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Defect of irreducible plane curves with simple singularities

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Abstract. We focus on the defect of singular plane curves, which was recently introduced by Dimca. Roughly speaking, the defect of a reduced plane curve measures the deviation from being a free curve. We find lower bounds on the defect for certain classes of irreducible plane curves admitting nodes, ordinary cusps and ordinary triple points. The main result states that reduced simply singular plane curves with sufficiently high Arnold exponents are never free.

In this note we study the defect for some classes of irreducible plane curves. This notion has been recently introduced by Dimca [4] and there are many interesting questions revolving around this notion that one may want to study. Probably the most interesting conjecture relating to the defect is [5, Conjecture 3.7], which asserts that the defect for line arrangements in $\mathbb{P}^2_{\mathbb{C}}$ is combinatorially determined, and this should be understood as a broad generalization of Terao's freeness conjecture. In order to present further questions, we need a preparation based on [3].

Let $S := \mathbb{C}[x,y,z] = \bigoplus_k S_k$ be the graded polynomial ring. In this note we consider reduced and not necessarily irreducible curves $C \subset \mathbb{P}^2_{\mathbb{C}}$. We denote by $\partial_x, \partial_y, \partial_z$ the partial derivatives and we define $\mathrm{Der}(S) = \{\partial := a \cdot \partial_x + b \cdot \partial_y + c \cdot \partial_z : a, b, c \in S\}$, which is the free S-module of $\mathbb{C}$-linear derivations of the ring S. Now for a reduced curve $C = \{f = 0\}$ with $f \in S_d$ homogeneous, we define

$$\mathrm{D}(f) = \{\partial \in \mathrm{Der}(S) : \partial(f) \in \langle f \rangle\}.$$

This means that $\mathrm{D}(f)$ is the graded S-module of derivations preserving the ideal $\langle f \rangle$. Recall that for a reduced curve $C = \{f = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$ we have the

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following decomposition [3, pp. 151–152]:

$$\mathrm{D}(f) = \mathrm{D}_0(f) \oplus S \cdot \delta_E,$$

where $\delta_E = x\partial_x + y\partial_y + z\partial_z$ is the Euler derivation, and

$$\mathrm{D}_0(f) = \{\partial \in \mathrm{Der}(S) : \partial f = 0\},$$

i.e., the set of all $\mathbb{C}$-linear derivations of S killing the polynomial f. It is classically known (see for example [3, p. 151]) that $\mathrm{D}_0(f)$ can be identified with the S-module of all non-trivial Jacobian relations for the partials of f, namely

$$\mathrm{AR}(f) = \{(a, b, c) \in S^3 : a \cdot \partial_x f + b \cdot \partial_y f + c \cdot \partial_z f = 0\}.$$

We have some numerical invariants associated with a curve $C = \{f = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$; one of them is the minimal degree among derivations killing f, i.e.,

$$\mathrm{mdr}(f) = \min\{r \in \mathbb{N} : \mathrm{D}_0(f)_r \neq 0\} = \min\{r \in \mathbb{N} : \mathrm{AR}(f)_r \neq 0\}.$$

Sometimes we will write $\mathrm{mdr}(C)$ for a curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ defined by f.

For a homogeneous polynomial $f \in S$ of degree d we define its *Jacobian ideal* as $J_f := \langle \partial_x f, \partial_y f, \partial_z f\rangle$. Now we define the saturation of J_f with respect to the irrelevant ideal $\mathfrak{m} = \langle x, y, z\rangle$ as $I_f := \bigcup_{k\geq 0}(J_f : \mathfrak{m}^k)$. The *Jacobian module* of f is defined as

$$N(f) = I_f/J_f.$$

The Jacobian module provides information about the curve defined by $f \in S$. In order to show its strength, let us present the following definition (see [3, Definition 8.1] or [6, Remark 4.7]).

Definition 1. A reduced curve $C = \{f = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$ defined by a homogeneous polynomial $f \in S$ is *free* if $\mathrm{D}(f)$, or equivalently $\mathrm{D}_0(f)$, is a free graded S-module.

It turns out that the freeness of $C = \{f = 0\}$ is equivalent to the condition $N(f) = 0$, i.e., to the Jacobian ideal being saturated; for details see [13, Proposition 1.9].

For a reduced curve $C = \{f = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$ we set

$$n(f)_j = \dim N(f)_j,$$

and we define the invariant

$$\nu(C) = \max\{n(f)_j\}_j.$$

The invariant $\nu(C)$ is called the *defect*, or the *defect from the freeness property*. It is rather difficult to compute the defect of a given curve C by using the above definition. However, Dimca showed the following crucial result. Before we formulate it, let us introduce some notation. If $C = \{f(x, y) = 0\}$

is a germ of an isolated plane curve singularity at $p = (0,0)$, then we define its *local Tjurina number* as follows:

$$\tau_p(C) = \dim_{\mathbb{C}}\left(\mathbb{C}\{x,y\}\Big/\left\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle\right).$$

Now for a reduced curve $C = \{f = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$ we define its *total Tjurina number* as

$$\tau(C) = \sum_{p \in \mathrm{Sing}(C)} \tau_p(C),$$

where $\mathrm{Sing}(C)$ denotes the set of all singular points of C.

THEOREM 2 ([4, Theorem 1.2]). *Let $C = \{f = 0\}$ be a reduced plane curve of degree d and $r = \mathrm{mdr}(C)$. Then the following hold:*

- *If $r < (d-1)/2$, then $\nu(C) = (d-1)^2 - r(d-1-r) - \tau(C)$.*
- *If $r \geq (d-2)/2$, then*

$$\nu(C) = \left\lceil \frac{3}{4}(d-1)^2 \right\rceil - \tau(C).$$

There are many interesting and difficult open problems regarding the notion of defect and here we would like to recall the two most important conjectures. The first, mentioned at the very beginning of the introduction, can be seen as a vast generalization of Terao's freeness conjecture and concerns line arrangements.

CONJECTURE 3. *For a line arrangement $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ the defect $\nu(\mathcal{L})$ is determined by the intersection lattice of $\mathcal{L}$. More precisely, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line arrangements that have isomorphic intersection lattices, then $\nu(\mathcal{L}_1) = \nu(\mathcal{L}_2)$.*

This conjecture seems to be difficult and for more details about it we refer the reader to an excellent recent survey by Dimca [5]. In our note, we focus on irreducible plane curves, and in order to present the main motivation for our research we need two additional definitions.

DEFINITION 4. A *plane rational cuspidal curve* is a rational curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ having only unibranch singularities.

We also recall another class of curves, defined in [7].

DEFINITION 5. A reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ is *nearly free* if $\nu(C) = 1$.

Regarding the above notions, we have the following surprising conjecture.

CONJECTURE 6. *Any rational cuspidal curve C is either free or nearly free.*

In the present note, strongly motivated by the above conjecture, we want to continue the idea of studying the defect for some natural classes of irreducible plane curves, since, apart from the above conjecture, *we do not have any general predictions or results for such curves.*

Our first result concerns nodal curves.

Definition 7. We say that an irreducible and reduced curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d is *nodal* if every singular point of C_d is an ordinary double point, i.e., a singular point having the local normal form $x^2 + y^2 = 0$.

Remark 8. We will refer to ordinary double points as *nodes*. Furthermore, if $n_2(C_d)$ denotes the number of nodes of an irreducible and reduced plane curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 3$, then by the genus formula we have $n_2(C_d) \leq (d-1)(d-2)/2$.

Theorem A. *Let C_d be a nodal plane curve of degree $d \geq 4$. Then*

$$\nu(C_d) \geq \frac{1}{4}(d^2 - 1).$$

In particular, the defect for nodal curves can be arbitrarily large.

The next result concerns irreducible and reduced plane curves of genus zero admitting only nodes and ordinary triple points as singularities.

Theorem B. *There exists an irreducible and reduced plane curve K_{3k} of degree $d = 3k$ with $k \geq 3$, of genus zero, that admits exactly $2k$ ordinary triple points and nodes as singularities and such that*

$$\nu(K_{3k}) \geq \frac{1}{4}(9k + 1)(k - 1).$$

Here by an *ordinary triple point* we mean a singularity defined by the local normal form $y^2x + x^3 = 0$.

Finally, we focus on certain cuspidal curves constructed by Ivinskis [10].

Theorem C. *There exists an irreducible and reduced plane curve C_{6k} of degree $d = 6k$, with $k \geq 1$, that admits exactly $9k^2$ ordinary cusps and no other singularities, such that*

$$\nu(C_{6k}) = 9k^2 - 9k + 1 = g(C_{6k}),$$

where $g(C_{6k})$ denotes the genus of C_{6k}. In particular, C_6 is nearly free.

For completeness, recall that an *ordinary cusp* is a singularity defined by the local normal form $y^2 + x^3 = 0$.

Our results show that rational cuspidal plane curves are special and this allows us to justify the heuristic phenomenon that it is difficult to construct irreducible free or nearly free curves.

Before we present the proofs, we need to recall the tools that we are going to use. We start with the following crucial result [6, Theorem 2.1].

THEOREM 9 (Dimca–Sernesi). *Let $C = \{f = 0\}$ be a reduced curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ having only quasi-homogeneous singularities. Then*

$$\mathrm{mdr}(f) \geq \alpha_C \cdot d - 2,$$

where α_C denotes the Arnold exponent of C.

The *Arnold exponent* of a reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ is defined as the minimum log canonical threshold $\mathrm{lct}_p(C)$ over all $p \in \mathrm{Sing}(C)$. When the singularities are just ordinary, we have the following result [2, Theorem 1.3].

THEOREM 10. *Let C be a reduced curve in $\mathbb{C}^2$ which has degree m, and let $p \in \mathrm{Sing}(C)$. Then $\mathrm{lct}_p(C) \geq 2/m$, and equality holds if and only if C is a union of m lines passing through p.*

By the above, if $p \in \mathbb{C}^2$ is an ordinary singularity of multiplicity r of C, then

$$\mathrm{lct}_p(C) = \frac{2}{r}.$$

Furthermore, if $q \in C$ is an ordinary cusp, then by [2, Example 1.5] we have

$$\mathrm{lct}_q(C) = \frac{5}{6}.$$

Proof of Theorem A. The existence of nodal curves follows from a result due to Severi [12]. Since all singular points $p \in \mathrm{Sing}(C_d)$ are nodes, we have $\mathrm{lct}_p(C_d) = 1$, and

$$\alpha_{C_d} = 1.$$

Then by Theorem 9,

$$\mathrm{mdr}(C_d) \geq d - 2.$$

By assumption $d \geq 4$, so

$$d - 2 \geq \frac{d-2}{2},$$

which means that by Theorem 2,

$$\nu(C_d) = \left\lceil \frac{3}{4}(d-1)^2 \right\rceil - \tau(C_d).$$

Now we want to find an upper bound on $\tau(C_d)$. First of all, since all singularities of C_d are nodes, one has $\tau_p(C_d) = 1$ for every $p \in \mathrm{Sing}(C_d)$. Since the number of nodes of C_d is bounded from above by $(d-1)(d-2)/2$, we get

$$\tau(C_d) \leq \frac{(d-1)(d-2)}{2}.$$

Taking into account the above inequality, we finally obtain

$$\nu(C_d) \geq \frac{3}{4}(d-1)^2 - \frac{(d-1)(d-2)}{2} = \frac{1}{4}(d^2-1),$$

which completes the proof. ■

Proof of Theorem B. The existence of irreducible curves K_{3k} of genus zero with $n_3 = 2k$ ordinary triple points and nodes as the only singularities follows from [9, Theorem 3.4]. The genus zero condition means that the curve has exactly

$$n_2 = \frac{9k^2 - 21k + 2}{2}$$

nodes as singularities. Since K_{3k} admits only nodes and ordinary triple points as singularities, we get

$$\alpha_{K_{3k}} = \min\left\{1, \frac{2}{3}\right\} = \frac{2}{3},$$

and so by Theorem 9,

$$\mathrm{mdr}(K_{3k}) \geq \frac{2}{3} \cdot 3k - 2 = 2k - 2.$$

Since $k \geq 3$, we have

$$2k - 2 > \frac{3k-2}{2},$$

so by using Theorem 2, the defect of K_{3k} can be bounded from below:

$$\nu(K_{3k}) = \left\lceil \frac{3}{4}(3k-1)^2 \right\rceil - 4 \cdot 2k - \frac{9k^2 - 21k + 2}{2} \geq \frac{1}{4}(9k+1)(k-1),$$

which completes the proof. ■

Proof of Theorem C. We start by showing the existence of curves C_{6k} with $k \geq 1$. Ivinskis [10, Lemma 4.1.7] showed that there exists an irreducible and reduced curve C_{6k} of degree $6k$ with $k \geq 1$ having exactly $9k^2$ ordinary cusps. This curve is constructed using the Kummer cover $\kappa : \mathbb{P}^2_{\mathbb{C}} \ni (x,y,z) \mapsto (x^k, y^k, z^k) \in \mathbb{P}^2_{\mathbb{C}}$ applied to an irreducible and reduced sextic with exactly nine ordinary cusps. Recall that such an irreducible sextic is the dual curve to a smooth elliptic curve E, and the ordinary cusps correspond to the nine inflection points of E.

Since our curve C_{6k} admits only ordinary cusps as singularities,

$$\alpha_{C_{6k}} = \frac{5}{6},$$

and by Theorem 9 we have

$$\mathrm{mdr}(C_{6k}) \geq \frac{5}{6} \cdot 6k - 2 = 5k - 2.$$

Since for $k \geq 1$ one has

$$5k - 2 > \frac{6k-2}{2} = 3k - 1,$$

and $\tau(C_{6k}) = 2 \cdot 9k^2 = 18k^2$, by Theorem 2,

$$\nu(C_{6k}) = \left\lceil \frac{3}{4}(6k-1)^2 \right\rceil - 18k^2.$$

Observe that

$$\left\lceil \frac{3}{4}(6k-1)^2 \right\rceil = \left\lceil 27k^2 - 9k + \frac{3}{4} \right\rceil = 27k^2 - 9k + 1,$$

and thus

$$\nu(C_{6k}) = 27k^2 - 9k + 1 - 18k^2 = 9k^2 - 9k + 1.$$

In particular, for $k = 1$ our curve C_{6k} is an irreducible sextic with nine ordinary cusps with $\nu(C_6) = 1$, so C_6 is nearly free. ∎

REMARK 11. The curves C_{6k} considered above are obviously not rational since

$$g(C_{6k}) = 9k^2 - 9k + 1 \geq 1.$$

Moreover, this shows that $g(C_{6k}) = \nu(C_{6k})$, and it is surprising that these two values coincide.

Let us now present the main result of the note, concerning reduced simply singular plane curves, i.e., reduced plane curves with only ADE singularities.

THEOREM D (Non-freeness criterion). *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a reduced plane curve of even degree $d = 2m \geq 4$, admitting only* ADE *singularities. Assume furthermore that the Arnold exponent of C satisfies $\alpha_C \geq \frac{1}{2} + \frac{1}{m}$. Then*

$$\nu(C) \geq 1.$$

In particular, C is not free.

Proof. The condition $\alpha_C \geq \frac{1}{2} + \frac{1}{m}$ ensures that the defect of C can be computed via the second formula in Theorem 2, namely

$$\nu(C) = \left\lceil \frac{3}{4}(2m-1)^2 \right\rceil - \tau(C).$$

Observe that

$$\left\lceil \frac{3}{4}(2m-1)^2 \right\rceil = 3m^2 - 3m + 1,$$

so it remains to estimate is $\tau(C)$. By Theorem 9 we get $\mathrm{mdr}(C) \geq m$, which follows from the fact that $\alpha_C \geq \frac{1}{2} + \frac{1}{m}$. Now, using a result due to Du Plessis and Wall [8, Theorem 3.2], we see that

$$\tau(C) \leq \tau_{\max}(2m, r) := (2m-1)(2m-r-1) + r^2 - \binom{2r-2m+2}{2},$$

where $r := \mathrm{mdr}(C)$. Since $\tau_{\max}(2m, r)$ is strictly decreasing as a function of r on the interval $[m, 2m-1]$, we get

$$\tau(C) \leq \tau_{\max}(2m, m) = 3m^2 - 3m,$$

and so we finally obtain

$$\nu(C) = 3m^2 - 3m + 1 - \tau(C) \geq 3m^2 - 3m + 1 - \tau_{\max}(2m, m) = 1,$$

which completes the proof. ∎

Now we present the following example to show that our main result is optimal.

Example 12. This example comes from [11, Lemma 7.5]. Fix an even integer $m \in \mathbb{Z}_{\geq 4}$ and consider the curve $\mathcal{C}_{2m} = \{C_1, C_2, C_3, C_4\} \subset \mathbb{P}^2_{\mathbb{C}}$, where

$$\begin{aligned} C_1 &: \quad x^{m/2} + y^{m/2} + z^{m/2} = 0, \\ C_2 &: -x^{m/2} + y^{m/2} + z^{m/2} = 0, \\ C_3 &: \quad x^{m/2} - y^{m/2} + z^{m/2} = 0, \\ C_4 &: \quad x^{m/2} + y^{m/2} - z^{m/2} = 0. \end{aligned}$$

Our curve $\mathcal{C}_{2m}$ is of degree $d = 2m$ and it has $3m$ singularities of type A_{m-1}; see [11, Lemma 7.5]. In particular, for $m = 4$ we obtain the arrangement of four conics that admits exactly twelve singularities of type A_3 – it is well-known that this arrangement is unique up to projective equivalence.

Since $\mathcal{C}_{2m}$ admits only singularities of type A_{m-1}, for each $p \in \mathrm{Sing}(\mathcal{C}_{2m})$ one has $\mathrm{lct}_p = \frac{1}{2} + \frac{1}{m}$, so the Arnold exponent of $\mathcal{C}_{2m}$ is equal to

$$\alpha_{\mathcal{C}_{2m}} = \frac{1}{2} + \frac{1}{m}.$$

By Theorem D,

$$\nu(\mathcal{C}_{2m}) \geq 1.$$

In fact, by [1, Theorem 3.12], our curve $\mathcal{C}_{2m}$ is nearly free, i.e., $\nu(\mathcal{C}_{2m}) = 1$.

Finally, let us present an application of Theorem D in the setting of line arrangements.

Example 13. Consider an arrangement $\mathcal{L}$ of $d = 2m$ lines which only admits double and triple intersections. Then $\alpha_{\mathcal{L}} = 2/3$, and if we assume that $m \geq 6$, then

$$\alpha_{\mathcal{L}} \geq \frac{1}{2} + \frac{1}{m}.$$

Using Theorem D we can conclude that there is no free arrangement of $d = 2m \geq 12$ lines with double and triple intersections.

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