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APPROXIMATION BY THE $[r]$-BASKAKOV SEQUENCE

Abstract. We modify the classical Baskakov sequence by using a positive integer parameter r to get a sequence that has a better order of approximation. We prove a convergence theorem for the modified sequence by using Korovkin's theorem. We then give a Voronovskaya-type asymptotic theorem for this sequence. Finally, we give two examples showing that the modified sequence gives better numerical results than the classical sequence.

1. Introduction. In 1957, Baskakov [5] introduced the sequence

$$M_n(\varphi; y) = \sum_{w=0}^{\infty} q_{n,w}(y)\, \varphi\left(\frac{w}{n}\right), \tag{1.1}$$

where

$$q_{n,w}(y) = \binom{n-1+w}{w} y^w (1+y)^{-n-w}, \qquad y \in [0,\infty), \tag{1.2}$$

and $\varphi \in C_\lambda[0,\infty) := \{\varphi \in C[0,\infty) : \varphi(t) = O((1+t)^\lambda)\}$ for some $\lambda > 0$, where $n \in \mathbb{N} := \{1, 2, \ldots\}$. The space $C_\lambda[0,\infty)$ is normed by

$$\|\varphi\|_{C_\lambda} = \sup_{t\in[0,\infty)} |\varphi(t)|(1+t)^{-\lambda}.$$

Many researchers have generalized and modified this sequence. Schurer [15] modified (1.1) by using a non-negative integer parameter p:

$$L_{n,p}(\varphi; y) = \sum_{w=0}^{\infty} q_{n+p,w}(y)\varphi\left(\frac{w}{n}\right). \tag{1.3}$$

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Sahai and Prasad [14] introduced a sum-of-integrals version of (1.1) for an integrable function φ on $[0, \infty)$:

$$Q_n(\varphi; y) = (n-1)\sum_{w=0}^{\infty} q_{n,w}(y)\int_0^{\infty} q_{n,w}(t)\varphi(t)\,dt. \tag{1.4}$$

Rempulska and Walczak [13] introduced a modification of (1.1) by using a parameter $s \in \mathbb{N}^0 := \{0, 1, \ldots\}$:

$$V_{n,s}(\varphi; y) = \sum_{w=0}^{\infty} q_{n,w}(y)\sum_{j=0}^{s} \frac{(y-w/n)^j}{j!}\varphi^{(j)}(w/n). \tag{1.5}$$

Aral and Erbay [2] introduced a generalization of (1.1) depending on a parameter α:

$$L_{n,\alpha}(\varphi; y) = \sum_{w=0}^{\infty} p_{n,w}^{(\alpha)}(y)\varphi\left(\frac{w}{n}\right), \tag{1.6}$$

where

$$p_{n,w}^{(\alpha)}(y) = \frac{y^{w-1}}{(1+y)^{n+w-1}}\left\{\frac{\alpha y}{1+y}\binom{n+w-1}{w} - (1-\alpha)(1+y)\binom{n+w-3}{w-2} + (1-\alpha)y\binom{n+w-1}{w}\right\}.$$

Mohammad et al. [10] studied a family of modified Baskakov sequences indexed by a parameter $s > -1/2$. Mohiuddine et al. [12] combined the Stancu sequence with the Baskakov–Kantorovich sequence to approximate functions on unbounded intervals dependent on the parameter $\alpha > 0$. Usta [16] constructed a version of the Baskakov sequence depending on two functions ξ_n and η_n:

$$R_n^{\sigma}(\varphi; y) = \sum_{w=0}^{\infty}\binom{w+n-1}{w}\frac{(\xi_n(y))^w}{(1+\eta_n(y))^{w+n}}(\varphi\circ\sigma^{-1})\left(\frac{w}{n}\right). \tag{1.7}$$

Abdul Samad and Mohammad [11] introduced rth powers of the rational Bernstein sequence. Cheng and Mohiuddine [7] constructed a new modification of the Baskakov sequence by using the second central moment:

$$F_n(\varphi; y) = \frac{1}{n}\sum_{w=0}^{\infty}\binom{n+w-1}{w}\frac{y^{w-1}}{(1+y)^{n+w+1}}(ny-k)^2\varphi\left(\frac{w}{n}\right). \tag{1.8}$$

The study of approximation using sequences has garnered significant interest. Various researchers have examined the convergence properties and practical applications of these sequences, illustrating their effectiveness in approximating functions from different function spaces (see, for example, [3, 4, 6, 8]).

The goal of this paper is to modify the classical Baskakov sequence, which is of order $O(n^{-1})$, to another sequence of order $O((rn)^{-1})$. This modification will reduce the arithmetic operations used in this sequence in numerical applications. The modified sequence is defined as

$$K_{n,r}(\varphi;y) = r\sum_{w=0}^{\infty} q_{rn,rw}(y)\varphi\left(\frac{w}{n}\right), \tag{1.9}$$

where $\varphi \in C_\lambda[0,\infty)$, $q_{rn,rw}(y)$ is defined by (1.2) and $r \in \mathbb{N}$. We call the sequence (1.9) the [r]-*Baskakov sequence*. In particular, when $r = 1$ then $K_{n,1}(\varphi;y) \equiv M_n(\varphi;y)$. Some preliminary results and lemmas on this modification are given in Section 2. Also, the convergence of this modification is proved, and a Voronovskaya-type asymptotic theorem is established in Section 3. Section 4 gives some numerical results for this sequence, comparing its performance with the classical sequence.

2. Preliminaries. We give some lemmas, which are used in the proofs of the main results in this paper.

LEMMA 2.1. *We have*

$$y(y+1)q'_{rn,rw}(y) = rn\left(\frac{w}{n} - y\right)q_{rn,rw}(y). \tag{2.1}$$

Proof. We calculate

$$\begin{aligned}(y^2+y)q'_{rn,rw}(y) &= \binom{rn+rw-1}{rw} y^{rw}(1+y)^{-(rn+rw)}(rw(y+1)-(rn+rw)y)\\ &= rn\left(\frac{w}{n}-y\right)q_{rn,rw}(y). \ \blacksquare\end{aligned}$$

LEMMA 2.2. *For $m \in \mathbb{N}^0$ and $n \in \mathbb{N}$, the mth order moments for the sequence $K_{n,r}(\cdot;y)$, defined by*

$$T_{n,m,w}(y) = K_{n,r}((t-y)^m;y) = r\sum_{w=0}^{\infty} q_{rn,rw}(y)\left(\frac{w}{n}-y\right)^m, \tag{2.2}$$

have the following properties:

(i) $T_{n,0,r}(y) = 1 + o(1)$,

(ii) $T_{n,1,r}(y) = o(1)$,

(iii) $T_{n,2,r}(y) = \dfrac{y^2+y}{rn} + o(1)$,

(iv) $T_{n,m+1,r}(y) = \dfrac{(y^2+y)(T'_{n,m,r}(y) + mT_{n,m-1,r}(y))}{rn}$,

(v) $T_{n,m,r}(y) = O\left(\dfrac{1}{(rn)^{[\frac{m+1}{2}]}}\right)$.

Proof. The proofs of (i)–(iii) are straightforward.
(iv) We have

$$T'_{n,m,r}(y) = r\sum_{w=0}^{\infty}\Big(q'_{rn,rw}(y)\Big(\frac{w}{n}-y\Big)^{m} - mq_{rn,rw}(y)\Big(\frac{w}{n}-y\Big)^{m-1}\Big)$$
$$= r\sum_{w=0}^{\infty}\Big(q'_{rn,rw}(y)\Big(\frac{w}{n}-y\Big)^{m}\Big) - mT_{n,m-1,r}(y).$$

Now, multiplying both sides by y^2+y and using Lemma 2.1, one has

$$(y^2+y)(T'_{n,m,r}(y)+mT_{n,m-1,r}(y)) = r\sum_{w=0}^{\infty} rn\Big(\frac{w}{n}-y\Big)q_{rn,rw}(y)\Big(\frac{w}{n}-y\Big)^{m}$$
$$= rnT_{n,m+1,r}(y).$$

(v) By induction, suppose the relation is true for $m=l$. From (iv), one has

$$rnT_{n,l+1,r}(y) = O\Big(\frac{1}{(rn)^{[\frac{l+1}{2}]}}\Big) + O\Big(\frac{1}{(rn)^{[\frac{l}{2}]}}\Big),$$
$$T_{n,l+1,r}(y) = O\Big(\frac{1}{(rn)^{[\frac{l+3}{2}]}}\Big) + O\Big(\frac{1}{(rn)^{[\frac{l+2}{2}]}}\Big).$$

If l is odd then

$$T_{n,l+1,r}(y) = O\Big(\frac{1}{(rn)^{[\frac{l+3}{2}]}}\Big).$$

If l is even then

$$T_{n,l+1,r}(y) = O\Big(\frac{1}{(rn)^{[\frac{l+2}{2}]}}\Big) = O\Big(\frac{1}{(rn)^{[\frac{(l+1)+1}{2}]}}\Big). \blacksquare$$

LEMMA 2.3. *The function $T_{n,m,r}(y)$ has the following properties:*

(i) $\lim_{n\to\infty} nT_{n,2,r}(y) = \dfrac{y^2+y}{r}$,

(ii) $\lim_{n\to\infty} nT_{n,m,r}(y) = 0$ *for* $m \neq 0, 2$.

Proof. (i) Use Lemma 2.2(iii).
(ii) By Lemma 2.2(v),

$$T_{n,m,r}(y) = O\Big(\frac{1}{(rn)^{[\frac{m+1}{2}]}}\Big).$$

Then

$$nT_{n,m,r}(y) = O\Big(\frac{n}{(rn)^{[\frac{m+1}{2}]}}\Big) = O\Big(\frac{1}{r^{[\frac{m+1}{2}]}n^{[\frac{m-1}{2}]}}\Big). \blacksquare$$

3. Main result. In this section we prove the convergence theorem and a Voronovskaya-type asymptotic theorem for the sequence $K_{n,r}(\varphi; y)$ using a modulus of continuity.

THEOREM 3.1 (Convergence theorem). *For every $\varphi \in C_\lambda[0,\infty)$, the sequence $K_{n,r}(\varphi; y)$ converges to $\varphi(y)$ as $n \to \infty$.*

Proof. Using Korovkin's theorem [9], this theorem is proved as follows:

$$K_{n,r}(1; y) = r\sum_{w=0}^{\infty} q_{rn,rw}(y) = 1 + o(1),$$

$$K_{n,r}(t; y) = r\sum_{w=0}^{\infty} q_{rn,rw}(y)\frac{w}{n} = y + o(1),$$

$$K_{n,r}(t^2; y) = r\sum_{w=0}^{\infty} q_{rn,rw}(y)\frac{w^2}{n^2} = \frac{rn+1}{rn}y^2 + \frac{1}{rn}y + o(1).$$

Hence, $K_{n,r}(\varphi; y) \to \varphi(y)$ as $n \to \infty$. ■

DEFINITION 3.2 (Modulus of continuity, [1]). For $\alpha > 0$, the modulus of continuity $\omega_\alpha(\varphi)$ is defined by

$$\omega_\alpha(\varphi) = \sup_{\substack{|t-y|\le\alpha \\ t,y\in[0,\infty)}} |\varphi(t) - \varphi(y)|. \tag{3.1}$$

THEOREM 3.3. *For $\varphi \in C_\lambda[0,\infty)$,*

$$|K_{n,r}(\varphi; y) - \varphi(y)| \le 2\omega_\alpha(\varphi) \tag{3.2}$$

where $\alpha = \sqrt{T_{n,2,r}T_{n,0,r}}$.

Proof. The modulus of continuity has the following property (see [1]):

$$|\varphi(t) - \varphi(y)| \le \omega_\alpha(\varphi)\left(\frac{|t-y|}{\alpha} + 1\right).$$

Now, by taking $K_{n,r}(.; y)$ of both sides, one gets

$$\begin{aligned}|K_{n,r}(\varphi(t); y) - \varphi(y)| &\le \omega_\alpha(\varphi)\left(\frac{K_{n,r}(|t-y|; y)}{\alpha} + 1\right)\\ &= \omega_\alpha(\varphi)\left(\left(\frac{r}{\alpha}\sum_{w=0}^{\infty} q_{rn,rw}(y)\left|\frac{w}{n} - y\right|\right) + 1\right).\end{aligned}$$

By using the Cauchy–Schwarz inequality and Lemma 2.2, one has

$$|K_{n,r}(\varphi;y)-\varphi(y)| \le \frac{\omega_\alpha(\varphi)}{\alpha}\left(r\sum_{w=0}^{\infty} q_{rn,rw}(y)\left(\frac{w}{n}-y\right)^2\right)^{1/2} \times\left(r\sum_{w=0}^{\infty} q_{rn,rw}(y)\right)^{1/2} + \omega_\alpha(\varphi)$$
$$= \omega_\alpha(\varphi)\left(\frac{1}{\alpha}(T_{n,2,r}(y))^{1/2}(T_{n,0,r}(y))^{1/2}+1\right).$$

Since $\alpha=\sqrt{T_{n,2,r}(y)T_{n,0,r}(y)}$, we get the conclusion. ∎

THEOREM 3.4. *If $\varphi\in C_\lambda[0,\infty)$ and φ'' exists and is continuous, then*

$$\lim_{n\to\infty} n(K_{n,r}(\varphi;y)-\varphi(y)) = \frac{y^2+y}{2r}\varphi''(y).$$

Proof. Taylor's expansion of $\varphi(t)$ at y is

$$\varphi(t)=\varphi(y)+(t-y)\varphi'(y)+\frac{(t-y)^2\varphi''(y)}{2}+(t-y)^2\zeta(t,y)$$

where $\zeta(t,y)\to 0$ as $t\to y$.

Now, by taking $K_{n,r}(\cdot;y)$ of both sides and using Lemma 2.2, one has

$$K_{n,r}(\varphi(t);y)=\varphi(y)K_{n,r}(1;y)+\varphi'(y)T_{n,1,r}(y)+\tfrac{1}{2}\varphi''(y)T_{n,2,r}(y) + K_{n,r}((t-y)^2\zeta(t,y);y).$$

Then

$$\lim_{n\to\infty} n(K_{n,r}(\varphi(t);y)-\varphi(y)) = \varphi'(y)\lim_{n\to\infty} nT_{n,1,r}(y) + \frac{1}{2}\varphi''(y)\lim_{n\to\infty} nT_{n,2,r}(y) + \lim_{n\to\infty} nK_{n,r}((t-y)^2\zeta(t,y);y).$$

By using Lemma 2.3, one has

$$\lim_{n\to\infty} n(K_{n,r}(\varphi(t);y)-\varphi(y)) = \frac{(y^2+y)\varphi''(y)}{2r} + \lim_{n\to\infty} nK_{n,r}((t-y)^2\zeta(t,y);y).$$

Therefore, if we show that

$$\lim_{n\to\infty} nK_{n,r}((t-y)^2\zeta(t,y);y)=0,$$

then the proof is complete.

We have

$$K_{n,r}(n(t-y)^2\zeta(t,y);y)=rn\sum_{w=0}^{\infty} q_{rn,rw}(y)\left(\frac{w}{n}-y\right)^2\zeta\left(\frac{w}{n},y\right).$$

By the Cauchy–Schwarz inequality,

$$K_{n,r}(n(t-y)^2\zeta(t,y);y) \le rn\bigg(\sum_{w=0}^{\infty} q_{rn,rw}(y)\bigg(\frac{w}{n}-y\bigg)^4\bigg)^{1/2} \times \bigg(\sum_{w=0}^{\infty} q_{rn,rw}(y)\zeta\bigg(\frac{w}{n},y\bigg)\bigg)^{1/2}.$$

Thus,

$$\lim_{n\to\infty} nK_{n,r}((t-y)^2\zeta(t,y);y) \le \lim_{n\to\infty} \sqrt{n}(nT_{n,4,r}(y))^{1/2}(K_{n,r}(\zeta^2(t,y);y))^{1/2}.$$

By using Korovkin's theorem [9] and Lemma 2.3,

$$K_{n,r}(\zeta^2(t,y);y) \to \zeta^2(y,y) = 0 \quad \text{as } n\to\infty$$

and

$$\lim_{n\to\infty} nT_{n,4,r}(y) = 0.$$

Consequently,

$$\lim_{n\to\infty} nK_{n,r}((t-y)^2\zeta(t,y);y) = 0. \ \blacksquare$$

4. Numerical examples. In this section we give examples of two test functions which show that the numerical results for the $[r]$-Baskakov sequence $K_{n,r}(\varphi;y)$ are better than for the classical sequence.

EXAMPLE 4.1. Set

$$\varphi(y) = \cos(6y)e^{-y}, \qquad y\in[0,2].$$

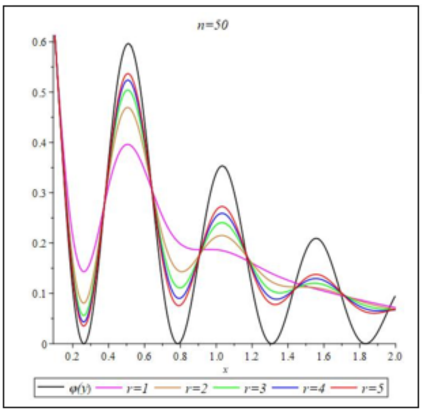

Fig. 1. The numerical convergence of $K_{n,r}(\varphi;y)$ with $r=1,2,3,4,5$ to $\varphi(y)$ in Example 4.1 with $n=50$.

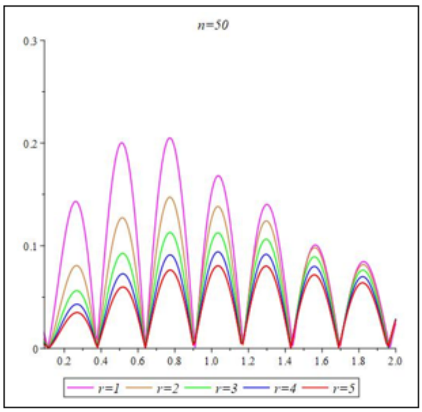

Fig. 2. The absolute error of the sequence $K_{n,r}(\varphi;y)$ in Example 4.1 with $n=50$ when $h=0.01$ and $r=1,2,3,4,5$.

Table 1. Some numerical values of $K_{n,r}(\varphi; y)$ and $\varphi(y)$ with $n = 25, 50$ for Example 4.1

n	y_i	$\varphi(y_i)$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
	0.1	0.616356	0.609165	0.608426	0.609829	0.610813	0.611711
25	0.4	0.364486	0.294011	0.313099	0.326324	0.334468	0.339841
	1.0	0.339157	0.188415	0.186771	0.198662	0.212444	0.225031
	0.1	0.616356	0.608329	0.610929	0.612389	0.613245	0.613800
50	0.4	0.364486	0.313099	0.334468	0.343617	0.348544	0.351604
	1.0	0.339157	0.186771	0.212444	0.235881	0.252959	0.265476

Table 2. The average absolute errors $\sum_{j=0}^{u} \frac{|K_{n,r}(\varphi; y_j) - \varphi(y_j)|}{u}$ with $h = 0.1$ in Example 4.1

r	$n = 25$	$n = 50$
1	0.149454	0.132629
2	0.132625	0.112542
3	0.121108	0.100386
4	0.112547	0.092082
5	0.105828	0.086029

Example 4.2. Set

$$\varphi(y) = \frac{1}{2} \sin^2(3\cos^3(-2x)), \quad y \in [0, 2].$$

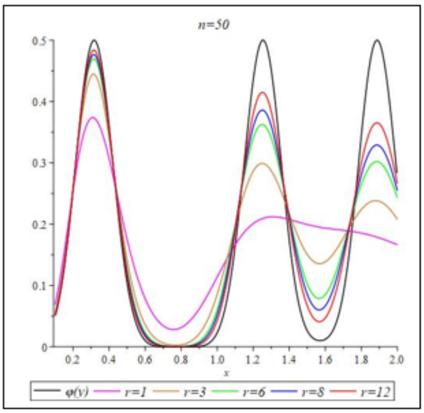

Fig. 3. The numerical convergence of $K_{n,r}(\varphi; y)$ with $r = 1, 3, 6, 8, 12$ to $\varphi(y)$ in Example 4.2 with $n = 50$.

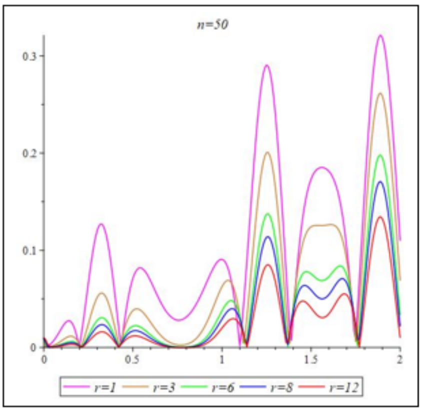

Fig. 4. The absolute error of the sequence $K_{n,r}(\varphi; y)$ in Example 4.2 with $n = 50$ when $h = 0.01$ and $r = 1, 3, 6, 8, 12$.

Table 3. Some numerical values of $K_{n,r}(\varphi;y)$ and $\varphi(y)$ with $n=25,50$ for Example 4.2

n	y_i	$\varphi(y_i)$	$r=1$	$r=3$	$r=6$	$r=8$	$r=12$
	0.3	0.493324	0.311287	0.403141	0.441708	0.453146	0.465483
25	1.0	0.023009	0.118944	0.107368	0.085545	0.074730	0.060591
	1.2	0.435315	0.166591	0.229567	0.286644	0.310030	0.340102
	0.3	0.493324	0.373014	0.441708	0.465482	0.472022	0.478831
50	1.0	0.023009	0.113758	0.085545	0.060591	0.052172	0.042923
	1.2	0.435315	0.200863	0.286644	0.340102	0.358616	0.380198

Table 4. The average absolute errors $\sum_{j=0}^{u} \frac{|K_{n,r}(\varphi;y_j)-\varphi(y_j)|}{u}$ with h=0.1 in Example 4.2

r	$n=25$	$n=50$
1	0.117661	0.100685
3	0.090147	0.067799
6	0.067799	0.045041
8	0.058012	0.036846
12	0.044998	0.027065

Based on the numerical data, the modified sequence becomes more accurate. As a result, the sequence is more effective than the classical sequence.

References

[1] M. Ali and R. B. Paris, *Generalization of Szász operators involving multiple Sheffer polynomials*, J. Anal. 31 (2023), 1–19.

[2] A. Aral and H. Erbay, *Parametric generalization of Baskakov operators*, Math. Comm. 24 (2019), 119–131.

[3] D. A. Ari and G. U. Yılmaz, *A note on Kantorovich type operators which preserve affine functions*, Fund. J. Math. Appl. 7 (2024), 53–58.

[4] R. Aslan, *Some approximation results on λ-Szász–Mirakjan–Kantorovich operators*, Fund. J. Math. Appl. 4 (2021), 150–158.

[5] V. A. Baskakov, *An instance of a sequence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk SSSR 113 (1957), 249–251 (in Russian).

[6] E. Baytunç, H. Aktuğlu and N. Mahmudov, *A new generalization of Szász–Mirakjan Kantorovich operators for better error estimation*, Fund. J. Math. Appl. 6 (2023), 194–210.

[7] W. T. Cheng and S. A. Mohiuddine, *Construction of a new modification of Baskakov operators on* $(0,\infty)$, Filomat 37 (2023), 139–154.

[8] H. Çiçek and A. İzgi, *Approximation by modified bivariate Bernstein–Durrmeyer and GBS bivariate Bernstein–Durrmeyer operators on a triangular region*, Fund. J. Math. Appl. 5 (2022), 135–144.

[9] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, 1960.

[10] A. J. Mohammad, S. A. Abdul-Hammed and T. A. Abdul-Qader, *Approximation of modified Baskakov operators based on parameter s*, Iraqi J. Sci. 62 (2021), 588–593.
[11] A. J. Mohammad and I. A. A. Samad, *The rth powers of the rational Bernstein polynomials*, Basrah J. Science 32 (2021), 179–191.
[12] S. A. Mohiuddine, N. Ahmad, F. Ozger, A. Alotaibi and B. Hazarika, *Approximation by the parametric generalization of Baskakov–Kantorovich operators linking with Stancu operators*, Iranian J. Sci. Technology Trans. A Sci. 45 (2021), 593–605.
[13] L. Rempulska and Z. Walczak, *On modified Baskakov operators*, Proc. A. Razmadze Math. Inst. 133 (2003), 109–117.
[14] A. Sahai and G. Prasad, *On simultaneous approximation by modified Lupas operators*, J. Approx. Theory 45 (1985), 122–128.
[15] F. Schurer, *On linear positive operators in approximation theory*, PhD thesis, TU Delft, 1965.
[16] F. Usta, *On approximation properties of a new construction of Baskakov operators*, Adv. Difference Equations 2021, art. 269, 13 pp.

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