 \, d\nu(t_1, \dots, t_n) = \int_X |f_1^{k_1} \cdots f_n^{k_n}|^2 \, d\mu \le \prod_{j=1}^n \|f_j^{2k_j}\|_n \le \prod_{j=1}^n \|f\|_{2nk_j}^{2k_j},$$

and

$$\int_{\mathbb{R}^n} |t_1^{k_1} \cdots t_n^{k_n}| h(t_1, \dots, t_n) \, d\nu(t_1, \dots, t_n) \le \prod_{j=1}^n \|f\|_{2nk_j}^{k_j} \cdot \|h\|_2.$$

Hence Φ is C^∞.

We denote by $D^{k_1,\dots,k_n}\Phi$ the derivative $\partial^{k_1+\dots+k_n}\Phi/\partial^{k_1}s_1 \cdots \partial^{k_n}s_n$,

$$D^{k_1,\dots,k_n}\Phi(s_1, \dots, s_n) = \int_{\mathbb{R}^n} \prod_{j=1}^n (it_j)^{k_j} \exp\Big(i \sum_{j=1}^n s_j t_j\Big) h(t_1, \dots, t_n) \, d\nu(t_1, \dots, t_n).$$

All these derivatives vanish at $(0, \dots, 0)$ and we have the bound

$$|D^{k_1,\dots,k_n}\Phi(s_1, \dots, s_n)| \le \prod_{j=1}^n \|f\|_{2nk_j}^{k_j} \cdot \|h\|_2 \quad \text{for all } s_1, \dots, s_n. \tag{4}$$

Given j $(1 \le j \le n)$, for every choice of k_ℓ $(\ell > j)$ and s_ℓ $(\ell < j)$, we consider the one-variable function

$$s \to D^{0,\dots,0,k_{j+1},\dots,k_n} \, \Phi(s_1, \dots, s_{j-1}, s, 0, \dots, 0),$$

and the sequence of its derivatives. By (4), the L^∞-norm of its kth derivative is bounded by

$$\Big(\prod_{\ell=j+1}^n \|f\|_{2nk_\ell}^{k_\ell} \|h\|_2\Big) \cdot M_k,$$

where $M_k = \|f\|_{2nk}^k$. This means, according to the definition in [22], that these functions belong to the class $C\{M_k\}$.

Now, by standard application of the Hölder inequality, the sequence $(\|f\|_p)$ is non-decreasing and $(\|f\|_p^p)$ is logarithmically convex. Logarithmic convexity is inherited by the arithmetic subsequence $(\|f\|_{2nk}^{2nk})$ and then by $(\|f\|_{2nk}^k)$, and the assumption that f belongs to $\mathcal{C}(\mu)$ together with the monotonicity of $(\|f\|_p)$ implies that

$$\sum_{k=1}^\infty (1/M_k)^{1/k} = \sum_{k=1}^\infty 1/\|f\|_{2nk} = \infty.$$

It follows from the Denjoy–Carleman Theorem that the class $C\{M_k\}$ is quasi-analytic, and thus if one of these functions vanishes at 0 together with all its derivatives then it is identically 0.

We conclude by an easy induction. For $j = 1$, given any $k_2, \ldots, k_n$, all the functions $s \mapsto D^{k,k_2,\ldots,k_n}\, \Phi(s, 0, \ldots, 0)$ for $k \geq 0$ vanish at 0, whence $D^{0,k_2,\ldots,k_n}\, \Phi(s_1, 0, \ldots, 0) = 0$ for all s_1 and all $k_2, \ldots, k_n$. Now, for a given $j \geq 1$, if

$$D^{0,\ldots,0,k_{j+1},\ldots,k_n}\Phi(s_1, \ldots, s_j, 0, \ldots, 0) = 0 \text{ for all } s_1, \ldots, s_j \text{ and } k_{j+1}, \ldots, k_n,$$

then similarly all functions $s \mapsto D^{0,\ldots,0,k_{j+2},\ldots,k_n}\Phi(s_1, \ldots, s_j, s, 0, \ldots, 0)$ are identically 0 and the induction hypothesis remains true for $j + 1$.

Finally, Φ itself is identically 0 and this implies $h = 0$ ν-a.e. So, the null function is the only function in $L^2(\nu)$ orthogonal to all polynomials, whence polynomials are dense in $L^2(\nu)$, and the proof is complete. ∎

Proof of Theorem 2. Assume that T is ergodic, let f be a non-zero function in $\mathcal{C}(\mu)$ whose spectral measure is continuous and concentrated on an independent compact set, and let U and V be two disjoint open sets in Γ. By Lemma 4, the spectral types of T on the factors generated by $\pi|_U f$ and $\pi|_V f$ are concentrated on $\mathrm{Gp}(U \cap \mathrm{supp}(\sigma_f))$ and $\mathrm{Gp}(V \cap \mathrm{supp}(\sigma_f))$, respectively. Since $\mathrm{supp}(\sigma_f)$ is independent, these groups have no common element other than 1. So, these factors are spectrally disjoint and a fortiori independent. The conclusion then follows from Lemma 3. ∎

4. The spectral projector on a Helson set. To prove Theorem 3 we need to approximate the indicator function of a compact Helson set by functions in $A(\Gamma)$ (here, we release the assumption that Γ itself is compact). The main tool is Drury's lemma, which was used to solve the problem of the union of two Sidon sets [6], and that of the union of two Helson sets by N. Varopoulos [25, 24]. We quote it in the version given by C. Herz [10] with a more convenient estimate of the norm of the functions obtained in $A(\Gamma)$.

Definition. For $0 < \varepsilon \leq 1$, let

$$\omega(\varepsilon) = \sup_{n\geq 1} \omega_n(\varepsilon),$$

where, denoting by E_n the canonical basis of $\mathbb{Z}^n$,

$$\omega_n(\varepsilon) = \inf\{\|\psi\|_{A(\mathbb{Z}^n)} : \psi \in A(\mathbb{Z}^n),\ \psi|_{E_n} = 1,\ |\psi| \leq \varepsilon \text{ on } \mathbb{Z}^n \setminus E_n\}.$$

Theorem (Drury, Varopoulos, Herz). *Let K be a compact Helson-α set of Γ. For each closed set F of Γ disjoint from K, for every $\varepsilon \in (0, 1]$ and every $\beta < \alpha^2$, there exists a function φ in $A(\Gamma)$ such that*

$$\varphi = 1 \text{ on } K, \quad |\varphi| \leq \varepsilon \text{ on } F \quad \text{and} \quad \|\varphi\|_{A(\Gamma)} \leq \beta^{-1}\omega(\beta\varepsilon).$$

REMARK. This is essentially [10, Theorem 2], except that it only states that there exists a function $\omega : (0,1] \to [1,\infty)$ with that property. Herz defines ω later on (after the statement of Proposition 1) as above, and he shows that it is suitable for the theorem. Also, the Helson constant is inverted there.

However, the estimates of $\omega(\varepsilon)$ given in [10] do not seem sufficient to prove Theorem 3. A little later, in a paper [17] on a slightly different problem, J.-F. Méla gave indirectly a nearly optimal bound.

THEOREM (Méla). *For all ε in $(0,1/2]$,*

$$\omega(\varepsilon) \le 2|\log\varepsilon| + 6.$$

Proof. As this result is not explicitly stated in [17], we explain briefly how to deduce it. Fix an arbitrary integer $n \ge 1$, and consider the construction in Section 7 when the group G is $\mathbb{T}^n$ and its dual is $\mathbb{Z}^n$. The measures denoted by ν_s in [17] are then the finite Riesz products on $\mathbb{T}^n \times \mathbb{T}$ admitting as density with respect to the Lebesgue measure the positive trigonometric polynomials

$$Q_s(z_1,\ldots,z_n,z) = \prod_{j=1}^{n}(1 + s(z_j z + \bar{z}_j \bar{z})), \quad s \in (0,1/2],$$

with $\|Q_s\|_{L^1(\mathbb{T}^{n+1})} = 1$ for all s.

For $s \in (0,1/2]$, the measure μ_s on $\mathbb{T}^n$ is then defined by $\widehat{\mu}_s(k_1,\ldots,k_n) = \widehat{\nu}_s(k_1,\ldots,k_n,1)$. It admits as density the factor P_s of $\bar{z}$ in the expansion of Q_s. On the canonical basis, its coefficients are all equal to s, all its other non-zero coefficients are odd powers s^{2k+1} of s with $k \ge 1$, and we still have $\|P_s\|_{L^1(\mathbb{T}^n)} \le 1$.

Now, the main idea is to construct, given $\varepsilon > 0$, a measure σ on $(0,1/2]$ of norm as small as possible with $\int s\,d\sigma = 1$ and $|\int s^{2k+1}\,d\sigma| \le \varepsilon$ for all $k \ge 1$. This is achieved by [17, Lemma 3], where J.-F. Méla shows that we can obtain $\|\sigma\| \le 2|\log\varepsilon| + 6$ (taking $a = \log 2 - 1/2$ in the bound given in [17]).

Then, integrating P_s with respect to σ, we get a trigonometric polynomial P on $\mathbb{T}^n$ whose Fourier transform ψ on $\mathbb{Z}^n$ satisfies $\psi = 1$ on E_n, $|\psi| \le \varepsilon$ elsewhere, and

$$\|\psi\|_{A(\mathbb{Z}^n)} = \|P\|_{L^1(\mathbb{T}^n)} \le \int \|P_s\|_{L^1(\mathbb{T}^n)}\,d|\sigma| \le 2|\log\varepsilon| + 6.$$

It follows that $\omega_n(\varepsilon) \le 2|\log\varepsilon| + 6$ for all $n \ge 1$. ■

REMARK. We also mention that this result by J.-F. Méla was used for a different problem in [13]. The bound is nearly optimal since it can also be deduced from [17] that $\omega(\varepsilon)/|\log\varepsilon|$ is bounded from below by a positive constant.

Proof of Theorem 3. Let a compact Helson set $K \subset \Gamma$ be given and denote by σ the spectral type of T.

A remarkable fact in the results by S. Drury, N. Varopoulos and C. Herz is that the bound for $\|\varphi\|_{A(\Gamma)}$ does not depend on the set F disjoint from K. Thanks to Méla's result, for $\varepsilon \le 1/2$, this bound is less than $c|\log \varepsilon|$, where c is a constant depending only on the Helson constant of K.

If we fix $\varepsilon \in (0, 1/2]$ and appply this to a non-decreasing sequence (F_n) of closed sets whose union is $\Gamma \setminus K$, we get a sequence $(\varphi_{n,\varepsilon})$ of functions with $A(\Gamma)$-norms $\le c|\log \varepsilon|$, hence also bounded in $L^\infty(\sigma)$. By extracting a subsequence if needed, we may assume that it converges to some function φ_ε in the weak* topology of the duality $(L^1(\sigma), L^\infty(\sigma))$. Then the operator $\varphi_\varepsilon(T)$ on $L^2(\mu)$ is the weak limit of the sequence $(\varphi_{n,\varepsilon}(T))$ and

$$\varphi_\varepsilon = 1 \ (\sigma|_K)\text{-a.e.} \quad \text{and} \quad |\varphi_\varepsilon| \le \varepsilon \ (\sigma|_{\Gamma \setminus K})\text{-a.e.}$$

For $2 \le p \le \infty$, as each $\varphi_{n,\varepsilon}(T)$ maps $L^p(\mu)$ into itself with operator norm $\le c|\log \varepsilon|$, we still find that $\varphi_\varepsilon(T)$ maps $L^p(\mu)$ into itself, and

$$\|\varphi_\varepsilon(T)\|_{\mathcal{L}(L^p(\mu))} \le c|\log \varepsilon|. \tag{5}$$

Let (ε_k) be a decreasing and summable sequence in $(0, 1/2]$. As $|\mathbf{1}_K - \varphi_{\varepsilon_k}| \le \varepsilon_k$ σ-a.e., we may write

$$\mathbf{1}_K = \varphi_{\varepsilon_1} + \sum_{k=1}^{\infty} (\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}) \ \sigma\text{-a.e.},$$

where the series converges in the $L^\infty(\sigma)$-norm, and the corresponding series

$$\varphi_{\varepsilon_1}(T) + \sum_{k=1}^{\infty} (\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)) \tag{6}$$

converges to π_K in $\mathcal{L}(L^2(\mu))$.

Moreover, for all $k \ge 1$, as $|\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}| \le 2\varepsilon_k$ σ-a.e.,

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^2(\mu))} = \|\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}\|_{L^\infty(\sigma)} \le 2\varepsilon_k \tag{7}$$

and, by (5),

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^\infty(\mu))} \le 2c|\log \varepsilon_{k+1}|. \tag{8}$$

Now, for $p \in [2, \infty)$, we have, by (7), (8) and the Riesz–Thorin interpolation theorem,

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^p(\mu))} \le (2\varepsilon_k)^{2/p} (2c|\log \varepsilon_{k+1}|)^{1-2/p}.$$

With $\varepsilon_k = \mathrm{e}^{-kp}$, we get

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^p(\mu))} \le 2^{2/p} \mathrm{e}^{-2k} \cdot 2c(k+1)p \le 4c(k+1)e^{-2k} p.$$

It follows that the series (6) converges in $\mathcal{L}(L^p(\mu))$. As $L^p(\mu) \subset L^2(\mu)$, this proves that π_K maps $L^p(\mu)$ into itself, and that

$$\|\pi_K\|_{\mathcal{L}(L^p(\mu))} \le cp + 4c\Big(\sum_{k=1}^{\infty}(k+1)\mathrm{e}^{-2k}\Big)p \le Cp,$$

where C is a constant depending only on the Helson constant of K.

The last assertion of the theorem is an immediate consequence. ∎

Proof of Theorem 1. Assume again that G is discrete, Γ is compact, and T is ergodic. Let f be a non-zero function in $L^2(\mu)$ whose spectral measure is continuous and concentrated on an independent Helson set K of Γ.

By Theorem 3, π_K maps $L^\infty(\mu)$ into $\mathcal{C}(\mu)$ and thus there is a dense subspace of $\pi_K(L^2(\mu))$ consisting of functions in $\mathcal{C}(\mu)$. When K does not contain any eigenvalue of T, these functions have continuous spectral mesures and so by Theorem 2 have a Gaussian distribution. It then follows that every function in $\pi_K(L^2(\mu))$ is Gaussian. Otherwise, as the set of eigenvalues is countable and σ_f is continuous, we can choose a sequence (K_n) of closed subsets of K which do not contain any eigenvalue, such that $\sigma_f(K\setminus K_n) \to 0$. Then each K_n is still a Helson set, so each $\pi_{K_n} f$ is Gaussian and it follows again that f is Gaussian. ∎

Proof of Corollary 1. We have to prove that $Z(f)$ is a Gaussian space, that is, that every non-zero function in $Z(f)$ has a Gaussian distribution. Under the assumptions of Theorem 1 or Theorem 2, every function h in $Z(f)$ has a continuous spectral measure concentrated on $K = \mathrm{supp}(\sigma_f)$. If K is a Helson set, it follows directly from Theorem 1 that h has a Gaussian distribution. If $f \in \mathcal{C}(\mu)$, we can apply Theorem 2 to functions $\varphi(T)f$ with $\varphi \in A(\Gamma)$, which all belong to $\mathcal{C}(\mu)$, and the result follows since these functions are dense in $Z(f)$. ∎

5. Complements

5.1. Foiaş and Strătilă measures and sets. We say that a positive continuous measure σ on Γ is an *FS measure* if, whenever the measure-preserving action $(T_g)_{g\in G}$ on $(X, \mathcal{B}, \mu)$ is ergodic and f is a complex function in $L^2(\mu)$ with $\sigma_f = \sigma$, then f is Gaussian; and we say that a Borel subset K of Γ is an *FS set* if every positive continuous measure concentrated on K is an FS measure. For symmetric sets or measures, these definitions match the definitions of [14, 15].

Recall that a compact subset K of Γ is a *Kronecker* set if every continuous function of modulus 1 on K is a uniform limit of characters. By Lemma 1 (see also [14]), any positive measure σ on Γ concentrated on a non-decreasing union of FS sets is an FS measure. It follows that the Foiaş and Strătilă theorem still holds for a *weak Kronecker set* in $\mathbb{T}$, that is, a closed subset K

such that, given any finite positive Borel measure σ on K, every continuous function of modulus 1 on K is a limit in σ-measure of characters, or equivalently such that any finite positive Borel measure on K is concentrated on a non-decreasing union of Kronecker sets [14, Lemma 1].

In the same way, the assumption on σ_f in Theorem 1 can be weakened: it is sufficient for σ_f to be continuous and concentrated on a non-decreasing union of independent Helson sets.

Now, a weak Kronecker set is a Helson-1 set and conversely a Helson-1 set is a translate of a weak Kronecker set [16, Chap. XIII]. Moreover, a weak Kronecker set is independent (in the strong sense). So, Theorem 1 extends the Foiaş and Strătilă theorem, and the actual extension consists in covering the case of Helson-α sets with $0 < \alpha < 1$.

Let us also recall the following result of [14] and [15] for measures on $\mathbb{T}$, which shows that in this case σ_f need not be concentrated on an independent set:

PROPOSITION.

(1) *If σ is an FS measure and $0 < \tau \ll \sigma$, then τ is an FS measure.*
(2) *Assume that σ_1 and σ_2 are mutually singular symmetric FS measures. Then $\sigma_1 + \sigma_2$ is a FS measure if and only if σ_2 is singular with respect to each translate of σ_1.*

REMARK. The only difficult point in this proposition is the "if" part of (2) [15, Corollary 10]: it is easy to see that $\sigma_1 + \sigma_2$ is FS if and only if T_{σ_1} and T_{σ_2} are disjoint but the result then requires the characterization of disjointness for GAG automorphisms established in [15].

5.2. Mildly mixing example. The Helson hypothesis forbids mixing for the corresponding Gaussian systems, since the upper bound $\sup_{g\in G} |\widehat{\sigma}(g)|$ in the definition of a Helson set K can be replaced by the upper limit at infinity (with a different constant [16, Chap. I, Prop. 5.2]). When $\alpha = 1$, in particular when K is a Kronecker set, it is easy to see that, given any positive Borel measure σ on K, there are non-trivial sequences of characters converging to 1 in $L^1(\sigma)$ (i.e. K is a *weak Dirichlet set*), which, in our context, implies rigidity: there are sequences (T_{g_j}), with $g_j \to \infty$, converging to the identity on the factor generated by f. In [14] non-rigid examples of FS sets are constructed by considering independent unions of Kronecker sets, but then the Gaussian system is generated by rigid factors.

On the contrary, Theorem 1 allows mild mixing, that is, absence of non-trivial rigid factors, and even a spectral form of partial mixing (usual partial mixing never occurs for Gaussian automorphisms which are not strongly mixing).

Indeed, by a result of T. Körner [11] (see [16, Chap. XIII, Theorem 3.14]), for each α in $(0,1)$, there exists a finite positive measure σ, concentrated on an independent Helson-α set K of $\mathbb{T}$, with the property

$$\text{(9)} \qquad \textit{for every Borel set } B \subset \mathbb{T}, \qquad \limsup |\widehat{\sigma|_B}(n)| = \alpha\sigma(B).$$

Then σ is continuous, since $\alpha < 1$, and $\widetilde{\sigma}$ shares the same property.

Consider the Gaussian automorphism $T_{\sigma+\widetilde{\sigma}}$, with spectral measure $\sigma_H = \sigma + \widetilde{\sigma}$ on its Gaussian space H. The spectral type of T on the kth chaos $H^{(k)}$ is the kth convolution power σ_H^{*k}, and the spectral type on the factor $\mathcal{B}(H)$ is the convolution exponential $\exp(\sigma_H) = \sum_{k\geq 0} \sigma_H^{*k}$. It follows from (9) that, for every positive integer k and every positive measure $\tau \ll \sigma_H^{*k}$, $\limsup |\widehat{\tau}(n)| \leq \alpha^k \|\tau\|$ (by density, it is sufficient to check this inequality for τ of the form $\sigma_1|_{B_1} * \cdots * \sigma_k|_{B_k}$ where $\sigma_j = \sigma$ or $\sigma_j = \widetilde{\sigma}$ and B_j is a Borel subset of $\mathbb{T}$, for $1 \leq j \leq k$), and therefore $\limsup |\widehat{\tau}(n)| \leq \alpha \|\tau\|$ for every continuous positive measure $\tau \ll \exp(\sigma_H)$.

So, we obtain:

Corollary 2. *For every α in $(0,1)$ there exists an FS measure σ on $\mathbb{T}$ such that the Gaussian automorphism $T = T_{\sigma+\widetilde{\sigma}}$ satisfies the property: for every square integrable zero-mean function h,*

$$\limsup |(T^n h \mid h)| = \limsup |\widehat{\sigma}_h(n)| \leq \alpha \|\sigma_h\| = \alpha \|h\|_2^2,$$

and in particular T is mildly mixing.

5.3. Independence in measure. From now on, it will be more convenient to restrict ourselves to the action of a single automorphism, so we take $G = \mathbb{Z}$, and $\Gamma = \mathbb{T}$ (identified with $\mathbb{S}^1$).

A natural question is whether the assumption of support independence in Theorem 1 can be replaced by a notion of independence "in measure". Such a property appears in the spectral analysis of Gaussian automorphisms. If σ is a continuous symmetric measure on $\mathbb{T}$, then T_σ has simple spectrum iff the following condition holds:

(10) *For all $n \geq 1$, there exists a set of full $\sigma^{\otimes n}$-measure on which the product map $(z_1, \ldots, z_n) \mapsto z_1 \cdots z_n$ is one-to-one modulo coordinate permutations.*

Indeed, this is equivalent to saying that for every $n \geq 1$ the cartesian power $T_\sigma^{\otimes n}$ restricted to the sub-σ-algebra consisting of sets invariant under coordinate permutations has simple spectrum, hence that T_σ restricted to the nth chaos has simple spectrum, and it also implies that the convolution powers of σ are mutually singular (see e.g. [12]), i.e., the chaoses are spectrally disjoint.

In [1], O. Ageev proved property (10) for the reduced spectral type (i.e. the spectral type restricted to zero-mean functions) of a class of rank 1 transformations, including the classical Chacon transformation, and it is clear that $L^2(\mu)$ contains zero-mean functions which are not Gaussian, e.g. zero-mean bounded functions. So, Theorem 1 fails if the only assumptions are that T be ergodic and σ_f be a continuous measure satisfying (10).

In the other direction, Theorem 1 implies that the spectral type of the Chacon transformation cannot be concentrated on an independent Helson set. But we do not know if it is concentrated on some independent set, and we leave open the question whether Theorem 1 is valid if we only assume that σ_f is continuous and concentrated on an independent set. We do not know either if the result holds under the assumption that σ_f is continuous, satisfies (10) and is concentrated on a Helson set.

5.4. Poisson suspensions. Poisson suspensions are particularly interesting in our context, because they appear together with Gaussians systems in the theory of processes with independent increments, and their spectral properties are similar. We briefly recall the results already discussed by E. Roy [20], to which we refer for a detailed exposition (see also [4]).

Let T be a measure-preserving automorphism of a standard σ-finite measure space $(X, \mathcal{B}, \mu)$, where there is no invariant set $E \in \mathcal{B}$ with $0 < \mu(E) < \infty$. This assumption is equivalent to saying that its spectral type σ, which can be assumed to be symmetric, is continuous. Then its Poisson suspension T_* is ergodic and spectrally isomorphic to the Gaussian automorphism T_σ.

More precisely, its L^2-space has a similar decomposition in an orthogonal sum $\bigoplus_{n\geq 0} H^{(n)}$ of chaoses where, up to a normalizing constant, $H^{(n)}$ is isometric to the tensor product $L^2(\mu)^{\otimes n}$ restricted to functions invariant under coordinate permutations, the action of T_* on $H^{(n)}$ being conjugate to $T^{\otimes n}$ and thus admitting the convolution power σ^{*n} as spectral type. The spectral isomorphism with T_σ sends each chaos $H^{(n)}$ of the suspension onto the corresponding chaos of the Gaussian system.

Poisson suspensions may have simple spectrum. Ageev's proof can be extended to some infinite measure-preserving transformations obtained by *cutting and stacking*, and such systems never have invariant sets of finite positive measure. In [3], A. Danilenko and V. Ryzhikov construct examples where moreover the Poisson suspension is mixing.

We thus obtain Poisson suspensions T_* with simple spectrum whose spectral type restricted to the chaos $H^{(1)}$ is the spectral type σ of T. However, E. Roy proves a disjointness property between Gaussian systems and Poisson suspensions [20, Prop. 4.11]. In particular, no function from the chaos $H^{(1)}$ can have a Gaussian distribution, so σ cannot be an FS measure.

More generally, the spectral type of a σ-finite measure-preserving automorphism T must be singular to every positive FS measure whenever there is no invariant set $E \in \mathcal{B}$ with $0 < \mu(E) < \infty$ [20, Theorem 4.13]. We have the following consequence:

COROLLARY 3. *Let T be a measure-preserving automorphism of a space $(X, \mathcal{B}, \mu)$ of σ-finite measure with no invariant set of non-zero finite measure, and let σ be its spectral type. Then $\sigma(K) = 0$ for every independent Helson set $K \subset \mathbb{T}$.*

REMARK. The last results can easily be extended to actions of locally compact second countable groups.

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