ACTA ARITHMETICA
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Character sums and the Riemann Hypothesis

by

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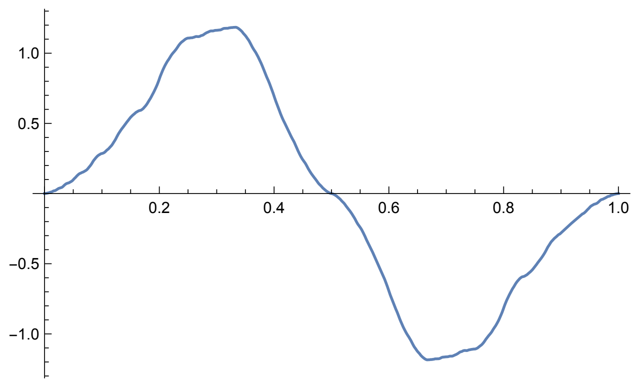
Dedicated to Henryk on his semisesquicentennial

Abstract. We prove that an innocent looking inequality implies the Riemann Hypothesis and show a way to approach this inequality through sums of Legendre symbols.

Introduction. Let

$$f(x)=\sum_{n=1}^{\infty}\frac{\lambda(n)\sin 2\pi nx}{n^2}$$

where λ is the Liouville lambda-function ([1]). Since $|\lambda(n)|=1$, this series is absolutely convergent for real x, so that f is continuous, odd and periodic with period 1 on $\mathbb{R}$. Here is a plot of $f(x)$ for $0\leqslant x\leqslant 1$ using 1000 terms of the series defining f:



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([1]) λ is completely multiplicative and takes the value -1 on primes so that $\lambda(p_1^{e_1}\dots p_r^{e_r})=(-1)^{e_1+\dots+e_r}$.

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Theorem 1. *If* $f(x) \geqslant 0$ *for* $0 \leqslant x \leqslant 1/4$, *then the Riemann Hypothesis is true.*

Theorem 1 is deceptive in that it looks like it should be a simple matter to prove that $f(x)$ is non-negative. A problem is that it is not clear whether $f(x)$ is differentiable or not, and even if it is, it would be difficult to estimate the derivative. So, proving that $f(x) > 0$ at some point does not immediately tell us about $f(x)$ at nearby points.

The "1/4" in Theorem 1 can be replaced by any positive constant. So the real issue is trying to prove that $f(x) > 0$ for small positive x.

Note that

$$\Big|\sum_{n=N+1}^{\infty}\frac{\lambda(n)\sin 2\pi nx}{n^2}\Big| < \int_N^\infty u^{-2}\,du = \frac{1}{N}$$

so that if for some x there is an N such that

$$\sum_{n=1}^{N}\frac{\lambda(n)\sin 2\pi nx}{n^2} \geqslant \frac{1}{N} \tag{1}$$

then it must be the case that $f(x) > 0$. We will use this idea a little later.

We can give an "explicit formula" for f in terms of the zeros $\rho = \beta + i\gamma$ of ζ:

Theorem 2. *Assuming the Riemann Hypothesis,*

$$f(x) = -\frac{4\pi^2 x^{3/2}}{3\zeta(1/2)} - \frac{8\pi^2}{3}x^{3/2}\sum_{n\leqslant 4x}\frac{\ell(n)}{\sqrt{n}}\Big(1-\frac{n}{4x}\Big)^{3/2}$$
$$+\pi\lim_{T\to\infty}\sum_{\substack{\rho=1/2+i\gamma\\|\gamma|\leqslant T}}\operatorname*{Res}_{z=\rho-1}\frac{X(1-z)\zeta(2z+2)x^{1-z}}{(1-z)\zeta(z+1)}.$$

Here $\ell(n)$ is defined through its generating function

$$\sum_{n=1}^{\infty}\ell(n)n^{-s} = \frac{\zeta(2s-1)}{\zeta(s)}$$

for $\Re s > 1$. Also, $X(s)$ is the factor from the functional equation for $\zeta(s)$ which can be defined by

$$X(s)^{-1} = X(1-s) = \frac{\zeta(1-s)}{\zeta(s)} = 2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

Note that if the zeros of $\zeta(s)$ are simple, then the term with the sum over the zeros of ζ becomes

$$\pi\sum_{\rho}\frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}.$$

Theorem 2 is nearly a converse to Theorem 1 in the sense that if RH is true and all the zeros are simple and

$$\sum_{\rho}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| \leqslant -\frac{4\pi}{3\zeta(1/2)} \tag{2}$$

then $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$. Note that

$$-\frac{4\pi}{3\zeta(1/2)} = 2.86834\ldots \quad \text{and} \quad \sum_{|\gamma|\leqslant 1000}\left|\frac{X(2-\rho)\zeta(2\rho)}{(2-\rho)\zeta'(\rho)}\right| = 0.264954\ldots$$

so that the inequality (2) seems plausible.

Finally, we remark that the formula of Theorem 2 for $f(x)$ hides very well the fact that $f(x)$ is periodic with period 1!

1. Prior results. There has been quite a lot of work connecting partial weighted sums of the Liouville lambda-function and the Riemann Hypothesis. We refer to [BFM] for a nice description of past work. In that paper the authors prove that the smallest value of x for which

$$\sum_{n\leqslant x}\frac{\lambda(n)}{n} < 0$$

is $x = 72185376951205$.

2. Character sums. A possible approach to proving that $f(x) > 0$ for small $x > 0$ lies in the fact that λ is completely multiplicative and takes the values ± 1. This scenario resembles quadratic Dirichlet characters (for simplicity think Legendre symbols) except that Dirichlet characters can also take the value 0. By the Chinese Remainder Theorem, for any N there is a prime number q such that $\lambda(n) = \left(\frac{n}{q}\right)$ for all $n \leqslant N$, where $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol (2) modulo q. As an example,

$$\lambda(n) = \left(\frac{n}{163}\right)$$

for all $n \leqslant 40$, but they differ at $n = 41$.

Let

$$f_q(x) = \sum_{n=1}^{\infty}\frac{\left(\frac{n}{q}\right)\sin 2\pi n x}{n^2}$$

be the Fourier sine series with $\lambda(n)$ replaced by $\left(\frac{n}{q}\right)$. If $f_q(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$ for a sufficiently large set of q, then it must also be the case that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$. (The proof is that if $f(x_0) < 0$ for some $0 < x_0 < 1/4$,

(2) $\left(\frac{n}{q}\right) = 0$ if $(n, q) > 1$; $\left(\frac{n}{q}\right) = +1$ if n is a square modulo q; and $\left(\frac{n}{q}\right) = -1$ if n is not a square modulo q.

then we can find a q such that $\left(\frac{n}{q}\right) = \lambda(n)$ for all $n \leqslant N$ where N is chosen so large that $|f(x_0)| > 1/N$; then it must be the case by the analogue of (1) for f_q that $f_q(x_0) < 0$.) The same assertion but with q restricted to primes congruent to 3 modulo 8 is also valid, since the Legendre symbols for these q can also imitate $\lambda(n)$ for arbitrarily long stretches $1 \leqslant n \leqslant N$. We can express this as follows:

THEOREM 3. *If*

$$f_q(x) \geqslant 0$$

for all $0 \leqslant x \leqslant 1/4$ *and all primes* q *congruent to* 3 *modulo* 8, *then the Riemann Hypothesis is true.*

REMARK 1. We could just as well have stated this theorem for $q \equiv 3 \bmod 4$. However, the intention is that we are interested in q for which χ_q imitates λ. Insisting that $\chi_q(2) = -1$ leads to the condition that $q \equiv 3 \bmod 8$.

The sums $f_q(x)$ still have the same problem in that it is tricky to prove for sure that they are positive for small positive x. However, the analogue of Theorem 2 above is much simpler, is unconditional, and leads to a straightforward way to check, for any given fixed q, that $f_q(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/4$.

THEOREM 4. *Let* $x \geqslant 0$. *Let* $q \equiv 3 \bmod 8$ *be squarefree. Then*

$$f_q(x) = 2\pi x L_q(1) - \frac{2\pi^2 x}{\sqrt{q}} \sum_{n \leqslant xq} \left(\frac{n}{q}\right)\left(1 - \frac{n}{xq}\right)$$

where

$$L_q(1) = \sum_{n=1}^{\infty} \frac{\left(\frac{n}{q}\right)}{n}.$$

Now Dirichlet's class number formula enters the picture. Let $K = \mathbb{Q}(\sqrt{-q})$ be the imaginary quadratic field obtained by adjoining $\sqrt{-q}$ to the rationals $\mathbb{Q}$. Let $h(q)$ be the class number ([3]) of K. Then Dirichlet's formula reads

$$h(q) = \frac{\sqrt{q}}{\pi} L_q(1)$$

for squarefree $q \equiv 3 \bmod 4$ and $q > 3$ (see [D] or [IK]). Thus, the theorem above can be rephrased in terms of $h(q)$. Moreover, we can express $L_q(1)$ as a finite character sum:

$$L_q(1) = -\frac{\pi}{q^{3/2}} \sum_{n=1}^{q} n \left(\frac{n}{q}\right).$$

([3]) The class number is a measure of how close to unique factorization the integers of K are; $h(q) = 1$ means the integers of K can be factored into primes in only one way.

Since $\left(\frac{n}{q}\right)$ is an odd function of q, we also have

$$L_q(1) = -\frac{2\pi}{q^{3/2}} \sum_{n=1}^{(q-1)/2} n \left(\frac{n}{q}\right)$$

and

$$h(q) = S_q\left(\frac{q}{2}\right) \quad \text{where} \quad S_q(N) := \sum_{n \leqslant N} \left(\frac{n}{q}\right)\left(1 - \frac{n}{N}\right).$$

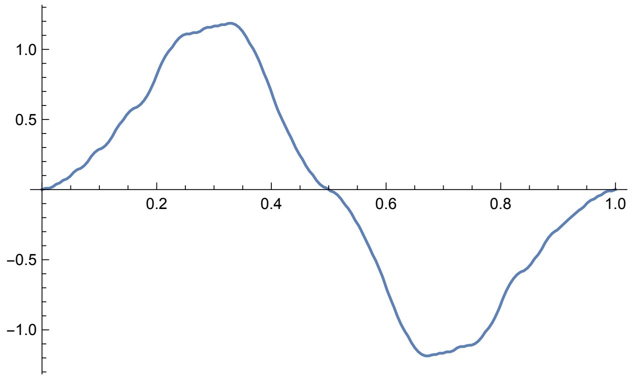
Corollary 1. *Let* $q > 3$ *be squarefree with* $q \equiv 3 \bmod 8$. *Then*

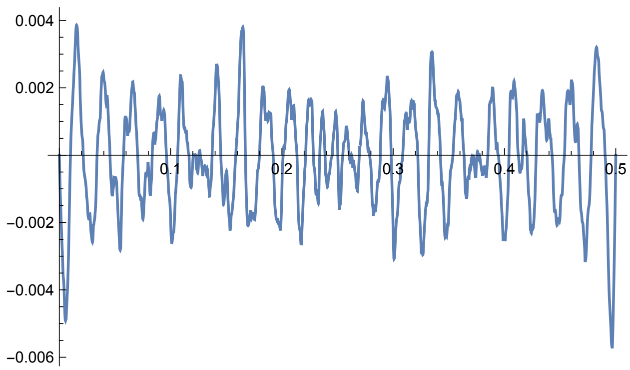
$$f_q(x) = \frac{2\pi^2 x}{\sqrt{q}}\left(S_q\left(\frac{q}{2}\right) - S_q(qx)\right).$$

Here is a plot of

$$f_{163}(x) = \frac{2\pi^2 x}{\sqrt{163}}\left(S_{163}\left(\frac{163}{2}\right) - S_{163}(163x)\right)$$

for $0 \leqslant x \leqslant 1$ and a plot of the difference $f(x) - f_{163}(x)$:





We can use the corollary to prove that $f_{163}(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/2$ and consequently that $f(x) \geqslant 0$ for $1/4 > x \geqslant 0.043$ as follows:

$$\begin{aligned} f(x) &= \sum_{n=1}^{40} \frac{\lambda(n)\sin 2\pi nx}{n^2} + \frac{\Theta}{40} = \sum_{n=1}^{40} \frac{\chi_{163}(n)\sin 2\pi nx}{n^2} + \frac{\Theta}{40} \\ &= f_{163}(x) + \frac{\Theta}{20} = \frac{2\pi^2 x}{\sqrt{163}}\left(S_{163}\left(\frac{163}{2}\right) - S_{163}(163x)\right) + \frac{\Theta}{20} \end{aligned}$$

where Θ denotes a number with absolute value at most 1, not necessarily the same at each occurrence. Now for a an integer, $S_{163}(163x)$ is constant for x in the interval $\left[\frac{a}{163}, \frac{a+1}{163}\right)$. Therefore, $f_{163}(x) \geqslant \min\left\{f_{163}\left(\frac{a}{163}\right), f_{163}\left(\frac{a+1}{163}\right)\right\}$ for x in this interval. We can tabulate these values:

a	1	2	3	4	5	6	7	8	9	10
$f_{163}\left(\frac{a}{163}\right)$	0.0095	0.0095	0.019	0.038	0.047	0.066	0.076	0.095	0.12	0.14

Since $\frac{\Theta}{20} \leqslant 0.05$, it follows from (1) that $f(x) \geqslant 0$ for $0.25 \geqslant x \geqslant \frac{7}{163} = 0.043$.

Corollary 2. *$f(x) \geqslant 0$ for $0.043 \leqslant x \leqslant 0.25$.*

It seems clear that for any given $\epsilon > 0$ we could replace 0.043 by ϵ in this inequality with enough computation time. Also, if we use Euler products instead of Dirichlet series, we can show that $f(x) \geqslant 0$ for $1/4 \geqslant x \geqslant 0.011$.

The following conjecture seems surprising.

Conjecture 1. *If $q \equiv 3 \bmod 8$ is squarefree, then $f_q(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/2$.*

Remark 2. J. Bober has checked that this inequality is true for all primes $q \equiv 3 \bmod 8$ up to 10^9.

Now we turn to the proofs.

3. Useful lemmas

Lemma 1. *For $y > 0$ we have*

$$\frac{1}{2\pi i} \int_{(c)} \frac{X(1-s)y^{1-s}}{1-s}\, ds = \frac{\sin 2\pi y}{\pi}$$

for any c satisfying $0 < c < 1$ where (c) denotes the path from $c - i\infty$ to $c + i\infty$.

The integrand has simple poles at $s = 0, -2, -4, \ldots$ with the residue at $s = -2n$ equal to

$$\frac{1}{\pi}\, \frac{(-1)^n (2\pi y)^{2n+1}}{(2n+1)!}.$$

Summing these leads to the desired formula. See also [T1]; the above is the integral of formula (7.9.5) in [T1].

LEMMA 2. *If $c > 0$ and $\Re a > 0$, then*

$$\frac{1}{2\pi i}\int\limits_{(c)} \frac{\Gamma(s)\Gamma(a)}{\Gamma(s+a)} x^{-s}\, ds = \begin{cases} (1-x)^{a-1} & \text{if } 0 < x < 1, \\ 0 & \text{if } x \geqslant 1. \end{cases}$$

This is formula (7.7.14) of [T1].

LEMMA 3. *If $c > 0$, then*

$$\frac{1}{2\pi i}\int\limits_{(c)} \frac{x^s}{s(s+1)}\, ds = \begin{cases} 1-\frac{1}{x} & \text{if } x > 1, \\ 0 & \text{if } 0 < x \leqslant 1. \end{cases}$$

This lemma is well-known and is easy to verify.

4. Proofs of theorems

Proof of Theorem 1. This assertion is a consequence of Landau's Theorem: "If $g(n) \geqslant 0$ then the rightmost singularity of $\sum_{n=1}^{\infty} g(n)n^{-s}$ is real." This is Theorem 10 of [HR] and Theorem 1.7 of [MV2]. What we actually need is an integral version of this theorem: "If $g(x) \geqslant 0$ then the rightmost singularity of $\int_1^{\infty} g(x)x^{-s}\, dx$ is real." The proof of this version is essentially the same as that of the first version (see [MV2, Lemma 15.1]). The application to our situation is slightly subtle. We argue as follows. Since

$$\sum_{n=1}^{\infty} \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)},$$

it follows from Lemma 1 that

$$\frac{f(x)}{\pi} = \frac{1}{2\pi i}\int\limits_{(c)} \frac{X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} x^{1-s}\, ds$$

where $0 < c < 1$. The integral is absolutely convergent for $0 < c < 1/2$. By Mellin inversion we have

$$\frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} = \int\limits_0^{\infty} f(x)x^{s-2}\, dx.$$

We split the integral into two integrals at $x = 4$ so that

$$\frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)} = \int\limits_0^{4} f(x)x^{s-2}\, dx + \int\limits_4^{\infty} f(x)x^{s-2}\, dx = I_1(s) + I_2(s),$$

say. The integral defining $I_1(s)$ is absolutely convergent for $\sigma > 1$ and the second integral is absolutely convergent for $\sigma < 1$. Using the periodicity of f we can show that the second integral converges for $\sigma < 2$. Indeed, let

$$F(x) = \int\limits_0^{x} f(t)\, dt.$$

Then $F(n) = 0$ for all integers n and F is bounded. Therefore,

$$\begin{aligned} I_2(s) &= \sum_{n=4}^{\infty} \int_n^{n+1} f(x)x^{s-2}\,dx \\ &= \sum_{n=4}^{\infty}\Big(F(x)x^{s-2}|_{x=n}^{x=n+1} - (s-2)\int_n^{n+1} F(x)x^{s-3}\,dx\Big) \\ &= -(s-2)\int_4^{\infty} F(x)x^{s-3}\,dx. \end{aligned}$$

This integral converges for $\Re s < 2$. So, we now have I_2 analytic for $\Re s < 2$. Clearly, $I_1 + I_2$ is analytic for $\Re s > \max\{-1/2, \rho - 1\}$, i.e. for $\Re s > 0$. (The pole of $X(1-s)$ at $s = 0$ is canceled by the zero of $1/\zeta(s+1)$ at $s = 0$.) It follows that $I_1(s) = (I_1(s) + I_2(s)) - I_2(s)$ is analytic for $\Re s > 0$. Hence $I_2(s)$ is also analytic for $\Re s > 0$, and since we already knew it was analytic for $\Re s < 2$, it follows that $I_2(s)$ is entire. Now, we can write I_1 as

$$I_1(s) = \int_{1/4}^{\infty} f(1/x)x^{-s}\,dx.$$

Recall we have assumed that $f(1/x) \geqslant 0$ for $x \geqslant 4$. Therefore, by Landau's Theorem, the rightmost singularity of $I_1(s)$ is real. Since I_2 is entire, it follows that the rightmost pole of $I_1(s) + I_2(s)$ must also be real. But the rightmost real pole of

$$I_1(s) + I_2(s) = \frac{\pi X(1-s)}{1-s}\,\frac{\zeta(2s+2)}{\zeta(s+1)}$$

is at $s = -1/2$. This must be the rightmost pole. Therefore the poles at $\rho - 1$ must all have their real parts less than or equal to $-1/2$. In particular, $\Re\rho \leqslant 1/2$, which is RH.

Proof of Theorem 2. We start again from

$$\frac{f(x)}{\pi} = \frac{1}{2\pi i}\int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\,ds$$

where $0 < c < 1/2$. The integrand has poles only at $s = -1/2$ and at $s = \rho - 1$ where ρ is a complex zero of $\zeta(s)$, and nowhere else in the s-plane. The residue at $s = -1/2$ is

$$\frac{X\left(\frac{3}{2}\right)}{\frac{3}{2}\zeta\left(\frac{1}{2}\right)}x^{3/2} = -\frac{4\pi}{3\zeta\left(\frac{1}{2}\right)}x^{3/2}.$$

Assuming that the zeros are simple, the residue at $s = \rho - 1$ is

$$\frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)}.$$

We (carefully) move the path of integration to (c) where $-2 < c < -1$. To do this we have to cross through a field of poles arising from the zeros of the zeta-function. We use Theorem 14.16 of [T1] (see also [R]) to find a path on which $1/\zeta(s+1) \ll T^{\epsilon}$ where we can safely cross. Using the bounds $|X(1-s)| \ll T^{\sigma-1/2}$ and $\zeta(2s+2) \ll T^{-1/2-\sigma}$ we can get the sum of the residues arising from the zeros up to height T together with an error term that tends to 0 as $T \to \infty$. Thus, assuming the zeros are simple,

$$\begin{aligned}\frac{f(x)}{\pi} &= -\frac{4\pi x^{3/2}}{3\zeta(1/2)} + \sum_{\rho} \frac{X(2-\rho)\zeta(2\rho)x^{2-\rho}}{(2-\rho)\zeta'(\rho)} \\ &\quad + \frac{1}{2\pi i}\int_{(c)} \frac{X(1-s)\zeta(2s+2)x^{1-s}}{(1-s)\zeta(s+1)}\,ds.\end{aligned}$$

If the zeros are not simple, we modify the sum over zeros appropriately. We make the change of variable $s \mapsto -s$ in the integral. Using the functional equation for the ζ-function and functional relations for the Γ-function, we see that the new integrand is

$$\frac{X(1+s)\zeta(2-2s)x^{1+s}}{(1+s)\zeta(1-s)} = -\pi^{3/2}2^{2s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+2)}\frac{\zeta(2s-1)}{\zeta(s)}x^{1+s}.$$

By Lemma 2,

$$\frac{1}{2\pi i}\int_{(c)} \pi^{3/2}2^{2s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+2)}\frac{\zeta(2s-1)}{\zeta(s)}x^{1+s} = \frac{8\pi}{3}x^{3/2}\sum_{n\leqslant 4x}\frac{\ell(n)}{\sqrt{n}}\left(1-\frac{n}{4x}\right)^{3/2}.$$

Then Theorem 2 follows.

Proof of Theorem 4. We denote $\chi_q(n) = \left(\frac{n}{q}\right)$. By Lemma 1,

$$f_q(x) = \frac{\pi}{2\pi i}\int_{(c)} L(s+1,\chi_q)X(1-s)x^{1-s}\,\frac{ds}{1-s} \tag{3}$$

where $0 < c < 1$. Since χ_q is odd, we find that the integrand has a pole at $s = 0$ and nowhere else in the complex plane. We move the path of integration to (c) where $c < -1$ to see that

$$f_q(x) = 2\pi x L(1,\chi_q) + \frac{\pi}{2\pi i}\int_{(c)} L(s+1,\chi_q)X(1-s)x^{1-s}\,\frac{ds}{1-s}.$$

Now let $s \mapsto -s$ in the integral and use the functional equation (see [D], [IK] or [MV2])

$$L(1-s,\chi_q) = 2q^{s-1/2}(2\pi)^{-s}\Gamma(s)\sin\frac{\pi s}{2}L(s,\chi_q).$$

After simplification, the integral above is

$$\frac{-2\pi^2}{2\pi i} \int_{(c)} q^{s-1/2} x^{1+s} L(s,\chi_q)\, \frac{ds}{s(s+1)}.$$

By Lemma 3, this integral is

$$\frac{-2\pi^2 x}{\sqrt{q}} \sum_{n\leqslant xq} \chi_q(n)\left(1-\frac{n}{xq}\right).$$

The proof of Theorem 4 is complete.

Remark 3. Note that the non-negativity, for $0<x<1/4$, of the right-hand side of (3) implies the Riemann Hypothesis. This condition only involves Dirichlet L-functions with quadratic characters. Thus, information solely about Dirichlet L-functions potentially gives the Riemann Hypothesis. This example shows that different L-functions somehow know about each other.

5. Further remarks. Since

$$h(q) \gg_\epsilon q^{1/2-\epsilon},$$

we see that

$$f_q(x) \geqslant 0 \quad \text{for } a \ll x \ll q^{-1/2-\epsilon}.$$

In particular,

$$f_q(a/q) \geqslant 0 \quad \text{for } a \ll q^{1/2-\epsilon}.$$

But this does not give information about $f(x)$.

Also, the Pólya–Vinogradov inequality tells us that

$$\max_N \Big|\sum_{n=1}^{N} \chi_q(n)\Big| \ll q^{1/2}\log q$$

and the work of Montgomery and Vaughan [MV1] shows that the Riemann Hypothesis for $L(s,\chi)$ implies that

$$\max_N \Big|\sum_{n=1}^{N} \chi_q(n)\Big| \ll q^{1/2}\log\log q.$$

Moreover, it is known that the right-hand side here cannot be replaced by any function that goes to infinity slower. It is also known, assuming the Riemann Hypothesis for $L(s,\chi)$, that

$$L(1,\chi) \ll \log\log q.$$

Our desired inequality can be expressed in terms of $L(1,\chi)$ as

$$\max_{N\leqslant q/4}\sum_{n=1}^{N}\chi(n)\Big(1-\frac{n}{N}\Big)\leqslant\frac{\sqrt{q}}{\pi}L(1,\chi). \tag{4}$$

It appears that both sides of this inequality can be as big as $\sqrt{q}\log\log q$.

A question is whether the converse of Theorem 1 is true. It might be possible to approach this by showing that the "3/2" derivative of $f(x)$ is positive at $x = 0$ so that there is a small interval to the right of 0 for which $f(x)\geqslant 0$. This method, or trying to prove (2) directly, would involve explicit estimates (assuming RH) for $1/\zeta(s)$ in the critical strip; see [MV2, Section 13.2] for a good approach to such explicit estimates.

Finally, we mention that $f(x)$ can be evaluated at a rational number $x = a/q$ as an average involving Dirichlet L-functions $L(s,\chi)$ where χ is a character modulo q.

6. Evaluation of $f_q(a/p)$. Let $p < q$ and $(a,p) = 1$. We explicitly evaluate $f_q(a/p)$ as a sum over characters modulo p as follows. We have

$$\begin{aligned}f_q(a/p)&=\sum_{n=1}^{\infty}\frac{\chi_q(n)\sin\frac{2\pi an}{p}}{n^2}=\sum_{n=1}^{\infty}\frac{\chi_q(n)}{n^2}\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi\bmod p}\tau(\psi)\overline{\psi}(an)\Big\}\\&=\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi\bmod p}\tau(\psi)\overline{\psi}(a)\sum_{n=1}^{\infty}\frac{\chi_q(n)\psi(n)}{n^2}\Big\}\\&=\frac{1}{\phi(p)}\Im\Big\{\sum_{\psi\bmod p}\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\Big\}.\end{aligned}$$

Now, if ψ is even then

$$\overline{\tau(\psi)}=\sum_{n=1}^{p}\overline{\psi(n)}e(-an/p)=\sum_{n=1}^{p}\overline{\psi}(-n)e(an/p)=\sum_{n=1}^{p}\overline{\psi}(n)e(an/p)=\tau(\overline{\psi}),$$

while if ψ is odd then

$$\overline{\tau(\psi)}=-\tau(\overline{\psi}).$$

Thus, for even ψ,

$$\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})+\tau(\overline{\psi})\psi(a)L(2,\chi_q\psi)\}=0,$$

and for odd ψ,

$$\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})+\tau(\overline{\psi})\psi(a)L(2,\chi_q\psi)\}=2\Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}.$$

Therefore, using the fact that $\tau(\chi_p) = i\sqrt{p}$ when $p \equiv 3 \bmod 4$, we have

$$f_q(a/p) = \frac{1}{\phi(p)} \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1 \\ \psi^2 \neq \psi_0}} \Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}$$
$$+ \delta(p \equiv 3 \bmod 4)\frac{\sqrt{p}}{\phi(p)}\Re\{\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}.$$

We use this to prove that

$$f_q(1/3) > 0 \quad \text{and} \quad f_q(1/5) > 0$$

for any q. By the formula above we have

$$f_q(1/3) = \frac{\sqrt{3}}{2}L(2,\chi_q\chi_3) > 0$$

and

$$f_q(1/5) = \frac{2}{\phi(5)}\Im\{(-1.17557 + 1.90211i)L(2,\chi_q\psi_1)\} = 1.9\alpha - 1.17\beta$$

where $\psi_1 = \{1, i, -i, -1, 0\}$ with $\tau(\psi_1) = -1.17557 + 1.90211i$ and

$$\alpha + i\beta = L(2,\chi_q\psi_1) = 1 + \frac{\chi_q(2)i}{2^2} - \frac{\chi_q(3)i}{3^2} - \frac{\chi_q(4)}{4^2} + \cdots.$$

Now

$$\alpha \geqslant 1 - \frac{1}{4^2} - \frac{1}{5^2} - \cdots = 0.716\ldots \quad \text{and} \quad |\beta| < \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 0.64\ldots.$$

Thus,

$$f_q(1/5) > 0.6.$$

A couple of formulas may help us move forward here. One is that if θ_1 and θ_2 are characters with coprime moduli m_1 and m_2 respectively, then (see [IK, (3.16)])

$$\tau(\theta_1\theta_2) = \theta_1(m_2)\theta_2(m_1)\tau(\theta_1)\tau(\theta_2).$$

The other is that

$$L(1-r,\theta) = -\frac{m^{r-1}}{r}\sum_{b=1}^{m}\theta(b)B_r(b/m)$$

for a character θ modulo m and a positive integer r where B_r is the rth Bernoulli polynomial (see [Wa, Theorem 4.2]). Recall the functional equation (see [D]) for a primitive character θ modulo m:

$$L(1-s,\theta) = \left(\frac{m}{2\pi}\right)^s \Gamma(s)(e^{\pi is/2} + \theta(-1)e^{-\pi is/2})L(s,\overline{\theta})/\tau(\overline{\theta}).$$

It follows that for an even $\theta = \chi_q\psi$, with $q \equiv 3 \bmod 4$ and ψ an odd character modulo p, we have

$$\begin{aligned} L(2,\chi\overline{\psi}) &= -\pi\left(\frac{pq}{2\pi}\right)^{-1} L(-1,\theta)/\tau(\theta) \\ &= -\pi\left(\frac{pq}{2\pi}\right)^{-1} L(-1,\chi\psi)/(\chi(p)\psi(q)\tau(\psi)i\sqrt{q}). \end{aligned}$$

Therefore,

$$\begin{aligned} \Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\} &= \Re\left\{\frac{2\pi^2\chi_q(p)\overline{\psi}(aq)}{pq^{3/2}}L(-1,\chi_q\psi)\right\} \\ &= -\Re\left\{\frac{\pi^2\chi_q(p)\overline{\psi}(aq)}{\sqrt{q}}\sum_{b=1}^{pq}\chi_q(b)\psi(b)B_2(b/(pq))\right\}. \end{aligned}$$

We sum this equation over the odd characters modulo p using

$$\begin{aligned} \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \psi\left(\frac{b}{aq}\right) &= \frac{1}{2}\sum_{\psi \bmod p}\left(\psi\left(\frac{b}{aq}\right) - \psi\left(-\frac{b}{aq}\right)\right) \\ &= \frac{\phi(p)}{2}\begin{cases} 1 & \text{if } b \equiv aq \bmod p, \\ -1 & \text{if } b \equiv -aq \bmod p. \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} &\sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\} \\ &= -\frac{\phi(p)}{2}\,\frac{\pi^2\chi_q(p)}{\sqrt{q}}\Bigg(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}}\chi_q(a)B_2(b/(pq)) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}}\chi_q(a)B_2(b/(pq))\Bigg). \end{aligned}$$

Note that

$$B_2(x) = x^2 - x + 1/6.$$

Also,

$$\begin{aligned} \sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}}\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}}\chi_q(b) &= 0, \\ \sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}}b\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}}b\chi_q(b) &= 0. \end{aligned}$$

Thus,

$$\sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \Im\{\tau(\psi)\overline{\psi}(a)L(2,\chi_q\overline{\psi})\}$$
$$= -\frac{\pi^2\chi_q(p)}{2p^2q^{5/2}}\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big).$$

Hence, we have

Theorem 5. *For primes p and q both congruent to 3 modulo 4 and for $1 \leqslant a < p/2$ we have*

$$f_q(a/p) = -\frac{\pi^2\chi_q(p)}{2p^2q^{5/2}}\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big).$$

As a consequence we also have

Corollary 3. *If*

$$\text{test}_a(p,q) := -\chi_q(p)\Big(\sum_{\substack{b\leqslant pq \\ b\equiv aq \bmod p}} b^2\chi_q(b) - \sum_{\substack{b\leqslant pq \\ b\equiv -aq \bmod p}} b^2\chi_q(b)\Big) > 0 \tag{5}$$

for all primes $p < q$ congruent to 3 modulo 8 and all $0 < a < p/2$, then the Riemann Hypothesis follows.

We note that by these techniques one can show

Theorem 6.

$$f_q(a/q) = \frac{\pi^2}{2\sqrt{q}}\left(\chi_q(a) - \frac{1}{q^2}\sum_{c=1}^{q-1} c^2(\chi_q(c-a) - \chi_q(c+a))\right).$$

When this formula is compared with our earlier formula

$$f_q\left(\frac{a}{q}\right) = \frac{2\pi^2}{q^{3/2}}\left(\frac{a}{3}\sum_{n\leqslant \frac{q-1}{2}} \chi_q(n) - \sum_{n=1}^{a}(a-n)\chi_q(n)\right),$$

we deduce the identity

$$\frac{a}{3}\sum_{n\leqslant (q-1)/2} \chi_q(n) - \sum_{n=1}^{a}(a-n)\chi_q(n)$$
$$= \frac{q}{4}\left(\chi_q(a) - \frac{1}{q^2}\sum_{c=1}^{q-1} c^2(\chi_q(c-a) - \chi_q(c+a))\right)$$

for $q \equiv 3 \bmod 4$.

Now we indicate another possible direction.

PROPOSITION 1. *If*

$$f_q(x) = 0$$

then x is a rational number.

Proof. By Corollary 1, $f_q(x) = 0$ implies that $S_q(q/2) - S_q(qx) = 0$. But $S_q(q/2) = h(q)$ is an integer. So $f_q(x) = 0$ implies that $S_q(qx)$ is a rational number. Now

$$S_q(qx) = \sum_{n \leqslant [qx]} \chi_q(n)\left(1 - \frac{n}{qx}\right) = \sum_{n \leqslant [qx]} \chi_q(n) - \frac{\sum_{n \leqslant [qx]} n\chi_q(n)}{qx}.$$

This has the shape integer $-$ $\frac{\text{integer}}{qx}$, which can only be rational if x is a rational number. ■

So, it suffices to show that $f_q(x)$ has no rational zeros; perhaps a congruence argument could work. However, Theorem 5 is not of much use here because the hypothetical x for which $f_q(x) = 0$ would likely have a denominator that is divisible by q, so the conditions of Theorem 5 do not hold.

We remark that there are rational values of x for which the numerator of $f_q(x)$ is congruent to 0 modulo q; for example

$$f_{19}\left(\tfrac{25}{76}\right) = \tfrac{19}{25}, \quad f_{19}\left(\tfrac{29}{190}\right) = \tfrac{19}{29}, \quad f_{19}\left(\tfrac{30}{209}\right) = \tfrac{19}{30}.$$

These examples, which all seem to have an x with denominator divisible by q, might be worth studying further.

Here is one final formula that may or may not be useful. Suppose that $f_q(x) = 0$. Let $y = xq$. Then either

$$\sum_{n \leqslant y} \chi_q(n) = h(q) \quad \text{and} \quad \sum_{n \leqslant y} n\chi_q(n) = 0$$

or else

$$y = \frac{\sum_{n \leqslant [y]} n\chi_q(n)}{\sum_{n \leqslant [y]} \chi_q(n) - h(q)}.$$

The first alternative seems unlikely as in that case there would be an interval on which $f_q(x)$ would be identically 0.

7. Conclusion. Conjecture 1 has been checked for primes up to 10^9 and it holds for those primes. However, probabilistic grounds call into question its truth for all primes $q \equiv 3 \bmod 8$. Of course, one only needs its truth for a set of characters χ_q for which $\chi_q(n) = \lambda(n)$ for all $n \leqslant N_q$ where $N_q \to \infty$ with q. Presumably something like this is correct (and should be equivalent to RH), but it is not clear how to proceed. But the results of Section 6 suggest a slightly alternative way forward which may have a more arithmetic flavor.

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