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Singular parabolic problems in the half-space

by

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Abstract. We study elliptic and parabolic problems governed by singular elliptic operators

$$\mathcal{L}=\sum_{i,j=1}^{N+1} q_{ij}D_{ij}+\frac{c}{y}D_y$$

in the half-space $\mathbb{R}^{N+1}_+=\{(x,y):x\in\mathbb{R}^N,\ y>0\}$ under Neumann boundary conditions at $y=0$. More general operators and oblique derivative boundary conditions are also considered.

1. Introduction. In this paper we study solvability and regularity of elliptic and parabolic problems associated to the degenerate operators

$$\mathcal{L}=\sum_{i,j=1}^{N+1} q_{ij}D_{ij}+\frac{c}{y}D_y \quad \text{and} \quad D_t-\mathcal{L} \tag{1}$$

in the half-space $\mathbb{R}^{N+1}_+=\{(x,y):x\in\mathbb{R}^N,\ y>0\}$ under Neumann boundary conditions at $y=0$. Here $c\in\mathbb{R}$ and $Q=(q_{ij})_{i,j=1}^{N+1}$ is a constant, real, positive definite matrix.

The special case where

$$\mathcal{L}=\Delta_x+B_y, \qquad B_y=D_{yy}+\frac{c}{y}D_y \tag{2}$$

has been extensively studied in [21]. In this situation B_y is a one-dimensional Bessel operator and Δ_x and B_y commute. These operators play a major role in the investigation of the fractional powers $(-\Delta_x)^s$ and $(D_t-\Delta_x)^s$, $s=(1-c)/2$, through the "extension procedure" of Caffarelli and Silvestre [4]. Nowadays this method has been extended to more general situations, first

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in [29] for self-adjoint operators, then in [13] for generators of semigroups, and in [3] with precise regularity results in the Hilbertian case.

The operators $\mathcal{L}$ in the general form (1) have been considered in [9, 11] and, in particular in [8] with $|c| < 1$ and in [10] for $c > -1$, as in our assumptions. The authors show solvability and regularity of related elliptic and parabolic problems in weighted L^p spaces, even for variable coefficients satisfying a VMO condition, using tools from linear PDEs and Muckhenoupt weights.

In this paper we use semigroup theory and operator-valued harmonic analysis to prove similar results in weighted L^p spaces, where the weight is a power y^m, $m \in \mathbb{R}$. Even though the most important cases are $m = 0$ and $m = c/\gamma$ where $\gamma = q_{N+1,N+1}$, which correspond to the Lebesgue measure and the symmetrizing measure, our methods work for all m satisfying $0 < (m+1)/p < c+1$.

In the language of semigroup theory, we prove that $\mathcal{L}$, endowed with Neumann boundary conditions generates an analytic semigroup in $L^p_m = L^p(\mathbb{R}^{N+1}_+; y^m dxdy)$; we characterize its domain as a weighted Sobolev space and show that it has maximal regularity, which means that both $D_t v$ and $\mathcal{L}v$ have the same regularity as $(D_t - \mathcal{L})v$. In comparison to [10], we prove the solvability of the problem $\lambda u - \mathcal{L}u = f$ also for complex λ.

The operator $\mathcal{L}$ is a model case of a more general situation which we explain below and which will be treated in a subsequent paper.

Let Ω be a smooth open bounded set in $\mathbb{R}^k$ and let $d := \text{dist}(\,\cdot\,, \partial\Omega)$ be the distance from $\partial\Omega$. Let us consider operators of the form

$$\mathcal{L} = \text{tr}(A(z)D^2) + \text{dist}(z, \partial\Omega)^{-1}\,(b(z), \nabla)\,, \qquad z \in \Omega,$$

under Neumann boundary condition at $\partial\Omega$. Here $A = (a_{ij})$ is uniformly elliptic and $A, b \in C\left(\overline{\Omega}\right)$. Standard localization and freezing of the coefficients reduce the last operator to the form (1) in the half-space $\mathbb{R}^{N+1}_+$. However, the form (2) is not sufficient, since mixed derivatives and oblique boundary conditions appear in the localization procedure, unless heavy restrictions are assumed.

Unfortunately, dealing with such additional terms is a serious complication, one important reason being the loss of commutativity. Indeed, generation properties and kernel estimates are easily proved for $\Delta_x + B_y$ from the same properties of the commuting blocks Δ_x and B_y, and this strategy clearly fails for operators like (1).

Moreover, the general case cannot be reduced to the form (2) by a linear change of variables, as is usually done for second order equations with constant coefficients. In fact, a linear change of variables which transforms $\sum_{ij} q_{ij} D_{ij}$ into $\Delta_x + D_{yy}$ acts also on $\frac{c}{y} D_y$, introducing an additional singular term like $\frac{a}{y}\cdot\nabla_x$, $a\in\mathbb{R}^N$, and changing the Neumann boundary condition (6) to the oblique one (47); see Section 8.

We prove both elliptic and parabolic estimates

$$\|D_{x_ix_j}u\|_{L^p_m}+\|D_{x_iy}u\|_{L^p_m}+\|D_{yy}u\|_{L^p_m}+\|y^{-1}D_yu\|_{L^p_m}\le C\|\mathcal{L}u\|_{L^p_m}, \tag{3}$$

and

$$\|D_tu\|_{L^p_m}+\|\mathcal{L}u\|_{L^p_m}\le C\|(D_t-\mathcal{L})u\|_{L^p_m}, \tag{4}$$

where the L^p norms are taken over $\mathbb{R}^{N+1}_+$ and on $(0,\infty)\times\mathbb{R}^{N+1}_+$ respectively.

In order to obtain (3) and (4) and show solvability, we use tools from vector-valued harmonic analysis; let us explain the main ideas.

Assume, without loss of generality as explained later, that

$$\mathcal{L}u=\Delta_xu+2a\cdot\nabla_xD_yu+B_yu=f.$$

Taking the Fourier transform with respect to x we obtain

$$-|\xi|^2\hat{u}(\xi,y)+2ia\cdot\xi D_y+B_y\hat{u}(\xi,y)=\hat{f}(\xi,y)$$

and so

$$\begin{aligned}|\xi|^2\hat{u}(\xi,y)&=-|\xi|^2(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}\hat{f}(\xi,y),\\ \xi D_y\hat{u}(\xi,y)&=\xi D_y(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}\hat{f}(\xi,y).\end{aligned}$$

Denoting by $\mathcal{F}$ the Fourier transform with respect to x we get

$$\begin{aligned}\Delta_x\mathcal{L}^{-1}&=-\mathcal{F}^{-1}\left(|\xi|^2(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}\right)\mathcal{F},\\ \nabla_xD_y\mathcal{L}^{-1}&=\mathcal{F}^{-1}D_y\left(i\xi(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}\right)\mathcal{F}.\end{aligned}$$

The boundedness of $\Delta_x\mathcal{L}^{-1}$ is equivalent to that of the multiplier

$$\mathbb{R}^N\ni\xi\mapsto|\xi|^2(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}$$

in $L^p(\mathbb{R}^N;L^p_m(0,\infty))=L^p_m$. Similarly, the boundedness of $\nabla_xD_y\mathcal{L}^{-1}$ is equivalent to that of the multiplier

$$\mathbb{R}^N\ni\xi\mapsto\xi D_y(|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1}.$$

To prove solvability, one more multiplier is needed, namely

$$\mathbb{R}^N\ni\xi\mapsto\lambda(\lambda+|\xi|^2-2ia\cdot\xi D_y-B_y)^{-1},$$

where λ belongs to a sector in the complex plane.

The proof of the boundedness of these multipliers uses a vector-valued Mikhlin multiplier theorem, which rests on square function estimates. The strategy for proving (4) is similar after taking the Fourier transform with respect to t.

We refer to [16, 7] and the new books [14, 15] for this approach, which we recall in Section 2.

A first crucial step consists in the study of the one-dimensional operator $L = B_y + ibD_y$, which is of independent interest. Since $b = 2a \cdot \xi$ in the multipliers above, we need precise dependence on b in all estimates.

The paper is organized as follows. In Section 2 we briefly recall the tools of vector-valued harmonic analysis we need.

In Sections 3 and 4 we recall some results concerning weighted Sobolev spaces and the one-dimensional Bessel operator B_y.

In Section 5 we define the $1d$-operator L through a quadratic form in $L^2((0,\infty), y^c\,dy)$. We prove heat kernel estimates for real times by domination and then we extend them to complex times, via Davies–Gaffney estimates.

In Section 6 we prove the boundedness of the multipliers introduced above.

In Section 7, which is the core of the paper, we prove generation results, maximal regularity and domain characterization for the operator $\mathcal{L}$, under Neumann boundary conditions.

Finally, in Section 8, we extend our results by considering operators of the form

$$\mathcal{L} = \sum_{i,j=1}^{N+1} q_{ij} D_{ij} + \frac{c}{y} D_y + \frac{a \cdot \nabla_x}{y}$$

and oblique derivative boundary conditions.

Notation. For $N \geq 0$, $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N,\ y > 0\}$. We write $\nabla u, D^2 u$ for the gradient and the Hessian matrix of a function u with respect to all x, y variables and $\nabla_x u, D_y u, D_{x_i x_j} u, D_{x_i y} u$ etc. to distinguish the role of x and y.

For $m \in \mathbb{R}$ we consider the measure $y^m dx dy$ in $\mathbb{R}^{N+1}_+$ and we write $L^p_m(\mathbb{R}^{N+1}_+)$, and often only L^p_m when $\mathbb{R}^{N+1}_+$ is understood, for the weighted space $L^p(\mathbb{R}^{N+1}_+; y^m dx dy)$.

Similarly, $W^{k,p}_m = W^{k,p}_m(\mathbb{R}^{N+1}_+) = \{u \in L^p_m : \partial^\alpha u \in L^p_m,\ |\alpha| \leq k\}$. We use $\hat{C}^k([0,\infty))$ for the space of uniformly continuous, k-times differentiable functions on $[0,\infty)$, tending to zero at infinity with all derivatives.

$\mathbb{C}^+$ stands for $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and, for $|\theta| \leq \pi$, we denote by Σ_θ the open sector $\{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\operatorname{Arg}(\lambda)| < \theta\}$.

Given $a, b \in \mathbb{R}$, $a \wedge b$ and $a \vee b$ denote their minimum and maximum. We write $f(x) \simeq g(x)$ for x in a set I and positive f, g, if for some $C_1, C_2 > 0$,

$$C_1\, g(x) \leq f(x) \leq C_2\, g(x), \quad x \in I.$$

2. Vector-valued harmonic analysis. Regularity properties for $\mathcal{L}$ and $D_t - \mathcal{L}$ follow once we prove the estimates

$$\|D^2u\|_p \le C\|\mathcal{L}u\|_p, \quad \|D_t u\|_p + \|D^2u\|_p \le C\|(D_t - \mathcal{L})u\|_p, \tag{5}$$

for u in an appropriate Sobolev space; this is equivalent to saying that $D^2\mathcal{L}^{-1}$ and $D_t(D_t - \mathcal{L})^{-1}$ are bounded operators. We use the strategy which arose first in the study of maximal regularity of parabolic problems, that is, for the equation $u_t = Au + f$, $u(0) = 0$, where A is the generator of an analytic semigroup on a Banach space X. Estimates like

$$\|u_t\|_p + \|Au\|_p \le \|f\|_p$$

are equivalent to the boundedness of the operator $A(D_t - A)^{-1}$.

This strategy relies on Mikhlin's vector-valued multiplier theorems which we now recall here, referring the reader to [7, 28, 16] for the proofs.

Let $\mathcal{S}$ be a subset of $B(X)$, the space of all bounded linear operators on a Banach space X. Then $\mathcal{S}$ is $\mathcal{R}$*-bounded* if there is a constant C such that

$$\Big\|\sum_i \varepsilon_i S_i x_i\Big\|_{L^p(\Omega;X)} \le C\Big\|\sum_i \varepsilon_i x_i\Big\|_{L^p(\Omega;X)}$$

for every finite sum as above, where $(x_i) \subset X$, $(S_i) \subset \mathcal{S}$ and $\varepsilon_i : \Omega \to \{-1, 1\}$ are independent and symmetric random variables on a probability space Ω. In particular, $\mathcal{S}$ is a bounded subset of $B(X)$. The smallest constant C for which the above definition holds is the $\mathcal{R}$*-bound* of $\mathcal{S}$, denoted by $\mathcal{R}(\mathcal{S})$. It is well-known that this definition does not depend on $1 \le p < \infty$ (however, the constant $\mathcal{R}(\mathcal{S})$ does) and that $\mathcal{R}$-boundedness is equivalent to boundedness when X is a Hilbert space. When X is an $L^p(\Sigma)$ space (with respect to any σ-finite measure defined on a σ-algebra Σ), testing $\mathcal{R}$-boundedness is equivalent to proving square function estimates [16, Remark 2.9].

PROPOSITION 2.1. *Let $\mathcal{S} \subset B(L^p(\Sigma))$, $1 < p < \infty$. Then $\mathcal{S}$ is $\mathcal{R}$-bounded if and only if there is a constant $C > 0$ such that for any finite families $(f_i) \subset L^p(\Sigma)$ and $(S_i) \subset \mathcal{S}$,*

$$\Big\|\Big(\sum_i |S_i f_i|^2\Big)^{1/2}\Big\|_{L^p(\Sigma)} \le C\Big\|\Big(\sum_i |f_i|^2\Big)^{1/2}\Big\|_{L^p(\Sigma)}.$$

The best constant C for which the above square function estimates hold satisfies $\kappa^{-1}C \le \mathcal{R}(\mathcal{S}) \le \kappa C$ for a suitable $\kappa > 0$ (depending only on p). Using the proposition above, $\mathcal{R}$-boundedness follows from domination by a positive $\mathcal{R}$-bounded family.

COROLLARY 2.2. *Let $\mathcal{S}, \mathcal{T} \subset B(L^p(\Sigma))$, $1 < p < \infty$, and assume that $\mathcal{T}$ is an $\mathcal{R}$-bounded family of positive operators and that for every $S \in \mathcal{S}$ there exists $T \in \mathcal{T}$ such that $|Sf| \le T|f|$ pointwise for every $f \in L^p(\Sigma)$. Then $\mathcal{S}$ is $\mathcal{R}$-bounded.*

We also need the following result about the integral mean of an $\mathcal{R}$-bounded family of operators; we state it in the version we use.

PROPOSITION 2.3 ([16, Corollary 2.14]). *Let X be a Banach space and let $\mathcal{F} \subset B(X)$ be an $\mathcal{R}$-bounded family of operators. For every strongly measurable $N : \Sigma \to B(X)$ on a σ-finite measure space (Σ, μ) with values in $\mathcal{F}$ and every $h \in L^1(\Sigma, \mu)$, we define an operator $T_{N,\mathcal{F}} \in B(X)$ by*

$$T_{N,\mathcal{F}}x = \int_\Sigma h(\omega)N(\omega)x\, d\mu(\omega), \qquad x \in X.$$

Then the family

$$\mathcal{C} = \{T_{N,\mathcal{F}} : \|h\|_{L^1} \le 1,\ N \text{ as above}\}$$

is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{C}) \le 2\mathcal{R}(\mathcal{F})$.

Let $(A, D(A))$ be a sectorial operator in a Banach space X; this means that $\rho(-A) \supset \Sigma_{\pi-\phi}$ for some $\phi < \pi$ and $\lambda(\lambda+A)^{-1}$ is bounded in $\Sigma_{\pi-\phi}$. The infimum of all such ϕ is called the *spectral angle* of A and denoted by ϕ_A. Note that $-A$ generates an analytic semigroup if and only if $\phi_A < \pi/2$. The definition of an $\mathcal{R}$-sectorial operator is similar, substituting boundedness of $\lambda(\lambda + A)^{-1}$ with $\mathcal{R}$-boundedness in $\Sigma_{\pi-\phi}$. As above, one denotes by ϕ_A^R the infimum of all ϕ for which this happens; since $\mathcal{R}$-boundedness implies boundedness, we have $\phi_A \le \phi_A^R$.

The $\mathcal{R}$-boundedness of the resolvent characterizes the regularity of the associated inhomogeneous parabolic problem, as we explain now.

An analytic semigroup $(e^{-tA})_{t\ge0}$ on a Banach space X with generator $-A$ has *maximal regularity of type L^q* $(1 < q < \infty)$ if for each $f \in L^q([0,T];X)$ the function $t \mapsto u(t) = \int_0^t e^{-(t-s)A} f(s)\, ds$ belongs to $W^{1,q}([0,T];X) \cap L^q([0,T];D(A))$. This means that the mild solution of the evolution equation

$$u'(t) + Au(t) = f(t), \qquad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. A characterization of maximal regularity is available in UMD Banach spaces, through the $\mathcal{R}$-boundedness of the resolvent in a suitable sector $\omega + \Sigma_\phi$, with $\omega \in \mathbb{R}$ and $\phi > \pi/2$ or, equivalently, of the scaled semigroup $e^{-(A+\omega')t}$ in a sector around the positive axis. In the case of L^p spaces it can be restated in the following form (see [16, Theorem 1.11]):

THEOREM 2.4. *Let $(e^{-tA})_{t\ge0}$ be a bounded analytic semigroup in $L^p(\Sigma)$, $1 < p < \infty$, with generator $-A$. Then $T(\cdot)$ has maximal regularity of type L^q if and only if the set $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi/2+\phi}\}$ is $\mathcal{R}$-bounded for some $\phi > 0$ if and only if there are constants $0 < \phi < \pi/2$ and $C > 0$ such that* *for any finite sequences* $(\lambda_i) \subset \Sigma_{\pi/2+\phi}$ *and* $(f_i) \subset L^p$,

$$\Big\| \Big(\sum_i |\lambda_i(\lambda_i + A)^{-1} f_i|^2 \Big)^{1/2} \Big\|_{L^p(\Sigma)} \le C \Big\| \Big(\sum_i |f_i|^2 \Big)^{1/2} \Big\|_{L^p(\Sigma)}$$

if and only if there are constants $0 < \phi' < \pi/2$ *and* $C' > 0$ *such that for any finite sequences* $(z_i) \subset \Sigma_{\phi'}$ *and* $(f_i) \subset L^p$,

$$\Big\| \Big(\sum_i |e^{-z_i A} f_i|^2 \Big)^{1/2} \Big\|_{L^p(\Sigma)} \le C' \Big\| \Big(\sum_i |f_i|^2 \Big)^{1/2} \Big\|_{L^p(\Sigma)}.$$

Finally, we state a version of the operator-valued Mikhlin multiplier theorem in the N-dimensional case (see e.g. [15, Corollary 8.3.22]).

THEOREM 2.5. *Let* $1 < p < \infty$ *and* $M \in C^N(\mathbb{R}^N \setminus \{0\}; B(L^p(\Sigma)))$ *be such that the set*

$$\{\xi^\alpha D_\xi^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ \alpha \in \{0,1\}^N\}$$

is $\mathcal{R}$*-bounded. Then the operator* $T_M = \mathcal{F}^{-1} M \mathcal{F}$ *is bounded in* $L^p(\mathbb{R}^N, L^p(\Sigma))$, *where* $\mathcal{F}$ *denotes the Fourier transform.*

3. Weighted Sobolev spaces. We collect in this section the main results concerning weighted Sobolev spaces, referring to [22] for further details and proofs, even in more general situations.

For $p > 1$ and $m > -1$, we define the Sobolev space

$$W^{2,p}_m = \{u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}_+) : u, D_{x_i}u, D_y u, D_{x_i x_j}u, D_{x_i y}u, D_{yy}u \in L^p_m\},$$

which is a Banach space equipped with the norm

$$\|u\|_{W^{2,p}_m} = \|u\|_{L^p_m} + \sum_{i=1}^N \|D_{x_i}u\|_{L^p_m} + \|D_y u\|_{L^p_m} + \sum_{i,j=1}^N \|D_{x_i x_j}u\|_{L^p_m} + \|D_{yy}u\|_{L^p_m} + \sum_{i=1}^N \|D_{x_i y}u\|_{L^p_m}.$$

Next, we add a Neumann boundary condition for $y = 0$ in the form $y^{-1}D_y u \in L^p_m$ and set

$$W^{2,p}_{m,\mathcal{N}} = \{u \in W^{2,p}_m : y^{-1}D_y u \in L^p_m\} \tag{6}$$

with the norm

$$\|u\|_{W^{2,p}_{m,\mathcal{N}}} = \|u\|_{W^{2,p}_m} + \|y^{-1}D_y u\|_{L^p_m}.$$

REMARK 3.1. With obvious changes we also consider the analogous Sobolev spaces $W^{2,p}_m$ and $W^{2,p}_{m,\mathcal{N}}$ on $\mathbb{R}_+$.

The next result clarifies in which sense the condition $y^{-1}D_y u \in L^p_m$ is a Neumann boundary condition.

Proposition 3.2.

(i) *If* $(m+1)/p>1$, *then* $W^{2,p}_{m,\mathcal{N}}=W^{2,p}_m$.

(ii) *If* $(m+1)/p<1$, *then*

$$W^{2,p}_{m,\mathcal{N}}=\Big\{u\in W^{2,p}_m:\lim_{y\to 0}D_yu(x,y)=0 \text{ for a.e. } x\in\mathbb{R}^N\Big\}.$$

In both cases (i) *and* (ii), *the norm of* $W^{2,p}_{m,\mathcal{N}}$ *is equivalent to that of* $W^{2,p}_m$.

We provide an equivalent description of $W^{2,p}_{m,\mathcal{N}}$, adapted to the operator $D_{yy}+cy^{-1}D_y$.

Proposition 3.3. *Let* $0<(m+1)/p<c+1$. *Then*

$$W^{2,p}_{m,\mathcal{N}}=\Big\{u\in W^{2,p}_{\rm loc}(\mathbb{R}^{N+1}_+):u,\Delta_xu\in L^p_m,D_{yy}u+c\frac{D_yu}{y}\in L^p_m \text{ and } \lim_{y\to 0}y^cD_yu(x,y)=0 \text{ for a.e. } x\in\mathbb{R}^N\Big\},$$

and the norms $\|u\|_{W^{2,p}_{m,\mathcal{N}}}$ *and*

$$\|u\|_{L^p_m}+\|\Delta_xu\|_{L^p_m}+\|(D_{yy}u+cy^{-1}D_yu)\|_{L^p_m}$$

are equivalent on $W^{2,p}_{m,\mathcal{N}}$.

The next results show the density of smooth functions in $W^{2,p}_{m,\mathcal{N}}$. Let

$$\mathcal{D}=\{u\in C_c^\infty([0,\infty)):D_yu(y)=0 \text{ for } y\le\delta \text{ and some } \delta>0\} \tag{7}$$

and (with finite sums)

$$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}=\Big\{u(x,y)=\sum_i u_i(x)v_i(y):u_i\in C_c^\infty(\mathbb{R}^N),\ v_i\in\mathcal{D}\Big\}. \tag{8}$$

Theorem 3.4. *If* $(m+1)/p>0$ *then* $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ *is dense in* $W^{2,p}_{m,\mathcal{N}}$.

Note that the condition $m+1>0$ is necessary for the inclusion $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}\subset W^{2,p}_{m,\mathcal{N}}$.

4. The Bessel operator B. In this section we consider for $c>-1$ the one-dimensional Bessel operator

$$B=D_{yy}+\frac{c}{y}D_y$$

in the space $L^p_m=L^p_m(\mathbb{R}_+)$, under Neumann boundary conditions at $y=0$. We summarize below all the main results we need, referring the reader to [21, Section 3], [20, Section 4], [25] for further details and to [19, 18, 17, 26] for analogous results in the multidimensional case.

Setting $H^1_c = \{u \in L^2_c : u' \in L^2_c\}$, the operator B (with Neumann boundary conditions) is associated to the nonnegative, symmetric and closed form in L^2_c,

$$\mathfrak{a}(u,v) := \int_0^\infty D_y u D_y \overline{v}\, y^c\, dy, \qquad D(\mathfrak{a}) = H^1_c.$$

If $1 < p < \infty$, we recall that

$$W^{2,p}_{\mathcal{N}}(m) = \{u \in W^{2,p}_{\rm loc}(\mathbb{R}_+) : u, D_{yy}u, D_y u, y^{-1}D_y u \in L^p_m\} \tag{9}$$

and for $p = \infty$ we define

$$\hat{C}^2_{\mathcal{N}} = \{u \in \hat{C}^2([0,\infty)) : D_y u(0) = 0\} \tag{10}$$

(we recall that the "hat" means that the functions and the derivatives tend to 0 at infinity, see Notation). The Neumann boundary condition, denoted by the subscript $\mathcal{N}$, is enclosed in the requirement $y^{-1}D_y u \in L^p_m$. This last requirement is redundant when $(m+1)/p > 1$ and equivalent to $D_y u(y) \to 0$ as $y \to 0$, when $(m+1)/p < 1$; see Proposition 3.2.

Consequently, we write B^n or, more pedantically $B^n_{m,p}$ if necessary, for the operator B endowed with the domain $W^{2,p}_{\mathcal{N}}(m)$. This time the superscript n indicates the Neumann boundary condition at $y = 0$.

THEOREM 4.1. *If $1 < p < \infty$ and $0 < (m+1)/p < c+1$, then B endowed with domain $W^{2,p}_{\mathcal{N}}(m)$ generates a bounded analytic semigroup e^{zB} of angle $\pi/2$ on L^p_m, which is positive for $z > 0$.*

If $p = \infty$, then B with domain $\hat{C}^2_{\mathcal{N}}$ generates an analytic semigroup of angle $\pi/2$ in $\hat{C}([0,\infty))$.

Proof. See [21, Propositions 3.3, 5.3] and [24, Proposition 3.4] for $p = \infty$. ∎

PROPOSITION 4.2. *Let $c > -1$. The semigroup $(e^{zB^n})_{z\in\mathbb{C}_+}$ consists of integral operators. Its heat kernel p_B, written with respect the measure $\rho^c d\rho$, satisfies, for every $\varepsilon > 0$, $z \in \Sigma_{\pi/2-\varepsilon}$ and some $C_\varepsilon, \kappa_\varepsilon > 0$,*

$$|p_B(z,y,\rho)| \le C_\varepsilon |z|^{-1/2}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa_\varepsilon |z|}\right),$$

$$|D_y p_B(z,y,\rho)| \le C_\varepsilon |z|^{-1}\rho^{-c}\left(\frac{y}{|z|^{1/2}} \wedge 1\right)\left(\frac{\rho}{|z|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa_\varepsilon |z|}\right).$$

Proof. See [21, Propositions 2.8, 2.9]. ∎

4.1. $\mathcal{R}$-boundedness of some families of operators associated to B^n. We consider a two-parameter family $(S_{\alpha,\beta}(t))_{t>0}$ of integral operators on L^p_m, defined for $\alpha,\beta\in\mathbb{R}$ and $t>0$ by

$$S^{\alpha,\beta}(t)f(y)=t^{-1/2}\Big(\frac{y}{\sqrt{t}}\wedge 1\Big)^{-\alpha}\int_0^\infty\Big(\frac{z}{\sqrt{t}}\wedge 1\Big)^{-\beta}\exp\Big(-\frac{|y-z|^2}{\kappa t}\Big)f(z)\,dz,$$

where κ is a positive constant. We omit the dependence on κ even though in some proofs we need to vary it.

We also define the families

$$\Gamma(\lambda)=\int_0^\infty e^{-\lambda t}S^{0,-c}(t)\,dt,\qquad \Psi(\lambda)=\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}}S^{0,-c}(t)\,dt,\qquad \lambda>0. \tag{11}$$

By Proposition 4.2 and the results in [21, 17], $S^{0,-c}(t)$ and $\Gamma(\lambda)$ sharply estimate the behaviour of the semigroup e^{tB^n} and of the resolvent of B^n, respectively, in the sense that, for $f\ge 0$,

$$e^{tB^n}f\simeq S^{0,-c}(t)f,\quad t>0,\qquad (\lambda-B^n)^{-1}f\simeq\Gamma(\lambda)f,\quad \lambda>0.$$

Similarly, $S^{0,-c}(t)/\sqrt{t}$ and $\Psi(\lambda)$ estimate the behaviour of the spatial derivative of the semigroup and of the resolvent, that is,

$$|D_y e^{tB^n}f|\le C\frac{S^{0,-c}(t)}{\sqrt{t}}|f|,\qquad t>0,$$
$$|D_y(\lambda-B^n)^{-1}f|\le C\Psi(\lambda)|f|,\qquad \lambda>0.$$

In the following proposition we summarize the main properties of the above families, referring to [21, Section 7] and [19, Section 4] for further details.

PROPOSITION 4.3. *Let $1<p<\infty$ and $\alpha,\beta\in\mathbb{R}$.*

(i) *If $\alpha<(m+1)/p<1-\beta$, then the family $(S^{\alpha,\beta}(t))_{t\ge 0}$ is $\mathcal{R}$-bounded on L^p_m.*
(ii) *If $0<(m+1)/p<c+1$, then the family $(\lambda\Gamma(\lambda))_{\lambda>0}$ is $\mathcal{R}$-bounded on L^p_m.*
(iii) *If $0<(m+1)/p<c+1$, then the family $(\sqrt{\lambda}\Psi(\lambda))_{\lambda>0}$ is $\mathcal{R}$-bounded on L^p_m.*

Proof. Property (i) follows from [21, Theorem 7.7], and (ii) from (i) and Proposition 2.3, with $h_\lambda(t)=\lambda e^{-\lambda t}$. Property (iii) follows similarly by (i) and Proposition 2.3, with $g_\lambda(t)=\sqrt{\lambda/t}\,e^{-\lambda t}$. ■

5. The operator $L_b=B+ibD_y-b^2/4$. In this section we prove generation properties in L^p_m and heat kernel bounds for the operator

$$L_b:=B+ibD_y-b^2/4,$$

following the method of [5, Sections 3, 4]. Note that $L_0=B$.

Recalling that the operator B^n is associated to the form in L^2_c,

$$\mathfrak{a}(u,v):=\int_0^\infty D_yuD_y\overline{v}\,y^c\,dy,\qquad D(\mathfrak{a})=H^1_c,$$

we consider the perturbed form $\mathfrak{a}_b$ defined on $D(\mathfrak{a}_b) = H^1_c$ by

$$\mathfrak{a}_b(u,v) = \mathfrak{a}(u,v) - ib\langle D_y u, v\rangle_{L^2_c} + \frac{b^2}{4}\langle u,v\rangle_{L^2_c} \tag{12}$$

and define L_b in L^2_c as the operator associated to the form $\mathfrak{a}_b$, that is,

$$D(L_b) = \Big\{u \in D(\mathfrak{a}_b) : \exists f \in L^2_c \text{ such that } \mathfrak{a}_b(u,v) = \int_0^\infty f\overline{v}y^c\,dy \text{ for every } v \in D(\mathfrak{a}_b)\Big\}, \tag{13}$$
$$L_b u = -f.$$

5.1. The auxiliary operator $B - i\frac{bc}{2y}$. For technical reasons we also consider the form $\tilde{\mathfrak{a}}_b$ defined on $D(\tilde{\mathfrak{a}}_b) = H^1_c$ by

$$\begin{aligned}\tilde{\mathfrak{a}}_b(u,v) &= \mathfrak{a}(u,v) - i\frac{b}{2}\langle u, D_y v\rangle_{L^2_c} - i\frac{b}{2}\langle D_y u, v\rangle_{L^2_c} \\ &= \int_{\mathbb{R}_+}\Big(D_y u D_y\overline{v} - i\frac{b}{2}D_y(u\overline{v})\Big)y^c\,dy\end{aligned} \tag{14}$$

and its associated operator A_b in L^2_c. Since for smooth functions with compact support away from the origin

$$c\int_0^\infty u\overline{v}y^{c-1}\,dy = \int_0^\infty u\big(D_y(y^c\overline{v}) - y^c D_y\overline{v}\big)\,dy = -\int_0^\infty D_y(u\overline{v})y^c\,dy,$$

the operator A_b is defined on smooth functions by

$$A_b := B - i\frac{bc}{2y}, \quad A_0 = B^n.$$

In the following proposition we collect the main properties satisfied by $\tilde{\mathfrak{a}}_b$.

PROPOSITION 5.1. *If $c+1>0$, then $\tilde{\mathfrak{a}}_b$ is accretive and closed in L^2_c. Moreover,*

(i) *the adjoint form $\tilde{\mathfrak{a}}_b^* : (u,v) \mapsto \overline{\tilde{\mathfrak{a}}_b(v,u)}$ satisfies $\tilde{\mathfrak{a}}_b^* = \tilde{\mathfrak{a}}_{-b}$;*
(ii) *its real part is $\operatorname{Re}\tilde{\mathfrak{a}}_b := \frac{\tilde{\mathfrak{a}}_b + \tilde{\mathfrak{a}}_b^*}{2} = \mathfrak{a}$;*
(iii) *for any $\epsilon > 0$ and $u \in H^1_c$,*

$$|\operatorname{Im}\tilde{\mathfrak{a}}_b(u,u)| \le \epsilon\Big(\mathfrak{a}(u,u) + \frac{b^2}{4\epsilon^2}\|u\|^2_{L^2_c}\Big).$$

Proof. Properties (i) and (ii) are immediate consequences of the definition. Since $\operatorname{Re}\tilde{\mathfrak{a}}_b(u,u) = \mathfrak{a}(u,u) \ge 0$, $\tilde{\mathfrak{a}}_b$ is accretive, and furthermore the norm induced by the form $\tilde{\mathfrak{a}}_b$ coincides with the one induced by $\mathfrak{a}$ and so $\tilde{\mathfrak{a}}_b$ is closed.

To prove (iii) it is enough to observe that for any $\epsilon > 0$ and $u \in H^1_c$ one has

$$|\mathrm{Im}\, \tilde{\mathfrak{a}}_b(u,u)| = \left| -b \int_0^\infty \mathrm{Re}(\overline{u} D_y u) y^c \, dy \right| \le |b| \, \|D_y u\|_{L^2_c} \|u\|_{L^2_c}$$
$$= \epsilon \left(2 \, \|D_y u\|_{L^2_c} \left| \frac{b}{2\epsilon} \right| \|u\|_{L^2_c} \right) \le \epsilon \left(\mathfrak{a}(u,u) + \frac{b^2}{4\epsilon^2} \|u\|^2_{L^2_c} \right),$$

where we have used the elementary inequality $D_y(|u|^2) = 2\,\mathrm{Re}(\overline{u} D_y u)$. ∎

By standard theory of sesquilinear forms we have the following results.

PROPOSITION 5.2. *If $c+1 > 0$, then the operator A_b generates an analytic semigroup of angle $\pi/2$ in L^2_c which satisfies, for any $\epsilon > 0$,*

$$\|e^{zA_b} f\|_{L^2_c} \le e^{\frac{b^2}{4\epsilon^2} \mathrm{Re}\, z} \|f\|_{L^2_c}, \qquad \forall z \in \Sigma_{\pi/2 - \arctan \epsilon}.$$

Moreover,

(i) *The semigroup $(e^{tA_b})_{t \ge 0}$ is L^∞-contractive and it is dominated by e^{tB^n}, that is,*

$$|e^{tA_b} f| \le e^{tB^n} |f|, \qquad t > 0, \ f \in L^2_c.$$

(ii) *$(e^{tA_b})_{t\ge 0}$ is a semigroup of integral operators and its heat kernel $\tilde{p}_b$, taken with respect to the measure $\rho^c d\rho$, satisfies, for some constant C independent of b,*

$$|\tilde{p}_b(t,y,\rho)| \le C t^{-1/2} \rho^{-c} \left(\frac{\rho}{t^{1/2}} \wedge 1 \right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa t} \right) \quad \textit{for a.e. } y, \, \rho > 0.$$

(iii) *$A_b^* = \tilde{A}_{-b}$ and for any $s > 0$ this operator has the scaling property*

$$I_{1/s} \circ A_b \circ I_s = s^2 A_{b/s}, \qquad I_s u(y) := u(sy).$$

Proof. The generation properties follow by using Proposition 5.1 and [27, Teorems 1.52, 1.53]. To prove (i) we first observe that the operator B is associated with the form $\mathfrak{a}(u,v) = \langle D_y u, D_y v \rangle_{L^2_c}$ and the semigroup e^{tB} is sub-Markovian since $\mathfrak{b}$ satisfies the hypotheses of [27, Corollary 2.17]. The domination property for e^{tA_b} then follows from [27, Theorem 2.21]. In particular, e^{tA_b} inherits the L^∞-contractivity of e^{tB}.

(ii) is a consequence of [2, Proposition 1.9] since e^{tA_b} is dominated by the positive integral operator e^{tB^n} whose kernel satisfies the stated estimate; see [21, Proposition 2.8] where, however, the kernel is written with respect to the Lebesgue measure.

(iii) follows from Proposition 5.1(i) by elementary computation. ∎

5.2. Bounds for e^{tL_b}. The following elementary lemma relates L_b to A_b.

LEMMA 5.3. *The isometry*

$$T : L^2_c \to L^2_c, \qquad (Tu)(y) = e^{i\frac{b}{2}y}u(y),$$

preserves H^1_c and satisfies

$$\mathfrak{a}_b(u,v) = \tilde{\mathfrak{a}}_b(Tu,Tv), \quad \forall u,v \in H^1_C, \qquad L_b = T^{-1} \circ A_b \circ T. \tag{15}$$

Proof. This follows from the equality $D_yTu = T(D_yu + i\frac{b}{2}u)$. ∎

REMARK 5.4. It is easier to prove sectoriality and domination for the form $\tilde{\mathfrak{a}}_b$ and the operator A_b rather than for $\mathfrak{a}_b$ and L_b since $\mathrm{Re}\,\tilde{\mathfrak{a}}_b(u,u) = \|D_yu\|_2^2$ (and this is not true for $\mathfrak{a}_b$).

The following result follows easily from Proposition 5.2, since $L_b = T^{-1} \circ A_b \circ T$.

PROPOSITION 5.5. *Let $c+1>0$. Then the operator L_b generates an analytic semigroup of angle $\pi/2$ in L^2_c which satisfies, for any $\epsilon > 0$,*

$$\|e^{zL_b}f\|_{L^2_c} \le e^{\frac{b^2}{4\epsilon^2}\mathrm{Re}\,z}\|f\|_{L^2_c}, \qquad \forall z \in \Sigma_{\pi/2-\arctan\varepsilon}.$$

Moreover,

(i) *The semigroup $(e^{tL_b})_{t\ge 0}$ is L^∞-contractive and dominated by $(e^{tB^n})_{t\ge0}$, that is,*

$$|e^{tL_b}f| \le e^{tB^n}|f|, \qquad t>0,\ f \in L^2_c.$$

(ii) *$(e^{tL_b})_{t\ge0}$ is a semigroup of integral operators and its heat kernel p_b, taken with respect the measure $\rho^c d\rho$, satisfies, for some constant C independent of b,*

$$|p_b(t,y,\rho)| \le Ct^{-1/2}\rho^{-c}\left(1 \wedge \frac{\rho}{\sqrt{t}}\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa t}\right)$$

for a.e. $y,\rho>0$.

(iii) *For any $s>0$ this operator has the scaling property*

$$I_{1/s} \circ L_b \circ I_s = s^2 L_{b/s}, \qquad I_su(y) := u(sy).$$

Proof. All the stated properties follow from Proposition 5.2 and Lemma 5.3 by using the equalities

$$e^{tL_b}f(y) = e^{-i\frac{b}{2}y}\,e^{tA_b}(e^{i\frac{b}{2}\cdot}f)(y), \qquad p_b(t,y,\rho) = e^{i\frac{b}{2}(\rho-y)}\,\tilde{p}_b(t,y,\rho).$$ ∎

To extend the above heat kernel estimates to the half-plane $\mathbb{C}_+$ we need the following elementary lemma; see [20, Lemma 5.2] for a straightforward proof.

Lemma 5.6. *Let $c+1>0$ and for $y_0, r>0$,*

$$Q_c(y_0,r) := \int_{y_0}^{y_0+r} y^c\,dy.$$

Then

$$Q_c(y_0,r) = r^{c+1} Q_c\Big(\frac{y_0}{r},1\Big), \qquad Q_c(y_0,r) \simeq r^{c+1}\Big(\frac{y_0}{r}\Big)^c \Big(\frac{y_0}{r}\wedge 1\Big)^{-c}.$$

In particular, the function Q_c satisfies, for some constants $C\ge 1$, the doubling condition

$$\frac{Q_c(y_0,s)}{Q_c(y_0,r)} \le C\Big(1\vee \frac{s}{r}\Big)^{1\vee (c+1)}, \qquad \forall\, s,r>0.$$

We also need to rewrite the estimate in Proposition 5.5(ii) in an equivalent but more symmetric way.

Lemma 5.7. *The estimate in Proposition* 5.5(ii) *is equivalent* (*after modifying the constant in the exponential*) *to*

$$(16)\quad |p_b(t,y,\rho)| \le Ct^{-\frac{c+1}{2}}\Big(\frac{y}{\sqrt{t}}\Big)^{-c/2}\Big(1\wedge \frac{y}{\sqrt{t}}\Big)^{c/2}\Big(\frac{\rho}{\sqrt{t}}\Big)^{-c/2}\Big(1\wedge \frac{\rho}{\sqrt{t}}\Big)^{c/2} \times \exp\Big(-\frac{|y-\rho|^2}{\kappa t}\Big)$$
$$\simeq \frac{1}{\sqrt{Q_c(y,\sqrt{t})Q_c(\rho,\sqrt{t})}}\exp\Big(-\frac{|y-\rho|^2}{\kappa t}\Big).$$

Proof. This follows by [21, Lemma 10.2] with $\gamma_1=\gamma_2=-c/2$. ∎

5.3. Bounds for e^{zL_b}. In order to extend the above kernel estimates to complex z we use the standard machinery of [27, Chapter 6], relying on estimates like (16) where the terms $1/\sqrt{Q_c(y,\sqrt{t})Q_c(\rho,\sqrt{t})}$ are substituted by powers of t, due to the doubling property of Lemma 5.6.

We consider the isometry

$$(17)\qquad \varPhi : L^2(\mathbb{R}_+, d\mu_{Q_c}) \to L^2(\mathbb{R}_+, d\mu), \qquad f\mapsto \frac{f}{\sqrt{Q_c(\,\cdot\,,1)}},$$

where

$$d\mu_{Q_c} := \frac{d\mu}{Q_c(\,\cdot\,,1)} \simeq (y\wedge 1)^c\,dy,$$

by Lemma 5.6. The map $\varPhi$ defines a similar operator $\tilde{L}_b = \varPhi^{-1}L_b\varPhi$ which acts on $L^2(\mathbb{R}_+, d\mu_{Q_c})$. In the following proposition we collect the main properties of $\tilde{L}_b$ which follow, by construction, from Proposition 5.5.

PROPOSITION 5.8. *$\tilde{L}_b$ generates an analytic semigroup of angle $\pi/2$ in $L^2(\mathbb{R}_+, d\mu_{Q_c})$ which satisfies, for any $\epsilon > 0$,*

$$\|e^{z\tilde{L}_b} f\|_{L^2(\mathbb{R}_+, d\mu_{Q_c})} \le e^{\frac{b^2}{4\epsilon^2}\operatorname{Re} z}\|f\|_{L^2(\mathbb{R}_+, d\mu_{Q_c})}, \qquad \forall z \in \Sigma_{\pi/2-\arctan\epsilon}. \tag{18}$$

Moreover:

(i) *The semigroup $(e^{t\tilde{L}_b})_{t\ge 0}$ is dominated by $e^{t\tilde{L}_0}$, that is,*

$$|e^{t\tilde{L}_b} f| \le e^{t\tilde{L}_0}|f|, \qquad t > 0,\ f \in L^2(\mathbb{R}_+, d\mu_{Q_c}).$$

(ii) *$(e^{t\tilde{L}_b})_{t\ge 0}$ is a semigroup of integral operators and its heat kernel $p_{\tilde{L}_b}$, taken with respect the measure $d\mu_{Q_c}$, satisfies*

$$p_{\tilde{L}_b}(t, y, \rho) = \sqrt{Q_c(y, 1)Q_c(\rho, 1)}\, p_b(t, y, \rho). \tag{19}$$

The doubling condition of Q_c guarantees the ultracontractivity of $\{e^{z\tilde{L}_b} : z \in \mathbb{C}_+\}$. The following lemma is the main reason for introducing the new operator $\tilde{L}_b$. Its kernel, in fact, is bounded by a constant depending only on t.

LEMMA 5.9. *$(e^{t\tilde{L}_b})_{t\ge 0}$ satisfies, for some constant C independent of b,*

(i) $\|e^{t\tilde{L}_b}\|_{\mathcal{L}(L^1(d\mu_{Q_c}), L^\infty)} \le C\left(1 + \frac{1}{t}\right)^{\frac{1\vee(c+1)}{2}}$,

(ii) $\|e^{t\tilde{L}_b}\|_{\mathcal{L}(L^1(d\mu_{Q_c}), L^2(d\mu_{Q_c}))} \le C\left(1 + \frac{1}{t}\right)^{\frac{1\vee(c+1)}{4}}$,

(iii) $\|e^{t\tilde{L}_b}\|_{\mathcal{L}(L^2(d\mu_{Q_c}), L^\infty(d\mu_{Q_c}))} \le C\left(1 + \frac{1}{t}\right)^{\frac{1\vee(c+1)}{4}}$.

Proof. To prove (i) we observe that from (19) and Lemmas 5.7 and 5.6, one has

$$\begin{aligned} |p_{\tilde{L}_b}(t, y, \rho)| &= \sqrt{Q_c(y, 1)Q_c(\rho, 1)}\, |p_b(t, y, \rho)| \\ &\le C\left[\frac{Q_c(y, 1)Q_c(\rho, 1)}{Q_c(y, \sqrt{t})Q_c(\rho, \sqrt{t})}\right]^{1/2} \exp\left(-\frac{|y-\rho|^2}{\kappa t}\right) \\ &\le C\left(1 \vee \frac{1}{\sqrt{t}}\right)^{1\vee(c+1)} \exp\left(-\frac{|y-\rho|^2}{\kappa t}\right) \\ &\simeq C\left(1 + \frac{1}{t}\right)^{\frac{1\vee(c+1)}{2}} \exp\left(-\frac{|y-\rho|^2}{\kappa t}\right). \end{aligned} \tag{20}$$

In particular, $e^{t\tilde{L}_b}$ is ultracontractive and satisfies (i).

To prove (ii) we observe that for $f \in L^1(\mathbb{R}_+, d\mu_{Q_c})$, using Proposition 5.8(i) and then (i) of the present lemma with $b = 0$ (note that $\tilde{L}_0$ is, by construction, self-adjoint), we get

$$\|e^{t\tilde{L}_b}f\|_2^2 \le \|e^{t\tilde{L}_0}|f|\|_2^2 = \langle e^{2t\tilde{L}_0}|f|, |f|\rangle \le C\left(1+\frac{1}{t}\right)^{\frac{1\vee(c+1)}{2}}\|f\|_1^2.$$

Finally, (iii) follows by duality. ■

We can now prove heat kernel estimates for $\tilde{L}_b$ for complex times. We need the following lemma of Phragmén–Lindelöf type [27, Lemma 6.18].

Proposition 5.10. *Let $\psi \in (0,\pi/2]$ and let F be an analytic function on Σ_ψ. Assume that, for some $A,\gamma>0$,*

$$\text{(21)}\qquad \begin{aligned} |F(z)| &\le A(\operatorname{Re} z)^{-\nu}, \quad &\forall z\in\Sigma_\psi,\\ |F(t)| &\le At^{-\nu}e^{-\gamma/t}, \quad &\forall t>0.\end{aligned}$$

Then for any $0<\psi'<\psi$,

$$\text{(22)}\qquad |F(z)| \le A2^\nu(\operatorname{Re} z)^{-\nu}\exp\left(-\frac{\gamma}{2|z|}\sin(\psi-\psi')\right), \quad \forall z\in\Sigma_{\psi'}.$$

Proposition 5.11. *The semigroup $\{e^{z\tilde{L}_b} : z\in\mathbb{C}_+\}$ consists of integral operators,*

$$e^{z\tilde{L}_b}f(y) = \int_0^\infty p_{\tilde{L}_b}(z,y,\rho)f(\rho)\,d\mu_{Q_c}, \quad f\in L^2(\mathbb{R}_+, d\mu_{Q_c}),\ y>0.$$

Furthermore for every $\epsilon>0$ and $0<\delta<1$ there exist $C,k>0$ independent of b such that, for every $z\in\Sigma_{\pi/2-\arctan\epsilon}$ and almost every $y,\rho>0$,

$$|p_{\tilde{L}_b}(z,y,\rho)| \le Ce^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}}\exp\left(-\frac{|y-\rho|^2}{k|z|}\right).$$

Proof. Let us fix $\epsilon>0$ and $0<\delta<1$. Let us observe that

$$t+is\in\Sigma_{\pi/2-\arctan(\delta\epsilon)} \implies \delta t+is\in\Sigma_{\pi/2-\arctan(\delta^2\epsilon)}.$$

Then using the semigroup law and Lemma 5.9 we find that for any $z = t+is\in\Sigma_{\pi/2-\arctan(\delta\epsilon)}$ and for some positive constant $C=C(\epsilon,\delta)$ which may vary in each occurrence,

$$\text{(23)}\qquad \begin{aligned} &\|e^{z\tilde{L}_b}\|_{\mathcal{L}(L^1(\mathbb{R}_+,d\mu_{Q_c}),L^\infty(\mathbb{R}_+))}\\ &\le \|e^{\frac{1-\delta}{2}t\tilde{L}_b}\|_{\mathcal{L}(L^2(d\mu_{Q_c}),L^\infty)}\,\|e^{(\delta t+is)\tilde{L}_b}\|_{\mathcal{L}(L^2(d\mu_{Q_c}),L^2(d\mu_{Q_c}))}\\ &\quad\times\|e^{\frac{1-\delta}{2}t\tilde{L}_b}\|_{\mathcal{L}(L^1(d\mu_{Q_c}),L^2(d\mu_{Q_c}))}\\ &\le C\left(1+\frac{1}{\frac{1-\delta}{2}t}\right)^{\frac{1\vee(c+1)}{2}}e^{\frac{b^2}{4(\delta^2\epsilon)^2}\delta\operatorname{Re} z} \simeq C\left(1+\frac{1}{\operatorname{Re} z}\right)^{\frac{1\vee(c+1)}{2}}e^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}.\end{aligned}$$

The Dunford–Pettis theorem then yields the existence of a kernel $p_{\tilde{L}_b}(z,y,\rho)$ which satisfies the first claim of the proposition and such that

$$|p_{\tilde{L}_b}(z,y,\rho)|\le C_\epsilon\left(1+\frac{1}{\operatorname{Re} z}\right)^{\frac{1\vee(c+1)}{2}} e^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}, \qquad \forall z\in\Sigma_{\pi/2-\arctan(\delta\epsilon)},\ y,\rho>0.$$

Let us now consider the analytic function $\Gamma(f_1,f_2):\Sigma_{\pi/2-\arctan(\delta\epsilon)}\to\mathbb{C}$ defined by

$$\Gamma(f_1,f_2)(z)=\langle e^{zL_b}f_1,f_2\rangle_{L^2(d\mu_{Q_c})},$$

where $f_1\in L^2(F,d\mu_{Q_c})\cap L^1(F,d\mu_{Q_c})$, $f_2\in L^2(E,d\mu_{Q_c})\cap L^1(E,d\mu_{Q_c})$ and E,F are two disjoints compact subsets of $\mathbb{R}_+$. Let $r=d(E,F)$ be their distance. Then from (20), (23) we have

$$|\Gamma(f_1,f_2)(z)|\le C\left(1+\frac{1}{\operatorname{Re} z}\right)^{\frac{1\vee(c+1)}{2}} e^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}\|f_1\|_{L^1(d\mu_{Q_c})}\|f_2\|_{L^1(d\mu_{Q_c})}, \qquad z\in\Sigma_{\frac{\pi}{2}-\arctan(\delta\epsilon)};$$

$$|\Gamma(f_1,f_2)(t)|\le C\left(1+\frac{1}{t}\right)^{\frac{1\vee(c+1)}{2}}\exp\left(-\frac{r^2}{\kappa t}\right)\|f_1\|_{L^1(d\mu_{Q_c})}\|f_2\|_{L^1(d\mu_{Q_c})}, \qquad t\in\mathbb{R}^+,$$

(note that $|z|\simeq\operatorname{Re} z$, $\operatorname{Re}\frac{1}{z}\simeq\frac{1}{\operatorname{Re} z}$ for $z\in\Sigma_{\pi/2-\arctan(\delta\epsilon)}$).

Therefore the function $H(z)=\Gamma(f_1,f_2)(z)e^{-\frac{b^2z}{4\epsilon^2\delta^3}}(1+\frac{1}{z})^{-\frac{1\vee(c+1)}{2}}$ satisfies (21) with

$$\gamma=r^2/k,\quad A=\|f_1\|_{L^1(d\mu_{Q_c})}\|f_2\|_{L^1(d\mu_{Q_c})},\quad \nu=0.$$

Proposition 5.10 then implies that for some positive constant $C=C(\epsilon,\delta)$, $k=k(\epsilon,\delta)$, for any $z\in\Sigma_{\pi/2-\arctan\epsilon}$,

$$\begin{aligned}&|\Gamma(f_1,f_2)(z)|\\&\quad\le Ce^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}}\exp\left(-\frac{r^2}{\kappa|z|}\right)\|f_1\|_{L^1(d\mu_{Q_c})}\|f_2\|_{L^1(d\mu_{Q_c})}.\end{aligned}$$

Let us fix $y,\rho>0$ such that $|y-\rho|>2s>0$ and set $r=|y-\rho|-2s$. Then

$$\begin{aligned}|p_{\tilde{L}_b}(z,y,\rho)|&\le\sup\{|p_{\tilde{L}_b}(z,y',\rho')|:y'\in B(y,s),\ \rho'\in B(\rho,s)\}\\&=\sup\{|\Gamma(f_1,f_2)(z)|:\\&\qquad\qquad\|f_1\|_{L^1(B(y,s),d\mu_{Q_c})}=\|f_2\|_{L^1(B(\rho,s),d\mu_{Q_c})}=1\}\\&\le Ce^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}}\exp\left(-\frac{r^2}{\kappa|z|}\right).\end{aligned}$$

Recalling that $r = |y-\rho| - 2s$ and letting $s \to 0$ we obtain

$$|p_{\tilde{L}_b}(z,y,\rho)| \le Ce^{\frac{b^2}{4\epsilon^2\delta^3}\operatorname{Re} z}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}} \exp\left(-\frac{|y-\rho|^2}{k|z|}\right)$$

for any $z \in \Sigma_{\pi/2-\arctan\epsilon}$, which is equivalent to the statement, given the arbitrariness of δ. ■

Remark 5.12. We remark that in [6], the authors work in an abstract metric measure space (M,d,μ) and assume that the heat kernel p is continuous in the space variables. In that case, in fact,

$$\sup_{x\in U_1,\, y\in U_2} |p(z,x,y)| = \sup\Big\{ \int_M e^{-zL} f_1\overline{f_2}\, d\mu : \\ \|f_1\|_{L^1(U_1,d\mu)} = \|f_2\|_{L^1(U_2,d\mu)} = 1\Big\}.$$

In our setting the continuity assumption can be avoided since the proofs of [6, Theorem 4.1, Corollary 4.4] hold only assuming that for a.e. $x,y \in M$,

$$\begin{aligned} p(z,x,y) &= \lim_{s\to 0} \int_M e^{-zL} f_1\overline{f_2}\, d\mu \\ &= \lim_{s\to 0} \frac{1}{\mu(B(x,s))\mu(B(y,s))} \int_{B(x,s)\times B(y,s)} p(z,\bar{x},\bar{y})\, d\mu(\bar{x})\, d\mu(\bar{y}), \end{aligned}$$

where $f_1 = \frac{\chi_{B(x,s)}}{\mu(B(x,s))}$, $f_2 = \frac{\chi_{B(y,s)}}{\mu(B(y,s))}$.

This holds outside a set of zero measure, when the measure μ is doubling (as in our case), by the Lebesgue differentiation theorem.

Finally, we prove estimates for the heat kernel of L_b for complex times.

Theorem 5.13. *Let $c+1>0$ and $b\in\mathbb{R}$. The semigroup $\{e^{zL_b} : z \in \mathbb{C}_+\}$ consists of integral operators*

$$e^{zL_b} f(y) = \int_0^\infty p_b(z,y,\rho) f(\rho)\, \rho^c\, d\rho, \qquad f \in L^2_c,\ y>0.$$

Furthermore for every $\epsilon>0$ and $0<\delta<1$ there exist $C,k>0$ independent of b such that, for every $z \in \Sigma_{\pi/2-\arctan\epsilon}$ and almost every $y,\rho>0$,

$$|p_b(z,y,\rho)| \le C\, e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z} |z|^{-1/2}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right).$$

Proof. The existence of the kernel follows directly by using the isometry Φ (see (17)) and Proposition 5.11; in particular, (19) extends to complex time.

Moreover, for some positive constant $C = C(\epsilon, \delta)$ one has

$$
\begin{aligned}
|p_b(z,y,\rho)| &= \frac{1}{\sqrt{Q_c(y,1)Q_c(\rho,1)}}|p_{\tilde{L}_b}(z,y,\rho)| \\
&\le C\sqrt{\frac{Q_c(y,\sqrt{|z|})Q_c(\rho,\sqrt{|z|})}{Q_c(y,1)Q_c(\rho,1)}}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}} \\
&\quad\times \frac{1}{\sqrt{Q_c(y,\sqrt{|z|})Q_c(\rho,\sqrt{|z|})}}\, e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z}\exp\left(-\frac{|y-\rho^2|}{k|z|}\right).
\end{aligned}
$$

Lemma 5.6 then implies

$$
\begin{aligned}
|p_b(z,y,\rho)| &\le C(1+|z|)^{\frac{1\vee(c+1)}{2}}\left(1+\frac{1}{|z|}\right)^{\frac{1\vee(c+1)}{2}} \\
&\quad\times \frac{1}{\sqrt{Q_c(y,\sqrt{|z|})Q_c(\rho,\sqrt{|z|})}}\, e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z}\exp\left(-\frac{|y-\rho^2|}{k|z|}\right).
\end{aligned}
\tag{24}
$$

To get rid of the extra term $(1+|z|)^{\frac{1\vee(c+1)}{2}}(1+1/|z|)^{\frac{1\vee(c+1)}{2}}$ we use the scaling property (iii) of Proposition 5.5 and the fact that in the above estimate the constants involved do not depend on b. For any $z \in \Sigma_{\pi/2-\arctan\epsilon}$, we write $z = \omega|z|$ and observe that the scaling equalities of Proposition 5.5(iii),

$$
|z|L_b = I_{1/\sqrt{|z|}} \circ L_{b\sqrt{|z|}} \circ I_{\sqrt{|z|}}, \qquad e^{zL_b} = I_{1/\sqrt{|z|}} \circ e^{\omega L_{b|z|}} \circ I_{\sqrt{|z|}},
$$

imply

$$
p_b(z,y,\rho) = |z|^{-\frac{c+1}{2}} p_{b\sqrt{|z|}}\left(\omega, \frac{y}{\sqrt{|z|}}, \frac{\rho}{\sqrt{|z|}}\right).
$$

Applying (24) with b replaced by $b\sqrt{|z|}$ to this equality and using Lemma 5.6 again we get

$$
\begin{aligned}
|p_b(z,y,\rho)| &\le |z|^{-\frac{c+1}{2}}\left|p_{b\sqrt{|z|}}\left(\omega, \frac{y}{\sqrt{|z|}}, \frac{\rho}{\sqrt{|z|}}\right)\right| \\
&\le C|z|^{-\frac{c+1}{2}} \frac{1}{\sqrt{Q_c\big(\frac{y}{\sqrt{|z|}},1\big)Q_c\big(\frac{\rho}{\sqrt{|z|}},1\big)}}\, e^{\frac{b^2|z|}{4\epsilon^2\delta}\operatorname{Re}\frac{z}{|z|}}\exp\left(-\frac{|y-\rho^2|}{k|z|}\right) \\
&= C\frac{1}{\sqrt{Q_c(y,\sqrt{|z|})Q_c(\rho,\sqrt{|z|})}}\, e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z}\exp\left(-\frac{|y-\rho^2|}{k|z|}\right).
\end{aligned}
$$

This concludes the proof, by Lemma 5.7. ∎

5.4. Generation properties and domain characterization. First we prove that the semigroup e^{zL_b} extrapolates to the spaces L^p_m.

Proposition 5.14. *If $c>-1$, $1<p\le\infty$ and $0<(m+1)/p<c+1$, then (e^{zL_b}) is an analytic semigroup of angle $\pi/2$ in L^p_m and $\hat C([0,\infty))$. Furthermore for every $\epsilon>0$, $0<\delta<1$ there exists $C>0$, independent of b, such that*

$$(25)\qquad |e^{zL_b}f|\le Ce^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re}z}S^{0,-c}(|z|)|f|,\quad f\in L^p_m,\ |\arg z|<\frac{\pi}{2}-\arctan\epsilon,$$

where

$$S^{0,-c}(t)f(y)=t^{-1/2}\int_0^{+\infty}\left(\frac{\rho}{\sqrt t}\wedge 1\right)^c\exp\left(-\frac{|y-\rho|^2}{\kappa t}\right)f(\rho)\,d\rho$$

for a suitable $\kappa>0$.

Proof. All properties for $p=2$, $m=c$ are contained in Theorem 5.13. The boundedness of e^{zL_b} in L^p_m then follows from (25) and [21, Proposition 12.2], [24, Proposition 6.2], and (25) extends to L^p_m.

The semigroup law is inherited from the one of L^2_c via a density argument and we only have to prove the strong continuity at 0. Assume first that $p<\infty$ and let $f,g\in C_c^\infty(\mathbb{R}_+)$. Then as $z\to 0$ with $z\in\Sigma_{\pi/2-\arctan\epsilon}$,

$$\begin{aligned}\int_0^\infty (e^{zL_b}f)gy^m\,dy&=\int_0^\infty (e^{zL_b}f)gy^{m-c}y^c\,dy\\&\to\int_0^\infty fgy^{m-c}y^c\,dy=\int_0^\infty fgy^m\,dy,\end{aligned}$$

by the strong continuity of e^{zL_b} in L^2_c. Observe now, by (25) and [21, Proposition 12.2], that the family $\{e^{z(L_b-\frac{b^2}{4\epsilon^2\delta})}: z\in\Sigma_{\pi/2-\arctan\epsilon}\}$ is uniformly bounded on $\mathcal{B}(L^p_m)$. Up to rescaling, a density argument then proves that the previous limit holds for all $f\in L^p_m$ and $g\in L^{p'}_m$. The semigroup is then weakly continuous, hence strongly continuous.

When $p=\infty$ the proof of the strong continuity in $\hat C([0,\infty))$ follows from the domain characterization for $p<\infty$ of Theorem 5.16 below, as in [24, Proposition 2.3]. ∎

To characterize the domain of L_b we need the following interpolation inequality.

Lemma 5.15. *If $c>-1$ and $0<(m+1)/p<c+1$, then there exists $C>0$ such that for every $u\in W^{2,p}_{m,\mathcal{N}}$,*

$$\|D_yu\|_{L^p_m}\le\varepsilon\|Bu\|_{L^p_m}+\frac{C}{\varepsilon}\|u\|_{L^p_m}.$$

A similar estimate holds for $p=\infty$ if $u\in\hat C^2_{\mathcal{N}}$.

Proof. For $p<\infty$ we use the pointwise estimate

$$|D_y e^{tB^n} f| \le \frac{C}{\sqrt{t}} S_0^{-c}(t)|f|$$

which follows from Proposition 4.2 and the norm estimate $\|S_0^{-c}(t)\|_{L^p_m} \le C$ (see [21, Proposition 12.2] or Proposition 4.3). If $u \in W^{2,p}_{m,\mathcal{N}}$ and $f = \lambda u - B^n u$, $\lambda > 0$, then $u = \int_0^\infty e^{-\lambda t} e^{tB^n} f\,dt$ and so

$$\begin{aligned}\|D_y u\|_{m,p} &\le \int_0^\infty \|e^{-\lambda t} D_y e^{tB^n} f\|_{m,p}\,dt \le C\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}} \big\|S_0^{-c}|f|\big\|_{m,p}\,dt\\ &\le \frac{C}{\sqrt{\lambda}}\|f\|_{m,p} \le C\Big(\sqrt{\lambda}\|u\|_{m,p} + \frac{1}{\sqrt{\lambda}}\|Bu\|_{m,p}\Big).\end{aligned}$$

When $p=\infty$ the proof is similar [23, Corollary 4.7]. ∎

THEOREM 5.16. *If $c>-1$, $1<p<\infty$ and $0<(m+1)/p<c+1$, the generator of (e^{zL_b}) is the operator L_b with domain $W^{2,p}_{m,\mathcal{N}}$.*

When $p=\infty$ the generator is L_b with domain $\hat{C}^2_{\mathcal{N}}$.

Proof. By the lemma above, the operator D_y is a small perturbation of B^n (with domain $W^{2,p}_{m,\mathcal{N}}$ or $\hat{C}^2([0,\infty))$) and therefore, by [12, Chapter III, Theorem 2.10], $L_b = B^n + ibD_y$, with the same domain as B^n, generates an analytic semigroup. We have to show that this semigroup coincides with that constructed before.

We consider first the case $p<\infty$. Let $(L_{m,p}, D_{m,p})$ be the generator of (e^{tL_b}) in L^p_m and consider the set

$$\mathcal{D} = \{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\},$$

which is dense in $W^{2,p}_{m,\mathcal{N}}$ by Theorem 4.1.

By using the definition of L_b through the form $\mathfrak{a}_b$ (see (13)), it is easy to see that $\mathcal{D} \subset D_{c,2}$ and that $L_b = L_{c,2}$ on $\mathcal{D}$. Since $\mathcal{D}$ is dense in $W^{2,2}_{c,\mathcal{N}}$, and L_b is closed on $W^{2,2}_{c,\mathcal{N}}$ and $L_{c,2}$ is closed on $D_{c,2}$, it follows that $W^{2,2}_{c,\mathcal{N}} \subset D_{c,2}$ and so $W^{2,2}_{c,\mathcal{N}} = D_{c,2}$, $L_b = L_{c,2}$ since both operators are generators. This completes the proof in the special case $p=2, m=c$.

Take now $u \in \mathcal{D}$ and let $f = \lambda u - L_b u \in L^p_m \cap L^2_c$ for large λ. Let $v \in D_{m,p}$ solve $\lambda v - L_{m,p} v = f$. Since the semigroups are consistent, v coincides with the L^2_c solution, which, by the previous step, is u. This shows that $\mathcal{D} \subset D_{m,p}$ and $L_b = L_{m,p}$ on $\mathcal{D}$ and, as before, one concludes the proof for $p<\infty$.

When $p=\infty$, we change $\mathcal{D}$ to

$$\mathcal{D} \subset \tilde{\mathcal{D}} = \{u \in C^2([0,\infty]) : D_y u(0) = 0, u \text{ with compact support}\},$$

which is dense in $\hat{C}^2_{\mathcal{N}}$. We choose p such that $1/p < c+1$ and argue as above, using L^p (with respect to the Lebesgue measure) instead of L^2_c. ∎

Formula (15) and the previous results allow us to characterize the domain of $A_b = B - i\frac{bc}{2y}$. Note that the Neumann boundary condition for L_b, that is, $y^c v'(y) \to 0$ as $y \to 0$, translates into a (complex) Robin condition $y^c(u'(y) - i\frac{b}{2}u(y)) \to 0$ for A_b. We formulate this result only for $p < \infty$.

PROPOSITION 5.17. *If $c > -1$ and $0 < (m+1)/p < c+1$, then*

$$D(A_b) = \left\{ u \in W^{2,p}_m : \frac{1}{y}\left(D_y u - i\frac{b}{2}u\right) \in L^p_m \right\},$$

Proof. Observe that the isometry T preserves $W^{2,p}_m$ and for any $v \in W^{2,p}_m$, setting $u(y) = Tv(y) = e^{i\frac{b}{2}y}v(y)$, one has

$$D_y u = T\left(D_y v + i\frac{b}{2}v\right), \qquad D_{yy}u = T\left(D_{yy}v + ibD_y v - \frac{b^2}{4}v\right).$$

The conclusion then follows by using the equalities

$$A_b = T \circ L_b \circ T^{-1}, \qquad D(A_b) = T^{-1}(W^{2,p}_{m,\mathcal{N}}). \ \blacksquare$$

5.5. Bounds for $D_y e^{zL_b}$. We need the following regularity result which follows from the holomorphy of e^{zL_b} and the characterization of the domain of L_b.

LEMMA 5.18. *For every fixed $\rho > 0$ the kernel $p_b(z, y, \rho)$ is holomorphic in $z \in \mathbb{C}_+$, and $p(z, \cdot, \rho) \in \hat{C}^2_{\mathcal{N}}$ if $\mathrm{Re}\, z > 0$. Moreover, all derivatives are jointly continuous in $\mathbb{C}_+ \times [0, \infty)$.*

Proof. Fixing p such that $1/p < c+1$ we work in $L^p = L^p_0$. If $s > 0$, by Theorem 5.13, $p_b(s, \cdot, \rho) \in L^p$ and so $e^{zL_b}p_b$ belongs to the domain of L_b in L^p, since the semigroup is analytic. Since $e^{zL_b}p_b(s, y, \rho) = p_b(z+s, y, \rho)$ by the semigroup law, Theorem 5.16 show that $p(z+s, \cdot, \rho) \in W^{2,p} \subset \hat{C}([0, \infty))$. Repeating the argument in this last space we deduce by Theorem 5.16 with $p = \infty$ that $p(z + 2s, \cdot, \rho) \in \hat{C}^2_{\mathcal{N}}$.

The analyticity with respect to $z \in \mathbb{C}_+$ and the joint continuity of the derivatives follow again by the identity $e^{zL_b}p_b(s, y, \rho) = p_b(z+s, y, \rho)$, using the analyticity of the semigroup in $\hat{C}([0, \infty))$, since the domain is $\hat{C}^2_{\mathcal{N}}$. $\blacksquare$

The Cauchy formula for the derivatives of holomorphic functions allows us to estimate $D_z p_b$ and $L_b p_b$.

PROPOSITION 5.19. *Let $c + 1 > 0$ and $b \in \mathbb{R}$. Then for every $\epsilon > 0$ and $0 < \delta < 1$ there exist $C, k > 0$ independent of b such that, for every $z \in \Sigma_{\pi/2 - \arctan \epsilon}$ and almost all $y, \rho > 0$,*

$$|L_b p_b(z, y, \rho)| + |D_z p_b(z, y, \rho)| \\ \le C e^{\frac{b^2}{4\epsilon^2\delta}\mathrm{Re}\, z} |z|^{-3/2} \rho^{-c} \left(\frac{\rho}{|z|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right).$$

Proof. Since the kernel p_b satisfies $D_z p_b = L_b p_b$, it is sufficient to deal with $D_z p_b$. Fix $\epsilon > 0$ and $0 < \delta < 1$. Setting $r := \tan\left(\frac{\arctan\epsilon - \arctan(\delta\epsilon)}{2}\right) < 1$, observe that

$$B(z_0, r|z_0|) \subset \Sigma_{\pi/2-\arctan(\delta\epsilon)}, \qquad \forall z_0 \in \Sigma_{\pi/2-\arctan\epsilon}.$$

Using the Cauchy formula for the derivatives of holomorphic functions in the ball $B(z_0, r|z_0|)$, we get

$$|D_z p_b(z_0, y, \rho)| \le \frac{1}{r|z_0|} \max_{|z-z_0|=r|z_0|} |p_b(z, y, \rho)|, \qquad y, \rho > 0.$$

Applying the estimate of Theorem 5.13 in the sector $\Sigma_{\pi/2-\arctan(\delta\epsilon)}$ we obtain, for suitable C', κ',

$$|D_z p_b(z_0, y, \rho)| \le C' \frac{1}{|z_0|^{3/2}} e^{\frac{b^2}{4\epsilon^2\delta^3} \operatorname{Re} z_0} \rho^{-c} \left(\frac{\rho}{|z_0|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa'|z_0|}\right),$$

which, by the arbitrariness of δ, is equivalent to the statement. ∎

Before proving the estimates for the derivative of the kernel p_b, let us observe that when $b \neq 0$, using the scaling equalities of Proposition 5.5(iii), for any $z \in \Sigma_{\pi/2}$ we have

$$\frac{1}{b^2} L_b = I_{|b|} \circ L_{b/|b|} \circ I_{1/|b|}, \qquad e^{zL_b} = I_{|b|} \circ e^{b^2 z L_{b/|b|}} \circ I_{1/|b|}$$

and so for $y, \rho > 0$

$$\begin{aligned} p_b(z, y, \rho) &= |b|^{c+1}\, p_{b/|b|}(b^2 z, |b|y, |b|\rho), \\ D_y p_b(z, y, \rho) &= |b|^{c+2}\, D_y p_{b/|b|}(b^2 z, |b|y, |b|\rho). \end{aligned} \tag{26}$$

The last equalities allow us to assume $|b| = 1$ in what follows.

We start by proving some interpolation estimates with respect to the sup-norm $\|\cdot\|_\infty$.

LEMMA 5.20. *Let $c + 1 > 0$ and $|b| = 1$. Then there exists $C > 0$ such that for $\lambda > 0$ and $u \in \hat{C}^2_{\mathcal{N}}$,*

$$\lambda\|u\|_\infty + \lambda^{1/2}\|D_y u\|_\infty + \|D_{yy} u\|_\infty \le C(\|\lambda u - L_b u\|_\infty + \|u\|_\infty).$$

Proof. Since $(L_b, \hat{C}^2_{\mathcal{N}})$ generates a bounded semigroup, for $u \in \hat{C}^2_{\mathcal{N}}$ and $\lambda > 0$ we have

$$\lambda\|u\|_\infty \le C(\|\lambda u - L_b u\|_\infty), \quad \|D_{yy} u\|_\infty \le C(\|u\|_\infty + \|L_b u\|_\infty).$$

Then

$$\|D_{yy} u\|_\infty \le C(\|\lambda u - L_b u\|_\infty + (\lambda + 1)\|u\|_\infty) \le 2C(\|\lambda u - L_b u\|_\infty + \|u\|_\infty).$$

The estimate of the gradient term follows from

$$\|D_y u\|_\infty^2 \le C\|u\|_\infty \|D_{yy} u\|_\infty. \ ∎$$

Now we localize the gradient estimates above. For $y, r > 0$ we set $B^+(y,r) := B(y,r) \cap \mathbb{R}_+$.

PROPOSITION 5.21. *Let $c+1 > 0$ and $|b| = 1$. Then there exists a constant $C > 0$ such that for all $u \in \hat{C}^2_{\mathcal{N}}$ and $\lambda > 0$,*

$$\lambda^{1/2}\|D_y u\|_{L^\infty(B^+(y,r))} \le C\big(\|\lambda u - L_b u\|_{L^\infty(B^+(y,2r))} + \big(\tfrac{1}{r^2}+1\big)\|u\|_{L^\infty(B^+(y,2r))}\big).$$

Proof. Set $r_n = r\sum_{k=1}^n 2^{-k}$. Then $r_1 = r/2$, $r_\infty = r$, and $r_{n+1} - r_n = r2^{-(n+1)}$.

Let $B_n^+ = B^+(y, r_n)$, $B_r^+ = B^+(y,r)$ and so on, and choose cut-off functions $\eta_n \in C_c^\infty(\mathbb{R})$ such that $\eta_n(y) = \eta_n(-y)$, $0 \le \eta_n \le 1$, $\eta_n = 1$ in B_n^+, $(\operatorname{supp} \eta_n) \cap \mathbb{R}_+ \subset B_{n+1}^+$, $|D_y\eta_n| \le \frac{C}{r}2^n$, $|D_{yy}\eta_n| \le \frac{C}{r^2}4^n$ for some constant $C > 0$ independent of n. Then also $|y^{-1}D_y\eta_n| \le \frac{C}{r^2}4^n$, since $D_y\eta_n(0) = 0$.

If $u \in \hat{C}_{\mathcal{N}}$ then $\eta_n u \in \hat{C}_{\mathcal{N}}$ and we have

$$L_b(\eta_n u) = \eta_n L_b u + 2D_y\eta_n D_y u + u\left(D_{yy}\eta_n + c\frac{D_y\eta_n}{y} + ibD_y\eta_n\right).$$

Applying Lemma 5.20 to $\eta_n u$ and using the inequality $s \le 1 + s^2$ we get

$$\begin{aligned}
&\lambda\|\eta_n u\|_\infty + \sqrt{\lambda}\|D_y(\eta_n u)\|_\infty + \|D_{yy}(\eta_n u)\|_\infty \\
&\quad \le C(\|(\lambda - L_b)(\eta_n u)\|_\infty + \|\eta_n u\|_\infty) \\
&\quad \le C\Bigg(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \frac{2^n}{r}\|D_y u\|_{L^\infty(B^+_{r_{n+1}})} \\
&\qquad\qquad + \|u\|_{L^\infty(B_r^+)}\left\|D_{yy}\eta_n + c\frac{D_y\eta_n}{y} + ibD_y\eta_n\right\|_\infty + \|u\|_{L^\infty(B_r^+)}\Bigg) \\
&\quad \le C\left(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \frac{2^n}{r}\|D_y(\eta_{n+1}u)\|_\infty + \left(\frac{4^n}{r^2}+1\right)\|u\|_{L^\infty(B_r^+)}\right).
\end{aligned}$$

Applying the interpolation inequalities for the gradient we get

$$\begin{aligned}
&\lambda\|\eta_n u\|_\infty + \sqrt{\lambda}\|D_y(\eta_n u)\|_\infty + \|D_{yy}(\eta_n u)\|_\infty \\
&\le C\Bigg(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \epsilon\frac{2^n}{r}\|D_{yy}(\eta_{n+1}u)\|_\infty \\
&\qquad\qquad + \frac{2^n}{\epsilon r}\|\eta_{n+1}u\|_\infty + \left(\frac{4^n}{r^2}+1\right)\|u\|_{L^\infty(B_r^+)}\Bigg) \\
&\le C\left(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \epsilon\frac{2^n}{r}\|D_{yy}(\eta_{n+1}u)\|_\infty + \left(\frac{2^n}{\epsilon r} + \frac{4^n}{r^2} + 1\right)\|u\|_{L^\infty(B_r^+)}\right).
\end{aligned}$$

Setting $\xi := C2^n\varepsilon r^{-1}$, we get

$$\lambda\|\eta_n u\|_\infty + \sqrt{\lambda}\|D_y(\eta_n u)\|_\infty + \|D_{yy}(\eta_n u)\|_\infty$$
$$\leq C\left(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \left(\frac{4^n}{\xi r^2} + \frac{4^n}{r^2} + 1\right)\|u\|_{L^\infty(B_r^+)}\right) + \xi\|D_{yy}(\eta_{n+1}u)\|_\infty.$$

It follows that

$$\xi^n(\sqrt{\lambda}\|D_y u\|_{L^\infty(B^+_{\frac{r}{2}})} + \|D_{yy}(\eta_n u)\|_\infty)$$
$$\leq C\left(\xi^n\|\lambda - L_b u\|_{L^\infty(B_r^+)} + \xi^n\left(\frac{4^n}{\xi r^2} + \frac{4^n}{r^2} + 1\right)\|u\|_{L^\infty(B_r^+)}\right)$$
$$+ \xi^{n+1}\|D_{yy}(\eta_{n+1}u)\|_\infty.$$

By choosing $\varepsilon = \varepsilon_n$ so that $\xi = 1/8$ and summing the previous inequality over $n \in \mathbb{N}$ we get

$$\sqrt{\lambda}\|D_y u\|_{L^\infty(B^+_{r/2})} + \sum_{n=1}^\infty \xi^n\|D_{yy}(\eta_n u)\|_\infty$$
$$\leq C\left(\|\lambda - L_b u\|_{L^\infty(B_r^+)} + \left(\frac{1}{r^2} + 1\right)\|u\|_{L^\infty(B_r^+)}\right) + \sum_{n=1}^\infty \xi^{n+1}\|D_{yy}(\eta_{n+1}u)\|_\infty.$$

Cancelling equal terms on both sides we arrive at

$$\sqrt{\lambda}\|D_y u\|_{L^\infty(B^+_{r/2})} \leq C\left(\|\lambda u - L_b u\|_{L^\infty(B_r^+)} + \left(\frac{1}{r^2} + 1\right)\|u\|_{\infty,r}\right). \ \blacksquare$$

To prove estimates for the derivative of the kernel p_b we also use the following basic estimate.

LEMMA 5.22. *Let $y_0, \rho > 0$. Then*

$$\sup_{y\in B(y_0,\sqrt{t})} \exp\left(-\frac{|y-\rho|^2}{t}\right) \leq e^{16}\exp\left(-\frac{9}{16}\,\frac{|y_0-\rho|^2}{t}\right).$$

Proof. If $|y_0 - \rho| \leq 4\sqrt{t}$ then for every $y > 0$,

$$\exp\left(-\frac{|y-\rho|^2}{t}\right) \leq e^{16}\exp\left(-\frac{|y_0-\rho|^2}{t}\right).$$

If $|y_0 - \rho| > 4\sqrt{t}$ and $y \in B(y_0, \sqrt{t})$, then $|y - \rho| \geq |y_0 - \rho| - |y - y_0| \geq |y_0 - \rho| - \frac{1}{4}|y_0 - \rho| = \frac{3}{4}|y_0 - \rho|$ and

$$\exp\left(-\frac{|y-\rho|^2}{t}\right) \leq \exp\left(-\frac{9}{16}\frac{|y_0-\rho|^2}{t}\right). \ \blacksquare$$

THEOREM 5.23. *Let $c + 1 > 0$ and $b \in \mathbb{R}$. Then for every $\epsilon > 0$ and $0 < \delta < 1$ there exist $C, k > 0$ independent of b such that, for every $z \in$*

$\Sigma_{\pi/2-\arctan\epsilon}$ *and almost all* $y,\rho>0$,

$$|D_y p_b(z,y,\rho)| \le C e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z}\frac{1}{|z|}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}}\wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right).$$

Proof. If $b=0$ this follows from Proposition 4.2. Let now $b\neq 0$; using the scaling property (26), we may assume that $|b|=1$. Then applying Proposition 5.21 to the function $u=p_b(z,\,\cdot\,,\rho)$ in $B^+(y_0,r)$ with $r=\sqrt{|z|}$ we get, for any $\lambda>0$,

$$\lambda|D_y p_b(z,y_0,\rho)| \le C\left(\|L_b u\|_{L^\infty(B^+(y,2r))} + \left(\frac{1}{r^2}+1+\lambda^2\right)\|u\|_{L^\infty(B^+(y,2r))}\right).$$

Using Theorem 5.13 and Proposition 5.19 with δ' such that $0<\delta<\delta'<1$, for suitable $C,\kappa>0$ we get

$$\begin{aligned}&\|u\|_{L^\infty(B^+(y_0,2r))}\\ &\quad\le C e^{\frac{1}{4\epsilon^2\delta'}\operatorname{Re} z}|z|^{-1/2}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}}\wedge 1\right)^c \sup_{y\in B^+(y_0,2r)}\exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right),\\ &\|L_b u\|_{L^\infty(B^+(y_0,2r))}\\ &\quad\le C e^{\frac{1}{4\epsilon^2\delta'}\operatorname{Re} z}|z|^{-3/2}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}}\wedge 1\right)^c \sup_{y\in B^+(y_0,2r)}\exp\left(-\frac{|y-\rho|^2}{\kappa|z|}\right).\end{aligned}$$

Lemma 5.22 then implies (for suitable $C',\kappa'>0$)

$$\|u\|_{L^\infty(B^+(y_0,2r))}\le A(z,y_0,\rho),\qquad \|L_b u\|_{L^\infty(B^+(y_0,2r))}\le \frac{1}{|z|}A(z,y_0,\rho),$$

where

$$A(z,y_0,\rho):=C' e^{\frac{1}{4\epsilon^2\delta'}\operatorname{Re} z}|z|^{-1/2}\rho^{-c}\left(\frac{\rho}{|z|^{1/2}}\wedge 1\right)^c \exp\left(-\frac{|y_0-\rho|^2}{\kappa'|z|}\right).$$

The previous inequalities then imply

$$|D_y p_b(z,y_0,\rho)| \le \left(\frac{1}{\lambda}\left(\frac{2}{|z|}+1\right)+\lambda\right)A(z,y_0,\rho).$$

Minimizing the last inequality over $\lambda>0$ we get, for a suitable $C>0$,

$$|D_y p_b(z,y_0,\rho)| \le 2\frac{\sqrt{2+|z|}}{\sqrt{|z|}}A(z,y_0,\rho)\le C\frac{e^{\frac{1}{4\epsilon^2}\left(\frac{1}{\delta}-\frac{1}{\delta'}\right)\operatorname{Re} z}}{\sqrt{|z|}}A(z,y_0,\rho),$$

which is the statement for $|b|=1$. ∎

Corollary 5.24. *Let* $1<p\le\infty$, $0<(m+1)/p<c+1$, *and let* $f\in L^p_m$ *or* $f\in \hat C([0,\infty))$. *For every* $\epsilon>0$, $0<\delta<1$ *there exists* $C>0$ *independent of* b, *such that, for every* $z\in\Sigma_{\pi/2-\arctan\epsilon}$, $e^{zL_b}f$ *is differentiable in* $(0,\infty)$

and satisfies

$$D_y e^{zL_b} f = \int_0^{+\infty} D_y p_b(z,\cdot,\rho) f(\rho)\rho^c \, d\rho \tag{27}$$

$$|D_y e^{zL_b} f| \le \frac{C}{|z|^{1/2}} e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z} S^{0,-c}(|z|)|f|. \tag{28}$$

Proof. Let $y_0, r > 0$ with $0 \notin [y_0 - r, y_0 + r]$. By Theorem 5.23, for almost all $y \in (y_0 - r, y_0 + r)$ and $\rho \in (0,\infty)$,

$$|D_y p_b(z,y,\rho)|\rho^c \le C e^{\frac{b^2}{4\epsilon^2\delta}\operatorname{Re} z} \frac{1}{|z|}\left(\frac{\rho}{|z|^{1/2}} \wedge 1\right)^c \exp\left(-\frac{\rho^2}{\kappa|z|}\right) \tag{29}$$

for suitable C and κ depending also on r and y_0. Then (27) follows by differentiating under the integral sign since the right hand side of (29) belongs to $L^{p'}_m$. Finally, (28) is consequence of Theorem 5.23. ∎

6. Multipliers. In this section we investigate the boundedness of some multipliers related to the degenerate operator

$$\mathcal{L} = \Delta_x + 2\sum_{i=1}^N a_i D_{iy} + D_{yy} + \frac{c}{y} D_y, \qquad a \in \mathbb{R}^N,\ |a| < 1.$$

Assuming that

$$\Delta_x u + 2a \cdot \nabla_x D_y u + B_y u = f$$

and taking the Fourier transform (denoted by $\mathcal{F}$ or $\hat{\cdot}$) with respect to x (with covariable ξ) we obtain

$$-|\xi|^2\, \hat{u}(\xi,y) + i2a\cdot\xi D_y \hat{u}(\xi,y) + B_y \hat{u}(\xi,y) = \hat{f}(\xi,y).$$

We introduce the quadratic form

$$Q_a(\xi) = |\xi|^2 - |a\cdot\xi|^2, \qquad (1-|a|^2)|\xi|^2 \le Q_a(\xi) \le |\xi|^2, \qquad \xi \in \mathbb{R}^N, \tag{30}$$

and consider the operator L_b of Section 5 with $b = 2a\cdot\xi$. The above computation shows that formally

$$(\lambda - \mathcal{L})^{-1} = \mathcal{F}^{-1}(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1}\mathcal{F}.$$

In order to prove that $\mathcal{L}$ generates an analytic semigroup and to prove regularity for the associated parabolic problem, we investigate the boundedness of the operator-valued multiplier

$$\mathbb{R}^N \ni \xi \mapsto R_\lambda(\xi) = (\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1}.$$

To characterize the domain of $\mathcal{L}$ we also consider the multipliers $|\xi|^2 R_\lambda$, $\xi D_y R_\lambda$ associated respectively with $\Delta_x(\lambda - \mathcal{L})^{-1}$, $D_{xy}(\lambda - \mathcal{L})^{-1}$. In the what follows we prove that the above multipliers satisfy the hypotheses of Theorem 2.5.

The following lemma is a reformulation of the heat kernel bounds of the previous section, adapted to the multipliers above. For any $|a| \le \delta < 1$ we set

$$\theta_\delta = \arctan \frac{|a|}{\sqrt{\delta^2 - |a|^2}} \in \left(\arctan \frac{|a|}{\sqrt{1-|a|^2}}, \frac{\pi}{2}\right]$$

where $\theta_{|a|} := \pi/2$.

LEMMA 6.1. *Let* $1 < p < \infty$ *be such that* $0 < (m+1)/p < c+1$. *For every* $|a| \le \delta < \delta' < 1$, *there exists* $C > 0$ *such that for* $f \in L^p_m$ *and* $z \in \Sigma_{\pi/2-\theta_\delta}$,

$$\begin{aligned}|e^{z(L_{2a\cdot\xi} - Q_a(\xi))} f| &\le C e^{-(Q_{\delta' \frac{a}{|a|}}(\xi)) \operatorname{Re} z} S^{0,-c}(|z|)|f| \\ &\le C e^{-(1-\delta'^2)|\xi|^2 \operatorname{Re} z} S^{0,-c}(|z|)|f|\end{aligned}$$

and

$$|D_y e^{z(L_{2a\cdot\xi} - Q_a(\xi))} f| \le C e^{-(1-\delta'^2)|\xi|^2 \operatorname{Re} z} \frac{S^{0,-c}(|z|)}{\sqrt{|z|}} |f|.$$

Proof. We use Proposition 5.14 with $\varepsilon = \frac{|a|}{\sqrt{\delta^2 - |a|^2}}$, $0 < \gamma = \frac{\delta^2 - |a|^2}{\delta'^2 - |a|^2} < 1$ and (30) to get

$$\begin{aligned}|e^{z(L_{2a\cdot\xi} - Q_a(\xi))} f| &\le C e^{-(Q_a(\xi) - \frac{|a\cdot\xi|^2}{\epsilon^2\gamma}) \operatorname{Re} z} S^{0,-c}(|z|)|f| \\ &= C e^{-(Q_{\delta' \frac{a}{|a|}}(\xi)) \operatorname{Re} z} S^{0,-c}(|z|)|f| \le C e^{-(1-\delta'^2)|\xi|^2 \operatorname{Re} z} S^{0,-c}(|z|)|f|\end{aligned}$$

for any $f \in L^p_m$ and $|\arg z| < \pi/2 - \theta_\delta$. The estimate for $D_y e^{z(L_{2a\cdot\xi} - Q_a(\xi))}$ follows similarly using Theorem 5.23. ∎

The following formulas follow since the resolvent is the Laplace transform of the semigroup. We state them to have precise bounds and to show that a similar representation holds for the gradient of the resolvent.

LEMMA 6.2. *Let* $|a| \le \delta < \delta' < 1$. *Then any* $f \in L^p_m$ *and* $\lambda \in \Sigma_{\pi-\theta_\delta}$,

$$(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} = e^{-i\theta} \int_0^\infty e^{-e^{-i\theta}\lambda t} e^{e^{-i\theta} t (L_{2a\cdot\xi} - Q_a(\xi))} f \, dt,$$

$$D_y(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} = e^{-i\theta} \int_0^\infty e^{-e^{-i\theta}\lambda t} D_y e^{e^{-i\theta} t (L_{2a\cdot\xi} - Q_a(\xi))} f \, dt.$$

Here $\theta = \frac{|\arg\lambda|}{\arg\lambda}(\pi/2 - \theta_{\delta'})$.

Proof. If $\lambda \in \Sigma_{\pi-\theta_\delta}$, then $\mu := e^{-i\theta}\lambda \in \Sigma_{\pi/2-\theta_\delta+\theta_{\delta'}}$. By Lemma 6.1,

$$|e^{t(e^{-i\theta}(L_{2a\cdot\xi} - Q_z(\xi)))} f| \le C e^{-(1-\delta')\cos\theta t} S^{0,-c}(t)|f|.$$

This implies, by standard results on analytic semigroups,

$$(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} = e^{-i\theta}\big(e^{-i\theta}\lambda - e^{-i\theta}(L_{2a\cdot\xi} - Q_a(\xi))\big)^{-1} = e^{-i\theta}\int_0^\infty e^{-e^{-i\theta}\lambda t} e^{e^{-i\theta}t(L_{2a\cdot\xi}-Q_a(\xi))} f\,dt.$$

The remaining equality follows similarly by differentiating the integrand (note that both integrals converge). ■

THEOREM 6.3. *Let $1 < p < \infty$ be such that $0 < (m+1)/p < c+1$. For every $|a| \le \delta < 1$ there exist $C, k > 0$ such that for $f \in L^p_m$ and $\lambda \in \Sigma_{\pi-\theta_\delta}$,*

$$|(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| \le C\Gamma(k(|\lambda| + |\xi|^2))|f|, \tag{31}$$

$$|D_y(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| \le C\Psi(k(|\lambda| + |\xi|^2))|f|. \tag{32}$$

Proof. For $\lambda \in \Sigma_{\pi-\theta_\delta}$ let us choose $\delta_2 < \delta_2'$ such that

$$|a| \le \delta < \delta_2 < \delta_2' < 1, \qquad \theta_\delta - \theta_{\delta_2} \le \pi/2,$$

and set $\theta = \frac{|\arg\lambda|}{\arg\lambda}(\pi/2 - \theta_{\delta_2})$ so that $\mu := e^{-i\theta}\lambda \in \Sigma_{\pi/2-\theta_\delta+\theta_{\delta_2}}$. Then using Lemmas 6.1 and 6.2 we get, for some C depending on δ_2,

$$\begin{aligned}|(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| &= \big|\big(\mu - e^{-i\theta}(L_{2a\cdot\xi} - Q_a(\xi))\big)^{-1} f\big| \\ &= \Big|\int_0^\infty e^{-\mu t} e^{e^{-i\theta}t(L_{2a\cdot\xi}-Q_a(\xi))} f\,dt\Big| \\ &\le C\int_0^\infty e^{-\operatorname{Re}\mu t} e^{-Q_{\delta_2'\frac{a}{|a|}}(\xi)\cos\theta t} S^{0,-c}(t)|f|\,dt \\ &\le C\int_0^\infty e^{-|\lambda|\sin(\theta_\delta-\theta_{\delta_2})t} e^{-Q_{\delta_2'\frac{a}{|a|}}(\xi)\sin\theta_{\delta_2}t} S^{0,-c}(t)|f|\,dt.\end{aligned}$$

Then using (30) we obtain for some positive constant k depending on δ,

$$|(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| \le C\int_0^\infty e^{-k(|\lambda|+|\xi|^2)} S^{0,-c}(t)|f|\,dt = C\Gamma(k(|\lambda| + |\xi|^2))|f|,$$

which is (31). The proof of (32) is similar: using Lemmas 6.1 and 6.2 and proceeding as before we get

$$\begin{aligned}|D_y(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| &= \big|\big(\mu - e^{-i\theta}D_y(L_{2a\cdot\xi} - Q_a(\xi))\big)^{-1} f\big| \\ &\le C\int_0^\infty e^{-k_\delta(|\lambda|+|\xi|^2)} S^{0,-c}(t)|f|\,dt = C\Psi(k(|\lambda| + |\xi|^2))|f|. \ ■\end{aligned}$$

The following result is a consequence of Lemma 6.1 and Theorem 6.3.

COROLLARY 6.4. *Let $1 < p < \infty$ be such that $0 < (m+1)/p < c+1$. For every $|a| \le \delta < 1$ the families of operators*

$$\{e^{z(L_{2a\cdot\xi} - Q_a(\xi))} : z \in \Sigma_{\pi/2-\theta_\delta},\ \xi \in \mathbb{R}^N \setminus \{0\}\},$$

$$\{\lambda(\lambda - L_{2a\cdot\xi} + Q_a(\xi))^{-1},\ \sqrt{\lambda} D_y(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} : \lambda \in \Sigma_{\pi-\theta_\delta},\ \xi \in \mathbb{R}^N \setminus \{0\}\}$$

are $\mathcal{R}$-bounded in L^p_m.

Proof. The $\mathcal{R}$-boundedness of $e^{z(L_{2a\cdot\xi} - Q_a(\xi))}$ follows from Lemma 6.1 and by domination using Corollary 2.2, since the family $(S^{\alpha,-c}(t))_{t\ge0}$ is $\mathcal{R}$-bounded by Proposition 4.3(i). The $\mathcal{R}$-boundedness of the families involving the resolvent follows from domination again, using Proposition 4.3(ii, iii), since by (31) one has

$$|\lambda(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} f| \le C|\lambda|\,\Gamma(k(|\lambda| + |\xi|^2))|f| \le C|\lambda|\,\Gamma(k|\lambda|)|f|,$$

and similarly for $\sqrt{\lambda} D_y(\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1}$. ∎

From now on we denote by $R_\lambda(\xi)$ the operator

$$(\lambda - L_{2a\cdot\xi} + Q_a(\xi))^{-1}$$

whenever it is defined.

To apply the Mikhlin multiplier theorem, we need a formula for the derivatives of the above functions with respect to ξ. In the following lemma, $\mathcal{S}_n$ denotes the set of permutations of n elements.

LEMMA 6.5. *Let $1 < p < \infty$ be such that $0 < (m+1)/p < c+1$, and consider, for any fixed $\lambda \in \Sigma_{\pi - \arctan\frac{|a|}{\sqrt{1-|a|^2}}}$, the map*

$$\mathbb{R}^N \ni \xi \mapsto R_\lambda(\xi) = (\lambda - L_{2a\cdot\xi} + Q_a(\xi))^{-1} \in B(L^p_m).$$

Then $R_\lambda, D_y R_\lambda \in C^\infty(\mathbb{R}^N \setminus \{0\}; B(L^p_m))$ and for any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \{0,1\}^N$ with $|\alpha| = n$ one has

$$\begin{aligned} D^\alpha_\xi R_\lambda(\xi) &= \sum_{\sigma\in\mathcal{S}_n} R_\lambda(\xi) \prod_{j=1}^n \big(2ia_{\sigma(j)} D_y R_\lambda(\xi) - 2\xi_{\sigma(j)} R_\lambda(\xi)\big), \\ D^\alpha_\xi D_y R_\lambda(\xi) &= \sum_{\sigma\in\mathcal{S}_n} D_y R_\lambda(\xi) \prod_{j=1}^n \big(2ia_{\sigma(j)} D_y R_\lambda(\xi) - 2\xi_{\sigma(j)} R_\lambda(\xi)\big). \end{aligned} \tag{33}$$

Furthermore for every $|a| \le \delta < 1$ there exist $C, k > 0$, depending only on δ such that, setting $\mu = k(|\lambda| + |\xi|^2)$ one has

$$\begin{aligned} |D^\alpha_\xi R_\lambda(\xi) f| &\le C\Gamma(\mu)(\Psi(\mu) + |\xi|\Gamma(\mu))^n |f|, \quad f \in L^p_m,\ \lambda \in \Sigma_{\pi-\theta_\delta}, \\ |D^\alpha_\xi D_y R_\lambda(\xi) f| &\le C\Psi(\mu)(\Psi(\mu) + |\xi|\Gamma(\mu))^n |f|, \quad f \in L^p_m,\ \lambda \in \Sigma_{\pi-\theta_\delta}. \end{aligned} \tag{34}$$

Proof. Fix $|a| \le \delta < 1$ and $\lambda \in \Sigma_{\pi - \arctan \frac{|a|}{\sqrt{\delta - |a|^2}}}$. Let us prove the first equality in (33) for $n = 1$, that is, for $j = 1, \dots, N$,

$$(35)\qquad \frac{\partial}{\partial \xi_j}(R_\lambda(\xi)) = R_\lambda(\xi)(2ia_j D_y - 2\xi_j)R_\lambda(\xi), \qquad \xi \in \mathbb{R}^N \setminus \{0\}.$$

Indeed, let us write, for $|h| \le 1$,

$$(36)\qquad \begin{aligned} &R_\lambda(\xi + he_j) - R_\lambda(\xi) \\ &= (\lambda + Q_a(\xi + he_j) - L_{2a\cdot(\xi+he_j)})^{-1} - (\lambda + Q_a(\xi) - L_{2a\cdot\xi})^{-1} \\ &= R_\lambda(\xi)\big[(\lambda + Q_a(\xi) - L_{2a\cdot\xi})(\lambda + Q_a(\xi+he_j) - L_{2a\cdot(\xi+he_j)})^{-1} - I\big] \\ &= R_\lambda(\xi)(L_{2a\cdot(\xi+he_j)} - L_{2a\cdot\xi} + Q_a(\xi) - Q_a(\xi + he_j)) \\ &\quad \cdot (\lambda + Q_a(\xi + he_j) - L_{2a\cdot(\xi+he_j)})^{-1} \\ &= R_\lambda(\xi)(2ia\cdot he_j D_y + |\xi|^2 - |\xi + he_j|^2)R_\lambda(\xi + he_j) \\ &= R_\lambda(\xi)(2ia_j hD_y - 2\xi_j h - h^2)R_\lambda(\xi + he_j). \end{aligned}$$

Applying the previous inequality we get

$$\begin{aligned} &\frac{R_\lambda(\xi + he_j) - R_\lambda(\xi)}{h} - R_\lambda(\xi)(2ia_jD_y - 2\xi_j)R_\lambda(\xi) \\ &\quad = R_\lambda(\xi)(2ia_jD_y - 2\xi_j)(R_\lambda(\xi + he_j) - R_\lambda(\xi)) - hR_\lambda(\xi)R_\lambda(\xi + he_j) \\ &\quad := A + F. \end{aligned}$$

From now on we write C, k to denote some positive constants which depend only on δ.

The term F tends to 0 as h tends to 0 in the norm of $B(L^p_m)$ since by Theorem 6.3 we have, for any $f \in L^p_m$,

$$|Ff| \le C\,|h|\,\Gamma(k(|\lambda| + |\xi|^2))\Gamma\big(k(|\lambda| + |\xi + he_j|^2)\big)|f| \le C|h|\Gamma(k(|\lambda|))^2|f|$$

where in the last inequality we have used the fact that $\Gamma(\lambda)f$ is decreasing in $\lambda > 0$ if $f \ge 0$. This shows that $\|F\|_{L^p_m} \le C|h|$. Concerning A, we apply again (36) to write

$$A = R_\lambda(\xi)(2ia_jD_y - 2\xi_j)R_\lambda(\xi)(2ia_jhD_y - 2\xi_jh - h^2)R_\lambda(\xi + he_j).$$

Using again Theorem 6.3 and since $\Gamma(\lambda)f, \Psi(\lambda)f$ are decreasing in $\lambda > 0$ if $f \ge 0$, we obtain

$$\begin{aligned} |Af| &\le Ch^2|R_\lambda(\xi)D_yR_\lambda(\xi)D_yR_\lambda(\xi + he_j)f| + |h|\,|\xi|\,|R^2_\lambda(\xi)D_yR_\lambda(\xi + he_j)f| \\ &\quad + C(h^2(|\xi| + |h|)|R_\lambda(\xi)D_yR_\lambda(\xi)R_\lambda(\xi + he_j)f| \\ &\qquad + |h|\,|\xi|(|\xi| + |h|)|R^2_\lambda(\xi)R_\lambda(\xi + he_j)f|) \\ &\le C\big[h^2\,\Gamma(k|\lambda|)(\Psi(k|\lambda|))^2 + |\xi|\,|h|(\Gamma(k|\lambda|))^2\Psi(k|\lambda|) \\ &\quad + |h|^2(|\xi|+|h|)\Gamma(k|\lambda|)\Psi(k|\lambda|)\Gamma(k|\lambda|) + |h|\,|\xi|(|\xi|+|h|)(\Gamma(k|\lambda|))^3\big]|f|. \end{aligned}$$

Since $\Gamma(k|\lambda|), \Psi(k|\lambda|)$ are bounded in L^p_m by Proposition 4.3, the last inequality shows that A tends to 0 in the norm of $B(L^p_m)$ as $h \to 0$. This proves (35).

The proof of the second equality in (33) for $n = 1$, that is, for $j = 1, \dots, N$,

$$\frac{\partial}{\partial \xi_j}(D_y R_\lambda(\xi)) = D_y R_\lambda(\xi)(2ia_j D_y - 2\xi_j)R_\lambda(\xi), \qquad \xi \in \mathbb{R}^N \setminus \{0\}, \tag{37}$$

is similar and we only sketch the main steps. As in (36), we write

$$\begin{aligned}
&\frac{D_y R_\lambda(\xi + he_j) - D_y R_\lambda(\xi)}{h} - D_y R_\lambda(\xi)(2ia_j D_y - 2\xi_j)R_\lambda(\xi) \\
&\quad = D_y R_\lambda(\xi)(2ia_j D_y - 2\xi_j)(R_\lambda(\xi + he_j) - R_\lambda(\xi)) - hD_y R_\lambda(\xi)R_\lambda(\xi + he_j) \\
&\quad := A + F.
\end{aligned}$$

Proceeding as before we then get, for $f \in L^p_m$,

$$\begin{aligned}
|Ff| &\le C\,|h|\,\Psi(k|\lambda|)\Gamma(k|\lambda|)|f|, \\
|Af| &\le C\big[h^2\,(\Psi(k|\lambda|))^3 + |h|\,|\xi|\Psi(k|\lambda|)\Gamma(k|\lambda|)\Psi(k|\lambda|) \\
&\qquad + |h|^2(|\xi| + |h|)\Psi(k|\lambda|)^2\Gamma(k|\lambda|) \\
&\qquad + |h|\,|\xi|(|\xi| + |h|)\Psi(k|\lambda|)(\Gamma(k|\lambda|))^2\big]\,|f|
\end{aligned}$$

and conclude as before. This proves (37).

In particular, from (35) and (37), using again Theorem 6.3, we get

$$\begin{aligned}
\left|\frac{\partial}{\partial \xi_j}(R_\lambda(\xi))f\right| &\le C\big(\Gamma(\mu)\Psi(\mu) + |\xi|(\Gamma(\mu))^2\big), \\
\left|\frac{\partial}{\partial \xi_j}(D_y R_\lambda(\xi))f\right| &\le C((\Psi(\mu))^2 + |\xi|\Psi(\mu)\Gamma(\mu)),
\end{aligned}$$

which is (34) for $n = 1$.

Finally, (33) for $n > 1$ follows by induction. For example if $n = 2$ and $l \neq j$ one has

$$\begin{aligned}
\frac{\partial^2}{\partial \xi_l \partial \xi_j}(R_\lambda(\xi)) &= \frac{\partial}{\partial \xi_l}[R_\lambda(\xi)(2ia_j Dy - 2\xi_j)R_\lambda(\xi)] \\
&= \frac{\partial}{\partial \xi_l}(R_\lambda(\xi))(2ia_j Dy - 2\xi_j)R_\lambda(\xi) + R_\lambda(\xi)(2ia_j Dy - 2\xi_j)\frac{\partial}{\partial \xi_l}(R_\lambda(\xi)) \\
&= R_\lambda(\xi)(2ia_l Dy - 2\xi_l)R_\lambda(\xi)(2ia_j Dy - 2\xi_j)R_\lambda(\xi) \\
&\quad + R_\lambda(\xi)(2ia_j Dy - 2\xi_j)R_\lambda(\xi)(2ia_l Dy - 2\xi_l)R_\lambda(\xi).
\end{aligned}$$

The estimates (34) now follow using again (33) and Theorem 6.3. ∎

Now we can finally prove that the multiplier $\lambda R_\lambda(\xi)$ associated with the operators $\lambda(\lambda - \mathcal{L})^{-1}$ satisfies the hypotheses of Theorem 2.5. This is crucial for proving that $\mathcal{L}$ generates an analytic semigroup in L^p_m.

THEOREM 6.6. *Let* $1 < p < \infty$ *be such that* $0 < (m+1)/p < c+1$. *Then for every* $|a| \le \delta < 1$ *the family*

$$\{\xi^\alpha D_\xi^\alpha(\lambda R_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ \alpha \in \{0,1\}^N,\ \lambda \in \Sigma_{\pi-\theta_\delta}\}$$

is $\mathcal{R}$*-bounded in* L^p_m.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \{0,1\}^N$, $|\alpha| = n$ and $|a| \le \delta < 1$. Using (34) we get, for some positive constants $C, k > 0$ and for any $f \in L^p_m$ and $\lambda \in \Sigma_{\pi-\theta_\delta}$,

$$|D_\xi^\alpha R_\lambda(\xi) f| \le C\Gamma(\mu)(\Psi(\mu) + \sqrt{\mu}\Gamma(\mu))^n |f|, \tag{38}$$

where $\mu = k(|\lambda| + |\xi|^2)$. In particular

$$\begin{aligned} |\xi^\alpha \lambda D_\xi^\alpha R_\lambda(\xi) f| &\le C\mu^{n/2}\mu\,\Gamma(\mu)(\Psi(\mu) + \sqrt{\mu}\,\Gamma(\mu))^n |f| \\ &= C\mu\Gamma(\mu)(\sqrt{\mu}\Psi(\mu) + \mu\,\Gamma(\mu))^n |f|. \end{aligned}$$

The $\mathcal{R}$-boundedness of $\xi^\alpha D_\xi^\alpha(\lambda R_\lambda)(\xi)$ then follows by composition and domination using Proposition 4.3 and Corollary 2.2. ∎

The next two theorems show that the multipliers $|\xi|^2 R_\lambda$, $\xi D_y R_\lambda$ associated with the operators $\Delta_x(\lambda - \mathcal{L})^{-1}$, $D_{xy}(\lambda - \mathcal{L})^{-1}$ satisfy the hypotheses of Theorem 2.5. This is essential for characterizing the domain of $\mathcal{L}$.

THEOREM 6.7. *Let* $1 < p < \infty$ *be such that* $0 < (m+1)/p < c+1$. *Then for every* $|a| \le \delta < 1$ *the family*

$$\{\xi^\alpha D_\xi^\alpha(|\xi|^2 R_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ \alpha \in \{0,1\}^N,\ \lambda \in \Sigma_{\pi-\theta_\delta}\}$$

is $\mathcal{R}$*-bounded in* L^p_m.

Proof. Let us observe that for any $\alpha \in \{0,1\}^N$ with $|\alpha| = n$ there exist $\beta^j \in \{0,1\}^N$ with $|\beta^j| = n-1$ such that

$$D_\xi^\alpha(|\xi|^2 R_\lambda)(\xi) = \sum_{j:\,\alpha_j = 1} 2\xi_j D_\xi^{\beta^j} R_\lambda(\xi) + |\xi|^2 D_\xi^\alpha R_\lambda(\xi).$$

Using (34) and proceeding as in Theorem 6.6, the equality above implies that for some $C, k > 0$ and any $f \in L^p_m$,

$$\begin{aligned} |\xi^\alpha D_\xi^\alpha(|\xi|^2 R_\lambda)(\xi) f| \le{}& C\mu^{\frac{n+1}{2}}\,\Gamma(\mu)(\Psi(\mu) + \sqrt{\mu}\,\Gamma(\mu))^{n-1}|f| \\ &+ C\mu^{n/2+1}\,\Gamma(\mu)(\Psi(\mu) + \sqrt{\mu}\,\Gamma(\mu))^n |f| \end{aligned}$$

with $\mu = k(|\lambda| + |\xi|^2)$. The conclusion now follows as at the end of the proof of Theorem 6.6. ∎

THEOREM 6.8. *Let* $1 < p < \infty$ *be such that* $0 < (m+1)/p < c+1$. *Then for every* $|a| \le \delta < 1$ *the family*

$$\{\xi^\alpha D_\xi^\alpha(\xi D_y R_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ \alpha \in \{0,1\}^N,\ \lambda \in \Sigma_{\pi-\theta_\delta}\}$$

is $\mathcal{R}$*-bounded in* L^p_m.

Proof. Fix $j = 1, \dots, N$ and let $\alpha \in \{0,1\}^N$ with $|\alpha| = n$; then there exists $\beta \in \{0,1\}^N$ with $|\beta| = n-1$ such that

$$D^\alpha_\xi(\xi_j D_y R_\lambda)(\xi) = \xi_j D^\alpha_\xi D_y R_\lambda(\xi) + \alpha_j D^\beta_\xi D_y R_\lambda(\xi).$$

Using (34) and proceeding as in Theorem 6.6 we get, for some $C, k > 0$ and any $f \in L^p_m$,

$$\begin{aligned}|\xi^\alpha D^\alpha_\xi(\xi_j D_y R_\lambda)(\xi) f| &\le C\big(\mu^{\frac{n+1}{2}}\Psi(\mu)(\Psi(\mu) + \sqrt{\mu}\,\Gamma(\mu))^n|f| \\ &\qquad + \mu^{n/2}\Psi(\mu)(\Psi(\mu) + \sqrt{\mu}\,\Gamma(\mu))^{n-1}|f|\big) \\ &= C\big(\sqrt{\mu}\Psi(\mu)(\sqrt{\mu}\Psi(\mu) + \mu\,\Gamma(\mu))^n|f| \\ &\qquad + \sqrt{\mu}\Psi(\mu)(\sqrt{\mu}\Phi(\mu) + \mu\,\Gamma(\mu))^{n-1}|f|\big),\end{aligned}$$

where $\mu = k(|\lambda| + |\xi|^2)$, and the rest is similar. ∎

7. Domain and maximal regularity for $\mathcal{L} = \Delta_x + 2a\cdot\nabla_x D_y + B_y u$. In this section we prove generation results, maximal regularity and domain characterization for the degenerate operator

$$\begin{aligned}\mathcal{L} &:= \Delta_x + 2\sum_{i=1}^N a_i D_{x_i y} + D_{yy} + \frac{c}{y}D_y \\ &= \Delta_x u + 2a\cdot\nabla_x D_y u + B_y u, \qquad a = (a_1, \dots, a_N) \in \mathbb{R}^N,\ |a| < 1,\end{aligned} \tag{39}$$

in L^p_m. The condition $|a| < 1$ is equivalent to the ellipticity of the top order coefficients. More general operators will be treated in the next section, based on this model case. We start with the L^2 theory.

7.1. The operator $\mathcal{L}$ in L^2_c. We use the Sobolev space $H^1_c := \{u \in L^2_c : \nabla u \in L^2_c\}$ equipped with the inner product

$$\langle u, v\rangle_{H^1_c} := \langle u, v\rangle_{L^2_c} + \langle \nabla u, \nabla v\rangle_{L^2_c}.$$

We consider the form in L^2_c

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle y^c\,dx\,dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u\, a\cdot\nabla_x \overline{v}\, y^c\,dx\,dy,\ D(\mathfrak{a}) = H^1_c$$

and its adjoint $\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)}$,

$$\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)} := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle y^c\,dx\,dy + 2\int_{\mathbb{R}^{N+1}_+} a\cdot\nabla_x u D_y \overline{v}\, y^c\,dx\,dy.$$

Proposition 7.1. *The forms $\mathfrak{a}$, $\mathfrak{a}^*$ are continuous, accretive and sectorial.*

Proof. We consider only the form $\mathfrak{a}$; the adjoint form can be handled similarly. If $u \in H^1_c$ then

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(u,u) &\geq \|\nabla_x u\|^2_{L^2_c} + \|D_y u\|^2_{L^2_c} - 2|a|\,\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c}\\ &\geq (1-|a|)(\|\nabla_x u\|^2_{L^2_c} + \|D_y u\|^2_{L^2_c}).\end{aligned}$$

By the ellipticity assumption $|a| < 1$, accretivity follows. Moreover,

$$\begin{aligned}|\operatorname{Im}\mathfrak{a}(u,u)| &\leq 2|a|\,\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c}\\ &\leq |a|(\|\nabla_x u\|^2_{L^2_c} + \|D_y u\|^2_{L^2_c}) \leq \frac{|a|}{1-|a|}\operatorname{Re}\mathfrak{a}(u,u).\end{aligned}$$

This proves the sectoriality and then the continuity of the form. ∎

We define the operators $\mathcal{L}$ and $\mathcal{L}^*$ associated respectively to the forms $\mathfrak{a}$ and $\mathfrak{a}^*$ by

$$(40)\qquad \begin{aligned}&D(\mathcal{L}) = \Big\{u \in H^1_c : \exists f \in L^2_c \text{ with } \mathfrak{a}(u,v) = \int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for all } v \in H^1_c\Big\},\\ &\mathcal{L}u = -f;\end{aligned}$$

$$(41)\qquad \begin{aligned}&D(\mathcal{L}^*) = \Big\{u \in H^1_c : \exists f \in L^2_c \text{ with } \mathfrak{a}^*(u,v) = \int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for all } v \in H^1_c\Big\},\\ &\mathcal{L}^*u = -f.\end{aligned}$$

If u, v are smooth functions with compact support in the closure of $\mathbb{R}^{N+1}_+$ (so that they do not need to vanish on the boundary), it is easy to see integrating by parts that

$$-\mathfrak{a}(u,v) = \langle \Delta_x u + 2a\cdot\nabla_x D_y u + B_y u, \overline{v}\rangle_{L^2_c}$$

if $\lim_{y\to 0} y^c D_y u(x,y) = 0$. This means that $\mathcal{L}$ is the operator $\Delta_x + 2a\cdot\nabla_x D_y + B_y$ with Neumann boundary conditions at $y = 0$. On the other hand,

$$-\mathfrak{a}^*(u,v) = \left\langle \Delta_x u + 2a\cdot\nabla_x D_y u + 2c\frac{a\cdot\nabla_x u}{y} + B_y u, \overline{v}\right\rangle_{L^2_c}$$

if $\lim_{y\to 0} y^c(D_y u(x,y) + 2a\cdot\nabla_x u(x,y)) = 0$ and therefore $\mathcal{L}^*$ is the operator $\Delta_x + 2a\cdot\nabla_x D_y + 2c\frac{a\cdot\nabla_x u}{y} + B_y$ with the above oblique condition at $y = 0$.

Proposition 7.2. *$\mathcal{L}$ and $\mathcal{L}^*$ generate contractive analytic semigroups $e^{z\mathcal{L}}$, $e^{z\mathcal{L}^*}$, $z \in \Sigma_{\pi/2-\arctan\frac{|a|}{1-|a|}}$, in L^2_c. Moreover, the semigroups $(e^{t\mathcal{L}})_{t\geq 0}, (e^{t\mathcal{L}^*})_{t\geq 0}$ are positive and L^p_c-contractive for $1 \leq p \leq \infty$.*

Proof. We argue only for $\mathcal{L}$. Generation follows immediately from Proposition 7.1 and [27, Theorem 1.52]. Positivity follows by [27, Theorem 2.6] after observing that if $u \in H^1_c$ is real, then $u^+ \in H^1_c$ and

$$\mathfrak{a}(u^+,u^-) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u^+, \nabla u^- \rangle y^c \, dx\, dy + 2 \int_{\mathbb{R}^{N+1}_+} D_y u^+ a \cdot \nabla_x u^- y^c \, dx\, dy = 0.$$

Finally, L^∞-contractivity follows by [27, Corollary 2.17] after observing that if $0 \le u \in H^1_{\alpha,c}$, then $1 \wedge u, (u-1)^+ \in H^1_{\alpha,c}$, and since $\nabla(1\wedge u) = \chi_{\{u<1\}} \nabla u$ and $\nabla(u-1)^+ = \chi_{\{u>1\}}\nabla u$, one has

$$\mathfrak{a}(1 \wedge u, (u-1)^+) = 0. \ \blacksquare$$

The Stein interpolation theorem then shows that the above semigroups are analytic in L^p_c for $1 < p < \infty$ (see [27, Proposition 3.12]) and a result by Lamberton yields maximal regularity in the same range (see [16, Theorem 5.6]). Since our results are more general, we do not state these consequences here.

Our aim is to characterize the domain of $\mathcal{L}$ in L^2_c, but first we identify a core.

Proposition 7.3. *If $c+1>0$ then the set $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ defined in* (8) *is a core for $\mathcal{L}$ in L^2_c.*

Proof. First, $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ is contained in H^1_c. Moreover, integrating by parts one sees that any $u \in C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ satisfies (40) with $\mathcal{L}u = \Delta_x u + 2a \cdot \nabla_x D_y u + B_y u \in L^2_c$. This yields $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D} \subseteq D(\mathcal{L})$.

Since $I - \mathcal{L}$ is invertible we have to show that $(I-\mathcal{L})(C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D})$ is dense in L^2_c, or equivalently $((I-\mathcal{L})(C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}))^\perp = \{0\}$. Let $v \in L^2_c$ be such that

$$\int_{\mathbb{R}^{N+1}_+} (I-\mathcal{L}) f \bar{v} \, dx \; y^c \, dy = 0, \qquad \forall f \in C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}.$$

Let us choose $f = a(x)u(y) \in C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$. Taking the Fourier transform with respect to x we get $\hat{f}(\xi,y) = \hat{a}(\xi)u(y)$ and

$$\int_{\mathbb{R}^{N+1}_+} \big[u(y) + |\xi|^2 u(y) - 2ia\cdot\xi D_y u(y) - B_y u(y)\big] \hat{a}(\xi)\bar{\hat{v}}(\xi,y)\, d\xi \; y^c\, dy = 0,$$

that is,

$$\int_{\mathbb{R}^{N+1}_+} \big[u(y) + Q_a(\xi)u(y) - L_{2a\cdot\xi}\, u(y)\big] \hat{a}(\xi)\bar{\hat{v}}(\xi,y)\, d\xi \; y^c\, dy = 0. \tag{42}$$

Fix $\xi_0 \in \mathbb{R}^N$, $r > 0$ and let $w(\xi) = \frac{1}{|B(\xi_0,r)|}\chi_{B(\xi_0,r)} \in L^2(\mathbb{R}^N)$. Let $(a_n)_n \subset C_c^\infty(\mathbb{R}^N)$ be a sequence of test functions such that $a_n \to \check{w}$ in $L^2(\mathbb{R}^N)$; then $\hat{a}_n \to w$ in $L^2(\mathbb{R}^N)$ and writing (42) with $\hat{a}$ replaced by $\hat{a}_n$ and letting $n \to \infty$

we obtain

$$\frac{1}{|B(\xi_0,r)|}\int\limits_{B(\xi_0,r)} d\xi \int\limits_0^\infty [u(y)+Q_a(\xi)u(y)-L_{2a\cdot\xi}\,u(y)]\bar{\hat{v}}(\xi,y)y^c\,dy=0.$$

Letting $r\to 0$ and using the Lebesgue differentiation theorem we find that for a.e. $\xi_0\in\mathbb{R}^N$,

$$\int\limits_0^\infty (I+Q_a(\xi_0)-L_{2a\cdot\xi_0})u(y)\bar{\hat{v}}(\xi_0,y)y^c\,dy=0,\qquad \forall u\in\mathcal{D}.$$

Since, by Theorems 3.4 and 5.16, $\mathcal{D}$ is a core for $L_{2a\cdot\xi_0}$ in $L^2_c(\mathbb{R}_+)$, the last equality implies $\hat{v}(\xi_0,\cdot)=0$ for a.e. $\xi_0\in\mathbb{R}^N$ and the proof is complete. ∎

THEOREM 7.4. *If* $c+1>0$ *then* $D(\mathcal{L})=W^{2,2}_{c,\mathcal{N}}$.

Proof. Observe that

$$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}\subset W^{2,2}_{c,\mathcal{N}}\cap D(\mathcal{L})$$

is a core for $\mathcal{L}$ by Proposition 7.3 and is dense in $W^{2,2}_{c,\mathcal{N}}$ by Theorem 3.4.

We have to show that the graph norm and the norm of $W^{2,2}_{c,\mathcal{N}}$ are equivalent on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$. Since the second is obviously stronger, we have to show the converse.

We use Proposition 3.3 and endow $W^{2,2}_{c,\mathcal{N}}$ with the equivalent norm

$$\|u\|_W=\|u\|_{L^2_c}+\|\Delta_x u\|_{L^2_c}+\|\nabla_x D_y u\|_{L^2_c}+\|B_y u\|_{L^2_c}.$$

Let $u\in C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ and $f=u-\mathcal{L}u$, so that $\|u\|_{L^2_c}\le\|f\|_{L^2_c}$. By taking the Fourier transform with respect to x (with co-variable ξ) we obtain

$$(1+Q_a(\xi)-L_{2a\cdot\xi})\hat{u}(\xi,\cdot)=\hat{f}(\xi,\cdot)$$

and therefore

$$\begin{aligned}|\xi|^2\hat{u}(\xi,\cdot)&=|\xi|^2(1+Q_a(\xi)-L_{2a\cdot\xi})^{-1}\hat{f}(\xi,\cdot),\\ \xi_i D_y\hat{u}(\xi,\cdot)&=\xi_i D_y(1+Q_a(\xi)-L_{2a\cdot\xi})^{-1}\hat{f}(\xi,\cdot).\end{aligned}\tag{43}$$

This means that

$$\Delta_x u=-\mathcal{F}^{-1}|\xi|^2R_1(\xi)\mathcal{F}f,\qquad i\frac{\partial}{\partial x_i}D_y u=\mathcal{F}^{-1}\xi D_y R_1(\xi)\mathcal{F}f.$$

The estimates $\|\Delta_x u\|_{L^2_c}\le C\|f\|_{L^2_c}$, $\|\nabla_x D_y u\|_{L^2_c}\le C\|f\|_{L^2_c}$ then follow from the boundedness of the multipliers $|\xi|^2R_1(\xi)$ and $\xi D_y R_1(\xi)$ in $L^2(\mathbb{R}^N;L^2_c(\mathbb{R}_+))=L^2_c$, proved in Theorems 6.7 and 6.8, and yield $\|B_y u\|_{L^2_c}\le C\|f\|_{L^2_c}$, by taking the difference.

This shows the equivalence of the graph norm and of the norm of $W^{2,2}_{c,\mathcal{N}}$ on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ and concludes the proof. ∎

7.2. The operator $\mathcal{L}$ in L^p_m. In this section we prove a domain characterization and maximal regularity for $\mathcal{L}$ in L^p_m. For clarity we often write $\mathcal{L}_{m,p}$ to emphasize the space on which the operator acts.

We shall use extensively the set $\mathcal{D}$ defined in (7). In particular, $\mathcal{L}$ is well-defined on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ when $m+1>0$.

In Proposition 7.2 and Theorem 7.4 we proved generation of an analytic semigroup in L^2_c in the sector $\Sigma_{\pi-\arctan\frac{|a|}{1-|a|}}$ and characterized the domain of the generator through the boundedness in L^2_c of the operators

$$(44)\quad (\lambda-\mathcal{L}_{c,2})^{-1},\ \Delta_x(\lambda-\mathcal{L}_{c,2})^{-1},\ \nabla_x D_y(\lambda-\mathcal{L}_{c,2})^{-1},\quad \lambda\in\Sigma_{\pi-\arctan\frac{|a|}{1-|a|}}.$$

On the other hand, the multipliers above are bounded in the larger sector

$$\Sigma_{\pi-\omega_a}:=\Sigma_{\pi-\arctan\frac{|a|}{\sqrt{1-|a|^2}}},\qquad \omega_a:=\arctan\frac{|a|}{\sqrt{1-|a|^2}},$$

by the results in Section 6.

In the next lemma we extend the families (44) to bounded operators on L^p_m on the bigger sector $\Sigma_{\pi-\omega_a}$. In particular, we prove that the resolvents $(\lambda-\mathcal{L}_{c,2})^{-1}$, $\lambda\in\Sigma_{\pi-\arctan\frac{|a|}{1-|a|}}$, initially constructed via the form method in L^2_c, extend to the larger sector $\Sigma_{\pi-\omega_a}$.

We recall the notation of Section 6, where $|a|\le\delta\le 1$,

$$\theta_\delta=\arctan\frac{|a|}{\sqrt{\delta^2-|a|^2}}\in\left(\arctan\frac{|a|}{\sqrt{1-|a|^2}},\frac{\pi}{2}\right],\qquad \theta_1=\omega_a.$$

LEMMA 7.5. *Let $0<(m+1)/p<c+1$. Then the operators*

$$(\lambda-\mathcal{L}_{c,2})^{-1},\quad \Delta_x(\lambda-\mathcal{L}_{c,2})^{-1},\quad \nabla_x D_y(\lambda-\mathcal{L}_{c,2})^{-1},\quad B_y(\lambda-\mathcal{L}_{c,2})^{-1}$$

initially defined on $L^p_m\cap L^2_c$ and for $\lambda\in\Sigma_{\pi-\arctan\frac{|a|}{1-|a|}}$, extend for $\lambda\in\Sigma_{\pi-\omega_a}$ to bounded operators on L^p_m, which we denote by $\mathcal{R}(\lambda)$, $\Delta_x\mathcal{R}(\lambda)$, $\nabla_x D_y\mathcal{R}(\lambda)$, $B_y\mathcal{R}(\lambda)$, respectively. Moreover for every $|a|\le\delta<1$ the family $\{\lambda\mathcal{R}(\lambda):\lambda\in\Sigma_{\pi-\theta_\delta}\}$ is $\mathcal{R}$-bounded on L^p_m.

Proof. Let $\lambda\in\Sigma_{\pi-\omega_a}$, $u\in C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ and $f=\lambda u-\mathcal{L}u$. By taking the Fourier transform with respect to x we obtain

$$(\lambda+Q_a(\xi)-L_{2a\cdot\xi})\hat{u}(\xi,\cdot)=\hat{f}(\xi,\cdot),\qquad \hat{u}(\xi,\cdot)=(\lambda+Q_a(\xi)-L_{2a\cdot\xi})^{-1}\hat{f}(\xi,\cdot).$$

This means $u=\mathcal{F}^{-1}R_\lambda(\xi)\mathcal{F}f$, where

$$R_\lambda(\xi)=(\lambda+Q_a(\xi)-L_{2a\cdot\xi})^{-1}.$$

Theorems 2.5 and 6.6 yield the boundedness of the Fourier multiplier R_λ in $L^p(\mathbb{R}^N,L^p_m(\mathbb{R}_+))=L^p_m$ and therefore the existence of a bounded operator $\mathcal{R}(\lambda)=\mathcal{F}^{-1}R_\lambda(\xi)\mathcal{F}\in\mathcal{B}(L^p_m)$.

Furthermore, [28, Theorem 4.3.9] and the $\mathcal{R}$-boundedness with respect to λ of $\lambda R_\lambda(\xi)$ and its ξ-derivatives (see again Theorem 6.6) imply that the family $\{\lambda\mathcal{R}(\lambda) : \lambda \in \Sigma_{\pi-\theta_\delta}\}$ is $\mathcal{R}$-bounded for any $|a| \le \delta < 1$.

Since $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ is a core for $\mathcal{L}_{c,2}$, this shows in particular that

$$(\lambda - \mathcal{L}_{c,2})^{-1} = \mathcal{F}^{-1}R_\lambda(\xi)\mathcal{F} = \mathcal{R}(\lambda)$$

for $\lambda \in \Sigma_{\pi-\arctan\frac{|a|}{1-|a|}}$, where both operators exist.

However, the previous equality extends to $\lambda \in \Sigma_{\pi-\omega_a}$. In fact, the set $E = \Sigma_{\pi-\omega_a} \cap \rho(\mathcal{L}_{c,2})$ (ρ denotes the resolvent) is open in $\Sigma_{\pi-\omega_a}$ and $(\lambda - \mathcal{L}_{c,2})^{-1} = \mathcal{R}(\lambda)$ for $\lambda \in E$, by the argument above. But E is also closed in $\Sigma_{\pi-\omega_a}$ and hence coincides with it. In fact, if $(\lambda_n) \subset E$ converges to $\lambda_0 \in \Sigma_{\pi-\omega_a}$, then $(\lambda_n - \mathcal{L}_{c,2})^{-1} = \mathcal{R}(\lambda_n)$ is uniformly bounded and $\lambda_0 \in E$, by elementary spectral theory.

The proofs for $\Delta_x\mathcal{R}(\lambda)$, $\nabla_x D_y\mathcal{R}(\lambda)$ are similar. As for (43) in Theorem 7.4, we have

$$\Delta_x(\lambda - \mathcal{L}_{c,2})^{-1} = -\mathcal{F}^{-1}|\xi|^2R_\lambda(\xi)\mathcal{F},$$
$$\nabla_x D_y(\lambda - \mathcal{L}_{c,2})^{-1} = -\mathcal{F}^{-1}\xi D_y R_\lambda(\xi)\mathcal{F},$$

and we can use Theorems 6.7 and 6.8 to obtain the boundedness of the multipliers in $L^p_m = L^p(\mathbb{R}^N; L^p_m(\mathbb{R}_+))$. The boundedness of $B_y\mathcal{R}(\lambda)$ then follows by taking the difference. ∎

LEMMA 7.6. *If $0 < (m+1)/p < c+1$, then an extension $\mathcal{L}_{m,p}$ of the operator $\mathcal{L}$, initially defined on $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$, generates an analytic semigroup $\{e^{z\mathcal{L}_{m,p}} : z \in \Sigma_{\pi/2-\omega_a}\}$ in L^p_m which is bounded on $\Sigma_{\pi/2-\theta_\delta}$ for any $|a| \le \delta < 1$. Moreover, the semigroup has maximal regularity and it is consistent with the semigroup generated by $\mathcal{L}_{c,2}$ in L^2_c.*

Proof. Let $|a| \le \delta < 1$ and consider the $\mathcal{R}$-bounded family of operators $\{\lambda\mathcal{R}(\lambda) : \lambda \in \Sigma_{\pi-\theta_\delta}\}$ defined by Lemma 7.5. In particular, it satisfies

$$\|\lambda\mathcal{R}(\lambda)\|_{\mathcal{B}(L^p_m)} \le C, \qquad \forall\lambda \in \Sigma_{\pi-\theta_\delta}.$$

By construction, $\mathcal{R}(\lambda)$ coincides with $(\lambda - \mathcal{L}_{c,2})^{-1}$ when restricted to $L^p_m \cap L^2_c$. Hence, by density, the family $\{\mathcal{R}(\lambda) : \lambda \in \Sigma_{\pi-\theta_\delta}\}$ satisfies the resolvent equation

$$\mathcal{R}(\lambda) - \mathcal{R}(\mu) = (\mu - \lambda)\mathcal{R}(\lambda)\mathcal{R}(\mu), \quad \forall\lambda, \mu \in \Sigma_{\pi-\theta_\delta},$$

in L^p_m and therefore it is a pseudoresolvent [12, Section 4.a]. Furthermore, $\mathrm{rg}(\mathcal{R}(\lambda))$ is dense in L^p_m for every $\lambda \in \Sigma_{\pi-\omega_a}$, since it contains $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$.

Let us prove that $\mathcal{R}(\lambda)$ is injective for every $\lambda \in \Sigma_{\pi-\theta_\delta}$. Let $f \in L^p_m$ with $\mathcal{R}(\lambda)f = 0$ for some $\lambda \in \Sigma_{\pi-\theta_\delta}$. Since $\mathrm{Ker}(\mathcal{R}(\lambda)) = \mathrm{Ker}(\mathcal{R}(\mu))$ for any $\lambda, \mu \in \Sigma_{\pi-\theta_\delta}$ [12, Lemma 4.5], we have $\mathcal{R}(\lambda)f = 0$ for every $\lambda > 0$. Given

$\epsilon > 0$, choose $g \in L^p_m \cap L^2_c$ such that $\|f-g\|_{L^p_m} < \epsilon$. Then

$$\lambda R(\lambda)g = \lambda R(\lambda)(g-f), \quad \|\lambda R(\lambda)g\|_{L^p_m} \le C\epsilon, \quad \forall \lambda > 0.$$

Since $\lambda R(\lambda)g = \lambda(\lambda - \mathcal{L}_{c,2})^{-1}g \to g$ as $\lambda \to \infty$ we may suppose that, up to a subsequence, $\lambda R(\lambda)g \to g$ a.e. Then Fatou's lemma yields

$$\|g\|_{L^p_m} \le \liminf_{\lambda\to\infty} \|\lambda\mathcal{R}(\lambda)g\|_{L^p_m} \le C\epsilon,$$

which implies $\|f\|_{L^p_m} \le \|f-g\|_{L^p_m} + \|g\|_{L^p_m} \le (1+C)\epsilon$, hence $f=0$ by the arbitrariness of ϵ, which proves the injectivity of $\mathcal{R}(\lambda)$.

At this point, the arbitrariness of δ and [12, Proposition 4.6] yield the existence of a densely defined closed operator $\mathcal{L}_{m,p}$ such that $\Sigma_{\pi-\omega_a} \subseteq \rho(\mathcal{L}_{m,p})$ and $\mathcal{R}(\lambda) = (\lambda - \mathcal{L}_{m,p})^{-1}$ for any $\lambda \in \Sigma_{\pi-\omega_a}$. By construction, $(\mathcal{L}_{m,p}; D(\mathcal{L}_{m,p}))$ extends $(\mathcal{L}, C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D})$ and one has

$$\|\lambda(\lambda-\mathcal{L}_{m,p})^{-1}\|_{\mathcal{B}(L^p_m)} \le C, \quad \lambda \in \Sigma_{\pi-\theta_\delta},\ |a| \le \delta < 1.$$

Then from standard results of semigroup theory (see for example [1, Section AII, Theorem 1.14]), $(\mathcal{L}_{m,p}, D(\mathcal{L}_{m,p}))$ generates an analytic semigroup $(e^{z\mathcal{L}_{m,p}})_{z\in\Sigma_{\pi/2-\omega_a}}$ in L^p_m which is bounded on $\Sigma_{\pi-\theta_\delta}$ for any $|a| \le \delta < 1$.

The maximal regularity of the semigroup follows, using Theorem 2.4, from the $\mathcal{R}$-boundedness of the resolvent family $\{\lambda(\lambda-\mathcal{L}_{m,p})^{-1} : \lambda \in \Sigma_{\pi-\theta_\delta}\}$. Finally, the semigroup is consistent with that in L^2_c, since the resolvents are consistent. ∎

We characterize the domain of $\mathcal{L}_{m,p}$ and collect in one theorem all the results proved in this section.

Theorem 7.7. *If $0 < (m+1)/p < c+1$, then the operator $\mathcal{L}_{m,p}$ with domain*

$$D(\mathcal{L}_{m,p}) = W^{2,p}_{m,\mathcal{N}}$$

generates an analytic semigroup $\{e^{z\mathcal{L}_{m,p}} : z \in \Sigma_{\pi/2-\omega_a}\}$ in L^p_m which is bounded on $\Sigma_{\pi/2-\theta_\delta}$ for any $|a| \le \delta < 1$. Moreover, $C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}$ is a core for $\mathcal{L}_{m,p}$ and the semigroup has maximal regularity.

Proof. In view of Lemma 7.6 we only have to show that $D(\mathcal{L}_{m,p}) = W^{2,p}_{m,\mathcal{N}}$. With the notation above, $D(\mathcal{L}_{m,p}) = \mathcal{R}(1)(L^p_m)$. Let $u = \mathcal{R}(1)f = (I-\mathcal{L}_{c,2})^{-1}f$ with $f \in L^2_c \cap L^p_m$. Then Lemma 7.5 yields

$$\|\Delta_x u\|_{L^p_m} + \|B_y u\|_{L^p_m} + \|\nabla_x D_y u\|_{L^p_m} \le C(\|\mathcal{L}u\|_{L^p_m} + \|u\|_{L^p_m}). \tag{45}$$

Using Theorems 4.1 and 7.4, we deduce that $u(x,\cdot) \in D(B^n_{c,2})$ for a.e. $x \in \mathbb{R}^N$. Moreover, $u(x,\cdot)$, $B_y u(x,\cdot) \in L^p_m(\mathbb{R}_+)$ for a.e. $x \in \mathbb{R}^N$.

Let us show that $u(x,\cdot) \in D(B^n_{m,p})$. In fact, setting $f := u(x,\cdot) - B_y u(x,\cdot) \in L^p_m(\mathbb{R}_+) \cap L^2_c(\mathbb{R}_+)$ we have $u = (I-B^n)^{-1}f \in D(B^n_{m,p}) \cap D(B^n_{c,2})$ by the consistency of the resolvent $(I-B^n)^{-1}$ in $L^p_m(\mathbb{R}_+)$ and in $L^2_c(\mathbb{R}_+)$.

Theorem 4.1 then implies

$$\|D_{yy}u\|_{L^p_m(\mathbb{R}_+)} + \|y^{-1}D_yu\|_{L^p_m} + \|D_yu\|_{L^p_m(\mathbb{R}_+)} \le C\|u - B_yu\|_{L^p_m(\mathbb{R}_+)}.$$

Then, raising to the power p, integrating over $\mathbb{R}^N$ and using Lemma 7.5 for the last inequality we get

$$\begin{aligned}\|D_{yy}u\|_{L^p_m} + \|D_yu\|_{L^p_m} + \|y^{-1}D_yu\|_{L^p_m} &\le C(\|u\|_{L^p_m} + \|B_yu\|_{L^p_m})\\ &\le C(\|u\|_{L^p_m} + \|\mathcal{L}u\|_{L^p_m}).\end{aligned} \tag{46}$$

By the density of $L^2_c \cap L^p_m$ in L^p_m, (45) and (46) hold for every $u \in D(\mathcal{L}_{m,p})$ and this last space is contained in $W^{2,p}_{m,\mathcal{N}}$, by Proposition 3.3.

Moreover, since the graph norm is clearly weaker than the norm of $W^{2,p}_{m,\mathcal{N}}$, (45) and (46) again show that these norms are equivalent on $D(\mathcal{L}_{m,p})$, in particular on $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ which is dense in $W^{2,p}_{m,\mathcal{N}}$, by Theorem 3.4.

Therefore $D(\mathcal{L}_{m,p}) = W^{2,p}_{m,\mathcal{N}}$ and in particular $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ is a core. ∎

COROLLARY 7.8. *Under the hypotheses of Theorem* 7.7, *for every* $u \in W^{2,p}_{m,\mathcal{N}}$,

$$\|D_{x_ix_j}u\|_{L^p_m} + \|D_{x_iy}u\|_{L^p_m} + \|D_{yy}u\|_{L^p_m} + \|y^{-1}D_yu\|_{L^p_m} \le C\|\mathcal{L}u\|_{L^p_m}.$$

Proof. By Theorem 7.7 the above inequality holds if $\|u\|_{L^p_m}$ is added to the right hand side. Applying it to $u_\lambda(x,y) = u(\lambda x, \lambda y)$, $\lambda > 0$, we obtain

$$\begin{aligned}\|D_{x_ix_j}u\|_{L^p_m} + \|D_{x_iy}u\|_{L^p_m} + \|D_{yy}u\|_{L^p_m} + \|y^{-1}D_yu\|_{L^p_m}&\\ \le C(\|\mathcal{L}u\|_{L^p_m} + \lambda^{-2}\|u\|_{L^p_m})&\end{aligned}$$

and the assertion follows by letting $\lambda \to \infty$. ∎

8. General operators and oblique derivative. Results for more general operators and boundary conditions follow by a linear change of variables, as we explain below. Let us first remove the assumption on the special form of $\mathcal{L} = \Delta_x + 2a \cdot \nabla_x D_y + B_y u$ by considering the general form in $\mathbb{R}^{N+1}_+$:

$$\mathcal{L} = \mathrm{Tr}(QD^2u) + \frac{c}{y}D_y = \sum_{i,j=1}^N q_{ij}D_{x_ix_j} + 2\sum_{i=1}^N q_iD_{x_iy} + \gamma D_{yy} + \frac{c}{y}D_y.$$

If Q_1 is the $N \times N$ matrix (q_{ij}) and $q = (q_1, \ldots, q_N)$ we assume that the quadratic form $Q(\xi,\eta) = Q_1(\xi,\xi) + \gamma\eta^2 + 2q\cdot\xi\,\eta$ is positive definite. Through a linear change of the x variables, the term $\sum_{i,j=1}^N q_{ij}D_{x_ix_j}$ is transformed into $\gamma\Delta_x$ and all the results of Section 7 hold, replacing c with c/γ in the statements (the condition $|a| < 1$ of Section 7 is satisfied since the change of variables preserves ellipticity). The addition of first order terms like $\alpha\cdot\nabla_x + \beta D_y$ is easily treated by standard perturbation theory of analytic semigroups and maximal regularity; the case of variable coefficients can also be handled by freezing the coefficients and will be done in the future to deal with degenerate problems in bounded domains.

A further change of variables allows one to deal with the operator

$$\mathcal{L} = \mathrm{Tr}(QD^2u) + \frac{v\cdot\nabla}{y} = \sum_{i,j=1}^{N+1} q_{ij}D_{ij} + \frac{b\cdot\nabla_x}{y} + \frac{cD_y}{y}, \qquad c\neq 0,$$

where $v=(b,c)$ and Q is positive definite. We impose an oblique derivative boundary condition $v\cdot\nabla u(x,0)=0$ in the integral form

$$\frac{v\cdot\nabla u}{y} = \frac{b\cdot\nabla_x u + cD_y u}{y} \in L^p_m \tag{47}$$

and therefore define

$$W^{2,p}_{m,v} = \{u\in W^{2,p}_m : y^{-1}v\cdot\nabla u \in L^p_m\}.$$

We transform $\mathcal{L}$ into a similar operator with $b=0$ and Neumann boundary conditions by defining the following isometry of L^p_m:

$$T\,u(x,y) := u\left(x-\frac{b}{c}y, y\right), \qquad (x,y)\in\mathbb{R}^{N+1}_+. \tag{48}$$

Lemma 8.1. *Let* $1<p<\infty$ *and* $v=(b,c)\in\mathbb{R}^{N+1}$, $c\neq 0$. *Then for* $u\in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$,

(i) $$T^{-1}\left(\mathrm{Tr}(QD^2u)+\frac{v\cdot\nabla u}{y}\right)Tu = \mathrm{Tr}(\tilde{Q}D^2u) + \frac{c}{y}D_y u,$$

where $\tilde{Q}$ *is a uniformly elliptic symmetric matrix defined by*

$$\tilde{Q} = \left(\begin{array}{c|c} Q_N - \frac{2}{c}b\otimes q + \frac{\gamma}{c^2}b\otimes b & q^t - \frac{\gamma}{c}b^t \\ \hline q-\frac{\gamma}{c}b & \gamma \end{array}\right)$$

and $\gamma = q_{N+1,N+1}$.

(ii) $T\,(W^{2,p}_{m,\mathcal{N}}) = W^{2,p}_{m,v}$.

Proof. This follows by a straightforward computation. ∎

As in Proposition 3.3 one can provide an equivalent description of $W^{2,p}_{m,v}$ through the oblique boundary condition $\lim_{y\to 0} y^c v\cdot\nabla u = 0$. For brevity we do not write out this result which is an easy translation of Proposition 3.3, using the previous lemma.

Finally, we can also deduce results for the last operator which we state only in the parabolic setting. The proof follows directly from the above lemma, the general theory of Section 2 and standard semigroup theory.

Theorem 8.2. *Let* $0<(m+1)/p<c/\gamma+1$, $v=(b,c)$ *with* $c\neq 0$, Q *uniformly elliptic and*

$$\mathcal{L} = \mathrm{Tr}(QD^2u) + \frac{v\cdot\nabla}{y}$$

with domain $W^{2,p}_{m,v}$. *Then for each* $1 < q < \infty$, $T > 0$, $u_0 \in W^{2,p}_{m,v}$ *and* $f \in L^q([0,T]; L^p_m)$, *the problem*

$$\frac{\partial}{\partial t} u(t,x,y) - \mathcal{L}u(t,x,y) = f(t,x,y), \qquad t > 0, \qquad u(0,x,y) = u_0(x,y),$$

admits a unique solution $u \in W^{1,q}([0,T]; L^p_m) \cap L^q([0,T]; W^{2,p}_{m,v})$.

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