STUDIA MATHEMATICA
Online First version

On the inverse problem for free quasiconformality in Banach spaces

by

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Abstract. It is known that the inverse of a quasiconformal homeomorphism of domains in $\mathbb{R}^n$ is also quasiconformal. This paper focuses on the inverse problem for free quasiconformality in Banach spaces. We first show that the inverse of a fully semisolid homeomorphism is fully semisolid under an additional coarsely Lipschitz condition in the quasihyperbolic metric. This gives several partial answers to two open problems posed by Väisälä. Next, we prove that the inverse of a locally quasisymmetric homeomorphism is also locally quasisymmetric. As applications, we obtain new characterizations of freely quasiconformal mappings in Banach spaces, and study the relation between freely quasiconformal mappings and quasisymmetric mappings between uniform domains.

1. Introduction and main results. Gehring and Palka [7] introduced the quasihyperbolic metric in proper subdomains of Euclidean spaces $\mathbb{R}^n$, which is an analogue of the hyperbolic metric. It has become an important tool in geometric function theory and analysis in metric spaces; see e.g. [2, 3, 11, 13, 15, 22, 24]. For example, Bonk et al. [2] investigated the negative curvature of the quasihyperbolic metric in terms of Gromov hyperbolicity and established an analogue of the Riemann Mapping Theorem for Gromov hyperbolic domains of $\mathbb{R}^n$. We refer to the recent works [1, 29, 30] for more discussions along this line.

Gehring and Osgood [6, Theorem 3] proved a Schwarz–Pick type result by showing the quasi-invariance of the quasihyperbolic distance under quasiconformal mappings between proper subdomains of $\mathbb{R}^n$. This result has been generalized by Soultanis and Williams [16] to metric measure spaces with bounded geometry. Huang and Liu [11] obtained a result similar to [6, Theorem 3] for weakly quasisymmetric mappings in quasiconvex metric spaces.

2020 *Mathematics Subject Classification*: Primary 30C65; Secondary 30C20.
Key words and phrases: quasiconformal homeomorphism, quasihyperbolic metric, quasisymmetric homeomorphism, solid map.
Received 11 October 2023; revised 4 April 2024.
Published online 17 July 2024.

DOI: 10.4064/sm231011-19-4

From late 1980's onwards, Väisälä [20, 22] expanded on the idea of Gehring–Palka–Osgood and developed a theory of quasiconformal mappings in infinite-dimensional Banach spaces. Recently, the research on freely quasiconformal mappings has aroused substantial interest. Examples include relative quasimöbius invariance of uniform domains [25, 28], local properties of quasihyperbolic mappings [12, 24, 26], quasisymmetry of freely quasiconformal mappings in the quasihyperbolic metric [8], and applications of Gromov hyperbolicity in Väisälä's free quasiworld [27].

We assume throughout this paper that E and E' are real Banach spaces with dimension at least 2, and that $G \subsetneq E$ and $G' \subsetneq E'$ are domains, i.e., nonempty connected open sets. A property of a homeomorphism $f\colon G \to G'$ will be qualified by "fully" (e.g. "fully φ-semisolid") if for every subdomain D of G, the restriction $f|_D$ of f has this property in D.

Let φ be a homeomorphism of $[0, +\infty)$, and let $M \geq 1$ and $C \geq 0$. Following the notation of [22], we say that a homeomorphism $f\colon G \to G'$ is

- *φ-semisolid* if $k_{G'}(f(z_1), f(z_2)) \leq \varphi(k_G(z_1, z_2))$ for all $z_1, z_2 \in G$;
- *φ-solid* if both f and f^{-1} are φ-semisolid;
- *freely φ-quasiconformal* if f is fully φ-solid;
- *C-coarsely M-Lipschitz in the quasihyperbolic metric* if for all $z_1, z_2 \in G$,

$$k_{G'}(f(z_1), f(z_2)) \leq M k_G(z_1, z_2) + C.$$

Here, k_G and $k_{G'}$ are the quasihyperbolic metrics of G and G', respectively; see Subsection 2.2 for the precise definition.

For a homeomorphism between proper subdomains of $\mathbb{R}^n$, it follows from [6, Theorem 3] and [19, Theorem 6.12] that K-quasiconformality is equivalent to free φ-quasiconformality with the parameters K and φ depending on each other and n. It is an open problem whether the above result can be made independent of the dimension n, and more generally if it holds in infinite-dimensional Banach spaces; see [22, Problem 13.2.1]. We remark that there are various other conformal metrics in geometric function theory, such as Apollonian metric, Ferrand's modulus metric, triangular ratio metric, and visual angle metric, which can be used to characterize conformality and quasiconformality. See [4, 9, 17, 23] and the references therein for more discussion and background information.

A deep theorem states that the inverse of a K-quasiconformal mapping between proper subdomains of $\mathbb{R}^n$ is K'-quasiconformal with K' depending on K and n. It is an interesting problem whether this result is valid with K' depending only on K. For metric spaces, as pointed out by Heinonen and Koskela [10], it is still unclear whether the inverse of a given quasiconformal mapping is also quasiconformal under the metric definition. The purpose of this paper is to consider the following inverse problem in Banach spaces, raised by Väisälä ([20, Problem 7.2.1], [22, Problem 7.5 or Problem 13.2.5]).

Problem 1.1. *Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism. Is f freely φ_1-quasiconformal with φ_1 depending only on φ?*

The "fully" in Problem 1.1 cannot be removed, because there is a semisolid mapping which is not solid (see [22, Example 7.4]) and solid mappings are not always quasiconformal (see [19, Subsection 6.10]). It is well-known that a K-quasiconformal homeomorphism between proper subdomains of $\mathbb{R}^n$ is η-quasisymmetric on all Whitney balls in the domain with η dependent of n and K, and vice versa. See [18] for the definition of quasisymmetric mappings.

In Banach spaces, it was shown by Väisälä in [22, Theorem 7.9] that a homeomorphism $f\colon G \to G'$ is fully φ-semisolid if and only if f is q-locally η-quasisymmetric for some q, i.e., there exist a constant $0 < q \leq 1$ and a self-homeomorphism η of $[0, +\infty)$ such that f is η-quasisymmetric on each Whitney ball $B(x, q\, d_G(x))$ for $x \in G$. Here, $d_G(x)$ denotes the norm distance between x and the boundary ∂G of G. In view of this result, one observes that Problem 1.1 is equivalent to the following inverse problem for local quasisymmetry on Whitney balls.

Problem 1.2. *Suppose that $f\colon G \to G'$ is a q-locally η-quasisymmetric homeomorphism. Is f^{-1} q_1-locally η_1-quasisymmetric with q_1 and η_1 depending only on q and η?*

Note that the inverse of a quasisymmetric mapping is also quasisymmetric. However, the local situation is somewhat complicated. In $\mathbb{R}^n$, the solution to Problem 1.2 is known with η_1 depending on n. It is unclear whether this result holds in infinite-dimensional Banach spaces. For more discussion in the metric spaces setting we refer to [11, 13, 16].

As our first main result, we give a partial solution to Problem 1.1 assuming that f^{-1} satisfies a coarse condition in the quasihyperbolic metric, which is necessary due to [22, Theorem 7.9].

Theorem 1.3. *Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism. If $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric, then f is freely φ_1-quasiconformal with φ_1 depending only on φ, M, and C.*

We state some applications of Theorem 1.3. Firstly, we know from [22, Theorem 7.6] that the notion of relativity introduced by Gehring [5] is equivalent to semisolidity. For the definition of relative mappings we refer to [22, Definition 7.2]. By [22, Lemma 2.3 and Theorem 3.4(2)], semisolidity implies coarsely Lipschitz property in the quasihyperbolic metric. Inspired by these results, we obtain more partial solutions to Problem 1.1 from Theorem 1.3, which can be regarded as new characterizations of freely quasiconformal mappings. Moreover, the following is an improvement of [22, Corollary 7.12] (for the notion of quantitative equivalence, see [22, Section 2.4] and Definition 2.4). Here φ, ϕ, ψ denote homeomorphisms of $[0, +\infty)$ and $\theta\colon [0,1) \to [0,+\infty)$ is an embedding with $\theta(0) = 0$.

COROLLARY 1.4. *Suppose that $f\colon G \to G'$ is a homeomorphism. Then the following are quantitatively equivalent:*

(1) *f is fully φ-semisolid and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric.*
(2) *f is fully φ-semisolid and $f^{-1}\colon G' \to G$ is ϕ-semisolid.*
(3) *f is fully φ-semisolid and $f^{-1}\colon G' \to G$ is θ-relative.*
(4) *f is freely ψ-quasiconformal.*

Secondly, we apply Theorem 1.3 and [22, Theorem 7.9] to investigate the inverse problem for locally quasisymmetric mappings in a quantitative way. We provide several partial solutions to Problem 1.2 and show the following:

COROLLARY 1.5. *Suppose that $f\colon G \to G'$ is a q-locally η-quasisymmetric homeomorphism. If one of the following conditions holds:*

(1) *$f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric;*
(2) *$f^{-1}\colon G' \to G$ is φ-semisolid;*
(3) *$f^{-1}\colon G' \to G$ is θ-relative,*

then $f^{-1}\colon G' \to G$ is q_1-locally η_1-quasisymmetric with q_1 and η_1 depending only on q, η, and the parameters in the above conditions.

It is an important problem to characterize in which situations quasiconformality and quasisymmetry are equivalent. This infinitesimal-to-global principle was investigated by several researchers; see e.g. [3, 10, 21, 22, 27]. As a third application of Theorem 1.3, we study the relation between freely quasiconformal mappings and quasisymmetric mappings between uniform domains. Our result in this direction is as follows.

THEOREM 1.6. *Let G and G' be c-uniform domains. Suppose $f\colon G \to G'$ is a fully φ-semisolid homeomorphism and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric. Then:*

(1) *f is θ-quasimöbius with θ depending only on c, φ, M, and C.*
(2) *f is η-quasisymmetric if and only if $f(\infty) = \infty$ or there is a number $\tau \geq 1$ and a point $z_0 \in G$ such that* $\operatorname{diam}(G) \leq \tau d_G(z_0)$ *and* $\operatorname{diam}(G') \leq \tau d_{G'}(f(z_0))$ *with η and τ depending only on each other and c, φ, M, and C.*

REMARK 1.7. We remark that Theorem 1.6(1) is an improvement of [22, Theorem 11.12], and Theorem 1.6(2) shows that the converse to [22, Theorem 6.33] is valid in a quantitative way.

Next, we explain the connection between Problem 1.1 and the following problem posed by Väisälä [20, Problem 7.2.2].

Problem 1.8. *Suppose that $f\colon G \to G'$ is a homeomorphism and that each point in G has a neighborhood in which f is freely φ-quasiconformal. Is f freely φ_1-quasiconformal with φ_1 depending only on φ?*

From the metric definition of quasiconformal mappings, we know that the shape of infinitesimal balls is distorted by a uniformly bounded amount. It follows that a homeomorphism between subdomains of $\mathbb{R}^n$ is quasiconformal if it is merely quasiconformal on a neighborhood of each point in the domain. This means that the answer to Problem 1.8 is affirmative for quasiconformal mappings of Euclidean spaces. Using Theorem 1.3 and [22, Theorem 7.9], we obtain the following partial solution to Problem 1.8 in Banach spaces.

Corollary 1.9. *Let φ be a homeomorphism of $[0, +\infty)$, and let $0 < q < 1$, $M \geq 1$ and $C \geq 0$. Suppose that a homeomorphism $f\colon G \to G'$ is freely φ-quasiconformal on each Whitney ball $B(x, q\, d_G(x))$ for every $x \in G$. If $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric, then f is freely φ_1-quasiconformal with φ_1 depending only on φ, q, M, and C.*

Finally, we study Problem 1.2 in the extremal case when $q = 1$. Surprisingly, we establish the following solution to Problem 1.2.

Theorem 1.10. *Suppose that $f\colon G \to G'$ is 1-locally η-quasisymmetric. Then f^{-1} is q-locally η_1-quasisymmetric for each $0 < q < 1$ with η_1 depending only on q and η. In particular, f is freely φ_1-quasiconformal with φ_1 depending only on η.*

From Theorems 1.6 and 1.10, we obtain global quasisymmetry from local quasisymmetry in uniform domains of Banach spaces.

Corollary 1.11. *Suppose that $f\colon G \to G'$ is a 1-locally η-quasisymmetric homeomorphism between c-uniform domains. If $f(\infty) = \infty$ or there is a number $\tau \geq 1$ and a point $x_0 \in G$ such that $\operatorname{diam}(G) \leq \tau d_G(x_0)$ and $\operatorname{diam}(G') \leq \tau d_{G'}(f(x_0))$, then f is η_1-quasisymmetric with η_1 depending only on c, η, and τ.*

The paper is organized as follows. In Section 2 we recall some definitions and auxiliary results. The proofs of Theorems 1.3, 1.6, and 1.10 are given in Section 3.

2. Preliminaries and auxiliary results

2.1. Notation. Let E be a real Banach space with dimension at least 2. The norm of a vector x in E is denoted by $|x|$. We use $|x_1 - x_2|$ to denote the norm distance between x_1, $x_2 \in E$. The line segment between x_1 and x_2 is denoted by $[x_1, x_2]$. The diameter of a bounded set W in the norm is denoted by $\operatorname{diam}(W)$. The distance between two subsets W_1 and W_2 of E is denoted by $\operatorname{dist}(W_1, W_2)$. Let $B(z,r)$ and $\overline{B}(z,r)$ be, respectively, the open ball and closed ball of radius r centered at the point z in E. We use $S(z,r)$ to denote the sphere of radius r centered at z. The length of an arc σ in the norm is denoted by $\ell(\sigma)$.

The one-point extension of E is the Hausdorff space $\dot{E} = E \cup \{\infty\}$, where the neighborhoods of ∞ are the complements of closed bounded subsets of E. Let $W \subset E$. We write ∂W and $\overline{W}$ for the boundary and the closure of W in the topology of $\dot{E}$.

2.2. Quasihyperbolic metric and uniform domains. Let G be a proper domain in E. Following the notation of [22], the *quasihyperbolic length* of a rectifiable arc σ in G is defined by

$$\ell_{k,G}(\sigma) = \int_\sigma \frac{ds}{d_G(z)},$$

where ds is the length element in the norm. For each pair of points x_1, x_2 in G, the *quasihyperbolic distance* in G between them is defined as

$$k_G(x_1, x_2) = \inf_\sigma \ell_{k,G}(\sigma),$$

where the infimum is taken over all rectifiable arcs σ in G joining x_1 and x_2. The quasihyperbolic diameter of a set $W \subset G$ is denoted by $\operatorname{diam}_{k_G}(W)$.

From [22, Theorem 3.4], one finds that the space (G, k_G) is a length metric space, and thus is c-quasiconvex for all $c \geq 1$. That is, for any $x_1, x_2 \in G$, there is an arc σ in G joining them with quasihyperbolic length $\ell_{k,G}(\sigma) \leq c k_G(x_1, x_2)$. In Euclidean spaces, we know that the space (G, k_G) is geodesic due to the Arzelà–Ascoli Theorem; see also [6]. For more discussion in infinite-dimensional Banach spaces we refer to [15] and the references therein.

Next, we introduce some estimates on the quasihyperbolic metric.

Lemma 2.1 ([22, Theorem 3.9]). *Suppose that $G \subset E$ is a domain with $x, y \in G$. If either $|x-y| \leq \frac{1}{2} d_G(x)$ or $k_G(x,y) \leq 1$, then*

$$\frac{1}{2}\frac{|x-y|}{d_G(x)} < k_G(x,y) \leq 2\frac{|x-y|}{d_G(x)}.$$

Lemma 2.2 ([20, Lemma 2.2]). *Suppose that $G \subset E$ is a domain.*

(1) *For all $z_1, z_2 \in G$,*

$$k_G(z_1, z_2) \geq \log\left(1 + \frac{|z_1 - z_2|}{\min\{d_G(z_1), d_G(z_2)\}}\right) \geq \left|\log \frac{d_G(z_1)}{d_G(z_2)}\right|$$

and

$$\frac{|z_1 - z_2|}{d_G(z_1)} \leq e^{k_G(z_1, z_2)} - 1.$$

(2) *For* $w \in G$, $0 < t < 1$ *and* $z_1, z_2 \in \overline{B}(w, td_G(w))$,

$$k_G(z_1, z_2) \le \frac{1}{1-t} \frac{|z_1 - z_2|}{d_G(w)}.$$

The class of uniform domains was introduced by Martio and Sarvas [14], and it is an analogue of quasidisks in higher-dimensional Euclidean spaces. In fact, this concept can be generalized to Banach spaces and metric spaces, and is useful in quasiconformal mapping theory; see e.g. [1, 2, 21, 25, 27, 30].

We conclude this part with the definition of uniform domains. Let $c \ge 1$. A domain G in E is called *c-uniform* if any $x, y \in G$ can be connected with a rectifiable arc σ in G satisfying

(1) $\min\{\ell(\sigma[x,z]), \ell(\sigma[z,y])\} \le c\, d_G(z)$ for all $z \in \sigma$,
(2) $\ell(\sigma) \le c\,|x - y|$,

where $\sigma[x,z]$ is the part of σ between x and z. Note that $\ell(\sigma)$ denotes the length of σ in norm. Such a σ is said to be a *uniform arc.*

2.3. Mappings. There are various classes of mappings closely related to freely quasiconformal mappings in the quasiworld. These mappings are defined via the norm and quasihyperbolic metric, and do not depend on volume and conformal modulus of a family of curves. Recently, these concepts have been generalized to metric spaces and serve as important tools in many areas of study; see e.g. [2, 3, 10, 11, 12, 13, 26].

Let $0 < q \le 1$ and $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. Let $1 < \alpha \le \beta$ and $M > 0$. Suppose that $G \subsetneq E$ and $G' \subsetneq E'$ are domains and $f\colon G \to G'$ is a homeomorphism. Following [22],

(i) f is said to be *η-quasimöbius* if for all distinct points $x, y, z, w \in G$,

$$\frac{|f(x) - f(z)|\,|f(y) - f(w)|}{|f(x) - f(w)|\,|f(y) - f(z)|} \le \eta\Big(\frac{|x - z|\,|y - w|}{|x - w|\,|y - z|}\Big);$$

(ii) f is said to be *η-quasisymmetric* if for each $t > 0$ and each triple x, a, b of distinct points in G,

$$|x - a| = t|x - b| \quad \text{implies} \quad |f(x) - f(a)| \le \eta(t)|f(x) - f(b)|;$$

(iii) f is said to be *q-locally η-quasisymmetric* if the restriction $f|_{B(x,qd_G(x))}$ of f to every Whitney ball $B(x, qd_G(x))$ is η-quasisymmetric for $x \in G$;
(iv) f is said to have the *(M, α, β)-ring property* if

$$\operatorname{diam}\big(f(\overline{B}(x,r))\big) \le M \operatorname{dist}\big(f(\overline{B}(x,r)), \partial f(B(x,\alpha r))\big)$$

for $x \in G$ and $B(x, \beta r) \subset G$.

Next, we record auxiliary results, due to Väisälä, regarding the relations between free quasiconformality and the other properties mentioned earlier.

Lemma 2.3. *Suppose that $f\colon G \to G'$ is a φ-semisolid homeomorphism. Then there are constants $M \ge 1$ and $C \ge 0$ depending only on φ, such that f is C-coarsely M-Lipschitz in the quasihyperbolic metric.*

Proof. Use [22, Lemma 2.3 and Theorem 3.4(2)]. ∎

Definition 2.4. We say that *condition $\mathfrak{C}_1$ implies condition $\mathfrak{C}_2$ quantitatively* if $\mathfrak{C}_1$ implies $\mathfrak{C}_2$ and the data $\Phi(\mathfrak{C}_2)$ of $\mathfrak{C}_2$ depends only on the data $\Phi(\mathfrak{C}_1)$ of $\mathfrak{C}_1$ and other given quantities. We say that *condition $\mathfrak{C}_1$ is quantitatively equivalent to condition $\mathfrak{C}_2$* if $\mathfrak{C}_1$ implies $\mathfrak{C}_2$ quantitatively and also $\mathfrak{C}_2$ implies $\mathfrak{C}_1$ quantitatively.

Lemma 2.5 ([22, Theorem 7.9]). *Suppose that $f\colon G \to G'$ is a homeomorphism. Then the following conditions are quantitatively equivalent:*

(1) *f is fully φ-semisolid;*
(2) *f is fully C-coarsely M-Lipschitz in the quasihyperbolic metric;*
(3) *for every $0 < q < 1$ there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that f is q-locally η-quasisymmetric;*
(4) *there is a constant $0 < q_1 < 1$ and a homeomorphism $\eta_1\colon [0,\infty) \to [0,\infty)$ such that f is q_1-locally η_1-quasisymmetric;*
(5) *f has the (M,α,β)-ring property for some $1 < \alpha \le \beta$ and $M > 0$.*

3. Proofs of main results

3.1. Inverse problem for free quasiconformality. This subsection is devoted to the proof of Theorem 1.3. We begin with some preparations. In what follows we suppose that $G \subsetneq E$ and $G' \subsetneq E'$ are domains, and that $f\colon G \to G'$ is a homeomorphism. Let $x' \in G'$ be a given point and $r' > 0$ be such that $B(x', 4r') \subset G'$. For $i \in \{1,2,4\}$, we set $B_i' = B(x', ir')$ and $B_i = f^{-1}\big(B(x', ir')\big)$. Recall that for a subset $W \subset E$, we use ∂W and $\overline{W}$ to denote the boundary and the closure of W, respectively.

To prove Theorem 1.3, we first establish some technical lemmas.

Lemma 3.1. *Let $W' \subset G'$ be a set with $\operatorname{diam}_{k_{G'}}(W') < \infty$. If $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric, then $\overline{f^{-1}(W')} \subset G$, $\partial f^{-1}(W') = f^{-1}(\partial W')$, and $\overline{f^{-1}(W')} = f^{-1}(\overline{W'})$.*

Proof. Because $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric and $\operatorname{diam}_{k_{G'}}(W') < \infty$, we see that $\operatorname{diam}_{k_G}(f^{-1}(W')) < \infty$. This, together with Lemma 2.2(1), shows that $f^{-1}(W')$ is bounded and $\operatorname{dist}(f^{-1}(W'), \partial G) > 0$. Therefore, $\overline{f^{-1}(W')} \subset G$ and $\partial f^{-1}(W')$ is also the boundary of $f^{-1}(W')$ in the topology of G. As f^{-1} is a homeomorphism, it follows that $\partial f^{-1}(W') = f^{-1}(\partial W')$ and

$$\overline{f^{-1}(W')} = f^{-1}(W') \cup \partial f^{-1}(W') = f^{-1}(W') \cup f^{-1}(\partial W') = f^{-1}(\overline{W'}). \quad ∎$$

LEMMA 3.2. *Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric. For all $w \in \overline{B_1}$ and $u \in \partial B_2$,*

$$\frac{|w-u|}{\min\{d_{B_4}(w), d_{B_4}(u)\}} \geq \frac{\varphi^{-1}\big(\log\frac{3}{2}\big)}{1+\varphi^{-1}\big(\log\frac{3}{2}\big)} =: t_1 \in (0,1).$$

Proof. Suppose on the contrary that there exist $w \in \overline{B_1}$ and $u \in \partial B_2$ such that

$$\frac{|w-u|}{\min\{d_{B_4}(w), d_{B_4}(u)\}} < t_1.$$

On the one hand, we know from Lemma 2.2(2) that

$$k_{B_4}(w,u) \leq \frac{1}{1-t_1}\frac{|w-u|}{\min\{d_{B_4}(w), d_{B_4}(u)\}} < \frac{t_1}{1-t_1} = \varphi^{-1}\left(\log\frac{3}{2}\right). \tag{3.3}$$

On the other hand, since f^{-1} is C-coarsely M-Lipschitz in the quasihyperbolic metric, Lemma 3.1 shows that $f(\overline{B_1}) = \overline{B_1'}$ and $f(\partial B_2) = \partial B_2'$. Thus $f(w) \in \overline{B_1'}$ and $f(u) \in \partial B_2'$. This ensures that

$$d_{B_4'}(f(w)) \geq 3r' \quad \text{and} \quad d_{B_4'}(f(u)) = 2r'.$$

Now, using Lemma 2.2(1), we obtain

$$k_{B_4'}(f(w), f(u)) \geq \left|\log\frac{d_{B_4'}(f(w))}{d_{B_4'}(f(u))}\right| \geq \log\frac{3}{2}.$$

As f is fully φ-semisolid, one finds that $f|_{B_4}\colon B_4 \to B_4'$ is φ-semisolid. This, together with (3.3), implies that

$$\log\frac{3}{2} \leq k_{B_4'}(f(w), f(u)) \leq \varphi(k_{B_4}(w,u)) < \log\frac{3}{2},$$

which is absurd. ∎

Next, we shall prove that the relative distance between any two points of $\overline{B_1}$ with respect to ∂B_4 is bounded above.

LEMMA 3.4. *Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric. Then, there is a constant C_1 depending only on M, C, and φ, such that for all $z_1, z_2 \in \overline{B_1}$,*

$$\frac{|z_1-z_2|}{\min\{d_{B_4}(z_1), d_{B_4}(z_2)\}} \leq C_1.$$

Proof. Fix $z_1, z_2 \in \overline{B_1}$. Without loss of generality, we may assume that $d_{B_4}(z_1) \leq d_{B_4}(z_2)$. By Lemma 3.1, we know that $f(z_1), f(z_2) \in \overline{B_1'}$, which implies

$$\frac{|f(z_1)-f(z_2)|}{d_{G'}(f(z_1))} \leq \frac{2r'}{3r'} = \frac{2}{3}.$$

By Lemma 2.2(2), we have

$$k_{G'}(f(z_1), f(z_2)) \le \frac{1}{1-\frac{2}{3}} \frac{|f(z_1)-f(z_2)|}{d_{G'}(f(z_1))} \le 2. \tag{3.5}$$

Since $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric, it follows from Lemma 2.2(1) and (3.5) that

$$\frac{|z_1-z_2|}{d_G(z_1)} \le e^{k_G(z_1,z_2)} - 1 \le e^{Mk_{G'}(f(z_1),f(z_2))+C} - 1 \le e^{2M+C} - 1. \tag{3.6}$$

Suppose first that $\partial B_4 \cap B\big(z_1, \frac{3}{4}d_G(z_1)\big) = \emptyset$. Then $B\big(z_1, \frac{3}{4}d_G(z_1)\big) \subset B_4$, which leads to

$$d_{B_4}(z_1) \ge \tfrac{3}{4}d_G(z_1).$$

Furthermore, one finds from (3.6) that

$$\frac{|z_1-z_2|}{\min\{d_{B_4}(z_1), d_{B_4}(z_2)\}} = \frac{|z_1-z_2|}{d_{B_4}(z_1)} \le \frac{4}{3}\frac{|z_1-z_2|}{d_G(z_1)} \le \frac{4}{3}(e^{2M+C}-1), \tag{3.7}$$

as desired.

It remains to consider the situation where $\partial B_4 \cap B\big(z_1, \frac{3}{4}d_G(z_1)\big) \ne \emptyset$. Obviously, there is a point $u \in \partial B_4 \cap B\big(z_1, \frac{3}{4}d_G(z_1)\big)$ such that

$$|z_1-u| \le 2d_{B_4}(z_1). \tag{3.8}$$

We now consider two cases according to the distance between z_1 and z_2.

Case 1: $z_2 \in B\big(z_1, \frac{3}{4}d_G(z_1)\big)$. Since f is fully φ-semisolid, Lemma 2.5 shows that $f|_{B(z_1,\frac{3}{4}d_G(z_1))}$ is η_1-quasisymmetric with η_1 depending only on φ. It follows from (3.8) that

$$\frac{|f(z_1)-f(u)|}{|f(z_1)-f(z_2)|} \le \eta_1\left(\frac{|z_1-u|}{|z_1-z_2|}\right) \le \eta_1\left(\frac{2d_{B_4}(z_1)}{|z_1-z_2|}\right)$$

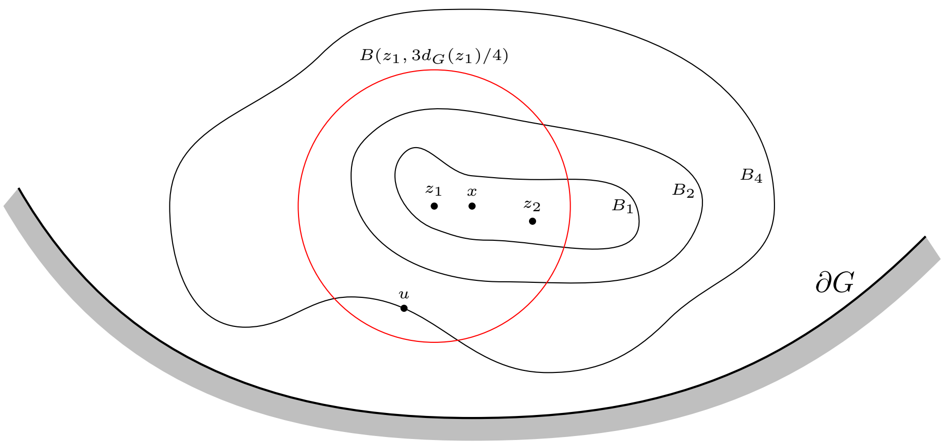

Fig. 1

and

$$\frac{|f(z_1)-f(u)|}{|f(z_1)-f(z_2)|} \geq \frac{3r'}{2r'} = \frac{3}{2}.$$

The above inequalities lead to

$$\frac{|z_1-z_2|}{\min\{d_{B_4}(z_1), d_{B_4}(z_2)\}} = \frac{|z_1-z_2|}{d_{B_4}(z_1)} \leq \frac{2}{\eta_1^{-1}(3/2)}. \tag{3.9}$$

Case 2: $z_2 \notin B\big(z_1, \frac{3}{4}d_G(z_1)\big)$. In this case, there exists a point $w \in S\big(z_1, \frac{1}{2}d_G(z_1)\big) \cap f^{-1}([f(z_1), f(z_2)])$. Since $f|_{B(z_1, \frac{3}{4}d_G(z_1))}$ is η_1-quasisymmetric, we have

$$\frac{|f(z_1)-f(u)|}{|f(z_1)-f(w)|} \leq \eta_1\left(\frac{|z_1-u|}{|z_1-w|}\right) \leq \eta_1\left(\frac{4d_{B_4}(z_1)}{d_G(z_1)}\right)$$

and

$$\frac{|f(z_1)-f(u)|}{|f(z_1)-f(w)|} \geq \frac{3r'}{2r'} = \frac{3}{2}.$$

Thus

$$\frac{d_{B_4}(z_1)}{d_G(z_1)} \geq \frac{1}{4}\eta_1^{-1}\left(\frac{3}{2}\right).$$

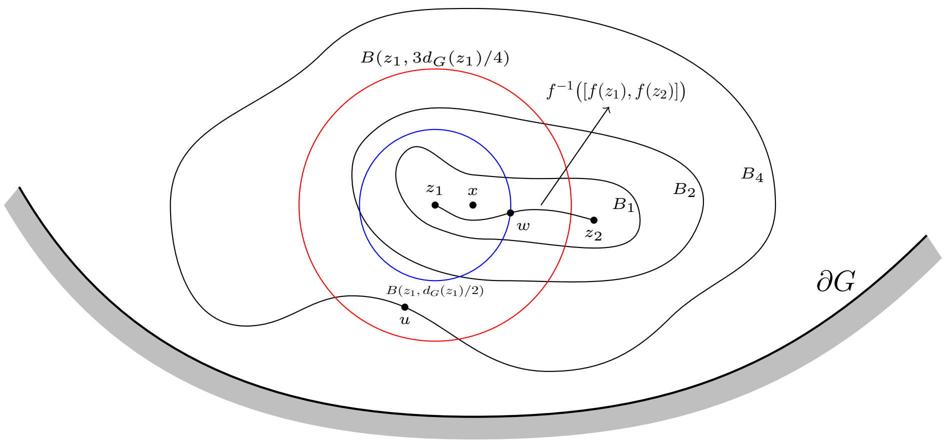

Fig. 2

By (3.6), it follows that

$$\frac{|z_1-z_2|}{\min\{d_{B_4}(z_1), d_{B_4}(z_2)\}} = \frac{|z_1-z_2|}{d_{B_4}(z_1)} \leq \frac{4}{\eta_1^{-1}(3/2)}\,\frac{|z_1-z_2|}{d_G(z_1)} \leq \frac{4(e^{2M+C}-1)}{\eta_1^{-1}(3/2)}. \tag{3.10}$$

By letting

$$C_1 = \max\left\{\frac{4}{3}, \frac{4}{\eta_1^{-1}(3/2)}\right\}(e^{2M+C}-1),$$

one deduces from (3.7), (3.9), and (3.10) that Lemma 3.4 is true. ∎

Lemma 3.11. *Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric. For all $z_1, z_2 \in \overline{B_1}$ and for every $w \in \partial B_2$, we have*

$$\frac{|z_1 - z_2|}{|z_1 - w|} \le \frac{2C_1}{t_1} =: C_2$$

where $0 < t_1 < 1$ and $C_1 \ge 1$ are the constants of Lemmas 3.2 and 3.4.

Proof. Fix $z_1, z_2 \in \overline{B_1}$ and $w \in \partial B_2$. We consider two cases. If $d_{B_4}(z_1) \le 2d_{B_4}(w)$, then one observes from Lemmas 3.2 and 3.4 that

$$\frac{|z_1 - z_2|}{|z_1 - w|} \le \frac{C_1}{t_1} \cdot \frac{d_{B_4}(z_1)}{\min\{d_{B_4}(z_1), d_{B_4}(w)\}} \le \frac{2C_1}{t_1}. \tag{3.12}$$

If $d_{B_4}(z_1) > 2d_{B_4}(w)$, then

$$|z_1 - w| \ge d_{B_4}(z_1) - d_{B_4}(w) > \frac{1}{2} d_{B_4}(z_1).$$

The above inequality, together with Lemma 3.4, guarantees that

$$\frac{|z_1 - z_2|}{|z_1 - w|} < 2\frac{|z_1 - z_2|}{d_{B_4}(z_1)} \le 2C_1. \tag{3.13}$$

From (3.12) and (3.13), Lemma 3.11 follows. ∎

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that $f\colon G \to G'$ is a fully φ-semisolid homeomorphism and $f^{-1}\colon G' \to G$ is C-coarsely M-Lipschitz in the quasihyperbolic metric.

To show that f is freely quasiconformal, it suffices to verify that $f^{-1}: G' \to G$ is also fully semisolid. By the equivalence of conditions (1) and (5) in Lemma 2.5, we only need to show that $f^{-1}\colon G' \to G$ has the $(M_1, 2, 4)$-ring property for some $M_1 > 0$. That is, we shall demonstrate that there is a positive constant M_1 depending only on M, C, and φ, such that

$$\operatorname{diam}(f^{-1}(\overline{B}(x', r'))) \le M_1 \operatorname{dist}(f^{-1}(\overline{B}(x', r')), \partial f^{-1}(B(x', 2r'))) \tag{3.14}$$

whenever $x' \in G'$ and $B(x', 4r') \subset G'$.

Towards this end, we let $x' \in G'$ be a given point and $r' > 0$ be such that $B(x', 4r') \subset G'$. We set $B'_i = B(x', ir')$ and $B_i = f^{-1}(B(x', ir'))$ for all $i \in \{1, 2, 4\}$. For any $z_1 \in \overline{B_1}$, there is a point $z_2 \in \overline{B_1}$ such that

$$\operatorname{diam}(\overline{B_1}) \le 3|z_1 - z_2|. \tag{3.15}$$

It follows from Lemma 3.11 that

$$\frac{|z_1 - z_2|}{|z_1 - w|} \le C_2 \tag{3.16}$$

for all $w \in \partial B_2$. Therefore, we see from (3.15) and (3.16) that

$$\operatorname{diam}(\overline{B_1}) \le 3C_2|z_1 - w|$$

for any $z_1 \in \overline{B_1}$ and $w \in \partial B_2$. This ensures that

$$\operatorname{diam}(\overline{B_1}) \leq 3C_2 \operatorname{dist}(\overline{B_1}, \partial B_2).$$

Finally, it follows from Lemma 3.1 that (3.14) holds with $M_1 = 3C_2$. ∎

Proof of Corollary 1.4. The implication (4)⇒(2) is trivial. For (2)⇒(1), we use [22, Lemma 2.3 and Theorem 3.4(2)]. The implication (1)⇒(4) follows from Theorem 1.3. Applying [22, Theorem 7.6], we obtain (2)⇔(3). ∎

Proof of Corollary 1.5. Use [22, Theorem 7.9] and Corollary 1.4. ∎

Proof of Corollary 1.9. Use [22, Theorem 7.9] and Theorem 1.3. ∎

Proof of Theorem 1.6. For (1), we see from Theorem 1.3 that f is freely φ_1-quasiconformal with φ_1 depending only on φ, M, and C. Using [22, Theorem 11.12], we deduce that $f\colon G \to G'$ is θ-quasimöbius with θ depending only on c, φ, M, and C.

Next, we verify (2). For the sufficiency, if $f(\infty) = \infty$, then f is θ-quasisymmetric by the first assertion and [22, (6.19)]. If G and G' are bounded domains, then the assumption ensures that there is a $\tau \geq 1$ and a point $z_0 \in G$ such that $\operatorname{diam}(G) \leq \tau d_G(z_0)$ and $\operatorname{diam}(G') \leq \tau d_{G'}(f(z_0))$. Therefore, it follows from [22, Theorem 6.33] that f is η-quasisymmetric with η depending only on c, φ, M, C, and τ.

For the necessity of (2), we assume that f is η-quasisymmetric. It follows from [22, Theorem 6.12] that f has a continuous extension onto $\overline{G}$, denoted by $f\colon \overline{G} \to \overline{G'}$, which is also η-quasisymmetric. If G is an unbounded domain, then [22, Theorem 6.11] asserts that f maps bounded sets onto bounded sets, and therefore $f(\infty) = \infty$.

It remains to consider the situation where G is a bounded domain. Choose a point $z_0 \in G$ with $d_G(z_0) \geq \frac{1}{2} \sup\{d_G(z) | \, z \in G\}$. Since G is a c-uniform domain, any two points z_1 and z_2 in G can be connected by a c-uniform arc σ. Pick a point $u \in \sigma$ such that $\ell(\sigma[z_1, u]) = \ell(\sigma[z_2, u])$. The uniformity of σ leads to

$$d_G(z_0) \geq \frac{1}{2} d_G(u) \geq \frac{1}{4c}\ell(\sigma) \geq \frac{1}{4c}|z_1 - z_2|,$$

which implies $\operatorname{diam}(G) \leq 4c\, d_G(z_0)$.

Next, we let $v_0 \in \partial G$ be a point such that $|f(v_0) - f(z_0)| \leq 2d_{G'}(f(z_0))$. Note that $\operatorname{diam}(G) \leq 4c\, d_G(z_0) \leq 4c|v_0 - z_0|$. As $f\colon \overline{G} \to \overline{G'}$ is η-quasisymmetric, one finds from [18, Theorem 3.5] that

$$|f(v_0) - f(z_0)| \geq \frac{\operatorname{diam}(G')}{2\eta(\operatorname{diam}(G)/|v_0 - z_0|)} \geq \frac{\operatorname{diam}(G')}{2\eta(4c)}.$$

This implies that

$$\operatorname{diam}(G') \leq 4\eta(4c)\, d_{G'}(f(x_0)),$$

finishing the proof. ∎

3.2. Inverse problem for local quasisymmetry. The purpose of this subsection is to prove Theorem 1.10. We begin with the following notation.

DEFINITION 3.17. Let $G \subset E$ be a domain. A half-open arc γ in G is called an *endcut* of G if $\overline{\gamma}$ is a closed arc with one endpoint in ∂G.

The concept of endcuts was introduced by Väisälä [21] who showed that each interior point in the domain can be connected to the boundary by a quasihyperbolic quasigeodesic endcut. This is easy whenever $\dim(E)$ is finite. In infinite-dimensional Banach spaces, Väisälä reached the goal by using a sequence of line segments [21, Theorem 3.10]. Recently, this concept has served as an important tool [24] in the study of an open problem posed by Väisälä regarding local properties of quasihyperbolic mappings.

By making use of endcuts, we next establish a technical lemma which is the key point in the proof of Theorem 1.10.

LEMMA 3.18. *Suppose that $f\colon G \to G'$ is a 1-locally η-quasisymmetric homeomorphism. Then there is a positive number H depending only on η such that for all $z \in G$,*

$$d_{G'}(f(z)) \le H d_{f(B(z,d_G(z)))}(f(z)). \tag{3.19}$$

Proof. Firstly, from the definition of local quasisymmetric mappings, the restriction $f|_{B(z,d_G(z))}$ of f to each Whitney ball $B(z, d_G(z))$ is η-quasisymmetric. Actually, it follows from [22, Theorem 6.12] that $f|_{B(z,d_G(z))}$ has a continuous extension onto $\overline{B}(z, d_G(z))$ for every $z \in G$, denoted by $f|_{\overline{B}(z,d_G(z))}$, which is also η-quasisymmetric.

Let $\varepsilon \in (0, 1/4)$ be small enough so that

$$\eta(1)\eta(\varepsilon) \le 1/2.$$

Set $\varepsilon_i = 2^{-i}\varepsilon$ for all positive integers i.

We recall the construction of endcuts given in [21, proof of Theorem 3.10]. Fix a point $z \in G$. Set $z_0 = z$. Choose a point $u_1 \in \partial G$ such that

$$|u_1 - z_0| < (1 + \varepsilon_1)d_G(z_0).$$

Let z_1 be the unique point in $[z_0, u_1] \cap S(z_0, d_G(z_0))$. We define inductively a sequence $\{z_n\}$ of points in $\overline{G}$ as follows. Assume that $z_0, \dots, z_i$ have been chosen. If $z_i \in \partial G$, then the process stops. Otherwise, if $z_i \in G$, then we pick a point $u_{i+1} \in \partial G$ with

$$|u_{i+1} - z_i| < (1 + \varepsilon_{i+1})d_G(z_i),$$

and let z_{i+1} be the unique point in $[z_i, u_{i+1}] \cap S(z_i, d_G(z_i))$. Therefore, the union γ of the segments $[z_i, z_{i+1})$ is an endcut emanating from z to a point $v \in \partial G$ such that the limit of z_n in the norm is v. Moreover, $d_G(z_{i+1}) < \varepsilon_{i+1} d_G(z_i)$.

If $z_1 \in \partial G$, then $d_{G'}(f(z)) \leq |f(z) - f(z_1)|$. Because $z_1 \in S(z, d_G(z))$ and f is η-quasisymmetric on $\overline{B}(z, d_G(z))$, we have

$$|f(z) - f(z_1)| \leq \eta(1)|f(z) - f(w)| \tag{3.20}$$

for every $w \in S(z, d_G(z))$. By the arbitrariness of $w \in S(z, d_G(z))$,

$$d_{G'}(f(z)) \leq \eta(1) d_{f(B(z,d_G(z)))}(f(z)).$$

Hence (3.19) is valid with $H = \eta(1)$ in this case.

Thus, we may assume without loss of generality that $z_{i+1} \in G$ for $i \geq 0$. Next, we prove that for all $i \in \mathbb{N}$,

$$|f(z_{i+1}) - f(z_{i+2})| \leq \tfrac{1}{2}|f(z_i) - f(z_{i+1})|. \tag{3.21}$$

Fix $i \in \mathbb{N}$. Since $\dim(E) \geq 2$, one observes that the intersection $S(z_{i+1}, d_G(z_{i+1})) \cap S(z_i, d_G(z_i))$ contains at least two points. Choose a point $w_{i+1} \neq z_{i+2}$ in this intersection. As f is η-quasisymmetric on $\overline{B}(z_{i+1}, d_G(z_{i+1}))$, we have

$$\frac{|f(z_{i+2}) - f(z_{i+1})|}{|f(w_{i+1}) - f(z_{i+1})|} \leq \eta\left(\frac{|z_{i+2} - z_{i+1}|}{|w_{i+1} - z_{i+1}|}\right) = \eta(1).$$

Similarly, because f is η-quasisymmetric on $\overline{B}(z_i, d_G(z_i))$ and $d_G(z_{i+1}) < \varepsilon_{i+1} d_G(z_i)$, we see that

$$\frac{|f(z_{i+1}) - f(w_{i+1})|}{|f(z_{i+1}) - f(z_i)|} \leq \eta\left(\frac{|z_{i+1} - w_{i+1}|}{|z_{i+1} - z_i|}\right) \leq \eta(\varepsilon_{i+1}).$$

One observes from these two inequalities that

$$\begin{aligned} |f(z_{i+1}) - f(z_{i+2})| &\leq \eta(1)|f(z_{i+1}) - f(w_{i+1})| \\ &\leq \eta(1)\eta(\varepsilon)|f(z_i) - f(z_{i+1})| \\ &\leq \tfrac{1}{2}|f(z_i) - f(z_{i+1})|, \end{aligned}$$

where the last inequality follows from the choice of ε. This gives (3.21).

Finally, using (3.21), we obtain

$$|f(z) - f(z_{n+1})| \leq \sum_{i=0}^{n} |f(z_i) - f(z_{i+1})| \leq 2|f(z) - f(z_1)|.$$

Since $\lim_{n\to\infty} z_n = v \in \partial G$, there is $v' \in \partial G'$ such that $\lim_{n\to\infty} f(z_n) = v'$. Therefore,

$$d_{G'}(f(z)) \leq |f(z) - v'| \leq 2|f(z) - f(z_1)|. \tag{3.22}$$

This inequality, together with (3.20), shows that (3.19) is true with $H = 2\eta(1)$. ∎

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. Because the "in particular" assertion follows from the first assertion and the equivalence of conditions (1) and (3) in Lemma 2.5, we only need to show the first assertion.

By the equivalence of conditions (2) and (3) in Lemma 2.5, it suffices to verify that f^{-1} is fully C-coarsely M-Lipschitz in the quasihyperbolic metric, that is, there are constants $M \geq 1$ and $C \geq 0$ depending only on η such that for any subdomain $G'_0 \subset G'$ and for all $x', y' \in G'_0$,

$$k_{G_0}(x,y) \leq M k_{G'_0}(x',y') + C, \tag{3.23}$$

where $f(x) = x'$, $f(y) = y'$ and $G_0 = f^{-1}(G'_0)$.

Since $f\colon G \to G'$ is 1-locally η-quasisymmetric, $f|_{G_0}\colon G_0 \to G'_0$ is also 1-locally η-quasisymmetric. It follows from Lemma 3.18 that there is a constant $H \geq 1$ depending only on η such that

$$d_{G'_0}(x') \leq H d_{f(B(x,d_{G_0}(x)))}(x') \tag{3.24}$$

for all $x \in G_0$. Furthermore, as $f|_{B(x,d_{G_0}(x))}$ is η-quasisymmetric, we know from [22, Theorem 6.3] that $(f|_{B(x,d_{G_0}(x))})^{-1}$ is η_0-quasisymmetric with η_0 depending only on η. This fact, together with Lemma 2.5, guarantees that there are constants $M_2 \geq 1$ and $C_2 \geq 0$ depending only on η such that

$$k_{B(x,d_{G_0}(x))}(x,y) \leq M_2 k_{f(B(x,d_{G_0}(x)))}(x',y') + C_2 \tag{3.25}$$

for all $x \in G_0$ and $y \in B(x, d_{G_0}(x))$.

On the other hand, $(G'_0, k_{G'_0})$ is a length metric space and thus is c-quasiconvex for all $c > 1$. In view of [22, Lemma 2.3], it is suffices to show that there is a constant $H_1 \geq 0$ depending only on η such that for all $x', y' \in G'_0$ with $k_{G'_0}(x', y') \leq 1/(6H)$,

$$k_{G_0}(x,y) \leq H_1. \tag{3.26}$$

Fix $x', y' \in G'_0$ with $k_{G'_0}(x',y') \leq 1/(6H)$. By Lemma 2.1 and (3.24), we obtain

$$|x' - y'| \leq \frac{1}{3H} d_{G'_0}(x') \leq \frac{1}{3} d_{f(B(x,d_{G_0}(x)))}(x').$$

This ensures that $y' \in f(B(x, d_{G_0}(x)))$ and $y \in B(x, d_{G_0}(x))$. Again by Lemma 2.1,

$$k_{f(B(x,d_{G_0}(x)))}(x',y') \leq 1.$$

This inequality, together with (3.25), shows that

$$\begin{aligned} k_{G_0}(x,y) &\leq k_{B(x,d_{G_0}(x))}(x,y) \\ &\leq M_2 k_{f(B(x,d_{G_0}(x)))}(x',y') + C_2 \leq M_2 + C_2 =: H_1. \end{aligned}$$

Hence (3.26) is valid, completing the proof. ∎

Proof of Corollary 1.11. Use Theorems 1.6 and 1.10. ∎

Acknowledgements. Yuehui He was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515111136). Qingshan Zhou was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515012441).

References

[1] A. Björn, J. Björn, and N. Shanmugalingam, *Bounded geometry and p-harmonic functions under uniformization and hyperbolization*, J. Geom. Anal. 270 (2021), 5259–5308.

[2] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Astérisque 270 (2001), viii+99 pp.

[3] T. Cheng, P. Jiang, and S. Yang, *Quasisymmetry and solidity of quasiconformal maps in metric spaces*, Studia Math. 272 (2023), 199–219.

[4] J. Ferrand, G. J. Martin, and M. K. Vuorinen, *Lipschitz conditions in conformally invariant metrics*, J. Anal. Math. 56 (1991), 187–210.

[5] F. W. Gehring, *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I Math. 336 (1963), no. 11, 21 pp.

[6] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasi-hyperbolic metric*, J. Anal. Math. 36 (1979), 50–74.

[7] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Anal. Math. 30 (1976), 172–199.

[8] T. Guan, B. Jiao, S. Ponnusamy, and Y. He, *Quasisymmetry of freely quasiconformal mappings in Banach spaces*, J. Math. Anal. Appl. 526 (2023), art. 127335, 8 pp.

[9] P. Hariri, R. Klén, and M. Vuorinen, *Conformally Invariant Metrics and Quasiconformal Mappings*, Springer Monogr. Math., Springer, Cham, 2020.

[10] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), 1–61.

[11] X. Huang and J. Liu, *Quasihyperbolic metric and quasisymmetric mappings in metric spaces*, Trans. Amer. Math. Soc. 367 (2015), 6225–6246.

[12] X. Huang, H. Liu, and J. Liu, *Local properties of quasihyperbolic mappings in metric spaces*, Ann. Acad. Sci. Fenn. Math. 41 (2016), 23–40.

[13] M. Huang, A. Rasila, X. Wang, and Q. Zhou, *Semisolidity and locally weak quasisymmetry of homeomorphisms in metric spaces*, Studia Math. 242 (2018), 267–301.

[14] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978), 383–401.

[15] A. Rasila and J. Talponen, *On quasihyperbolic geodesics in Banach spaces*, Ann. Acad. Sci. Fenn. Math. 39 (2014), 163–173.

[16] E. Soultanis and M. Williams, *Distortion of quasiconformal maps in terms of the quasihyperbolic metric*, J. Math. Anal. Appl. 402 (2013), 626–634.

[17] T. Sugawa, M. Vuorinen, and T. Zhang, *Conformally invariant complete metrics*, Math. Proc. Cambridge Philos. Soc. 174 (2023), 273–300.

[18] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 97–114.

[19] P. Tukia and J. Väisälä, *Lipschitz and quasiconformal approximation and extension*, Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981), 303–342.

[20] J. Väisälä, *Free quasiconformality in Banach spaces. I*, Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990), 355–379.

[21] J. Väisälä, *Free quasiconformality in Banach spaces. II*, Ann. Acad. Sci. Fenn. Ser. A I Math. 16 (1991), 255–310.

[22] J. Väisälä, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces*, in: Quasiconformal Geometry and Dynamics (Lublin, 1996), Banach Center Publ. 48, Inst. Math., Polish Acad. Sci., 1999, 55–118.

[23] X. Zhang, *Isometries for the modulus metric in higher dimensions are conformal mappings*, Sci. China Math. 64 (2021), 1951–1958.

[24] Q. Zhou, *Quasihyperbolic mappings in Banach spaces*, Ann. Fenn. Math. 46 (2021), 335–344.

[25] Q. Zhou, L. Li, S. Ponnusamy, and Y. He, *Relatively quasimöbius mappings in Banach spaces*, Proc. Amer. Math. Soc. 151 (2023), 4781–4792.

[26] Q. Zhou, Y. Li, and A. Rasila, *A note on Väisälä's problem concerning free quasiconformal mappings*, Monatsh. Math. 196 (2021), 607–616.

[27] Q. Zhou and S. Ponnusamy, *Gromov hyperbolicity in the free quasiworld. I*, Studia Math. 268 (2023), 23–49.

[28] Q. Zhou and A. Rasila, *Quasimöbius invariance of uniform domains*, Studia Math. 261 (2021), 1–24.

[29] Q. Zhou and A. Rasila, *Teichmüller's problem for Gromov hyperbolic domains*, Israel J. Math. 252 (2022), 399–427.

[30] Q. Zhou and A. Rasila, *Geometric characterizations of Gromov hyperbolic Hölder domains*, Forum Math. 34 (2022), 1621–1640.

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