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Norm attaining vectors and Hilbert points

by

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Abstract. Let H be a Hilbert space that can be embedded as a dense subspace of a Banach space X such that the norm of the embedding is 1. We consider the following statements for a nonzero vector φ in H:

(A) $\|\varphi\|_X = \|\varphi\|_H$.
(H) $\|\varphi + f\|_X \geq \|\varphi\|_X$ for every f in H such that $\langle f, \varphi \rangle = 0$.

We use duality arguments to establish that (A)⇒(H), before turning our attention to the special case when the Hilbert space in question is the Hardy space $H^2(\mathbb{T}^d)$ and the Banach space is either the Hardy space $H^1(\mathbb{T}^d)$ or the weak product space $H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d)$. If $d = 1$, then the two Banach spaces are equal and it is known that (H)⇒(A). If $d \geq 2$, then the Banach spaces do not coincide and a case study of the polynomials $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$ for $\alpha \geq 0$ illustrates that the statements (A) and (H) for these two Banach spaces describe four distinct sets of functions.

1. Introduction. The purpose of this paper is to introduce and study an abstract framework, containing as special cases the recently investigated concepts of minimal norm Hankel operators [3] and Hilbert points [5, 4], in addition to inner functions in Hardy spaces on polydiscs [10]. Our starting point is as follows.

DEFINITION. An *admissible pair* (H, X) is a Hilbert space H that can be embedded as a dense subspace of a Banach space X such that the norm of the embedding is 1. A nonzero vector φ in H is called *norm attaining* in X if $\|\varphi\|_X = \|\varphi\|_H$.

Suppose that (H, X) is an admissible pair and let X^* denote the dual space of X. Since H is a subspace of X and $\|f\|_X \leq \|f\|_H$ holds for every f in H, it is plain that every Ψ in X^* defines a bounded linear functional on H and $\|\Psi\|_{H^*} \leq \|\Psi\|_{X^*}$. It follows from the Riesz representation theorem

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that there is ψ in H such that

$$\Psi(f) = \langle f, \psi \rangle$$

for every f in H. This embeds X^* as a subspace of H and we say that a vector ψ in H is in X^* when ψ belongs to this subspace.

THEOREM 1. *Let (H, X) be an admissible pair and let φ be a nonzero vector in H. The following are equivalent:*

(a) *φ is norm attaining in X.*
(b) *φ is in X^* and $\|\varphi\|_{X^*} = \|\varphi\|_H$.*

The conditions of Theorem 1 capture two (equivalent) ways in which the Hilbert space properties of the vector in question are preserved under the embedding in X.

DEFINITION. Let (H, X) be an admissible pair. A nonzero vector φ in H is called a *Hilbert point* in X if

$$\|\varphi + f\|_X \geq \|\varphi\|_X$$

whenever f is in H and $\langle f, \varphi \rangle = 0$.

The reasoning behind the name is that if f and φ are in H and $\langle f, \varphi \rangle = 0$, then

$$\|\varphi + f\|_H = \sqrt{\|\varphi\|_H^2 + \|f\|_H^2} \geq \|\varphi\|_H,$$

by orthogonality. This definition attempts to capture that the geometry of X is locally like the geometry of H near the point φ.

THEOREM 2. *Let (H, X) be an admissible pair and let φ be a nonzero vector in H. The following are equivalent:*

(c) *φ is a Hilbert point in X.*
(d) *φ is in X^* and $\|\varphi\|_X \|\varphi\|_{X^*} = \|\varphi\|_H^2$.*

Since $\|\psi\|_H^2 \leq \|\psi\|_X \|\psi\|_{X^*}$ plainly holds for every ψ in X^*, the condition in Theorem 2(d) reformulates the geometric property of a Hilbert point as a statement about a general estimate that is attained. As a consequence, we have the following.

COROLLARY 3. *Let (H, X) be an admissible pair. If a nonzero vector φ in H is norm attaining in X, then φ is a Hilbert point in X.*

The proofs of Theorems 1 and 2 are fairly direct consequences of the Hahn–Banach theorem and the Hilbert space structure of H.

We are particularly interested in two classes of admissible pairs. To set the stage for the first class, let $\mathbb{T}$ denote the unit circle in the complex plane. The d-fold cartesian product $\mathbb{T}^d = \mathbb{T} \times \cdots \times \mathbb{T}$ becomes a compact abelian group under coordinatewise multiplication and its Haar measure coincides with the product measure generated by the normalized Lebesgue arc length measure on $\mathbb{T}$. For $1 \leq p < \infty$, we define the Hardy space $H^p(\mathbb{T}^d)$ as the closure in $L^p(\mathbb{T}^d)$ of the set of polynomials in d complex variables.

The first admissible pair of interest is (H, X) with $H = H^2(\mathbb{T}^d)$ and $X = H^1(\mathbb{T}^d)$. Since a nontrivial function in $H^1(\mathbb{T}^d)$ can only vanish on a set of measure 0 on $\mathbb{T}^d$ (see e.g. [10, Theorem 3.3.5]), it follows from the Cauchy–Schwarz inequality that φ is norm attaining in $H^1(\mathbb{T}^d)$ if and only if $|\varphi|$ is constant and nonzero almost everywhere on $\mathbb{T}^d$. This is equivalent to the assertion that $\varphi = CI$ for a constant $C \neq 0$ and an inner function I.

For this admissible pair, our definition of a Hilbert point is in agreement with the definition of Hilbert points in Hardy spaces from [5]. Hence Corollary 3 above supplies a simpler proof of the case $p = 1$ of [5, Corollary 2.5], which asserts that constant multiples of inner functions are Hilbert points in $H^1(\mathbb{T}^d)$. The results in [5] also demonstrate that the converse statement, that all Hilbert points in $H^1(\mathbb{T}^d)$ are constant multiples of inner functions, is true if and only if $d = 1$.

In our second admissible pair of interest, H is a functional Hilbert space on a nonempty set Ω [7, §36]. We will additionally assume that the constant functions (on Ω) are elements of H and that the multiplier algebra $M(H)$ is dense in H. Moreover, we will normalize the norm of H so that $\|1\|_H = 1$.

The Banach space X in this admissible pair will be the *weak product space* $H \odot H$, which is the collection of all functions f on Ω that admit a *weak factorization*

$$f = \sum_{j=1}^{\infty} g_j h_j, \tag{1.1}$$

for sequences $(g_j)_{j\geq 1}$ and $(h_j)_{j\geq 1}$ in H such that

$$\sum_{j=1}^{\infty} \|g_j\|_H \|h_j\|_H < \infty. \tag{1.2}$$

The norm of $H \odot H$ is the infimum of (1.2) over all possible weak factorizations (1.1). We refer to [1, Theorem 2.1] for a proof that $H \odot H$ is a Banach space.

We will say that a given weak factorization (1.1) is *optimal* should it attain this infimum. The additional assumptions on H ensure that

$$\|f\|_{H\odot H} \leq \|f\|_H$$

for every f in H and that $M(H)$ (and hence H) is dense in $H \odot H$, so $(H, H \odot H)$ is an admissible pair. It is plain that a function φ is norm attaining in $H \odot H$ if and only if $\varphi = \varphi \cdot 1$ is an optimal weak factorization of φ.

The assumptions on H also allow us to invoke [1, Theorem 2.5], which asserts that there is an antilinear isometric isomorphism from the dual space of $H \odot H$ to the space of all bounded Hankel operators on H. It follows from this and Theorem 1 that if $H = H^2(\mathbb{T}^d)$, then the requirement that $\varphi = \varphi \cdot 1$ is an optimal weak factorization of φ coincides with the definition of minimal norm Hankel operators from [3].

This point of view was utilized by Ortega-Cerdà and Seip in their counterexample to an infinite-dimensional analogue of Nehari's theorem [9]. Their work implies, and is qualitatively equivalent to, the fact that an optimal weak factorization of $\varphi(z) = z_1 + z_2$ in the weak product space $H^2(\mathbb{T}^2) \odot H^2(\mathbb{T}^2)$ is $\varphi = \varphi \cdot 1$.

It is a direct consequence of the well-known inner-outer factorization that $H^1(\mathbb{T}) = H^2(\mathbb{T}) \odot H^2(\mathbb{T})$ as sets and with equality of norms. The inner-outer factorization is also the key ingredient in the proof of [3, Theorem 1], which asserts that $\|\varphi\|_{(H^2(\mathbb{T})\odot H^2(\mathbb{T}))^*} = \|\varphi\|_{H^2(\mathbb{T})}$ if and only if φ is a constant multiple of an inner function. In the present context, this can be more easily deduced from Theorem 1.

For $d \geq 2$, it is an important open problem in harmonic analysis (see [8]) whether there is an absolute constant $C_d > 0$ such that

$$\|f\|_{H^1(\mathbb{T}^d)} \geq C_d \|f\|_{W(\mathbb{T}^d)}$$

for every f in $W(\mathbb{T}^d) = H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d)$.

The work of Ortega-Cerdà and Seip discussed above shows that $C_2 \leq 2\sqrt{2}/\pi < 1$. A minor improvement can be found in [3, Theorem 5]. Since plainly

$$\|f\|_{H^1(\mathbb{T}^d)} \leq \|f\|_{W(\mathbb{T}^d)} \leq \|f\|_{H^2(\mathbb{T}^d)}, \tag{1.3}$$

the open problem is to ascertain whether $H^1(\mathbb{T}^d)$ and $W(\mathbb{T}^d)$ are equal as sets. Note that (1.3) also shows that if φ is norm attaining in $H^1(\mathbb{T}^d)$, then φ is norm attaining in $W(\mathbb{T}^d)$. This inspires us to compare the admissible pairs $(H^2(\mathbb{T}^2), H^1(\mathbb{T}^2))$ and $(H^2(\mathbb{T}^2), W(\mathbb{T}^2))$ in detail. Our case study is concerned with the polynomials

$$\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$$

for $\alpha \geq 0$. In order to state our result, we let $\alpha_0 = 1.62420\ldots$ denote the unique (see Lemma 6) solution of the equation

$$\sqrt{4-\alpha^2} = \frac{2}{\alpha} \arcsin \frac{\alpha}{2}$$

on the interval $(0, 2)$.

THEOREM 4. *Suppose that $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$ for $\alpha \geq 0$. Then*

(i) *φ_α is never norm attaining in $H^1(\mathbb{T}^2)$;*
(ii) *φ_α is a Hilbert point in $H^1(\mathbb{T}^2)$ if and only if $\alpha = 0$ or if $\alpha = \alpha_0$;*
(iii) *φ_α is norm attaining in $W(\mathbb{T}^2)$ if and only if $0 \leq \alpha \leq 1/2$;*
(iv) *φ_α is a Hilbert point in $W(\mathbb{T}^2)$ if and only if $0 \leq \alpha \leq 1/2$ or if $\alpha = 2$.*

The main novelty of Theorem 4 is in assertions (ii) and (iv). Assertion (i) is trivial, since φ_α does not have constant modulus on $\mathbb{T}^2$. Taking into account Theorem 1, we note that Theorem 4(iii) is equivalent to [3, Theorem 10(a)].

As in the proof of Theorems 1 and 2, the main idea in our approach to Theorem 4 is duality. In the case of $X = H^1(\mathbb{T}^2)$ we will rely on the Riesz representation theorem for $L^1(\mathbb{T}^2)$, and in the case of $X = W(\mathbb{T}^2)$ our arguments will involve Hankel operators on $H^2(\mathbb{T}^2)$.

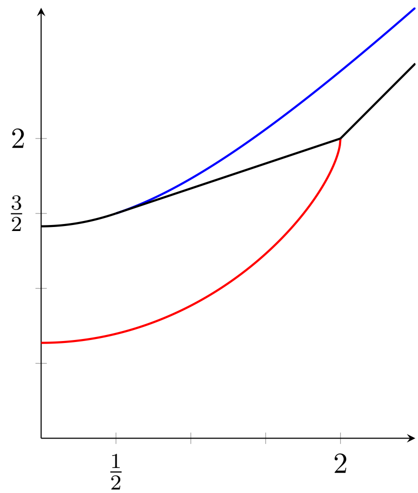

Fig. 1. Norms of $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$ for $0 \leq \alpha \leq 2.5$. From top to bottom: $H^2(\mathbb{T}^2)$, $H^2(\mathbb{T}^2) \odot H^2(\mathbb{T}^2)$, and $H^1(\mathbb{T}^2)$.

Our efforts towards the proof of Theorem 4 have two remarkable byproducts. First, we can determine for which $\alpha \geq 0$ either of the equalities in (1.3) are attained; see Figure 1. Second, we are able to find optimal weak factorizations of φ_α for every $\alpha \geq 0$. We defer the precise statements to Section 3 below.

Theorem 4 illustrates in a striking way how norm attaining vectors and Hilbert points for the two Banach spaces $H^1(\mathbb{T}^2)$ and $W(\mathbb{T}^2)$ describe four distinct classes of functions. This stands in stark contrast to the case $d = 1$ where the four classes all coincide (with constant multiples of inner functions). It is clear that the inner-outer factorization has a strong impact on the situation in the latter case.

If the functional Hilbert space H is a normalized complete Pick space, then H and $H \odot H$ enjoy an analogue of the inner-outer factorization (see [2, Theorems 1.4 and 1.12]). It would be interesting to know what can be said of the norm attaining vectors and Hilbert points in this context.

Organization. The present paper is organized into two further sections. The next section contains the proofs of Theorems 1 and 2. Section 3 is devoted to the case study of φ_α and culminates with the proof of Theorem 4.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. We begin with the easiest implication (b)⇒(a). Suppose that φ is in X^* and $\|\varphi\|_{X^*} = \|\varphi\|_H$. Then

$$\|\varphi\|_H = \|\varphi\|_{X^*} \geq \frac{|\langle \varphi, \varphi\rangle|}{\|\varphi\|_X} = \frac{\|\varphi\|_H^2}{\|\varphi\|_X},$$

so $\|\varphi\|_X \geq \|\varphi\|_H$, and consequently $\|\varphi\|_X = \|\varphi\|_H$.

For the implication (a)⇒(b), suppose that φ is in H and that $\|\varphi\|_X = \|\varphi\|_H$. By the Hahn–Banach theorem, there is some ψ in X^* such that $\|\psi\|_{X^*} = 1$ and $\langle \varphi, \psi\rangle = \|\varphi\|_X$. If g is in $\ker\psi$ (i.e. g is in X and $\langle g, \psi\rangle = 0$), then the properties of ψ ensure that

$$\|\varphi + g\|_X \geq |\langle \varphi + g, \psi\rangle| = \|\varphi\|_X.$$

This means that if g is in $H \cap \ker\psi$, then

$$\|\varphi + \alpha g\|_H \geq \|\varphi + \alpha g\|_X \geq \|\varphi\|_X = \|\varphi\|_H$$

for every complex number α. This is equivalent to

$$2\operatorname{Re}(\alpha\langle g, \varphi\rangle) + |\alpha|^2\|g\|_H^2 \geq 0,$$

which holds for all complex numbers α if and only if $\langle g, \varphi\rangle = 0$. Every function f in H may be decomposed as

$$f = \left(f - \frac{\langle f, \psi\rangle}{\|\varphi\|_X}\varphi\right) + \frac{\langle f, \psi\rangle}{\|\varphi\|_X}\varphi$$

by the assumption that φ is in H. The first term is in $H \cap \ker\psi$ since $\langle \varphi, \psi\rangle = \|\varphi\|_X$, and so it is orthogonal to φ by the above. This means that

$$|\langle f, \varphi\rangle| = \frac{|\langle f, \psi\rangle|}{\|\varphi\|_X}\|\varphi\|_H^2 \leq \|f\|_X\|\varphi\|_H,$$

where in the final estimate we have used the fact that $\|\psi\|_{X^*} = 1$ and $\|\varphi\|_X = \|\varphi\|_H$. Since H is dense in X, we infer that φ is in X^* and $\|\varphi\|_{X^*} \leq \|\varphi\|_H$. ■

Proof of Theorem 2. We begin with the proof that (d)⇒(c). Suppose that φ is in X^* and $\|\varphi\|_X\|\varphi\|_{X^*} = \|\varphi\|_H^2$. If f is in H and $\langle f, \varphi\rangle = 0$, then

$$\|\varphi + f\|_X \geq \frac{|\langle \varphi + f, \varphi\rangle|}{\|\varphi\|_{X^*}} = \frac{\|\varphi\|_H^2}{\|\varphi\|_{X^*}} = \|\varphi\|_X.$$

For the proof that (c)$\Rightarrow$(d), we suppose that φ is a Hilbert point in X. Since φ is in H by assumption, we can decompose any f in H as

$$f = \left(f - \frac{\langle f, \varphi\rangle}{\|\varphi\|_H^2}\varphi\right) + \frac{\langle f, \varphi\rangle}{\|\varphi\|_H^2}\varphi.$$

The first term is orthogonal to φ by construction, so the assumption that φ is a Hilbert point in X ensures that

$$\|f\|_X \geq \frac{|\langle f, \varphi\rangle|}{\|\varphi\|_H^2}\|\varphi\|_X.$$

Since H is dense in X, it follows that $\|\varphi\|_H^2 \geq \|\varphi\|_X \|\varphi\|_{X^*}$. ■

3. A case study. A small amount of preparation is required before we can approach the proof of Theorem 4. We begin by recalling that a function f in $L^1(\mathbb{T}^d)$ is uniquely determined by the *Fourier coefficients*

$$\widehat{f}(\kappa) = \int_{[0,2\pi]^d} f(e^{i\theta_1}, \ldots, e^{i\theta_d}) e^{-i(\kappa_1\theta_1 + \cdots + \kappa_d\theta_d)} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}, \tag{3.1}$$

where the multi-index $\kappa = (\kappa_1, \ldots, \kappa_d)$ runs over $\mathbb{Z}^d$. In particular, a function f in $L^1(\mathbb{T}^d)$ is in the Hardy space $H^1(\mathbb{T}^d)$ if and only if $\widehat{f}(\kappa) = 0$ whenever $\kappa_j < 0$ for at least one $1 \leq j \leq d$. The set $\{z^\kappa\}_{\kappa \in \mathbb{Z}^d}$ forms an orthonormal basis for the Hilbert space $L^2(\mathbb{T}^d)$ and we will call it the *standard basis*. Also, let P stand for the orthogonal projection from $L^2(\mathbb{T}^d)$ to $H^2(\mathbb{T}^d)$.

The following result is contained in [5, Theorem 2.2(a)], but we include a complete proof to illustrate its interaction with Theorem 2. In its statement, we will write $\operatorname{sgn} z = z/|z|$ if z is a nonzero complex number and $\operatorname{sgn} z = 0$ if $z = 0$.

Lemma 5. *A nontrivial function φ in $H^2(\mathbb{T}^d)$ is a Hilbert point in $H^1(\mathbb{T}^d)$ if and only if*

$$P(\operatorname{sgn} \varphi) = \frac{\|\varphi\|_{H^1(\mathbb{T}^d)}}{\|\varphi\|_{H^2(\mathbb{T}^d)}^2}\varphi. \tag{3.2}$$

Proof. Suppose that (3.2) holds. If f is in $H^2(\mathbb{T}^d)$ and $\langle f, \varphi\rangle = 0$, then $\langle f, \operatorname{sgn} \varphi\rangle = 0$. Consequently,

$$\|\varphi\|_{H^1(\mathbb{T}^d)} = \langle \varphi, \operatorname{sgn} \varphi\rangle = \langle \varphi + f, \operatorname{sgn} \varphi\rangle \leq \|\varphi + f\|_{H^1(\mathbb{T}^d)},$$

which demonstrates that φ is a Hilbert point in $H^1(\mathbb{T}^d)$.

Conversely, suppose that φ is a Hilbert point in $H^1(\mathbb{T}^d)$. By Theorem 2, we know that φ is in the dual space of $H^1(\mathbb{T}^d)$. If we consider $H^1(\mathbb{T}^d)$ as a subspace of $L^1(\mathbb{T}^d)$, then it follows from the Hahn–Banach theorem and the Riesz representation theorem for $L^1(\mathbb{T}^d)$ that there is a function ψ in

$L^\infty(\mathbb{T}^d)$ such that $P\psi = \varphi$ and $\|\psi\|_{L^\infty(\mathbb{T}^d)} = \|\varphi\|_{(H^1(\mathbb{T}^d))^*}$. When combined with Theorem 2, this shows that

$$\|\psi\|_{L^\infty(\mathbb{T}^d)} = \|\varphi\|_{(H^1(\mathbb{T}^d))^*} = \frac{\langle \varphi, \varphi\rangle}{\|\varphi\|_{H^1(\mathbb{T}^d)}} = \frac{\langle \varphi, \psi\rangle}{\|\varphi\|_{H^1(\mathbb{T}^d)}}. \tag{3.3}$$

Since φ is a nontrivial function in $H^1(\mathbb{T}^d)$ by assumption, it can only vanish on a set of measure 0 on $\mathbb{T}^d$ (see e.g. [10, Theorem 3.3.5]). Hence it follows from (3.3) that $\varphi\overline{\psi} = |\varphi| > 0$ almost everywhere on $\mathbb{T}^d$, and so there is a positive constant C such that $\psi = C \operatorname{sgn}\varphi$ almost everywhere on $\mathbb{T}^d$. The constant is determined by (3.3). ∎

Lemma 6. *If $0 \leq \alpha \leq 2$ and*

$$\frac{\alpha}{2}\sqrt{4-\alpha^2} = \arcsin\frac{\alpha}{2},$$

then $\alpha = 0$ or $\alpha = 1.62420\ldots$.

Proof. It is plain that the equation holds for $\alpha = 0$. If $\alpha > 0$, then we rewrite the equation as

$$\sqrt{4-\alpha^2} = \frac{2}{\alpha}\arcsin\frac{\alpha}{2}.$$

The left-hand side decreases from 2 to 0, while the right-hand side increases (because $x \mapsto x/\sin x$ is increasing on $[0, \pi/2]$) from 1 to $\pi/2$. It follows that there is a unique solution $0 < \alpha < 2$, which can easily be estimated. ∎

Let m be an integer. A function f in $L^1(\mathbb{T}^d)$ is called *m-homogeneous* if

$$f(e^{i\vartheta}z_1, \ldots, e^{i\vartheta}z_d) = e^{im\vartheta}f(z_1, \ldots, z_d)$$

for almost every z on $\mathbb{T}^d$. It follows from (3.1) that f is m-homogeneous if and only if $\widehat{f}(\kappa) = 0$ whenever $\kappa_1 + \cdots + \kappa_d \neq m$. Consequently, the Hardy space $H^1(\mathbb{T}^d)$ only contains nontrivial m-homogeneous functions with $m \geq 0$, and they are all polynomials. The following result corresponds to Theorem 4(ii).

Theorem 7. *If $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$ for $\alpha \geq 0$, then φ_α is a Hilbert point in $H^1(\mathbb{T}^2)$ if and only if $\alpha = 0$ or $\alpha = 1.62420\ldots$.*

Proof. We will use Lemma 5. Since φ_α is 2-homogeneous, it is plain that $\operatorname{sgn}\varphi_\alpha$ is also 2-homogeneous. Consequently, it follows that

$$P(\operatorname{sgn}\varphi_\alpha) = az_1^2 + bz_1z_2 + cz_2^2.$$

Since $\varphi_\alpha(z_2, z_1) = \varphi_\alpha(z_1, z_2)$, we must have $a = c$. Hence, Lemma 5 implies that φ_α is a Hilbert point in $H^1(\mathbb{T}^2)$ if and only if

$$\alpha\, \widehat{\operatorname{sgn}\varphi_\alpha}(0,2) = \widehat{\operatorname{sgn}\varphi_\alpha}(1,1). \tag{3.4}$$

We begin with the latter Fourier coefficient, which is slightly simpler to compute. Here, we have

$$(\operatorname{sgn}\varphi_\alpha(e^{i\theta_1}, e^{i\theta_2}))e^{-i(\theta_1+\theta_2)} = \operatorname{sgn}(\alpha + 2\cos(\theta_1 - \theta_2)),$$

which means that

$$\widehat{\operatorname{sgn}\varphi_\alpha}(1,1) = \int_0^{2\pi} \operatorname{sgn}(\alpha + 2\cos\vartheta)\,\frac{d\vartheta}{2\pi} = \begin{cases} \frac{2}{\pi}\arcsin\frac{\alpha}{2} & \text{if } 0 \le \alpha \le 2,\\ 1 & \text{if } \alpha > 2.\end{cases}$$

For the former Fourier coefficient, we have

$$(\operatorname{sgn}\varphi_\alpha(e^{i\theta_1}, e^{i\theta_2}))e^{-2i\theta_2} = e^{i(\theta_1-\theta_2)}\operatorname{sgn}(\alpha + 2\cos(\theta_1-\theta_2)),$$

which yields

$$\widehat{\operatorname{sgn}\varphi_\alpha}(0,2) = \int_0^{2\pi} e^{i\vartheta}\operatorname{sgn}(\alpha + 2\cos\vartheta)\,\frac{d\vartheta}{2\pi} = \begin{cases} \frac{1}{2\pi}\sqrt{4-\alpha^2} & \text{if } 0 \le \alpha \le 2,\\ 0 & \text{if } \alpha > 2.\end{cases}$$

We insert these formulas into (3.4). There are plainly no solutions if $\alpha > 2$. If $0 \le \alpha \le 2$, then we get precisely the equation considered in Lemma 6. ∎

Before we proceed to the second part of our case study, let us compute

$$\begin{aligned}\|\varphi_\alpha\|_{H^1(\mathbb{T}^2)} &= \int_0^{2\pi} |\alpha + 2\cos\vartheta|\,\frac{d\vartheta}{2\pi}\\ &= \begin{cases} \frac{2}{\pi}\left(\alpha\arcsin\frac{\alpha}{2} + \sqrt{4-\alpha^2}\,\right) & \text{if } 0 \le \alpha \le 2,\\ \alpha & \text{if } \alpha > 2.\end{cases}\end{aligned}$$

This computation and Theorem 11 below form the basis for Figure 1.

Let m be an integer and let P_m denote the orthogonal projection from $L^2(\mathbb{T}^d)$ to its subspace of m-homogeneous functions. By orthogonality, every f in $H^2(\mathbb{T}^d)$ satisfies the equation

$$\|f\|^2_{H^2(\mathbb{T}^d)} = \sum_{m=0}^{\infty} \|P_m f\|^2_{H^2(\mathbb{T}^d)}. \tag{3.5}$$

It is clear that P_m is densely defined on the weak product space $W(\mathbb{T}^d)$. Next, we show that it extends to a norm 1 operator on $W(\mathbb{T}^d)$, and consequently on its dual space. This result (in a slightly different context) can be found in [6, Theorem 5]. In order to make the present paper self-contained, we repeat the proof.

Lemma 8. *If m is a nonnegative integer, then P_m extends to a norm 1 operator on $W(\mathbb{T}^d)$ and on $(W(\mathbb{T}^d))^*$.*

Proof. The first assertion implies the other by duality, since P_m is self-adjoint in the pairing of $H^2(\mathbb{T}^d)$. The function $f(z) = z_1^m$ shows that $\|P_m\|_{W(\mathbb{T}^d)\to W(\mathbb{T}^d)} \ge 1$. Let f be a function in $W(\mathbb{T}^d)$ and let $f = \sum_{j\ge 1} g_j h_j$ be a weak factorization of f. Then

$$P_m f = \sum_{j=1}^{\infty}\sum_{n=0}^{m} P_n g_j P_{m-n} h_j,$$

and consequently

$$\begin{aligned}\|P_m f\|_{W(\mathbb{T}^d)} &\leq \sum_{j=1}^{\infty}\sum_{n=0}^{m} \|P_n g_j\|_{H^2(\mathbb{T}^d)}\|P_{m-n} h_j\|_{H^2(\mathbb{T}^d)} \\ &\leq \sum_{j=1}^{\infty} \|g_j\|_{H^2(\mathbb{T}^d)}\|h_j\|_{H^2(\mathbb{T}^d)},\end{aligned}$$

where we have used the Cauchy–Schwarz inequality in the inner sum and (3.5) twice. ∎

Lemma 9. *Suppose that m is a nonnegative integer. If φ is a nontrivial m-homogeneous polynomial, then there is an m-homogeneous polynomial ψ such that*

$$\|\varphi\|_{W(\mathbb{T}^d)} = \frac{\langle \psi, \varphi\rangle}{\|\psi\|_{(W(\mathbb{T}^d))^*}}. \tag{3.6}$$

Proof. Since φ is nontrivial, it follows from the Hahn–Banach theorem and the fact that $(W(\mathbb{T}^d))^*$ is embedded in $H^2(\mathbb{T}^d)$ that there is ψ in $H^2(\mathbb{T}^d)$ such that (3.6) holds. Since P_m is self-adjoint in the pairing of $H^2(\mathbb{T}^d)$ and since $P_m\varphi = \varphi$, it follows from Lemma 8 that (3.6) also holds if ψ is replaced by $P_m\psi$. ∎

Let $\overline{H^2}(\mathbb{T}^d)$ be the closed subspace of $L^2(\mathbb{T}^d)$ consisting of the complex conjugates of functions in $H^2(\mathbb{T}^d)$ and let $\overline{P}$ denote the orthogonal projection from $L^2(\mathbb{T}^d)$ to $\overline{H^2}(\mathbb{T}^d)$. Let ψ be a function in $H^2(\mathbb{T}^d)$. The formula

$$\mathbf{H}_\psi f = \overline{P}(\overline{\psi} f)$$

densely defines a Hankel operator $\mathbf{H}_\psi$ from $H^2(\mathbb{T}^d)$ to $\overline{H^2}(\mathbb{T}^d)$. In the present context, [1, Theorem 2.5] asserts that $\mathbf{H}_\psi$ extends to a bounded linear operator if and only if ψ is in $(W(\mathbb{T}^d))^*$ and that in this case $\|\mathbf{H}_\psi\| = \|\psi\|_{(W(\mathbb{T}^d))^*}$. If ψ is in $(W(\mathbb{T}^d))^*$ and f, g are in $H^2(\mathbb{T}^d)$, then

$$\langle \mathbf{H}_\psi f, \overline{g}\rangle = \langle fg, \psi\rangle.$$

This formula makes it easy to compute the matrix of $\mathbf{H}_\psi$ with respect to the standard basis that $H^2(\mathbb{T}^d)$ and $\overline{H^2}(\mathbb{T}^d)$ inherit from $L^2(\mathbb{T}^d)$.

Lemma 10. *If $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$ for $\alpha \geq 0$, then*

$$\|\varphi_\alpha\|_{(W(\mathbb{T}^2))^*} = \max(\sqrt{2+\alpha^2}, 1+\alpha).$$

Proof. The matrix of the Hankel operator $\mathbf{H}_{\varphi_\alpha}$ with respect to the standard basis of $H^2(\mathbb{T}^2)$ and $\overline{H^2}(\mathbb{T}^2)$, with rows and columns containing all zeros omitted, is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 1 & \alpha & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $(\mathbf{e}_j)_{j=1}^6$ be the standard basis of $\mathbb{C}^6$. Due to orthogonality and the block structure of the matrix, it is sufficient to let it act on the subspaces $\operatorname{span}\{\mathbf{e}_1\}$, $\operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$, and $\operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$. The norms are, respectively, $\sqrt{2+\alpha^2}$, $1+\alpha$, and $\sqrt{2+\alpha^2}$. ■

We mention in passing that the block structure of the matrix appearing in the proof of Lemma 10 is a special case of a general phenomenon that occurs for Hankel operators on $H^2(\mathbb{T}^d)$ with m-homogeneous symbols (see [3, Theorem 4]).

Lemma 10 allows us to compute one of the two nontrivial quantities in the condition of Theorem 2(d) for the polynomials φ_α. It is also the crucial ingredient in the following result.

Theorem 11. *Suppose that $\varphi_\alpha(z) = z_1^2 + \alpha z_1 z_2 + z_2^2$. Then*

$$\|\varphi_\alpha\|_{W(\mathbb{T}^2)} = \begin{cases} \sqrt{2+\alpha^2} & \text{if } 0 \leq \alpha \leq 1/2, \\ \frac{4+\alpha}{3} & \text{if } 1/2 < \alpha \leq 2, \\ \alpha & \text{if } \alpha > 2. \end{cases}$$

Proof. By Lemma 9 there is a 2-homogeneous polynomial

$$\psi(z) = az_1^2 + bz_1z_2 + cz_2^2$$

such that

$$\|\varphi_\alpha\|_{W(\mathbb{T}^2)} = \frac{\langle \psi, \varphi_\alpha \rangle}{\|\psi\|_{(W(\mathbb{T}^2))^*}}.$$

It follows from triangle inequality (for $(W(\mathbb{T}^2))^*$) that if this formula holds for ψ_1 and ψ_2, then it also holds for $\psi_1 + \psi_2$. Since the coefficients of φ_α are real, it follows that a, b, and c are real. Moreover, since $\varphi_\alpha(z_2, z_1) = \varphi_\alpha(z_1, z_2)$, we must have $a = c$. We consider first the case that $a = c \neq 0$, where we normalize ψ with $a = c = 1$ and $b = \beta \geq 0$. Using Lemma 10, we see that

$$\|\varphi_\alpha\|_{W(\mathbb{T}^2)} = \sup_{\beta \geq 0} F_\alpha(\beta) \quad \text{for} \quad F_\alpha(\beta) = \begin{cases} \frac{2+\alpha\beta}{\sqrt{2+\beta^2}} & \text{if } 0 \leq \beta \leq 1/2, \\ \frac{2+\alpha\beta}{1+\beta} & \text{if } \beta > 1/2. \end{cases}$$

There are three cases to consider:

(i) If $0 \leq \alpha \leq 1/2$, then F_α is increasing until $\beta = \alpha$ and then decreasing.
(ii) If $1/2 < \alpha \leq 2$, then F_α is increasing until $\beta = 1/2$ and then decreasing.
(iii) If $\alpha > 2$, then F_α is increasing.

Note that to attain supremum in (iii) we have to let $\beta \to \infty$. This is equivalent to the case $a = c = 0$ that we excluded above. The proof is completed by checking that

$$F_\alpha(\alpha) = \sqrt{2+\alpha^2}, \qquad F_\alpha(1/2) = \frac{4+\alpha}{3}, \qquad F_\alpha(\beta) \to \alpha \quad \text{as } \beta \to \infty. \blacksquare$$

The knowledge of $\|\varphi_\alpha\|_W$ from Theorem 11 makes it possible to guess an optimal weak factorization (1.1) of φ_α in the three cases.

(i) If $0 \leq \alpha \leq 1/2$, then an optimal weak factorization is $\varphi_\alpha = \varphi_\alpha \cdot 1$.
(ii) If $1/2 < \alpha < 2$, then an optimal weak factorization is

$$\varphi_\alpha(z) = \frac{2}{3}(\alpha - 1/2)(z_1+z_2)(z_1+z_2) + \frac{2}{3}(2-\alpha)\left(z_1^2 + \frac{z_1 z_2}{2} + z_2^2\right) \cdot 1.$$

(iii) If $\alpha > 2$, then an optimal weak factorization is

$$\varphi_\alpha(z) = \left(z_1 + \frac{\alpha + \sqrt{\alpha^2-4}}{2} z_2\right)\left(z_1 + \frac{\alpha - \sqrt{\alpha^2-4}}{2} z_2\right).$$

We conclude the paper by wrapping up the proof of Theorem 4.

Proof of Theorem 4. Statement (i) is trivial since no φ_α has constant modulus on $\mathbb{T}^2$, and—as noted above—statement (ii) is the same as Theorem 7. It is plain that $\|\varphi_\alpha\|_{H^2(\mathbb{T}^2)} = \sqrt{2+\alpha^2}$. To settle (iii) and (iv), we use, respectively, Theorems 1 and 2, which requires solving the equations

$$\|\varphi_\alpha\|_{(W(\mathbb{T}^2))^*} = \sqrt{2+\alpha^2} \quad \text{and} \quad \|\varphi_\alpha\|_{W(\mathbb{T}^2)}\|\varphi_\alpha\|_{(W(\mathbb{T}^2))^*} = 2+\alpha^2.$$

By Lemma 10, the first equality holds if and only if $0 \leq \alpha \leq 1/2$. We then use both Lemma 10 and Theorem 11 to see that the second equality holds if and only if $0 \leq \alpha \leq 1/2$ or $\alpha = 2$. $\blacksquare$

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