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On two uniform exponents of approximation related to Wirsing's problem

by

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Abstract. We aim to fill a gap in the proof of an inequality relating two exponents of uniform Diophantine approximation stated in a paper by Bugeaud. We succeed in verifying the inequality in several instances, in particular for small dimension. Moreover, we provide counterexamples to generalizations, which contrasts with the case of ordinary approximation where such phenomena do not occur. Our results contribute to the understanding of the discrepancy between small absolute values of a polynomial at a given real number and approximation to the number by algebraic numbers of absolutely bounded degree, a fundamental issue in the famous problem of Wirsing and its variants.

1. Introduction

1.1. Some exponents of approximation and basic properties. Let $n \geq 1$ be an integer and ξ be a transcendental real number. For a polynomial P, by its *height* H_P we mean the maximum modulus of its coefficients. For a parameter $X > 1$, we call the non-zero integer polynomial P_X of degree at most n and height at most X the *best approximation polynomial for* n, ξ *up to parameter* X if $P = P_X$ minimizes $|P(\xi)|$ among all such integer polynomials P. This is defined uniquely up to sign since ξ is transcendental. Similarly, write α_X for the algebraic number of degree at most n and height at most X that minimizes $H(\alpha_X) \cdot |\xi - \alpha_X|$. Here the *height* of α, denoted by $H(\alpha_X)$, is simply the height of its minimal polynomial over the integers with coprime coefficients.

Define the *local exponents of approximation* at parameter $X > 1$ by

$$w(n,\xi,X) = -\frac{\log |P_X(\xi)|}{\log X}, \qquad w^*(n,\xi,X) = -\frac{\log(H(\alpha_X) \cdot |\alpha_X - \xi|)}{\log X},$$

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and let us derive the classical exponents of approximation

(1) $$w_n(\xi) = \limsup_{X\to\infty} w(n,\xi,X), \qquad w_n^*(\xi) = \limsup_{X\to\infty} w^*(n,\xi,X)$$

and

(2) $$\widehat{w}_n(\xi) = \liminf_{X\to\infty} w(n,\xi,X), \qquad \widehat{w}_n^*(\xi) = \liminf_{X\to\infty} w^*(n,\xi,X).$$

The ordinary exponents (without "hat") may take the formal value $+\infty$, on the other hand the uniform exponents (with "hat") are bounded independently of ξ in terms of n; see Davenport and Schmidt [10], and also [8]. The exponents $w_n(\xi)$ date back to Mahler, and the exponents $w_n^*(\xi)$ were introduced by Wirsing [26] in his famous paper on approximation to real numbers by algebraic numbers of bounded degree. The uniform exponents $\widehat{w}_n(\xi), \widehat{w}_n^*(\xi)$ were introduced by Bugeaud and Laurent [7].

It is easily seen that all four types of exponents from (1), (2) form non-decreasing sequences in n, in particular we have

(3) $$w_1(\xi) \le w_2(\xi) \le \cdots, \qquad \widehat{w}_1^*(\xi) \le \widehat{w}_2^*(\xi) \le \cdots.$$

Several other relations are known, most notably Dirichlet's Theorem implies

(4) $$w_n(\xi) \ge \widehat{w}_n(\xi) \ge n, \qquad n \ge 1.$$

A famous open problem of Wirsing [26] asks if a strengthening of (4) of the form $w_n^*(\xi) \ge n$ holds as well. As a rather short argument (see Proposition 4.7 below) yields for any ξ the estimates

(5) $$\widehat{w}_n^*(\xi) \le \widehat{w}_n(\xi), \qquad w_n^*(\xi) \le w_n(\xi),$$

the conjectural inequality is indeed stronger, and only verified for $n \in \{1,2\}$ so far [9]. It was already noticed by Wirsing that for any n and almost all real numbers in the sense of Lebesgue measure we have $w_n^*(\xi) = n$, in fact $\widehat{w}_n^*(\xi) = n$ as well as a consequence of (28) below from [7], and of Sprindžuk's result [25]. However, for general ξ the best known lower bounds for $w_n^*(\xi)$ and large n of order $n/(2-\log 2)$ have very recently been obtained by Poëls [16], improving on $n/\sqrt{3}$ from [1]. Since $\widehat{w}_1(\xi) = \widehat{w}_1^*(\xi)$ for any transcendental real ξ, in fact both being equal to 1 for any irrational ξ by Khinchin [13], combining (3) and (4) we get

(6) $$w_n^*(\xi) \ge \widehat{w}_n^*(\xi) \ge 1, \qquad n \ge 1.$$

Let us further define

$$\kappa(n,\xi,X) := w(n,\xi,X) - w^*(n,\xi,X),$$

and accordingly

$$\underline{\kappa}_n(\xi) := \liminf_{X\to\infty} \kappa(n,\xi,X), \qquad \overline{\kappa}_n(\xi) := \limsup_{X\to\infty} \kappa(n,\xi,X).$$

These exponents have not been explicitly defined or studied before. Note that

$$\overline{\kappa}_n(\xi) \geq \max\left\{w_n(\xi) - w_n^*(\xi), \widehat{w}_n(\xi) - \widehat{w}_n^*(\xi)\right\} \tag{7}$$

and conversely

$$\underline{\kappa}_n(\xi) \leq \min\left\{w_n(\xi) - w_n^*(\xi), \widehat{w}_n(\xi) - \widehat{w}_n^*(\xi)\right\} \tag{8}$$

as the corresponding estimates are true for general pairs of real functions $f(X)$ and $g(X)$ in place of $w(n, \xi, X)$ and $w^*(n, \xi, X)$. Note that (7), (8) and [6, Theorem 2.10] combined yield $\underline{\kappa}_n(\xi) = \overline{\kappa}_n(\xi) = 0$ for any algebraic number ξ. Moreover, a generalization of both claims of (5) of the form

$$\underline{\kappa}_n(\xi) \geq 0 \tag{9}$$

holds, by the same Proposition 4.7. Note however that there is no reason for the quantities $\underline{\kappa}_n(\xi), \overline{\kappa}_n(\xi)$ to be monotonic in n, in particular an analogue of (3) does not hold. We believe that these exponents carry important information on the discrepancy between small values of polynomials and approximation to real numbers by algebraic numbers, closely related to Wirsing's problem and its variants, and deserve to be studied in detail.

1.2. An open problem and related questions. Let us return to the classical exponents. As stated in [5], conversely to (5) we have

$$w_n(\xi) - w_n^*(\xi) \leq n - 1. \tag{10}$$

This follows by combining Lemmas 4.1 and 4.2 below. The sharpness of (10) is open in general; for partial results see [4]. It was further stated in [6] that we similarly have

$$\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq n - 1. \tag{11}$$

The latter estimate is sharp, as it was noticed in [6] that any *Liouville number* ξ (i.e. ξ is transcendental and $w_1(\xi) = \infty$) satisfies for any $n \geq 1$ the identities

$$\widehat{w}_n(\xi) = n, \qquad \widehat{w}_n^*(\xi) = 1. \tag{12}$$

The left identity stems from [19]; see Theorem 4.6 below. In fact, it was shown in [6] that the interval $\left[n - 2 + \frac{1}{n}, n - 1\right]$ is contained in the spectrum of the difference $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi)$, and upon assuming some natural conjecture to be true, the spectrum contains $[0, n - 1]$.

However, there seems to be a considerable gap in the sketched proof of (11). Indeed, the proof is based on Lemma 4.2 below, which however only holds for separable P. In contrast to (10) for the ordinary exponents, where this problem can be overcome rather easily by Lemma 4.1 below due to Wirsing, for the uniform exponents it is a priori unclear why one can restrict oneself to this case. While at first sight this may appear to be a rather minor technicality, it turns out to cause great problems, as evidenced by the results in §3.2 below.

The purpose of this note is twofold: Firstly, we want to establish (11) rigorously. This will be the content of §2. Secondly, in §3, we study the related parametric functions $\kappa(n,\xi,X)$ and their extremal values $\underline{\kappa}_n(\xi), \overline{\kappa}_n(\xi)$. We once again stress that understanding these quantities is at the core of Wirsing's famous problem discussed in §1.1.

To this end, we again emphasize that the delicacy of proving (11) is reflected in our results in §3.2, where we provide counterexamples to stronger claims involving the exponents κ and their extremal values. In particular, Theorem 3.3 provides a counterexample to a parametric inequality in the spirit of (9) that if true would generalize both (10) and (11). This illustrates that indeed one has to be careful with non-separable polynomials.

2. Results towards a proof of (11). Despite the examples from §3.2 mentioned above that suggest otherwise, we still believe that (11) is true for all n. While we have been unable to prove (11) in full generality, we provide several weaker claims. First we verify it for small n.

THEOREM 2.1. *The claim* (11) *holds for* $n \le 5$ *and any transcendental real number* ξ.

While the conclusion for $n \le 3$ is immediate by combining results from [5, 19], the cases $n = 4$ and $n = 5$ require a more sophisticated approach. It is likely that with more effort the range for n can be extended by the arguments for the latter cases. However, we have not intensified our investigations in this direction, and it is doubtful that the strategy is sufficient for a proof for general n without further new ideas.

Next we want to state criteria implying (11) for arbitrary n. In this context we define another classical exponent of approximation. Let the *ordinary exponent of simultaneous approximation* denoted by $\lambda_n(\xi)$ be the supremum of λ such that

$$\max_{1\le j\le n} |q\xi^j - p_j| \le q^{-\lambda}$$

has infinitely many solutions in integer vectors $q, p_1, \ldots, p_n$.

THEOREM 2.2. *Let an integer* $n \ge 1$ *and a transcendental real number* ξ *satisfy any of the following conditions:*

(i) $\widehat{w}_n(\xi) = n$,
(ii) $\widehat{w}_n(\xi) > 2n - 3$,
(iii) $\widehat{w}_n^*(\xi) \ge n - 2$,
(iv) $\lambda_n(\xi) \in \left[\frac{1}{n}, \frac{1}{n-2}\right] \cup (1, \infty]$,

(v) $w_1(\xi) \geq n$,
(vi) $w_n(\xi) \leq \frac{2n-1+\sqrt{4n-3}}{2}$.

Then (11) *holds.*

It has been shown very recently by Poëls [17, Theorem 1.1] that $\widehat{w}_3(\xi) \leq 2+\sqrt{5}$ and $\widehat{w}_n(\xi) \leq 2n-2$ for $n \geq 4$, improving on [20] for $3 \leq n \leq 9$. Additionally [17, Theorem 1.2] shows that (ii) can only happen for small n, as for sufficiently large n indeed stronger bounds of order $\widehat{w}_n(\xi) \leq 2n-\sqrt[3]{n}/3$ were obtained. See also [24].

Notably, for $n \geq 3$ it is in fact not known whether counterexamples to (i) exist. By (4), one of the conditions (i), (ii) in particular applies to any ξ if $n \leq 3$ (and (iii) does as well), so part of Theorem 2.1 is directly implied. However, the claim of Theorem 2.1 for $n=2$ is in fact used in the proof of this case.

A corollary of part (ii) is the following unconditional but weaker bound that may be of some interest, especially for small n.

THEOREM 2.3. *For any integer $n \geq 3$ and any transcendental real number ξ we have*

$$\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq 2n-4.$$

Proof. If $\widehat{w}_n(\xi) > 2n-3$ then the stronger bound $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq n-1 \leq 2n-4$ for $n \geq 3$ applies by Theorem 2.2(ii). If otherwise $\widehat{w}_n(\xi) \leq 2n-3$ then (6) implies $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq \widehat{w}_n(\xi) - 1 \leq 2n-4$ as well. ∎

This slightly improves on the bound $2n-2$ that is immediate from Davenport and Schmidt [10], in fact $2n-3$ for $n \geq 4$ follows from the aforementioned [17, Theorem 1.1]. Note that for large n we infer the bound $2n - \sqrt[3]{n}/3 - 1$ from [17, Theorem 1.2] recalled above as well, in particular the expression is not of order $2n - O(1)$ as $n \to \infty$. It would be desirable to obtain a bound of the form $(2-\varepsilon)n$ for some explicit $\varepsilon > 0$.

3. On the exponents $\kappa(n, \xi, X)$

3.1. On $\overline{\kappa}_n$. Our next theorem gives another upper estimate for the difference $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi)$, in fact for the quantity $\overline{\kappa}_n(\xi)$. However, another exponent of approximation occurs in the formula. Write $\lfloor x \rfloor$ for the largest integer less than or equal to x.

THEOREM 3.1. *Let $n \geq 2$ be an integer and set*

$$k = \left\lfloor \frac{n}{2} \right\rfloor.$$

Then for any real number ξ we have

$$\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq \max\{n-1, w_k(\xi)\}. \tag{13}$$

In fact, the stronger inequality

$$\overline{\kappa}_n(\xi) \leq \max\{n-1, w_k(\xi)\} \tag{14}$$

holds.

By (7), estimate (14) indeed implies (13). The claim shows that the desired estimate (11) holds if $w_k(\xi) \leq n-1$. The classical formula of Bernik [3],

$$\dim_H(\{\xi : w_m(\xi) \geq \lambda\}) = \frac{m+1}{\lambda+1}, \qquad m \geq 1, \lambda \in [m, \infty], \tag{15}$$

where $\dim_H$ denotes Hausdorff dimension, applied for $m = k$ in particular implies

COROLLARY 3.2. *Given an integer $n \geq 2$, define the set of counterexamples to* (11) *as*

$$\Omega_n := \{\xi \in \mathbb{R} : \widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) > n-1\}$$

and let similarly

$$\Gamma_n := \{\xi \in \mathbb{R} : \overline{\kappa}_n(\xi) > n-1\} \supseteq \Omega_n.$$

Then

$$\dim_H(\Omega_n) \leq \dim_H(\Gamma_n) \leq \frac{1}{2} + o(1), \qquad n \to \infty.$$

By Theorem 2.1, the estimates for Ω_n are only relevant for $n \geq 6$. While the bound in Corollary 3.2 does not seem particularly strong, it is not clear if there is any other easy argument available that improves on it, even for Ω_n. Indeed, it seems that no improvement via Theorem 2.2 can be obtained when combining it with known metrical theory for classical exponents, and only (i) and (iii) have the potential to lead to such an improvement if the metrical theory is sufficiently well developed (for (iv) this is excluded for $n \geq 6$ by a metrical result from [2]). In this context note that the Hausdorff dimension of counterexamples to (iii) is at least $1/4 - o(1)$ as $n \to \infty$, as a consequence of [22, Theorem 2.5] (in fact, of its proof) and (15).

For a reverse positive lower bound for the Hausdorff dimension of Γ_n, see Corollary 3.4 below. For $n \in \{2, 3\}$ we will improve the bounds resulting from Corollary 3.2 in Corollary 6.2 below.

3.2. Counterexamples to stronger versions of (11). The following example shows that a parametric version of (11) does not hold in general, so that $w_k(\xi)$ cannot be dropped in (14) of Theorem 3.1. Recall that a Liouville number is a transcendental real number satisfying $w_1(\xi) = \infty$.

THEOREM 3.3. *Let $n \geq 2$ be an integer and ξ be a transcendental real number. Let $2 \leq k \leq n$ be another integer. If $w_1(\xi) > n + k - 1$ then*

$$\overline{\kappa}_n(\xi) \geq \left(1 - \frac{1}{k}\right) \cdot w_1(\xi). \tag{16}$$

Thus, if $w_1(\xi) > 2n-1$ *then*

$$\overline{\kappa}_n(\xi) \geq \left(1-\frac{1}{n}\right)\cdot w_1(\xi). \tag{17}$$

In particular, if ξ *is a Liouville number then*

$$\overline{\kappa}_n(\xi) = \infty. \tag{18}$$

The proof is based on Liouville's inequality (Theorem 4.4 below on the minimum distance between algebraic numbers). Another application of Theorem 3.3 is

Corollary 3.4. *We have*

$$\dim_H(\varGamma_n) \geq \dim_H(\varGamma_n \setminus \varOmega_n) \geq \frac{2-o(1)}{n+\sqrt{n}}, \qquad n\to\infty, \tag{19}$$

with $\varGamma_n$ *defined in Corollary* 3.2. *Moreover,* $\dim_H(\varGamma_n\setminus\varOmega_n) \geq 1/n$ *for all* n.

Proof. Let k be the smallest integer with $k^2-k+1>n$, of order $\sqrt{n}$. Then the right hand side of (16) exceeds $n-1$ as soon as $w_1(\xi) > n+k-1$, which also justifies application of (16). Then we can further use (15) for $m=1$ and any $\lambda > n+k-1$, which gives the lower bound of (19) for the set $\varGamma_n$. Finally, the hypothesis implies (11) for all ξ within our set from (15) by Theorem 2.2(v), hence the estimate is preserved when removing $\varOmega_n$. The bound $2/(2n) = 1/n$ follows similarly from (17) with $\lambda = 2n-1+\varepsilon$ via (15). ∎

Similarly, parametric variants of (19) for estimating

$$\dim_H(\{\xi\in\mathbb{R} : \overline{\kappa}_n(\xi) > \lambda\}), \qquad \lambda \geq \lambda_0(n), \tag{20}$$

from below with large enough $\lambda_0(n)$ can be obtained. Moreover, variants of Theorem 3.3 involving $w_\ell(\xi)$ for any $\ell \leq n/2$ can be derived, in particular we have a generalization of (18) of the form $\overline{\kappa}_n(\xi) = \infty$ when ξ is a U_ℓ*-number of index* $\ell \leq n/2$ in Mahler's classification of real numbers, meaning that $w_{\ell-1}(\xi) < \infty$ and $w_\ell(\xi) = \infty$.

As indicated in §1.1, inseparable polynomials cause problems when attempting to prove (11), the deeper reason being that we cannot apply Lemmas 4.2 and 6.1. This motivates the definition of the separable local exponent $w_{\mathrm{sep}}(n,\xi,X)$ like $w(n,\xi,X)$ but restricting to separable polynomials P. Let us define the uniform exponent

$$\widehat{w}_{n,\mathrm{sep}}(\xi) := \liminf_{X\to\infty} w_{\mathrm{sep}}(n,\xi,X).$$

Hence, in order to prove (11), one may try to prove for any real ξ the identity

$$\widehat{w}_{n,\mathrm{sep}}(\xi) = \widehat{w}_n(\xi). \tag{21}$$

Indeed, if (21) holds for some n, ξ, then (11) holds for this pair via

$$\begin{aligned}\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) &= \widehat{w}_{n,\mathrm{sep}}(\xi) - \widehat{w}_n^*(\xi) \\ &= \liminf_{X\to\infty} w_{\mathrm{sep}}(n,\xi,X) - \liminf_{X\to\infty} w^*(n,\xi,X) \\ &\le \limsup_{X\to\infty}(w_{\mathrm{sep}}(n,\xi,X) - w^*(n,\xi,X)) \le n-1,\end{aligned}$$

where for the last inequality we have used Lemma 4.2 below. For completeness, define similarly $w_{\mathrm{irr}}(n,\xi,X)$ and $\widehat{w}_{n,\mathrm{irr}}(\xi)$ when restricting to irreducible polynomials. It is clear that

$$w_{\mathrm{irr}}(n,\xi,X) \le w_{\mathrm{sep}}(n,\xi,X) \le w(n,\xi,X),$$

hence for any n, ξ we have

$$\widehat{w}_{n,\mathrm{irr}}(\xi) \le \widehat{w}_{n,\mathrm{sep}}(\xi) \le \widehat{w}_n(\xi).$$

We remark that one may accordingly define ordinary exponents $w_{n,\mathrm{irr}}(\xi)$ and $w_{n,\mathrm{sep}}(\xi)$ as well via upper limits, but they turn out to be equal to $w_n(\xi)$ by Lemma 4.1 below.

However, at least for $n = 2$ we can prove that (21) is in general false as well, so this avenue is not possible.

THEOREM 3.5. *Assume ξ satisfies $w_1(\xi) > 3$. Then*

$$\widehat{w}_{2,\mathrm{sep}}(\xi) \le 1 + \frac{2}{w_1(\xi)-1} < 2 = \widehat{w}_2(\xi). \tag{22}$$

In particular, for any Liouville number ξ we have

$$\widehat{w}_{2,\mathrm{irr}}(\xi) = \widehat{w}_{2,\mathrm{sep}}(\xi) = 1.$$

REMARK 1. A slightly stronger estimate for $\widehat{w}_{2,\mathrm{irr}}(\xi)$ when $w_1(\xi) \in (3,\infty)$ can be deduced by our method. We further remark that, as shown in [21], the exponent $\widehat{w}_{=n,\mathrm{irr}}(\xi)$ (denoted just $\widehat{w}_{=n}(\xi)$ in [21]) with polynomials irreducible of degree precisely n equals 0 for any Liouville number and any $n \ge 2$. The proof is similar to our proof of Theorem 3.5 below, based on Minkowski's Second Convex Body Theorem. In exact degree the associated ordinary exponents $w_{=n,\mathrm{irr}}(\xi)$ become more interesting as well. In particular, it is open for $n > 7$ if they are always at least n (for $n \le 7$ this was settled in [23, Theorem 1.3]).

We believe there is equality in the leftmost inequality of (22). If we denote by $\mathrm{spec}(\cdot) \subseteq \mathbb{R}$ the set of values taken by an exponent for all real numbers, this would imply

$$\mathrm{spec}(\widehat{w}_{2,\mathrm{sep}}) = [1,2] \cup \mathrm{spec}(\widehat{w}_2), \qquad \mathrm{spec}(\widehat{w}_2 - \widehat{w}_{2,\mathrm{sep}}) = [0,1]. \tag{23}$$

The spectrum of $\widehat{w}_2$ is known to be contained and dense in $[2,(3+\sqrt{5})/2]$. The inclusion of $[1,2]$ in the left identity of (23) follows directly from the equality in (22). On the other hand, if $\widehat{w}_2(\xi) > 2$ for some ξ then it follows as in [21] that all best approximations are irreducible of degree exactly 2, which implies $\widehat{w}_{2,\mathrm{sep}}(\xi) = \widehat{w}_2(\xi)$. This argument shows that the spectrum of $\widehat{w}_{2,\mathrm{spec}}(\xi)$ contains the spectrum of $\widehat{w}_2(\xi)$; more precisely, unconditionally we have

$$\mathrm{spec}(\widehat{w}_2) \cap (2,\infty) = \mathrm{spec}(\widehat{w}_{2,\mathrm{sep}}) \cap (2,\infty).$$

The point $\{2\}$ can be added by considering generic numbers with $w_2(\xi) = 2$; see Theorem 3.7 below. Hence the left identity of (23) holds under our assumption. The right identity of (23) uses additionally the fact that $w_1(\xi) \geq 3 > 2$ implies $\widehat{w}_2(\xi) = 2$ by Theorem 4.6 below.

We further believe that for larger n again Liouville numbers will provide counterexamples to (21), but the proof will be more involved so we do not embark on it.

3.3. Some remarks on $\underline{\kappa}_n$ and the limit of κ. Let us study $\underline{\kappa}_n(\xi)$. By (9), (10), (8) we get

$$0 \leq \underline{\kappa}_n(\xi) \leq w_n(\xi) - w_n^*(\xi) \leq n-1. \tag{24}$$

This motivates the following

Problem 1. Can we improve the bound $n-1$ for $\underline{\kappa}_n$ from (24) for $n \geq 2$ and all ξ? Is it in fact always 0?

Let us first consider $n = 2$. Then, complementary to the bound $n-1 = 1$ in (24), we get the following estimates in terms of the uniform exponent.

Theorem 3.6. *For any real number ξ we have*

$$\underline{\kappa}_2(\xi) \leq \frac{w_2(\xi)}{\widehat{w}_2(\xi)-1} - \widehat{w}_2(\xi) \tag{25}$$

and

$$\underline{\kappa}_2(\xi) \leq \frac{1}{\widehat{w}_2(\xi)-1}. \tag{26}$$

The bound in (26) is a refinement of (24) using (4). Note that the bound in (25) is always non-negative (see Jarník [11]). We enclose a few observations on Problem 1 for general n:

• If $\underline{\kappa}_n(\xi) = 0$ for all ξ, this would imply Wirsing's problem has an affirmative answer for given n. In fact $\underline{\kappa}_n(\xi) = 0$ implies $w_n^*(\xi) \geq \widehat{w}_n(\xi) \geq n$. While this is audacious to conjecture in general, for $n = 2$, Theorem 3.6 and the results from [9, 15] on which it is based indicate that it may be true.

• Constructing ξ with $\underline{\kappa}_n(\xi) > 0$ does not seem easy either, as *all* best approximations of large norm must have untypically small derivative. Moreover, it probably forces $\widehat{w}_n(\xi) > n$; see Conjecture 1 below.

• An analogue of Jarník's estimate on the logarithmic ratio of best approximation norms, as in the proof of Theorem 3.6 below, for higher dimension would generalize Theorem 3.6. However, this still seems unknown. While the results from [14] suggest such an effective estimate, it does not seem to be settled. Nevertheless, in view of Conjecture 1, this suggests that $n-1$ can be improved. Moreover, it seems that Wirsing's method and results [26] do not immediately imply anything better than $n-1$ either, for example if $w_n(\xi) = 2n-1$ (or larger).

The next metrical result on the limit of κ is an easy consequence of a claim from [7].

THEOREM 3.7. *Let $n \geq 2$ be an integer and ξ be a transcendental real number. If*

$$w_n(\xi) = n$$

then

$$\lim_{X\to\infty} \kappa(n,\xi,X) = 0. \tag{27}$$

In particular, we have (27) *for almost all ξ with respect to Lebesgue measure.*

Proof. We use the estimate of Bugeaud and Laurent [7, Theorem 2.1]

$$\widehat{w}_n^*(\xi) \geq \frac{w_n(\xi)}{w_n(\xi)-n+1}. \tag{28}$$

It follows that

$$\begin{aligned} \overline{\kappa}_n(\xi) &= \limsup_{X\to\infty} (w(n,\xi,X) - w^*(n,\xi,X)) \\ &\leq \limsup_{X\to\infty} w(n,\xi,X) - \liminf_{X\to\infty} w^*(n,\xi,X) \\ &= w_n(\xi) - \widehat{w}_n^*(\xi) \leq w_n(\xi) - \frac{w_n(\xi)}{w_n(\xi)-n+1}. \end{aligned} \tag{29}$$

For $w_n(\xi) = n$ the right hand side vanishes, thus together with (9) we deduce

$$0 \leq \underline{\kappa}_n(\xi) \leq \overline{\kappa}_n(\xi) \leq 0,$$

so the limit is 0. Finally, for the metrical claim we conclude by Sprindžuk's famous result [25]. ∎

More generally, combining (29) and (15) enables us to get an estimate of (20) from above for $\lambda \geq 0$, complementary to Corollary 3.2 where $\lambda = n-1$. For large λ, only small improvements compared to the bounds obtained via trivially estimating $\widehat{w}_n^*(\xi) \geq 1$ in place of (28) and using (15) are achieved. In particular, for $\lambda = n-1$ the bound becomes significantly weaker than the one from Corollary 3.2.

The proof of Theorem 3.7 and the dual inequality

$$w_n^*(\xi) \geq \widehat{w}_n(\xi)/(\widehat{w}_n(\xi)-n+1)$$

to (28) established in [7] suggest the following may be true.

CONJECTURE 1. We have

$$\underline{\kappa}_n(\xi) \le \widehat{w}_n(\xi) - \frac{\widehat{w}_n(\xi)}{\widehat{w}_n(\xi) - n + 1}. \tag{30}$$

In particular,

$$\widehat{w}_n(\xi) = n \quad \text{implies} \quad \underline{\kappa}_n(\xi) = 0.$$

However, even the last special claim is not as straightforward as Theorem 3.7, and we have not been able to prove it rigorously. Note also that $\widehat{w}_n(\xi) = n$ does not imply the stronger claim $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) = 0$, as any Liouville number ξ satisfies (12). Combining (26) with our unproved estimate (30) for $n = 2$ gives a conditional bound

$$\underline{\kappa}_2(\xi) \le \frac{1}{\sqrt{2}}.$$

4. Preparatory lemmas. We need some lemmas. The first is essentially due to Wirsing [26, Hilfssatz 4].

LEMMA 4.1 (Wirsing). *Let ξ be a transcendental real number and $n \ge 1$. Let $\varepsilon > 0$. Let P be an integer polynomial of degree at most n and large enough height, and assume for some $w > 0$ we have*

$$|P(\xi)| \le H_P^{-w}.$$

Then there exists an irreducible divisor Q of P such that

$$|Q(\xi)| \le H_Q^{-w+\varepsilon}.$$

We can assume $H_Q \to \infty$ as $H_P \to \infty$.

We note that this easily implies (10). The next result is part of Lemma A.8 in Bugeaud's book [5], attributed to Feldman and Diaz there.

LEMMA 4.2 (Feldman). *If P is a separable integer polynomial of degree at most n then it has a root α such that*

$$|\xi - \alpha| \ll_n |P(\xi)| \cdot H_P^{n-2}.$$

Thus, without restriction to separable polynomials, the claim (11) would be obvious. The next well-known claim is essential to the proof of Lemma 4.1 above.

LEMMA 4.3 (Gelfond's Lemma). *For any polynomials P, Q, R with $P = QR$ of degree at most n we have*

$$H_P \asymp_n H_Q H_R.$$

THEOREM 4.4 (Liouville inequality; see [5, Corollary A.2]). *If α is real algebraic of degree m and β is real algebraic of degree n and they have different minimal polynomials then*

$$|\alpha - \beta| \gg_{m,n} H(\alpha)^{-n} H(\beta)^{-m}.$$

Theorem 4.5 (Bugeaud, Schleischitz [8, Theorem 2.3]). *For any positive integers m,n and any real transcendental number ξ we have*

$$\min\{w_m(\xi), \widehat{w}_n(\xi)\} \leq m+n-1.$$

The next result is combination of [19, Theorems 1.12, 1.6].

Theorem 4.6 (Schleischitz). *Let $n\geq 1$ be an integer and ξ be a transcendental real number. If $w_1(\xi)\geq n$ then $\widehat{w}_n(\xi)=n$. If $\lambda_n(\xi)>1$ then $w_1(\xi)>2n-1\geq n$, so again $\widehat{w}_n(\xi)=n$.*

For completeness, we end this section with a well-known fact, although not explicitly used in the proofs below; see for example [23, Proposition 2.7] for a proof.

Proposition 4.7. *For any integer polynomial P of degree at most n, any of its roots α satisfies*

$$|\xi-\alpha| \gg_{n,\xi} H(\alpha)^{-1}|P(\xi)|.$$

Proposition 4.7 immediately proves (5), (9).

For the proofs below recall that $P=P_X$ minimizes $|P(\xi)|$ among all integer polynomials of degree at most n and height at most X. It satisfies

$$|P_X(\xi)| \leq X^{-\widehat{w}_n(\xi)+o(1)}, \qquad X\to\infty.$$

5. Proof of Theorem 3.1. Let $\varepsilon>0$. Let X be any large parameter and $P=P_X$ of height $H_P\leq X$ be the corresponding best approximation polynomial. For simplicity write $\tilde{w}=w(n,\xi,X)$. Define $\sigma=\sigma_X$ implicitly by

$$X=H_P^{\sigma}, \qquad \sigma=\sigma_X\geq 1.$$

Then

$$|P(\xi)|=H_P^{-\sigma\tilde{w}}.$$

By Wirsing's Lemma 4.1 and again assuming X was chosen sufficiently large, there is an irreducible (thus separable) divisor $Q=Q_X$ of P such that

$$|Q(\xi)|=H_Q^{-\sigma w}, \qquad w\geq \tilde{w}-\varepsilon.$$

We can assume $H_Q>1$ and therefore may define $\tau=\tau_X$ implicitly via

$$H_Q^{\tau}=X, \qquad \tau=\tau_X\geq 1-\varepsilon.$$

The lower bound is obtained from Gelfond's Lemma 4.3, which in fact implies the stronger claim $H_Q\ll_n H_P\leq X$. Let $m=m_X=\deg Q\leq n$ and $r=n-m=\deg R$, where $P=QR$.

We may assume

$$m>k=\left\lfloor\frac{n}{2}\right\rfloor \tag{31}$$

for large enough X. Indeed, if otherwise $m \leq k$ for arbitrarily large X then for some modified small $\varepsilon > 0$ we get

$$w_k(\xi) \geq w_m(\xi) \geq \sigma \tilde{w} - \varepsilon \geq \tilde{w} - \varepsilon,$$

which further implies

$$\tilde{w} \leq w_k(\xi) + \varepsilon \leq \max\{w_k(\xi), n-1\} + w^*(n,\xi,X) + \varepsilon,$$

and the claim follows by subtracting $w^*(n,\xi,X)$ from both sides as $X \to \infty$ and thus $\varepsilon \to 0$.

To simplify the presentation, from now on let us consider a sequence of P and assume $H_P \to \infty$, implying $X \to \infty$, and use the $o(1)$ notation as $X \to \infty$. Similar to the proof of Theorem 3.6, Feldman's Lemma 4.2 implies that the polynomial Q has a root α such that

$$\begin{aligned} |\alpha - \xi| &\ll_n H(\alpha)^{-1}|Q(\xi)|H_Q^{m-1} \\ &= H(\alpha)^{-1}X^{-\sigma w/\tau}X^{(m-1)/\tau} = H(\alpha)^{-1}X^{-(\sigma w - m + 1)/\tau}. \end{aligned}$$

Hence

$$w^*(n,\xi,X) \geq \frac{\sigma w - m + 1}{\tau} - o(1), \qquad X \to \infty. \tag{32}$$

On the other hand, if we assume $H_R \to \infty$ as $X \to \infty$, we get

$$|P(\xi)| = |Q(\xi)| \cdot |R(\xi)| \geq X^{-\sigma w/\tau} \cdot H_R^{-w_r(\xi)-o(1)}, \qquad X \to \infty.$$

If otherwise H_R is bounded on a subsequence as $X \to \infty$ then by transcendence of ξ we get stronger estimates from $|R(\xi)| \gg 1$. Now since by Gelfond's Lemma 4.3 we know that

$$H_R \asymp_n \frac{H_P}{H_Q} = X^{1/\sigma} \cdot X^{-1/\tau} = X^{(\tau-\sigma)/(\sigma\tau)}$$

we infer

$$w(n,\xi,X) \leq \frac{\sigma w}{\tau} + w_r(\xi)\frac{\tau - \sigma}{\sigma\tau} + o(1), \qquad X \to \infty. \tag{33}$$

Note that

$$r = n - m \leq n - k - 1 \leq k$$

by (31) and definition of k. Together with (3) this implies

$$w_r(\xi) \leq w_k(\xi).$$

Hence comparing the bounds (32), (33) we get

$$\begin{aligned} \kappa(n,\xi,X) = w(n,\xi,X) - w^*(n,\xi,X) &\leq w_r(\xi)\frac{\tau-\sigma}{\sigma\tau} + \frac{m-1}{\tau} + o(1) \\ &\leq w_k(\xi)\frac{\tau-\sigma}{\sigma\tau} + \frac{n-1}{\tau} + o(1). \end{aligned} \tag{34}$$

Here we consider $\tau \geq 1$, $\sigma \geq 1$ absolute so that the expression is maximized over all pairs ≥ 1. The right hand side clearly decays in σ, hence we can put

$\sigma = 1$ and get

$$\kappa(n,\xi,X) \le \frac{1}{\tau}((\tau-1)w_k(\xi)+n-1)+o(1) = w_k(\xi)+\frac{n-1-w_k(\xi)}{\tau}+o(1).$$

If $n-1 \ge w_k(\xi)$ then the bound decreases in τ, hence we may put $\tau = 1$ for all large X to get the bound $n-1$. If otherwise $n-1 \le w_k(\xi)$ then we have the bound $w_k(\xi)$. Since X was arbitrary, the stronger claim (14) follows as $X \to \infty$ and (13) is implied by (7).

6. Proof of Theorem 2.2

6.1. Another lemma. We first verify

LEMMA 6.1. *Assume that for given n, ξ and all large X the polynomial P_X is separable. Then* (11) *holds, in fact $\overline{\kappa}_n(\xi) \le n-1$. More generally, for every X with this property we have*

$$\kappa(n,\xi,X) \le n-1.$$

Proof. We may apply Lemma 4.2 to P_X. Proceeding as in the proof of Theorem 3.1 (or Theorem 3.6) with $P = Q$ and thus $\sigma = \tau$, and writing $w = w(n,\xi,X)$, we see that the polynomial P_X of degree $m \le n$ has a root α with

$$|\xi-\alpha| \ll_n H(\alpha)^{-1}X^{-(\sigma w-m+1)/\sigma}.$$

In other words,

$$w^*(n,\xi,X) \ge \frac{\sigma w-m+1}{\sigma} \ge w-\frac{n-1}{\sigma} \ge w-n+1.$$

The last claim of the lemma is now immediate, and the first follows by (7) if the separability hypothesis holds for all large X. ∎

We remark that thanks to this lemma we can improve the bound of Corollary 3.2 for very small n.

COROLLARY 6.2. *We have*

$$0 = \dim_H(\varOmega_2) \le \dim_H(\varGamma_2) \le \frac{2}{3}, \qquad 0 = \dim_H(\varOmega_3) \le \dim_H(\varGamma_3) \le \frac{1}{2}.$$

From the argument for Corollary 3.2 we would get the bounds 1 and 2/3 respectively.

Proof of Corollary 6.2. All but the upper bounds for $\dim_H(\varGamma_2)$ and $\dim_H(\varGamma_3)$ are clear by Theorem 2.1. If for all large X the best approximation P_X is separable, then Lemma 6.1 implies $\overline{\kappa}_n(\xi) \le n-1$, so we can assume the opposite. But for $n \le 3$, this means that P_X decomposes into linear factors for certain arbitrarily large X. This implies $w_1(\xi) \ge n$ by Wirsing's Lemma 4.1, and the desired estimates follow from (15) applied to $m = 1$. ∎

6.2. Proof of Theorem 2.2. If $\widehat{w}_n(\xi)=n$ then by (6) trivially

$$\widehat{w}_n^*(\xi)\geq 1=n-n+1=\widehat{w}_n(\xi)-n+1. \tag{35}$$

Now assume $\widehat{w}_n(\xi)>2n-3$. By Theorem 2.1 we may assume $n\geq 3$ (in fact $n\geq 4$). Assume P_X is not separable for some large X. Then it has the square of a polynomial as a factor. But by $n\geq 3$ this means that any irreducible factor has degree at most $\max\{\lfloor n/2\rfloor, n-2\}=n-2$. Thus Wirsing's Lemma 4.1 and (4) imply

$$w_{n-2}(\xi)\geq \widehat{w}_n(\xi)\geq n.$$

However by Theorem 4.5 this implies

$$\widehat{w}_n(\xi)=\min\{\widehat{w}_n(\xi), w_{n-2}(\xi)\}\leq n+(n-2)-1=2n-3.$$

Thus, assuming the reverse inequality $\widehat{w}_n(\xi)>2n-3$ we find that all best approximation polynomials of large norm are separable. Consequently, the claim follows from Lemma 6.1.

Assume in turn $\widehat{w}_n^*(\xi)\geq n-2$. Then the claim is obviously true if $\widehat{w}_n(\xi)\leq (n-2)+(n-1)=2n-3$. If this inequality is false then the implication follows from (ii).

Assume (iv) holds. If $\lambda_n(\xi)\leq 1/(n-2)$ then by the main result of [18] we have

$$\widehat{w}_n^*(\xi)\geq \frac{1}{\lambda_n(\xi)}\geq n-2,$$

which is hypothesis (iii). If $\lambda_n(\xi)>1$, then it follows from Theorem 4.6 that $\widehat{w}_n(\xi)=n$, so we have reduced this case to (i). This also handles the case of (v).

Finally, for the conclusion from (vi), combining (29) and (7) yields

$$\widehat{w}_n(\xi)-\widehat{w}_n^*(\xi)\leq \overline{\kappa}_n(\xi)\leq w_n(\xi)-\frac{w_n(\xi)}{w_n(\xi)-n+1}.$$

A short calculation shows that the right hand side is at most $n-1$ if $w_n(\xi)$ is bounded as in (vi).

Remark 2. While in the above proof we have used Theorem 2.1 in the deduction from some conditions, it was not applied when concluding from (v). This is important because we will use this very implication from Theorem 2.2(v) in the deduction of Theorem 2.1 below, so there is no circular reasoning in our argument.

7. Proof of Theorem 2.1

7.1. Proof for $n\leq 3$. Let $n\leq 3$. Then if P_X is not separable, it decomposes into linear factors. If this happens for certain arbitrarily large X, by Wirsing's Lemma 4.1 and (4) this implies that

$$w_1(\xi)\geq \widehat{w}_n(\xi)\geq n.$$

We conclude via Theorem 2.2(v) as mentioned in Remark 2. If otherwise there is no such sequence of X that tends to infinity, then (11) holds as well by Lemma 6.1.

7.2. Towards the proof for $n \in \{4,5\}$. A preparatory lemma. The cases $n=4$, $n=5$ are considerably more complicated. We prove an auxiliary fact for general n.

Let us first fix some notation. Let $X>1$ be any large number and $P=P_X$ the corresponding best approximation polynomial. Write

$$P=\prod_{i=1}^{\ell} Q_i^{\alpha_i}, \qquad \alpha_i \in \mathbb{N},$$

for its factorization into irreducible polynomials. We may assume $H_i := H(Q_i)>1$ for all i, since if otherwise some fixed Q occurs as a factor of P_X for arbitrarily large X, in the argument below we can instead of P_X consider P_X/Q, of comparable height and value at ξ, using for the latter the fact that ξ is transcendental. We omit the details. Let further

$$d_i = \deg Q_i \geq 1.$$

Note that

$$\sum_{i=1}^{\ell} \alpha_i d_i \leq n. \tag{36}$$

Let $\gamma_i>0$ be defined by

$$|Q_i(\xi)| = H_i^{-\gamma_i}.$$

Define $\beta_i>0$, $1\leq i\leq \ell$, by

$$H_i = X^{\beta_i}.$$

Gelfond's Lemma 4.3 implies $\prod X^{\alpha_i\beta_i} = \prod H_i^{\alpha_i} \ll_n H_P \leq X$ and thus

$$\sum_{i=1}^{\ell} \alpha_i\beta_i \leq 1+o(1). \tag{37}$$

All error terms are understood as $X\to\infty$. Now since

$$|P(\xi)| = \prod_{i=1}^{\ell} |Q_i(\xi)|^{\alpha_i} = \prod_{i=1}^{\ell} H_i^{-\alpha_i\gamma_i} = X^{-\sum \alpha_i\beta_i\gamma_i},$$

we get

$$w(n,\xi,X) \leq \sum_{i=1}^{\ell} \alpha_i\beta_i\gamma_i + o(1). \tag{38}$$

On the other hand, considering the separable polynomial

$$Q := \prod_{i=1}^{\ell} Q_i,$$

which satisfies

$$\deg Q = \sum_{i=1}^{\ell} d_i, \qquad |Q(\xi)| = \prod_{i=1}^{\ell} |Q_i(\xi)| = \prod_{i=1}^{\ell} H_i^{-\gamma_i},$$

we find via Lemma 4.2 and using $H_Q \asymp_n \prod_{i=1}^{\ell} H_i$ by Gelfond's Lemma 4.3 that Q has a root α with

$$|\alpha - \xi| \ll_n H(\alpha)^{-1} \cdot \prod_{i=1}^{\ell} H_i^{-\gamma_i+d_i-1} = H(\alpha)^{-1} \cdot X^{-\sum_{i=1}^{\ell} \beta_i(\gamma_i - d_i + 1)},$$

thus

$$w^*(n, \xi, X) \geq \sum_{i=1}^{\ell} \beta_i \cdot (\gamma_i - d_i + 1) - o(1). \tag{39}$$

Comparing (38) and (39) we get

$$\kappa(n, \xi, X) \leq \sum_{i=1}^{\ell} \alpha_i \beta_i \gamma_i - \sum_{i=1}^{\ell} \beta_i \cdot (\gamma_i - d_i + 1) + o(1).$$

We have thus proved the following local lemma.

LEMMA 7.1. *For any large X with induced parameters $\ell, d_i, \alpha_i, \beta_i, \gamma_i$ we have*

$$\kappa(n, \xi, X) \leq \sum_{i=1}^{\ell} (\alpha_i - 1)\beta_i\gamma_i + \beta_i(d_i - 1) + o(1) \qquad \text{as } X \to \infty.$$

Thus, if

$$\sum_{i=1}^{\ell} (\alpha_i - 1)\beta_i\gamma_i + \beta_i(d_i - 1) \leq -1 + \sum_{i=1}^{\ell} \alpha_i d_i \leq n - 1 \tag{40}$$

then

$$\kappa(n, \xi, X) \leq n - 1 + o(1) \qquad \text{as } X \to \infty. \tag{41}$$

REMARK 3. A similar argument can be applied to $Q := \prod_{j \in J} Q_j$ for any subset $J \subseteq \{1, \ldots, \ell\}$. However, it turns out one may restrict oneself to the full set $\{1, \ldots, \ell\}$.

We prove the cases $n = 4$ and $n = 5$ by using Lemma 7.1 and Theorems 4.5 and 4.6. By Lemma 6.1 we only have to consider parameters X for which P_X is not separable. Moreover, we can assume that for all large X,

P_X does not decompose into linear factors, as otherwise by Wirsing's Lemma 4.1 we have $w_1(\xi) \geq \widehat{w}_n(\xi) \geq n$ and the claim follows from Theorem 2.2(v).

We may assume P_X has exact degree n. Otherwise we could multiply it by the corresponding power of the variable T to get $\tilde{P}(T) = T^t P_X$ for $t = n -$ deg P_X, which keeps the height unaffected: $H_{\tilde{P}} = H_P$, and decreases its value at ξ, i.e. $|\tilde{P}(\xi)| < |P(\xi)|$ (as we may assume $\xi \in (0,1)$), contradicting the definition of P_X.

We will use the notation of the present §7.2 in what follows.

7.3. Proof for $n = 4$. First let $n = 4$. In view of the above restrictions, for X any large parameter, it remains to consider two cases.

Case A: We have

$$\ell = 2, \qquad P_X = Q_1^2 Q_2, \qquad d_1 = 1, \qquad d_2 = 2,$$

that is, Q_1 linear and Q_2 irreducible quadratic. Then (37) gives

$$2\beta_1 + \beta_2 \leq 1 + o(1), \tag{42}$$

in particular

$$\beta_1 \leq 1/2 + o(1).$$

The criterion (40) from Lemma 7.1 yields the sufficient condition

$$\beta_1\gamma_1 + \beta_2 \leq n - 1 = 3.$$

Using (42) this can be relaxed to $(\gamma_1 - 2)\beta_1 + 1 + o(1) \leq 3$ or equivalently

$$\gamma_1 \leq 2 + \frac{2 - o(1)}{\beta_1} - o(1).$$

In particular, $\gamma_1 \leq 6 - \varepsilon$ for small enough $\varepsilon > 0$ suffices to have (41) for the corresponding (large) X of Case A. Note that by definition of $\widehat{w}_n(\xi)$ this further implies

$$\widehat{w}_n(\xi) - w^*(n, \xi, X) \leq n - 1 + o(1), \qquad X \to \infty, \tag{43}$$

for those X.

However, if we let $\varepsilon = 1/2$ and assume on the contrary $\gamma_1 > 6 - \varepsilon$ for arbitrarily large X inducing Case A then $w_1(\xi) \geq 6 - 1/2 = 5.5 > 4 = n$, implying (11) by Theorem 2.2(v). In particular, we again have (43). So (43) holds for all X inducing Case A.

Case B: We have

$$P_X = Q_1^2, \qquad d_1 = 2.$$

Then Q_1 is irreducible quadratic and

$$\ell = 1, \qquad \alpha_1 = \alpha = 2, \qquad 0 < \beta_1 = \beta \leq 1/2 + o(1), \qquad d_1 = d = 2, \qquad \gamma_1 = \gamma.$$

Now the sufficient criterion (40) of Lemma 7.1 becomes

$$\beta(\gamma+1) \leq 3$$

or

$$\gamma \leq \frac{3}{\beta} - 1. \tag{44}$$

In particular,

$$\gamma \leq 5 - o(1) \tag{45}$$

suffices to prove

$$\kappa(4,\xi,X) = w(4,\xi,X) - w^*(4,\xi,X) \leq 3 + o(1) = n - 1 + o(1)$$

for X in question. We may thus assume otherwise:

$$\gamma > 5 + \varepsilon = n + d - 1 + \varepsilon > n + d - 1, \qquad \varepsilon > 0.$$

(While there is a small gap in the converse of (45), indeed if contrary to our claim (11) we have $w(4,\xi,X) - w^*(4,\xi,X) = n - 1 + \delta$ for some ξ and a strictly positive $\delta = \delta(\xi) > 0$ for certain large X uniformly, then the above argument easily implies we may take $\varepsilon = \varepsilon(\delta) = \delta/2 > 0$; we omit the details.) If this happens for arbitrarily large X, then by Theorem 4.5 we have

$$\widehat{w}_n(\xi) = \min\{\widehat{w}_n(\xi), w_d(\xi)\} \leq n + d - 1 = n + 1.$$

So it suffices to show

$$w^*(n,\xi,X) \geq 2 - o(1), \qquad X \to \infty, \tag{46}$$

for X inducing Case B, to deduce that for such X again (43) holds. Assume this is true. As we have obtained (43) for any large X inducing Case A as well, we can pass to the lower limit to indeed deduce $\widehat{w}_n(\xi) - \widehat{w}_n^*(\xi) \leq (n+1) - 2 = n - 1$, i.e. (11).

However, for X inducing Case B, by Feldman's Lemma 4.2 the quadratic $Q_1 = Q$ has a root α with

$$|\alpha - \xi| \ll H_Q^{-\gamma} = H(\alpha)^{-1} H_Q^{-(\gamma-1)} = H(\alpha)^{-1} X^{-\beta(\gamma-1)}.$$

Hence if

$$\beta(\gamma - 1) \geq 2 - o(1), \qquad X \to \infty,$$

or equivalently

$$\gamma \geq 1 + \frac{2 - o(1)}{\beta}, \tag{47}$$

we have (46) and we are done. So, combining the criteria (44) and (47), we see that the only problematic case is

$$\frac{3}{\beta} - 1 < \gamma < 1 + \frac{2 - \delta}{\beta},$$

for some fixed $\delta > 0$ and certain arbitrarily large X inducing Case B. But this implies $\beta > 1/2 + \delta/2$, a contradiction for large X.

REMARK 4. We stress again that Theorems 4.5 and 4.6 were vital ingredients of the proofs of both Cases A and B. Indeed, the stronger claim $\kappa(4, \xi, X) \leq n-1+o(1)$ is false in either case in general, by a generalization of Theorem 3.3. The same will be true for $n = 5$ below.

7.4. Proof for $n = 5$. We now prove the claim for $n = 5$. Here by the initial argument as for $n = 4$, it remains to consider four cases:

CASE A: We have

$$P_X = Q_1^2 Q_2, \qquad d_1 = 1, \qquad d_2 = 3.$$

Then

$$2\beta_1 + \beta_2 \leq 1 + o(1),$$

in particular

$$\beta_1 \leq \frac{1}{2} + o(1),$$

and (40) becomes

$$\beta_1\gamma_1 + 2\beta_2 \leq n - 1 = 4.$$

So a sufficient condition is

$$\gamma_1 \leq \frac{2 - o(1)}{\beta_1} + 4.$$

Hence

$$\gamma_1 \leq 7.5$$

suffices for large X. But if for arbitrarily large such X we have

$$\gamma_1 > 7.5 > 5 = n$$

then $w_1(\xi) \geq n$ and we conclude (43) for X in question by the same arguments as for $n = 4$.

CASE B: We have

$$P_X = Q_1^3 Q_2, \qquad d_1 = 1, \qquad d_2 = 2.$$

Then

$$3\beta_1 + \beta_2 \leq 1 + o(1),$$

thus

$$\beta_1 \leq \frac{1}{3} + o(1),$$

and (40) becomes

$$2\beta_1\gamma_1 + \beta_2 \leq 4 = n - 1.$$

Equivalently,

$$\beta_1(2\gamma_1 - 3) \leq 3 - o(1), \qquad \gamma_1 \leq \frac{\frac{3-o(1)}{1/3+o(1)} + 3}{2} = 6 - o(1),$$

so

$$\gamma_1 \leq 6 - \varepsilon, \qquad \varepsilon > 0,$$

suffices for large enough X. But if for arbitrarily large X and $\varepsilon = 1/2$ we have

$$\gamma_1 > 6 - \varepsilon = 5.5 > 5 = n$$

then again $w_1(\xi) \geq n$ and we conclude that (43) holds.

Case C:

$$P_X = Q_1^2 Q_2 Q_3, \qquad d_1 = d_2 = 1, \qquad d_3 = 2.$$

Then

$$2\beta_1 + \beta_2 + \beta_3 \leq 1 + o(1),$$

in particular

$$\beta_1 \leq \frac{1}{2} + o(1),$$

and (40) yields the criterion

$$\beta_1\gamma_1 + \beta_3 \leq 4 = n - 1,$$

hence as $\beta_2 \geq 0$ we get

$$\beta_1(\gamma_1 - 2) + 1 \leq 4, \qquad \gamma_1 \leq \frac{3}{\beta_1} + 2$$

as a sufficient condition. Now if conversely for arbitrarily large X we have

$$\gamma_1 > \frac{3}{\beta_1} + 2 - o(1) \geq 8 - o(1),$$

then again $w_1(\xi) \geq 8 - \varepsilon = 7.5 > n$ and we conclude that (43) holds.

Case D: We have

$$P_X = Q_1^2 Q_2, \qquad d_1 = 2, \qquad d_2 = 1.$$

Then again

$$2\beta_1 + \beta_2 \leq 1 + o(1),$$

in particular

$$\beta_1 \leq \frac{1}{2} + o(1),$$

and (40) becomes

$$\beta_1\gamma_1 + \beta_1 = \beta_1(\gamma_1 + 1) \leq n - 1 = 4$$

so a sufficient condition is

$$\gamma_1 \leq \frac{4}{\beta_1} - 1.$$

Hence

$$\gamma_1 \le 7 - \varepsilon,$$

so for uniform $\varepsilon > 0$ and large X it suffices to prove

$$w(5, \xi, X) - w^*(5, \xi, X) \le n - 1 + o(1).$$

But if for $\varepsilon = 1/2$ and arbitrarily large X we have

$$\gamma_1 > 7 - \varepsilon = 6.5 > 6 = n + d_1 - 1,$$

then by Theorem 4.5 we see that

$$\widehat{w}_n(\xi) = \min\{\widehat{w}_n(\xi), w_{d_1}(\xi)\} \le n + d_1 - 1 = n + 1.$$

So we only have to prove that

$$w^*(5, \xi, X) \ge 2 - o(1), \qquad X \to \infty,$$

for X inducing Case D, to conclude first (43) and finally (11) by the same argument as for $n = 4$. By Feldman's Lemma 4.2, for this again with the same argument as in Case B for $n = 4$ it suffices to have

$$\gamma_1 \ge 1 + \frac{2}{\beta_1}.$$

So the only bad case is

$$\frac{4}{\beta_1} - 1 < \gamma_1 < 1 + \frac{2}{\beta_1}.$$

But this implies $\beta_1 > 1$, a contradiction.

8. Proof of Theorem 3.3. We only need to prove (16); then (17) is just the case $k = n$ and (18) follows in turn for $w_1(\xi) = \infty$. Let $Q(T) = aT - b$ with $a > 0$ and $H_Q > 1$ be very small at ξ, say

$$|Q(\xi)| = H_Q^{-\lambda}$$

for some $\lambda > n + k - 1$. Clearly we can assume $|b| \asymp_\xi a$, so this implies

$$|\xi - b/a| = \frac{|Q(\xi)|}{a} \asymp_\xi H_Q^{-\lambda-1}. \tag{48}$$

Consider the parameter

$$X = H_{Q^k} \asymp_n H_Q^k,$$

where we have used Gelfond's Lemma 4.3. Consider $R = Q^k$, which has degree $k \le n$ and $H_R = X$. Moreover,

$$|R(\xi)| = H_Q^{-k\lambda} \ll_{n,\lambda} X^{-\lambda}.$$

Hence

$$w(n, \xi, X) \ge \lambda - o(\lambda)$$

as $H_Q \to \infty$.

On the other hand, any other algebraic number η of degree at most n and height at most X has distance from b/a at least

$$|\eta - b/a| \gg_n X^{-1}H_Q^{-n} \gg_n X^{-1-n/k}$$

by the Liouville inequality (Theorem 4.4). So by $\lambda > n+k-1$, this implies by (48) and the triangle inequality for large enough X that

$$|\xi - \eta| \geq |b/a - \eta| - |\xi - b/a| \geq \frac{1}{2}|b/a - \eta| \gg_n X^{-1-n/k}.$$

So the smallest value comes from b/a, the root of Q, for which we use (48), and estimating its right hand side as $\gg H(b/a)^{-1}X^{-\lambda/k-o(\lambda)}$ we get

$$w^*(n,\xi,X) \leq \frac{\lambda}{k} + o(\lambda).$$

Combining the two bounds we get

$$\kappa(n,\xi,X) \geq \lambda - \frac{\lambda}{k} - o(\lambda) = \left(1 - \frac{1}{k} - o(1)\right)\lambda.$$

As λ can be chosen arbitrarily close to $w_1(\xi)$ and by assumption $w_1(\xi) > n+k-1$, we can find infinitely many such Q and thus arbitrarily large X, and the claim follows.

9. Proof of Theorem 3.5. Let us first recall that $\widehat{w}_2(\xi) = 2$ is implied by the hypothesis $w_1(\xi) > 3$, as mentioned in the paragraph following Remark 1.

For the leftmost inequality of (22), we first prove an auxiliary lemma for approximation to a single number using a standard determinant argument.

Lemma 9.1. *Let $P(T) = cT + d$ of height $H_P > 1$ be a linear integer polynomial best approximation for $n = 1$ (essentially this means $-d/c$ is a convergent to ξ). Let $\lambda \geq 1$ be determined by*

$$|P(\xi)| = H_P^{-\lambda}.$$

Let $R(T) = aT + b$ be the linear integer polynomial that minimizes $|Q(\xi)|$ among all linear integer polynomials $Q(T)$ of height $H_Q \leq C \cdot H_P^{\lambda}$ for fixed $C \in \left(0, \frac{1}{2}\right)$ that are not a scalar multiple of P. Then

$$|R(\xi)| \asymp_C H_P^{-1}.$$

The implied constants are absolute.

Proof. By well-known estimates for continued fractions, for

$$|R(\xi)| \ll H_P^{-1}$$

it suffices to take $R(T) = Q(T) = aT + b$ the best approximation polynomial for the parameter $Y = CH_P$. This is clearly not a multiple of P as $C < 1$ and P has a primitive coefficient vector since it is a best approximation polynomial. Moreover, as $\lambda \geq 1$, clearly $H_Q \leq C \cdot H_P \leq C \cdot H_P^{\lambda}$, and by Dirichlet's Theorem and the definition of R indeed

$$H_R \leq C \cdot H_P, \qquad |R(\xi)| \leq |Q(\xi)| \leq (C \cdot H_P)^{-1} \ll_C H_P^{-1}.$$

(Usually R corresponds to the convergent $-b/a$ of ξ preceding $-d/c$ in the continued fraction algorithm.)

For the reverse inequality, considering the matrix formed by the integer coefficient vectors of P, R shows by linear independence and multi-linearity of the determinant that

$$1 \leq \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & R(\xi) \\ c & P(\xi) \end{vmatrix} \leq |P(\xi)| H_R + |R(\xi)| H_P \leq C + |R(\xi)| H_P,$$

and the claim follows after a short rearrangement using $0 < C < 1/2$. ■

Now we prove the theorem. Let P, R be as in Lemma 9.1, for arbitrary but fixed $C \in (0, 1/2)$, and assume $\lambda > 3$. By the lemma,

$$|P(\xi)| = H_P^{-\lambda}, \qquad |R(\xi)| \asymp H_P^{-1}.$$

Hence the polynomials

$$V_1 := P^2, \qquad V_2 := PR$$

are quadratic and satisfy

$$\max_{i=1,2} H_{V_i} \ll H_P^2, \qquad |V_1(\xi)| = H_P^{-2\lambda}, \qquad |V_2(\xi)| \asymp H_P^{-\lambda-1}. \tag{49}$$

Take the parameter

$$X = H_P^{\lambda-1-\varepsilon}$$

for small $\varepsilon \in (0, (\lambda-3)/2)$ and consider all linear or quadratic integer polynomials V_3 of height at most $H_{V_3} \leq X$. Clearly any polynomial in span $\{V_1\}$ is not separable, hence for our exponent we can restrict to the complementary set of polynomials. We will show that any such $V_3 \notin$ span $\{V_1\}$ with $H_{V_3} \leq X$ satisfies

$$|V_3(\xi)| \gg H_P^{-\lambda-1}. \tag{50}$$

If this is true then

$$w_{\mathrm{sep}}(2, \xi, X) \leq \frac{\lambda+1+o(1)}{\lambda-1-\varepsilon} = 1 + \frac{2+\varepsilon+o(1)}{\lambda-1-\varepsilon}.$$

The claim (22) of the theorem follows with $H_P \to \infty$ as we may take ε arbitrarily small and λ arbitrarily close to $w_1(\xi)$. The latter specialization for Liouville numbers follows as then the upper bound becomes 1, while by considering linear polynomials only we get the reverse inequality, to conclude

$$1 \geq \widehat{w}_{2,\mathrm{sep}}(\xi) \geq \widehat{w}_{2,\mathrm{irr}}(\xi) \geq \widehat{w}_{1,\mathrm{irr}}(\xi) = \widehat{w}_1(\xi) = 1.$$

So the inequalities must be equalities. To prove (50), we distinguish two cases.

CASE 1: $V_3 \in \operatorname{span}\{V_1, V_2\} \setminus \operatorname{span}\{V_1\}$. This means $V_3 = PS$ with linear $S \notin \operatorname{span}\{P\}$. Moreover, by Gelfond's Lemma 4.3 we have $H_S \ll H_{V_3}/H_P \leq X/H_P = H_P^{\lambda-2-\varepsilon}$, hence we may assume $H_S < C \cdot H_P^{\lambda}$ if we assume H_P is large enough. By Lemma 9.1 we conclude

$$|S(\xi)| \gg H_P^{-1}$$

and thus

$$|V_3(\xi)| = |P(\xi)| \cdot |S(\xi)| \gg H_P^{-\lambda-1},$$

so (50) holds.

CASE 2: $V_3 \notin \operatorname{span}\{V_1, V_2\}$. Assume contrary to (50) that some linear or quadratic integer polynomial V_3 (in fact it is not hard to show that V_3 must be irreducible quadratic) outside the span of $\{V_1, V_2\}$ satisfies the estimates

$$H_{V_3} \leq H_P^{\lambda-1-\varepsilon}, \qquad |V_3(\xi)| \leq H_P^{-\lambda-1}. \tag{51}$$

We proceed similarly to the proof of Lemma 9.1. Writing $V_i(\xi) = v_{i,0} + v_{i,1}\xi + v_{i,2}\xi^2$, $1 \leq i \leq 3$, from linear independence and multi-linearity of the determinant we get

$$1 \leq \begin{vmatrix} v_{1,2} & v_{1,1} & v_{1,0} \\ v_{2,2} & v_{2,1} & v_{2,0} \\ v_{3,2} & v_{3,1} & v_{3,0} \end{vmatrix} = \begin{vmatrix} v_{1,2} & v_{1,1} & V_1(\xi) \\ v_{2,2} & v_{2,1} & V_2(\xi) \\ v_{3,2} & v_{3,1} & V_3(\xi) \end{vmatrix}.$$

Now expanding the right hand side determinant and using (49) and (51) to estimate $|v_{i,j}|$ and $|V_i(\xi)|$ from above, we can estimate its absolute value as $\ll H_P^{-\delta}$ for $\delta = \min\{\varepsilon, \lambda - 3\} > 0$ using $\lambda > 3$. Together with the lower bound 1 we derive a contradiction for large enough H_P.

10. Proof of Theorem 3.6. For (25), note that the weaker estimate

$$\underline{\kappa}_2(\xi) \leq w_2(\xi) - w_2^*(\xi) \leq w_2(\xi) - \widehat{w}_2(\xi)^2 + \widehat{w}_2(\xi) \leq w_2(\xi) - 2$$

follows directly by combining [15, Theorem 2] (see also [9]) and (8). For the stronger claim, it follows more precisely from the proof in [15] that there are infinitely many best approximations satisfying

$$|P'(\xi)| \geq |P(\xi)| \cdot H_P^{\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi)+1},$$

which implies P has a root α with

$$|\xi - \alpha| \ll H(\alpha)^{-1} \cdot H_P^{-(\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi))+o(1)}.$$

Moreover, a result of Jarník [11, 12] implies that P remains a best approximation up to a parameter of size $X \geq H_P^{\sigma - o(1)}$ for $\sigma = \widehat{w}_2(\xi) - 1$. Indeed, this happens whenever P is linearly independent of its preceding and successive minimal polynomial, which occurs for our P above. Hence

$$|\xi - \alpha| \ll H(\alpha)^{-1} H_P^{-(\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi))+o(1)} = H(\alpha)^{-1} X^{-(\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi))/\sigma + o(1)}.$$

In other words,

$$w^*(2,\xi,X) \geq \frac{\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi)}{\sigma} - o(1),$$

hence, since clearly $w(2,\xi,X) \leq w_2(\xi)/\sigma + o(1)$, we get

$$\kappa(2,\xi,X) \leq w(2,\xi,X) - \frac{\widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi)}{\sigma} + o(1) \leq \frac{w_2(\xi)}{\widehat{w}_2(\xi)-1} - \widehat{w}_2(\xi) + o(1).$$

The claim follows as there are arbitrarily large such X.

For (26), we again use P and X as above and for simplicity write $w = w(2,\xi,X)$. By Lemma 4.2, P has a root α satisfying

$$|\xi-\alpha| \ll |P(\xi)| = X^{-w} = H(\alpha)^{-1}X^{-w}H_P = H(\alpha)^{-1}X^{-(\sigma w-1)/\sigma}.$$

In other words,

$$w^*(2,\xi,X) \geq w - \frac{1}{\sigma} - o(1).$$

Rearranging, for such X we have

$$\kappa(2,\xi,X) = w(2,\xi,X) - w^*(2,\xi,X) \leq \frac{1}{\sigma} + o(1) \leq \frac{1}{\widehat{w}_2(\xi)-1} + o(1),$$

so the claim follows.

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