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Cohomology on the centric orbit category of a fusion system

by

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Abstract. We study the higher derived limits of mod p cohomology on the centric orbit category of a saturated fusion system on a finite p-group. It is an open problem whether all such higher limits vanish. This is known in many cases, including for fusion systems realized by a finite group and for many classes of fusion systems which are not so realized. We prove that the higher limits of H^j vanish provided $j \leq p-2$, by showing that the same is true for the contravariant part of a simple Mackey composition factor of H^j under the same conditions.

1. Introduction. The purpose of this note is to show that $H^j(-,\mathbb{F}_p)$, considered as a contravariant functor on the centric orbit category of a saturated fusion system on a p-group, has vanishing higher derived limits provided $j \leq p-2$.

THEOREM 1.1. *Fix a prime p and a nonnegative integer $j \leq p-2$. For each saturated fusion system $\mathcal{F}$ on a finite p-group,*

$$\lim^i H^j(-,\mathbb{F}_p)|_{\mathcal{O}(\mathcal{F}^c)} = 0 \quad \textit{for all } i \geq 1.$$

Here $\mathcal{F}^c$ denotes the full subcategory of $\mathcal{F}$ with objects the $\mathcal{F}$-centric subgroups, and $\mathcal{O}(\mathcal{F}^c)$ denotes the corresponding orbit category. For example, the special case $j = 1$ of Theorem 1.1 states that the functor $H^1(-,\mathbb{F}_p)\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{F}_p\text{-}\mathsf{mod}$ is acyclic when p is odd. This was a case that motivated our work for reasons we outline below. But first we give context for this result, referring to [4, Section 2] and [1, Section III.5.6] for further details.

One way to study the classifying space of a finite group G at a prime p (or of a compact Lie group, or of a saturated fusion system over a p-group)

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is to recognize the space as glued from classifying spaces of collections of proper subgroups. Following Dwyer's uniform approach to such homotopy decompositions in the 1990s [7], this comes in the form of a map

$$\operatorname{hocolim}_{\mathcal{C}} F \to BG$$

inducing an isomorphism in mod p cohomology, where $\mathcal{C}$ is a small category and $F\colon \mathcal{C} \to \mathsf{Top}$ is some functor such that for each $c \in \mathcal{C}$, the composite $F(c) \to BG$ identifies $F(c)$ with the classifying space of a subgroup of G up to homotopy. The decomposition most relevant here is the *subgroup decomposition for the centric collection* [7]. In this case, $\mathcal{C} = \mathcal{O}_p^c(G)$ is the full subcategory of the orbit category of G on the p-centric subgroups, and $F(P) = \widetilde{B}P$ is an appropriate lifting to Top of the classifying space functor $B\colon \mathcal{O}_p^c(G) \to \mathsf{hoTop}$ to the homotopy category. For example, one can use $\widetilde{B}P = EG \times_G G/P$, the Borel construction applied to the orbit G/P considered as a G-space [3, Section 5.2].

On the other hand, if $\mathcal{F}$ is a saturated fusion system on a p-group S, there may be no finite group with Sylow p-subgroup S realizing the fusion in $\mathcal{F}$ (i.e., $\mathcal{F}$ may be exotic). Consequently there is no longer an obvious lifting of the classifying space functor. As was noticed by Broto, Levi, and Oliver [4, §2], the Dwyer–Kan obstructions to such a lifting [9] are the same as the obstructions to the existence and uniqueness of a centric linking system $\mathcal{L}$ associated with $\mathcal{F}$. By a theorem of Chermak, these obstructions vanish [5], and the p-completion $|\mathcal{L}|_p^\wedge$ is then regarded as a classifying space "$B\mathcal{F}$" for the fusion system. The subgroup decomposition in this context takes the form of a mod p cohomology isomorphism $\operatorname{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B} \to |\mathcal{L}|$, where $\widetilde{B}$ is the left homotopy Kan extension along the quotient functor $\tilde{\pi}\colon \mathcal{L} \to \mathcal{O}(\mathcal{F}^c)$ of the constant functor $\mathcal{L} \to *$ [1, Proposition III.5.29].

The most immediate application of the subgroup decomposition for a saturated fusion system is to the computation of cohomology of the linking system. The mod p cohomology $H^*(|\mathcal{L}|, \mathbb{F}_p)$ of the linking system is the abutment of the Bousfield–Kan spectral sequence for the homotopy colimit with E_2 page given by $E_2^{i,j} = \lim^i_{\mathcal{O}(\mathcal{F}^c)} H^j(-, \mathbb{F}_p)$. Note that by a general property of p-local functors on orbit categories, this page is bounded to the right [4, Corollary 3.4]. It is an open problem whether the functor $H^j(-, \mathbb{F}_p)$ is acyclic, i.e., whether $\lim^i_{\mathcal{O}(\mathcal{F}^c)} H^j(-, \mathbb{F}_p) = 0$ for all $i \geq 1$. For example, see [2, Problem 7.12] and [6, Conjecture]. Dwyer called a homology decomposition with this property "sharp". Sharpness would immediately yield the most natural application of the subgroup decomposition, which is that the cohomology of the linking system satisfies a Cartan–Eilenberg stable elements formula:

$$H^j(|\mathcal{L}|, \mathbb{F}_p) \cong \lim_{P \in \mathcal{O}(\mathcal{F}^c)} H^j(P, \mathbb{F}_p).$$

The stable elements formula was already shown by Broto, Levi, and Oliver [4, Theorem 5.8] (along with other important consequences for combinatorial descriptions mapping spaces), but the proof relies on difficult results from homotopy theory. It would be very nice to see the stable elements formula directly via sharpness of the subgroup decomposition.

Sharpness over the centric p-orbit category of a finite group G was shown by Dwyer [8, §10], and over the centric orbit category of $\mathcal{F}_S(G)$ by Díaz and Park [6, Theorem B]. Sharpness has since been established for certain families of exotic fusion systems, notably: those on p-groups with an abelian subgroup of index p [6], the smallest Benson–Solomon fusion system [13], all fusion systems of characteristic p-type/local characteristic p (in the sense of the CFSG) [13], and for the 27 exotic fusion systems on the Sylow p-subgroup of $G_2(p)$ [12]. The sharpness problem has been studied from a very general point of view also by Yalçın [17], who showed that sharpness for the subgroup and the normalizer decompositions are equivalent.

We follow Díaz and Park [6] by regarding cohomology as a Mackey functor for the fusion system, and we study the simple Mackey functors $S_{T,V}$ occurring as composition factors of H^j. Theorem 1.1 is ultimately deduced from the following stronger result. We write $k = \mathbb{F}_p$ for short.

THEOREM 1.2. *Fix a saturated fusion system $\mathcal{F}$ on a p-group S, a subgroup $T \leq S$, and a simple $k\,\mathrm{Out}_{\mathcal{F}}(T)$-module V. If the simple Mackey functor $S_{T,V}$ is a composition factor of $H^j(-,k)$ and $j \leq p-2$, then the restriction ${S_{T,V}}^*|_{\mathcal{O}(\mathcal{F}^c)}$ of the contravariant part of $S_{T,V}$ has vanishing higher derived limits.*

Theorem 1.2 is amenable to the standard technique of "pruning", where one filters ${S_{T,V}}^*$ by subquotient functors (not themselves contravariant parts of Mackey functors) that take the value 0 except on a single $\mathcal{F}$-conjugacy class of subgroups of S. Thus, our proof of Theorem 1.1 goes by filtering H^j first as a Mackey functor completely, and then second as a coefficient system.

One reason for our looking at this problem grew out of a loose analogy with the paper [10], where we studied higher limits of the center functor $\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$, $P \mapsto Z(P)$ in the context of Oliver's proof [15] of Chermak's Theorem [5] on centric linking systems. That proof proceeds by a reduction to the case where $\mathcal{F}$ is realizable by a finite p-constrained group Γ with normal centric p-subgroup Q. The observation of [10] was the relevance of finding a p-local subgroup H of Γ that controls fixed points on $Z(Q)$, i.e. that satisfies $C_{Z(Q)}(H) = C_{Z(Q)}(\Gamma)$, a problem that had been studied by the first author under the guise of "control of weak closure of elements". This motivated us to look at whether techniques for "controlling transfer" in finite groups could be useful in studying the higher limits of the functor $H^1(-,\mathbb{C}^\times)$ and its subfunctor $H^1(-,\mathbb{F}_p)$. The issue is that it appears difficult to get reductions similar to those for the center functor in order for these techniques to be applicable. Also, general techniques for controlling transfer are known only when $p \geq 5$, so in the end what Theorem 1.1 gives in the case $j = 1$ is stronger than what those methods seemingly would have yielded even if they had been applicable.

2. Background results

2.1. Nilpotent action on group cohomology. Let k be a commutative ring with identity. If G is a finite group, H is a subgroup of G, and M is a kG-module, then we use the usual notation $\mathrm{tr}_H^G\colon M^H \to M^G$ for fixed points and the relative trace map. If p is a rational prime which is zero in k, then the relative trace is zero in cases where there is an element of G outside H but normalizing H and acting with small nilpotence degree on M. We state this when $k = \mathbb{F}_p$, the only case we need.

LEMMA 2.1. *Let G be a finite group, p a prime, and V an $\mathbb{F}_p[G]$-module. Suppose g is an element of G of p-power order such that $(g-1)^{p-1}V = 0$. Then $\mathrm{tr}_H^G(V) = 0$ for every subgroup H of G with $g \in N_G(H) - H$.*

Proof. Decompose $\mathrm{tr}_H^G = \mathrm{tr}_{H\langle g\rangle}^G \mathrm{tr}_{H\langle g^p\rangle}^{H\langle g\rangle} \mathrm{tr}_H^{H\langle g^p\rangle}$. We have

$$\mathrm{tr}_{H\langle g^p\rangle}^{H\langle g\rangle}(v) = (1 + g + \cdots + g^{p-1})v = (g-1)^{p-1}v = 0$$

for each $v \in \mathrm{tr}_H^{H\langle g^p\rangle}(C_V(H))$. ∎

In this paper, if X and Y are two subsets of a group G, we write $[X, Y]$ for the subgroup of G generated by the set of commutators $[x, y] = xyx^{-1}y^{-1}$ with $x \in X$ and $y \in Y$. Our iterated commutators are right-associated: set $[X, Y; 1] = [X, Y]$, and inductively $[X, Y; i] = [X, [X, Y; i-1]]$ for $i \geq 2$. If G has a left action on some module V, the notation $[X, V; i]$ should be interpreted in the semidirect product of V by G, in which case $[X, V; i]$ is a subspace of V, and $[x, v; i] = (x-1)^i v$ for all $x \in G$ and $v \in V$.

The techniques we have generally take advantage of situations in a finite p-group in which some subgroup G normalizes another subgroup P and acts with small nilpotence degree on it, usually an action which is quadratic (or trivial): $[G, G, P] = 1$. Then the following lemma of Miyamoto [14, Lemma 2] provides a bound on the nilpotence degree of the action of G on $H^j(P, A)$ that is linear in j when A is finite abelian.

LEMMA 2.2. *Let P be a finite p-group, A a finite abelian group with trivial P-action, and G a finite group acting on P and A. Assume h and n are nonnegative integers such that $(g-1)^h$ acts as zero on each G-composition factor of P, and such that $(g-1)^n$ acts as zero on each G-composition factor* *of A. Then for all $j \geq 0$, $(g-1)^{(h-1)j+n}$ acts as zero on each G-composition factor of $H^j(P, A)$.*

2.2. Mackey functors for fusion systems. The notation we use for fusion systems follows [1]. We apply morphisms from right to left. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. A subgroup Q of S is *fully $\mathcal{F}$-normalized* (respectively, *fully $\mathcal{F}$-centralized*) if $|N_S(Q)| \geq |N_S(Q')|$ (respectively, $|C_S(Q)| \geq |C_S(Q')|$) for each conjugate Q' of Q in $\mathcal{F}$, i.e., for each subgroup of the form $\varphi(Q)$ with $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$. A subgroup Q of S is *$\mathcal{F}$-centric* if $C_S(Q') \leq Q'$, i.e., $C_S(Q') = Z(Q')$, for each $\mathcal{F}$-conjugate Q' of Q. The symbol $\mathcal{F}^c$ denotes the set of $\mathcal{F}$-centric subgroups, and also the full subcategory with the same objects. Note that a subgroup of S is $\mathcal{F}$-centric if and only if it is fully centralized and contains its centralizer in S [1, Definition I.3.1]. By one of the axioms for saturation, a fully normalized subgroup $P \leq S$ is also fully centralized and fully automized: $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ [1, Proposition I.2.5].

For each pair of subgroups $P, Q \leq S$, $\mathrm{Inn}(Q)$ acts on $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ by left composition. The orbit category $\mathcal{O}(\mathcal{F})$ of $\mathcal{F}$ is the category with the same objects as $\mathcal{F}$ and with morphism sets

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Inn}(Q) \backslash \mathrm{Hom}_{\mathcal{F}}(P, Q),$$

the orbits under this action. If $\mathcal{X}$ is a collection of subgroups of S which is closed under $\mathcal{F}$-conjugacy and also closed under passing to overgroups in S, then we abuse notation by using $\mathcal{X}$ also for the full subcategory of $\mathcal{F}$ with object set $\mathcal{X}$, and we write $\mathcal{O}(\mathcal{X})$ for the corresponding orbit category. Other than the full orbit category itself, we will only need to work with the centric orbit category, the case $\mathcal{X} = \mathcal{F}^c$.

Since the morphisms in a fusion system model conjugation of p-subgroups in a finite group, it is natural that there is a notion of a Mackey functor for fusion systems. We first want to recall from [6, Section 2] the definition of a Mackey functor in this setting in the form that is most useful later. Let k be a commutative ring with identity. Let $M = (M^*, M_*)$ be a pair of functors from $\mathcal{O}(\mathcal{F})$ to k-mod with M^* contravariant and M_* covariant. Set $\mathrm{r}_P^Q = M^*([\iota_P^Q])$, $\mathrm{t}_P^Q = M_*([\iota_P^Q])$, and $\mathrm{iso}([\varphi]) = M_*([\varphi])$ for each $P \leq Q \leq S$ and each isomorphism $[\varphi]\colon P \to \varphi(P)$ in $\mathcal{O}(\mathcal{F})$. Then M is a Mackey functor for $\mathcal{F}$ if the following conditions hold [6, Definition 2.1, Proposition 2.2]:

(1) $M(P) := M^*(P) = M_*(P)$ for each $P \leq S$,
(2) (Isomorphism) $M_*([\varphi]) := \mathrm{iso}(\varphi) = M^*([\varphi]^{-1})$ for each isomorphism $[\varphi]$ in $\mathcal{O}(\mathcal{F})$,
(3) (Mackey formula) for each $P, Q \leq R \leq S$,

$$r_Q^R \circ t_P^R = \sum_{x \in [Q \backslash R / P]} t_{Q \cap {}^xP}^Q \circ r_{Q \cap {}^xP}^{{}^xP} \circ \mathrm{iso}(c_x|_P).$$

A morphism $M \to N$ of Mackey functors is a family of k-module homomorphisms $\eta_P\colon M(P) \to N(P)$ such that $\eta = (\eta_P)$ is both a natural transformation from $M^* \to N^*$ and a natural transformation $M_* \to N_*$ simultaneously. A subfunctor of M is a subfunctor of M^* which is simultaneously a subfunctor of M_*, and quotient functors are defined objectwise.

2.3. Simple Mackey functors. The simple objects in $\mathrm{Mack}_k(\mathcal{F})$ are parametrized by pairs (T, V), where T is a subgroup of S taken up to $\mathcal{F}$-conjugacy, and where V is a simple (irreducible) $k\operatorname{Out}_{\mathcal{F}}(T)$-module taken up to isomorphism [6, Section 3]. When convenient we view V as a $k\operatorname{Aut}_{\mathcal{F}}(T)$-module via inflation. The corresponding simple Mackey functor $S_{T,V}$ has the property that $S_{T,V}(T) = V$ and $S_{T,V}(Q) = 0$ for all subgroups Q such that T is not $\mathcal{F}$-conjugate to a subgroup of Q.

Let $T \leq S$ and let V be a simple $k\operatorname{Out}_{\mathcal{F}}(T)$-module. We give the description of the functor $S_{T,V}$ on objects and isomorphisms in $\mathcal{O}(\mathcal{F})$ from [6, p. 153], since this will be important for the proof of Theorem 1.1, but interestingly the effect of $S_{T,V}$ on nonisomorphisms is not so important for our argument. For that we refer the interested reader to the description in [6]. Our treatment is a little different from (but equivalent to) that in [6], since we need to pay somewhat closer attention to precisely how $S_{T,V}(Q)$ decomposes as a direct sum of $k\operatorname{Out}_{\mathcal{F}}(Q)$-modules.

The set $\operatorname{Hom}_{\mathcal{F}}(T, Q)$ is an $\operatorname{Aut}_{\mathcal{F}}(Q)$-$\operatorname{Aut}_{\mathcal{F}}(T)$ biset with action on either side given by composition. The orbits $\operatorname{Hom}_{\mathcal{F}}(T, Q)/\operatorname{Aut}_{\mathcal{F}}(T)$ are in correspondence with the set of subgroups of Q that are $\mathcal{F}$-conjugate to T. Likewise the double orbits $\operatorname{Aut}_{\mathcal{F}}(Q)\backslash\operatorname{Hom}_{\mathcal{F}}(T, Q)/\operatorname{Aut}_{\mathcal{F}}(T)$ are in correspondence with the $\operatorname{Aut}_{\mathcal{F}}(Q)$-orbits of such subgroups. Let

$$A_{T,Q} = [\operatorname{Aut}_{\mathcal{F}}(Q)\backslash\operatorname{Hom}_{\mathcal{F}}(T, Q)/\operatorname{Aut}_{\mathcal{F}}(T)]$$

be a set of representatives for the double orbits.

For each $\alpha \in \operatorname{Hom}_{\mathcal{F}}(T, Q)$, we temporarily set $U = \alpha(T)$ and denote (formally) by $\alpha \otimes V$ the $k\operatorname{Aut}_{\mathcal{F}}(U)$-module, isomorphic to V as a k-module via $\alpha \otimes v \mapsto v$, with action

$$\varphi \cdot (\alpha \otimes v) = \alpha \otimes \alpha^{-1}\varphi\alpha v$$

for each $\varphi \in \operatorname{Aut}_{\mathcal{F}}(U)$ and $v \in V$. Since T acts trivially on V, U acts trivially on $\alpha \otimes V$. Set

$$W_\alpha = \operatorname{tr}_U^{N_Q(U)}(\alpha \otimes V), \tag{2.3}$$

the image of the relative trace; here $N_Q(U)$ acts on the $\operatorname{Out}_{\mathcal{F}}(U)$-module $\alpha \otimes V$ through the composite

$$N_Q(U) \twoheadrightarrow N_{\operatorname{Inn}(Q)}(U) \twoheadrightarrow \operatorname{Aut}_{N_Q(U)}(U) \to \operatorname{Aut}_{\mathcal{F}}(U)$$

with $UC_Q(U)$ acting trivially. Note W_α is a k-submodule of $\alpha \otimes V$. It has the structure of a $kN_{\operatorname{Aut}_{\mathcal{F}}(Q)}(U)$-module on which $N_Q(U)$ acts trivially.

Write U^Q for the Q-conjugacy class of U, and $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q)$ for the stabilizer of this class in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since $\mathrm{Inn}(Q)$ is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ that acts transitively on U^Q,

$$N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q) = N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U)\,\mathrm{Inn}(Q).$$

By construction, the subgroup $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U) \cap \mathrm{Inn}(Q) = N_{\mathrm{Inn}(Q)}(U)$ acts trivially on W_α, and we may regard W_α as a module for $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q)$ via the composite

$$N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q) \twoheadrightarrow N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q)/\mathrm{Inn}(Q) \cong N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U)/N_{\mathrm{Inn}(Q)}(U).$$

Set now

$$S_{T,V}(Q)_\alpha = W_\alpha\uparrow^{\mathrm{Aut}_{\mathcal{F}}(Q)}_{N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q)} = \bigoplus_{\varphi} W_{\varphi\alpha}, \tag{2.4}$$

where φ runs over a set of representatives for the left cosets of $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(U^Q)$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Then $S_{T,V}(Q)_\alpha$ is a $k\,\mathrm{Out}_{\mathcal{F}}(Q)$-module, that is, Q still acts trivially. The value of $S_{T,V}$ on the subgroup Q is then

$$S_{T,V}(Q) = \bigoplus_{\alpha\in A_{T,Q}} S_{T,V}(Q)_\alpha, \tag{2.5}$$

an $\mathrm{Out}_{\mathcal{F}}(Q)$-invariant direct sum decomposition.

Now let Q' be another subgroup of S and $\beta\colon Q \to Q'$ an isomorphism in $\mathcal{F}$. The bijection $\mathrm{Hom}(T,Q) \xrightarrow{\beta\circ-} \mathrm{Hom}(T,Q')$ determines the k-module isomorphism $\alpha \otimes V \cong \beta\alpha \otimes V$ intertwining the actions of $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\alpha(T))$ and $N_{\mathrm{Aut}_{\mathcal{F}}(Q')}(\beta(\alpha(T)))$ with respect to conjugation by β. It induces an isomorphism of k-modules

$$W_\alpha \cong W_{\beta\circ\alpha} \tag{2.6}$$

intertwining the actions of $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\alpha(T)^Q)$ and $N_{\mathrm{Aut}_{\mathcal{F}}(Q')}(\beta(\alpha(T))^{Q'})$. The component of $\mathrm{iso}(\beta)$ at α is the corresponding map of induced modules

$$\mathrm{iso}(\beta)_\alpha\colon S_{T,V}(Q)_\alpha \to S_{T,V}(Q')_{\beta\alpha},$$

and $\mathrm{iso}(\beta)$ is the sum of these maps.

Note that if V is a simple $\mathbb{F}_p\,\mathrm{Out}_{\mathcal{F}}(T)$-module, then $S_{T,V}$ takes values in $\mathbb{F}_p$-mod. Further, since V is simple, $S_{T,V}$ is a simple functor [6].

2.4. Higher limits of functors on orbit categories. Let k be a commutative $\mathbb{Z}_{(p)}$-algebra and let M be a contravariant functor from the orbit category of a group or a fusion system to the category of k-modules. A common technique for computing the higher limits of M (and especially for showing that such higher limits vanish) is to use a filtration of M each of whose successive quotient functors is atomic, namely a functor which vanishes except on a single conjugacy class of subgroups. This method does not always work as, for example, in the case of the center functor [16]. But the idea is often effective for making reductions even when it does not work directly. And ultimately it is all that is needed for the proof of the main theorem here.

Let G be a finite group and let A be a $\mathbb{Z}_{(p)}G$-module. Define a functor

$$F_A\colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

via $F_A(1) = A$ and $F_A(P) = 0$ when $P \neq 1$. The action of $G = \mathrm{Aut}_{\mathcal{O}_p(G)}(1)$ on $A = F_A(1)$ is the given one. The higher limits of F_A are denoted $\Lambda^i(G, A)$ and arise as the higher limits of atomic functors, as was first shown by Jackowski, McClure, and Oliver [1, Proposition 5.20].

Proposition 2.7. *Let F be any functor on the orbit category of a fusion system $\mathcal{F}$ which vanishes except on the $\mathcal{F}$-conjugacy class of a subgroup P. Then there is an isomorphism* $\lim^* F \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P), F(P))$.

Let $P_1, \dots, P_n$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of subgroups of S such that if $i < j$, then P_j is not conjugate to a subgroup of P_i. Then one can make a filtration $0 = M_0 \subset M_1 \subseteq \cdots \subset M_n = M$, in which M_j is the functor equal to M on the union of the conjugacy classes P_i with $i \leq j$, and 0 otherwise. Then $(M_i/M_{i-1})(P) = M(P)$ if P is conjugate to P_i, and it is zero otherwise, i.e., the quotient is atomic.

In general, we use the notation M_Q for the atomic subquotient functor of M corresponding to the $\mathcal{F}$-conjugacy class of Q, namely the functor with values $M_Q(P) = M(P)$ if P is $\mathcal{F}$-conjugate to Q, and $M_Q(P) = 0$ otherwise.

The next lemma is proved using long exact sequences on higher limits corresponding to short exact sequences of functors arising out of a filtration of the above type.

Lemma 2.8 ([1, Corollary 5.21(a)]). *Let M be a contravariant functor on the centric orbit category of a fusion system $\mathcal{F}$. Assume that M_Q is acyclic for all $Q \in \mathcal{F}^c$, i.e., $\Lambda^m(\mathrm{Out}_{\mathcal{F}}(Q), M(Q)) = 0$ for all $m \geq 1$. Then M is acyclic.*

The functors $\Lambda^*(G, M)$ vanish in many cases; see for example [1, Section III.5] for many results along these lines. In the next lemma we state two such vanishing results.

Recall that a radical p-chain of length m in the finite group G is a sequence $O_p(G) = P_0 < P_1 < \cdots < P_m$ such that $P_i = O_p(N_G(P_1, \dots, P_i))$ for each $i = 0, \dots, n$, where $N_G(P_1, \dots, P_i)$ denotes the intersection of the normalizers in G of the P_i.

Lemma 2.9. *Let G be a finite group, M a $\mathbb{Z}_{(p)}G$-module, and $m \geq 1$.*

(1) *If $O_p(G) \neq 1$, then $\Lambda^m(G, M) = 0$.*
(2) *If $\mathrm{tr}_1^{N_G(P_1,\dots,P_m)}(M) = 0$ for each radical p-chain $1 = P_0 < P_1 < \cdots < P_m$ of length m in G, then $\Lambda^m(G, M) = 0$.*

Proof. For (1), see [1, Proposition III.5.24(b)]. Then (2) is a restatement of [1, Proposition III.5.27], given (1). ∎

We want to show (Theorem 1.2) that the restriction of the contravariant part of each Mackey composition factor $S_{T,V}$ of $H^j(-,\mathbb{F}_p)$ is acyclic when $j \le p-2$. For doing this, Díaz and Park show we can restrict attention to composition factors $S_{T,V}$ with T not $\mathcal{F}$-centric.

LEMMA 2.10. *Let k be a field of characteristic p and $\mathcal{F}$ a saturated fusion system on the finite p-group S. For each $\mathcal{F}$-centric subgroup T and each simple $k\,\mathrm{Out}_{\mathcal{F}}(T)$-module V, the restriction ${S_{T,V}}^*|_{\mathcal{O}(\mathcal{F}^c)}$ of the contravariant part of $S_{T,V}$ is acyclic.*

Proof. Proposition 3.3 of [6] implies that when T is centric, ${S_{T,V}}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is an $\mathcal{F}^c$-restricted Mackey functor for $\mathcal{F}$ in the sense of [6, Definition 2.1]. The lemma then follows from Theorem A there. ∎

3. Proof of Theorem 1.1. Throughout this section we fix a prime p, and a saturated fusion system $\mathcal{F}$ on the p-group S. We set $k = \mathbb{F}_p$ for short. For fixed $j \ge 0$, we consider $H^j(-,k)$ as a Mackey functor on $\mathcal{O}(\mathcal{F})$ where the contravariant structure is induced by restrictions and conjugations, and where the covariant structure is induced by transfers and conjugations (as usual).

We first fix some additional notation that we keep for the remainder of the section.

Let $\mathcal{B}$ be the collection of all normal subgroups B of S such that

$$C_S(B) \le B \quad \text{and} \quad [B,B,S] := [B,[B,S]] = 1. \tag{3.1}$$

By [11, 5.3.12], each subgroup $B \le S$ maximal subject to being normal and abelian coincides with its centralizer in S. Since $[B,S] \le B$, we have $[B,B,S] \le [B,B] = 1$. Thus, $\mathcal{B}$ is nonempty. Further, since each member of $\mathcal{B}$ is normal in S, it is fully normalized, hence fully centralized by one of the saturation axioms for $\mathcal{F}$. This implies $\mathcal{B} \subseteq \mathcal{F}^c$.

DEFINITION 3.2. Let T be a subgroup of S. Define $\mathcal{Q}$ to be the set of pairs (Q,α) consisting of a centric subgroup $Q \in \mathcal{F}^c$ and a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(T,Q)$ having the property that there are $B \in \mathcal{B}$ and an isomorphism $\beta\colon Q \to Q'$ in $\mathcal{F}$ such that $B \cap \beta\alpha(T) < B \cap \beta(Q)$.

The proof of Theorem 1.1 is broken into two propositions. In the first one, we show that the $k\,\mathrm{Out}_{\mathcal{F}}(Q)$-submodule $S_{T,V}(Q)_\alpha$ of $S_{T,V}(Q)$ (see (2.4)) is 0 whenever $S_{T,V}$ is a composition factor of $H^j(-,\mathbb{F}_p)$, $(Q,\alpha) \in \mathcal{Q}$, and $j \le p-2$. In the second one, we use this to show that the atomic subquotient $({S_{T,V}}^*)_Q$ is acyclic for an arbitrary $\mathcal{F}$-centric subgroup Q when T is not $\mathcal{F}$-centric.

Proposition 3.3. *Fix a prime p, a saturated fusion system $\mathcal{F}$ on a finite p-group S, a nonnegative integer j, a subgroup $T \leq S$, and a simple $k\operatorname{Out}_{\mathcal{F}}(T)$-module V. If $j \leq p-2$ and $S_{T,V}$ is a composition factor of $H^j(-,k)$, then $S_{T,V}(Q)_\alpha = 0$ for all $(Q,\alpha) \in \mathcal{Q}$.*

Proof. Set $U = \alpha(T)$ and $W_\alpha = \operatorname{tr}_U^{N_Q(U)}(\alpha \otimes V)$. By (2.4), $S_{T,V}(Q)_\alpha$ takes the form

$$S_{T,V}(Q)_\alpha = W_\alpha \uparrow^{\operatorname{Aut}_{\mathcal{F}}(Q)}_{N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(U^Q)}.$$

That is, $S_{T,V}(Q)_\alpha$ is induced from W_α.

Let $(Q,\alpha) \in \mathcal{Q}$. By Definition 3.2 there is an $\mathcal{F}$-isomorphism $\beta\colon Q \to Q'$ and $B \in \mathcal{B}$ such that for $U' = \beta(U)$, we have $B \cap U' < B \cap Q'$. The map β induces an intertwining $W_\alpha \cong W_{\beta\circ\alpha}$. Because of this we may change to lighter notation by replacing Q by Q', U by U', and α by $\beta \circ \alpha$. Thus, $B \cap U < B \cap Q$ and we want to show $W_\alpha = 0$.

Set $B_Q = B \cap Q$ for short. Since B is normal in S, B_Q is normal in Q and $N_{B_Q}(U)$ is normal in $N_Q(U)$. Also, as $B \cap U < B_Q$, we have $U < UB_Q$. Hence $U < N_{UB_Q}(U) = UN_{B_Q}(U)$. Since $S_{T,V}$ is a composition factor of $H^j(-,k)$, we see that V is a $k\operatorname{Out}_{\mathcal{F}}(T)$-composition factor of $H^j(T,k)$, and $\alpha \otimes V$ is an $\operatorname{Out}_{\mathcal{F}}(U)$-composition factor of $H^j(U,k)$. We have $[N_{B_Q}(U), N_{B_Q}(U), U] \leq [B,B,S] = 1$; in particular $N_{B_Q}(U)$ acts quadratically on every $\operatorname{Aut}_{\mathcal{F}}(U)$-composition factor of U. The hypotheses of Lemma 2.2 thus hold with $h = 2$ and $n = 1$, and with U, k, and $\operatorname{Aut}_{\mathcal{F}}(U)$ in the roles of P, A, and G. As $\alpha \otimes V$ is a composition factor of $H^j(U,k)$, by the lemma we have $(b-1)^{j+1}(\alpha \otimes V) = 0$ for all $b \in N_{B_Q}(U)$. Since composition factors are always $\mathbb{F}_p$-vector spaces and $j+1 < p$, Lemma 2.1 implies $\operatorname{tr}_U^{UN_{B_Q}(U)}(\alpha \otimes V) = 0$. Hence

$$W_\alpha = \operatorname{tr}_U^{N_Q(U)}(\alpha \otimes V) = \operatorname{tr}_{UN_{B_Q}(U)}^{N_Q(U)}(\operatorname{tr}_U^{UN_{B_Q}(U)}(\alpha \otimes V)) = 0,$$

and this completes the proof. ∎

Proposition 3.4. *Fix a prime p, a saturated fusion system $\mathcal{F}$ on a finite p-group S, and a nonnegative integer j. Let $S_{T,V}$ be a composition factor of $H^j(-,k)$ such that T is not $\mathcal{F}$-centric. If $j \leq p-2$, then for all $Q \in \mathcal{F}^c$, the atomic subquotient functor $(S_{T,V}{}^*)_Q$ is acyclic.*

Proof. Fix $Q \in \mathcal{F}^c$. The functor $(S_{T,V}{}^*)_Q$ does not depend on Q, but only on the $\mathcal{F}$-conjugacy class of Q. By Proposition 2.7, we have

$$\lim\nolimits^*_{\mathcal{O}(\mathcal{F}^c)}(S_{T,V}{}^*)_Q \cong \Lambda^*(\operatorname{Out}_{\mathcal{F}}(Q'), S_{T,V}(Q')),$$

for any $\mathcal{F}$-conjugate Q' of Q. So we may assume Q to be fully normalized in $\mathcal{F}$. In particular, $R := \operatorname{Out}_S(Q)$ is a Sylow p-subgroup of $G := \operatorname{Out}_{\mathcal{F}}(Q)$. By Lemma 2.9(1), the proposition holds if $O_p(G) \neq 1$, so we are reduced to $O_p(G) = 1$.

Adopt the notation of Section 2.3. By (2.5) and additivity of the functors $\Lambda^*(G,-)$, we have

$$\Lambda^*(G, S_{T,V}(Q)) = \bigoplus_{\alpha\in A_{T,Q}} \Lambda^*(G, S_{T,V}(Q)_\alpha). \tag{3.5}$$

Fix arbitrary $B \in \mathcal{B}$ and $\alpha \in A_{T,Q}$. Set $B_Q = B \cap Q$ and $U = \alpha(T)$ for short, and let $X = \bigcap_{\varphi\in \mathrm{Aut}_{\mathcal{F}}(Q)} \varphi(U)$.

Assume that $S_{T,V}(Q)_\alpha$ is nonzero. By Proposition 3.3 and (2.6),

$$B \cap \varphi(U) = B_Q$$

for every choice of $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. In particular, $B_Q \leq X \leq B \cap U$.

Since T is not $\mathcal{F}$-centric, U and B_Q are not $\mathcal{F}$-centric, and $B_Q < B$. Thus $Q < QB$, and so $Q < N_{QB}(Q) = QN_B(Q)$. As B is normal in S, $N_B(Q)$ is normal in $N_S(Q)$. Let C be a normal subgroup of $N_S(Q)$ minimal subject to $B_Q < C \leq N_B(Q)$. Since C normalizes Q and B is normal in S, we have $[C, Q] \leq B_Q$, and hence

$$[C, \varphi(U)] \leq [C, Q] \leq B_Q \leq C \cap X \leq C \cap \varphi(U)$$

for each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. That is,

$$C \leq N_S(\varphi(U)) \quad \text{and} \quad \varphi(U) \leq N_S(C) \tag{3.6}$$

for each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$.

Fix a radical p-chain $1 = Q_0 < Q_1 < \cdots < Q_m$ of G with $m \geq 1$, and let $H = N_G(Q_1, \ldots, Q_m)$ be its normalizer. Conjugating in G in order to ensure that $Q_m \leq R$, we will show that the hypotheses of Lemma 2.9(2) hold for the conjugate chain, and then conjugating back, it holds for the one just fixed. In this way we are reduced to $Q_m \leq R$.

Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be arbitrary. Recall that C was chosen so that C/B_Q is a minimal normal subgroup of $N_S(Q)/B_Q$ (contained in $N_B(Q)/B_Q$). As a minimal normal subgroup of a finite p-group, $\bar{C} := QC/Q \cong C/B_Q$ is therefore of order p and contained in the center of $R = \mathrm{Out}_S(Q) = N_S(Q)/Q$, the last equality following because $Q \in \mathcal{F}^c$. Thus, $\bar{C} \leq H$. Since $C \leq B$, we have $[C, \varphi(U)] \leq \varphi(U) \cap B$ by (3.6), and so $[C, C, \varphi(U)] \leq [C, B] = 1$ as B is abelian. In particular, $[C, C, V_1] = 0$ for every $\mathrm{Aut}_{\mathcal{F}}(\varphi(U))$-composition factor V_1 of $\varphi(U)$. As $\varphi\alpha \otimes V$ is an $\mathrm{Aut}_{\mathcal{F}}(\varphi\alpha(U))$-composition factor of $H^j(\varphi\alpha(T), k)$, we have

$$(c-1)^{j+1}(\varphi\alpha \otimes V) = 0 \quad \text{for all } c \in C$$

by Lemma 2.2 applied with $\mathrm{Aut}_{\mathcal{F}}(\varphi(U))$, $\varphi(U)$, k, 1, and 2 in the roles of G, P, A, n and h. So $(c-1)^{j+1}W_{\varphi\alpha} = 0$ for all $c \in C$ as $W_{\varphi\alpha}$ is a k-submodule of $\varphi\alpha \otimes V$. Since this holds for all $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, by the direct sum decomposition (2.4) we have

$$(c-1)^{j+1}S_{T,V}(Q)_\alpha = 0 \quad \text{for all } c \in C.$$

Lemma 2.1 now applies with H in the role of G, with 1 in the role of H, and with g a generator of $\bar{C}$. As $j+1<p$, we have $\mathrm{tr}_1^H(S_{T,V}(Q)_\alpha)=0$ by that lemma. Therefore, Lemma 2.9(2) and (3.5) combine to give $\lim^m_{\mathcal{O}(\mathcal{F}^c)}(S_{T,V}{}^*)_Q \cong \Lambda^m(\mathrm{Out}_{\mathcal{F}}(Q), S_{T,V}(Q))=0$. ∎

Proof of Theorem 1.2. Let $S_{T,V}$ be a composition factor of $H^j(-,k)$ as a Mackey functor on $\mathcal{O}(\mathcal{F})$, and suppose that $j\le p-2$. If T is centric, then $S_{T,V}{}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic by Lemma 2.10. If T is not $\mathcal{F}$-centric, then Proposition 3.4 shows that the atomic functor $(S_{T,V}{}^*)_Q$ is acyclic for each $Q\in\mathcal{F}^c$, so again $S_{T,V}{}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic by Lemma 2.8. ∎

Proof of Theorem 1.1. As made explicit in the proof of [6, Proposition 4.3], given a filtration of $H^j(-,\mathbb{F}_p)$ whose successive quotients are simple Mackey functors for $\mathcal{F}$, the restrictions to $\mathcal{O}(\mathcal{F}^c)$ of the contravariant parts of the members of the filtration yield a filtration for $H^j(-,\mathbb{F}_p)|_{\mathcal{O}(\mathcal{F}^c)}$. So the theorem follows from Theorem 1.2. ∎

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