APPLICATIONES MATHEMATICAE
Online First version

Gerhard Ströhmer

ON THE EXISTENCE AND STABILITY OF CERTAIN EQUILIBRIUM STATES OF GAS BALLS

Abstract. The paper combines two results about rotating equilibrium states of one or more gas balls. One discusses the structural stability and its limits under changes of the angular momentum, the other deals with the stability of solutions obtained by replacing mass points solving a restricted m-body problem.

1. Introduction. In this paper we prove a number of results about equilibrium solutions of the equations of self-gravitating, viscous, barotropic fluids.

In [4] it was shown that given an equilibrium solution fulfilling a certain non-degeneracy condition, one can find solutions close to the given one with angular momenta in a small interval around that of the given solution, defining a short curve in the set of such solutions.

In Section 2, for fluids with equations of state in a fairly large class, we prove some a priori estimates that allow us to show that such curves can only terminate at points where the solutions lose their stability in the sense of Condition 2 below, which will be referred to as energy stability in a slight abuse of language. This is also of interest for a single gas ball, as it shows that one can increase the angular momentum of the gas ball until it becomes unstable in that sense.

In Section 3 we prove that we can replace systems of mass points forming a solution of a restricted m-body problem by gas balls, provided the points are far enough apart and fulfill the stability and non-degeneracy Conditions

2020 *Mathematics Subject Classification*: Primary 76U05; Secondary 76E20, 70E55, 76N10, 45G05.
Key words and phrases: rotating equilibrium states, multiple gas balls, self-gravitating, existence, stability, restricted m-body problem.
Received 2 July 2024.
Published online 12 February 2025.

DOI: 10.4064/am2541-10-2024

5 and 7, resulting in an equilibrium solution with m gas balls fulfilling Condition 2.

In Section 4 we then show that these conditions are fulfilled for solutions of the restricted two-body problem. This gives us examples of equilibrium solutions with two gas balls satisfying the assumptions of the main result in [7], and the existence for all times of solutions for any initial value close to that equilibrium. Indeed, showing that there are examples of equilibrium states with more than one gas ball satisfying the assumptions of [5, 7] is one of the main points of this paper.

Needing to go into detail to make these statements precise, we begin by discussing the nature of the gas we are considering. We assume it has the equation of state $p = p(\rho) \geq 0$, with $p \in C^1([0,\infty)) \cap C^4((0,\infty))$, $p(0) = 0$, $p'(\rho) > 0$ for $\rho > 0$ and $p(\rho)\rho^{-2}|(0,1) \in L^1((0,1))$. The function

$$e(\rho) = \int_0^\rho p(s)s^{-2}\,ds$$

then belongs to $C^0([0,\infty)) \cap C^5((0,\infty))$, and $e(0) = 0$. In physical terms this is the free energy density per unit mass, and $\varepsilon = \rho e$ is the corresponding density per unit volume. For $\rho > 0$ one can easily derive the equation

$$\rho\varepsilon''(\rho) = p'(\rho). \tag{1}$$

For some of our results we have to add more assumptions about the equation of state of the fluid.

The equilibrium solutions we are discussing here are characterized by the condition that the velocity is that of a rotation around a fixed axis at a fixed angular velocity. Using a suitable coordinate system, we may assume the gas is rotating around the x_3-axis in $\mathbb{R}^3$ with angular velocity ω, i.e., with velocity $v(x) = \omega e_3 \times x$. Then, in a coordinate system rotating around the x_3-axis with angular velocity ω, the fluid of variable density $\rho = \rho(x)$ fills an open subset Ω of $\mathbb{R}^3$ that is the union of $m \geq 1$ bounded domains Ω_k $(k = 1, \ldots, m)$ of class C^4 (see [4] for the regularity of Ω) such that the $\overline{\Omega}_k$ are pairwise disjoint, and $\rho(x) > 0$ in $\overline{\Omega}$, while $\rho(x) = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. The gravity potential V, the negative of the potential used in physics, produced by the mass distribution is

$$V(x) = \sum_{k=1}^{m} V_k(x)$$

with

$$V_k(x) = \mathsf{k} \int_{\Omega_k} \frac{\rho(y)}{|x-y|}\,dy,$$

where k is the gravity constant. As we assume there is an entity outside Ω

exerting a constant pressure $\mathsf{p}_0 > 0$ on the fluid, we obtain the boundary condition

$$\rho|\partial\Omega = \vartheta \tag{2}$$

with $p(\vartheta) = \mathsf{p}_0$. Then with $Px = [x_1, x_2, 0]^\dagger$ the equilibrium solutions satisfy the equations

$$\nabla\Big(\varepsilon'(\rho) - V - \frac{1}{2}\omega^2|Px|^2\Big) = 0, \qquad \varepsilon'(\rho) - V - \frac{1}{2}\omega^2|Px|^2 = C_k \tag{3}$$

in each Ω_k with (usually distinct) constants C_k. We can think of the tuple $(\rho, V, \Omega, \omega, C)$ as representing an equilibrium solution if they satisfy equation (3) and the boundary condition (2). Note that these solutions are already determined by the function $\rho : \mathbb{R}^3 \to \mathbb{R}$ alone, as

$$V(x) = \mathsf{k} \int\limits_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy$$

and $\overline{\Omega} = \{x \in \mathbb{R}^3 \mid \rho(x) > 0\}$, leading to overdetermined systems for ω and C_k. We refer to these functions, sets and numbers as the *equilibrium solution* given by ρ. If ρ is not the density of an equilibrium solution, this will of course fail. Also let

$$M_j = \int\limits_{\Omega_j} \rho(x)\, dx$$

be the mass of the gas contained in the jth domain, and let

$$X_j = \frac{1}{M_j} \int\limits_{\Omega_j} x\rho(x)\, dx$$

denote the center of gravity of the mass in Ω_j. From [4, Corollary 3.6] together with the possibility of translation in all or some directions we know we can choose our coordinate system so that

$$\sum_{j=1}^{m} M_j X_j = \int\limits_{\Omega} x\rho(x)\, dx = 0. \tag{4}$$

This means the center of gravity is located at $x = 0$, which we assume from now on.

To make the condition we call energy stability precise, we introduce a function vector $U = [u, \alpha, \beta]$ containing the variation of the velocity u, the variation of the density α and the variation β of $\partial\Omega$ in the normal direction. Then we define a scalar product $(\cdot,\cdot)_{\mathfrak{H}}$ on $\mathfrak{H} = L^2(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}) \times L^2(\partial\Omega, \mathbb{R})$ by

$$([u_1, \alpha_1, \beta_1], [u_2, \alpha_2, \beta_2])_{\mathfrak{H}} = \int\limits_{\Omega} (\rho u_1 u_2 + \alpha_1\alpha_2)\, dx + \vartheta \int\limits_{\partial\Omega} \beta_1\beta_2\, d\sigma.$$

With

$$\mathcal{W}_k^1 = [0, 1_\Omega, 1_{\partial\Omega}] \quad (k = 1, \ldots, m)$$

and $U = [u, \alpha, \beta]$ let $Q_1^k(U) = (\mathcal{W}_k^1, U)_{\mathfrak{H}}$, the linearization of the masses of the gas balls. Likewise $\mathcal{W}_\mu^4 = x_\mu[0, 1, 1]$ gives us the components of the product of the linearized center of gravity and the mass by $(Q_4)_\mu(U) = (\mathcal{W}_\mu^4, U)_{\mathfrak{H}}$, or

$$Q_4^k([u, \alpha, \beta]) = \int_{\Omega_k} x\alpha\, dx + \vartheta \int_{\partial\Omega_k} x\beta\, d\sigma.$$

With $\mathcal{W}_\mu^2 = [e_\mu, 0, 0]$ the components of the momentum linearize as

$$(Q_2)_\mu(U) = (\mathcal{W}_\mu^2, U)_{\mathfrak{H}}$$

for $\mu = 1, 2, 3$, or, equivalently,

$$Q_2(U) = \int_\Omega \rho u\, dx. \tag{5}$$

The linearized angular momentum is given by

$$\mathcal{W}_\mu^3 = [e_\mu \times x, 0, 0] + f_\mu[0, 1, 1],$$

with $f_\mu = \omega(\delta_{\mu 3}|x|^2 - x_3 x_\mu)$ and $(Q_3)_\mu(U) = (\mathcal{W}_\mu^3, U)_{\mathfrak{H}}$. Let $\mathfrak{n}$ be the unit outward normal to Ω, and

$$\alpha_r = -e_3 \times x \cdot \nabla^\dagger \rho, \qquad \beta_r = e_3 \times x \cdot \mathfrak{n} \tag{6}$$

and $U_r = [0, \alpha_r, \beta_r]$. These are equal to zero if the solution is rotationally symmetric around the e_3-axis. Finally, let $\mathcal{N}_1$ be the space spanned by $\mathcal{W}_k^1, \mathcal{W}_\mu^2, \mathcal{W}_\mu^3, \mathcal{W}_\mu^4$ for $k = 1, \ldots, m$ and $\mu = 1, 2, 3$, and $\mathcal{N}_2$ be the space spanned by $\mathcal{N}_1 \cup \{U_r\}$. In addition let $\mathfrak{H}_1 = \mathcal{N}_1^\perp$ and $\mathfrak{H}_2 = \mathcal{N}_2^\perp$, where $\perp$ indicates the orthogonal complement in $\mathfrak{H}$.

Definition 1. For $k = 1, \ldots, m$, $\alpha \in L^2(\Omega)$, $\beta \in L^2(\partial\Omega)$ let

$$I_{\Omega_k}(\alpha, \beta)(x) = \mathsf{k}\Big[\int_{\Omega_k} \alpha(y)|x - y|^{-1}\, dy + \vartheta \int_{\partial\Omega_k} \beta(y)|x - y|^{-1}\, d\sigma_y\Big],$$

and $I(\alpha, \beta)(x) = \sum_{k=1}^m I_{\Omega_k}(\alpha, \beta)(x)$. Then for $U = [u, \alpha, \beta] \in \mathfrak{H}$ define

$$\begin{aligned} E(U) = \int_\Omega (\rho|u|^2 + \varepsilon''(\rho)|\alpha|^2)\, dx - \int_{\partial\Omega} \frac{\partial p}{\partial \mathfrak{n}} |\beta|^2\, d\sigma \\ - \int_\Omega \alpha I(\alpha, \beta)\, dx - \vartheta \int_{\partial\Omega} \beta I(\alpha, \beta)\, d\sigma. \end{aligned}$$

Furthermore, let $A_2 = \{W \in \mathfrak{H}_2 \mid \|W\|_{\mathfrak{H}} = 1\}$ and

$$\lambda_\rho = \inf_{W \in A_2} E(W).$$

We can now finally give a precise form of the *energy stability* condition.

CONDITION 2. $\lambda_\rho > 0$.

If $\frac{\partial\rho}{\partial\mathfrak{n}} < 0$, then using [5, Lemma 3.14] it is easy to see that Condition 2 here and Condition 1.2 in [5] are equivalent. It is likewise easy to see that this is equivalent to the condition in [5, Lemma 7.1]. The new condition allows us to eliminate the condition $\frac{\partial\rho}{\partial\mathfrak{n}} < 0$.

In Section 2 we prove the homotopy property mentioned at the beginning. It is made precise in Definition 3 and Theorem 4.

DEFINITION 3. We say a function $\rho : I \to L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ defined on an open real interval I has *Property* 3 if the following statements are true.

(1) For every $L \in I$ the set $\overline{\Omega}^L = \{x \in \mathbb{R}^3 \mid \rho(L,x) > 0\}$ is the closure of a bounded open set with C^4 boundary, $\rho(L) \in C^4(\overline{\Omega}^L)$ is the density of an equilibrium solution with angular velocity ω^L,

$$L = \omega^L \int_{\mathbb{R}^3} \rho(L)|Py|^2\,dy,$$

and $\rho \in C^0(I, L^1(\mathbb{R}^3))$.

(2) For every compact subinterval $J \subset I$ and $L_1 \in J$ there is a constant C such that for $L \in J$ there is a bijective transformation $T^L : \overline{\Omega}^{L_1} \to \overline{\Omega}^L$ with $T^L \in C^4(\overline{\Omega}^{L_1}), T^{-1} \in C^4(\overline{\Omega}^L)$ and

$$\|T^L\|_{C^4(\overline{\Omega}^{L_1})} + \|(T^L)^{-1}\|_{C^4(\overline{\Omega}^L)} \le C$$

and we have $\|\rho(L)\|_{C^4(\overline{\Omega}^L)} \le C$, and the functions $L \to T^L$ and $L \to \rho(L) \circ T^L$ are continuous from J to $C^1(\overline{\Omega}^{L_1})$.

(3) $\lambda_{\rho(L)} > 0$ for all $L \in I$.

Then we have the following result, mentioned at the beginning, that ensures that the local equilibrium-valued function constructed in [4] can be continued at least until the solutions lose strict stability.

THEOREM 4. *Assume that with an $\eta_1 > 0$ we have $\varepsilon'(\rho) \ge \eta_1\rho^\beta$ with a $\beta > 1/3$ for $\rho \ge \vartheta$, and if $\beta \le 1/2$, then also assume $\varepsilon''(\rho) \ge \eta_1\rho^{\beta-1}$. Further, assume there is a ρ_0 representing an equilibrium solution such that*

$$L_0 = \omega_0 \int_{\mathbb{R}^3} \rho_0|Py|^2\,dy$$

with the angular momentum ω_0 of this solution, and that $\lambda_{\rho_0} > 0$. Then there exists a maximal open interval $I \subset \mathbb{R}$ with $L^0 \in I$ and a function $\rho = \rho(L)$ having Property 3 on I and $\rho(L_0) = \rho_0$. This function is unique in the sense that for any functions ρ_1, ρ_2 with Property 3 on intervals I_1, I_2 containing L_0 such that $\rho_k(L_0) = \rho_0$ we have $\rho_1|(I_1 \cap I_2) = \rho_2|(I_1 \cap I_2)$. In addition, if L_b is a finite boundary point of the interval I, then $\lambda_{\rho(L)} \to 0$ as $L \to L_b$.

Note the result in [1], which with $\mathsf{p}_0 = 0$, $m = 1$, and using different spaces, proves the existence of a family of solutions for all angular momenta without any reference to stability, as does the earlier result in [2].

Now we continue with describing our results about systems of gas balls close to systems of mass points. To this end we introduce the restricted m-body problem. Let Z_k ($k = 1, \dots, m$) be m distinct mass points in $\mathbb{R}^3$ with masses M_k, rotating around the x_3-axis with angular velocity $\omega \neq 0$, and combine these vectors into the $3 \times m$ matrix

$$Z = [Z_1, \dots, Z_m].$$

Then $M_0 = \sum_{k=1}^m M_k$ is the total mass and

$$Z_0 = M_0^{-1} \sum_{k=1}^{m} M_k Z_k$$

the center of gravity. The matrix Z is called a *solution of the restricted m-body problem* with angular velocity ω if it satisfies the system of equations $Q_\omega(Z) = 0$ with

$$Q_{k,\omega}(Z) = M_k \omega^2 P Z_k - \mathsf{k} \sum_{j=1,\, j \neq k}^{m} M_k M_j (Z_k - Z_j)|Z_k - Z_j|^{-3}, \tag{7}$$

and $Q_\omega(Z) = [Q_{1,\omega}(Z), \dots, Q_{m,\omega}(Z)]$. The physical meaning of this system is that if the mass points move with angular velocity ω around the x_3-axis, the centrifugal force is in equilibrium with the force of gravity. Note that only ω^2 enters the equation, so any Z solving equation (7) for ω also solves it for $-\omega$. After a translation in the x_3-direction we can make $Z_0 = 0$, which we assume from now on. As $(PZ_k)_3 = 0$ for all $k = 1, \dots, m$, by Lemma 10 we then have $Z_{3k} = 0$, and $PZ_k = Z_k$. Note that for all $\sigma > 0$ we have $Q_{\sigma^{-3/2}\omega}(\sigma Z) = \sigma^{-2} Q_\omega(Z)$, therefore if Z is a solution of $Q_\omega(Z) = 0$ for the angular velocity ω, then $Q_{\sigma^{-3/2}\omega}(\sigma Z) = 0$ as well. Thus we can obtain all solutions with $\omega > 0$ from those for $\omega = 1$. Assume from now on that the $3 \times m$ matrix Z solves the equation $Q_1(Z) = 0$. Then also $Q_{\sigma^{-3/2}}(\sigma Z) = 0$, which means that σZ is a solution with angular velocity $\omega = \sigma^{-3/2}$.

Assuming Condition 1.4 of [4], we can replace these points σZ for sufficiently large σ by gas balls that are slight deformations of single, non-rotating gas balls with the same mass in equilibrium, without obtaining any stability properties for the resulting solution.

We first restate the part of Condition 1.4 in [4] referring to the matrix Z here. Let V be the space of all $3 \times m$ matrices and V_1 be the set of all $3 \times m$ matrices with distinct columns, and let A be the matrix from [4] with the property that $Ax = e_3 \times x$ for $x \in \mathbb{R}^3$.

CONDITION 5. $Z \in V_1$ and $Q_1(Z) = 0$, and

$$\{dZ \in V \mid dQ_1(Z)(dZ) = 0\} \subset \{aAZ + b \mid a \in \mathbb{R},\, b \in \mathbb{R}^3\},$$

where dQ_1 is the first derivative of Q_1.

Now we proceed to the part of Condition 1.4 in [4] referring to the non-rotating gas balls. It was mostly proved in [8] for $p(\rho) = C\rho^{\varkappa}$ with $\varkappa > 4/3$, giving an energy-stable solution for a non-rotating gas ball. To obtain the necessary form of Condition 1.4 in [4] one needs to refer to [7, Theorem 12], and apply it to non-rotating gas balls.

CONDITION 6. For every $M > 0$ there is an $R > 0$ and a function $f : [0, R] \to (0, \infty)$ such that $\Omega = B_R$ and $\rho(x) = f(|x|)$ form a solution of equations (2) and (3) for $\omega = 0$ with mass M, also fulfilling Condition 2.

If we have a solution of the restricted m-body problem fulfilling Condition 5 and a fluid fulfilling Condition 6, then by the aforementioned results we can deduce that for sufficiently distant mass points, obtained by a proper scaling of Z, we can replace these points by gas balls. To obtain the stability properties of these solutions, we need to add the following stability Condition 7 for the solution of the restricted m-body problem with $\omega = 1$, also letting

$$L_3 = \sum_{k=1}^{m} M_k |Z_k|^2.$$

CONDITION 7. There is an $\eta_2 > 0$ such that for all $dZ_j \in \mathbb{R}^3$ $(j = 1, \ldots, m)$ that satisfy

$$\sum_{j=1}^{m} M_j dZ_j = 0, \qquad \sum_{j=1}^{m} (e_3 \times Z_j) \cdot dZ_j = 0 \quad (j = 1, \ldots, m),$$

we have

$$\begin{aligned}\mathcal{E}(dZ) &= 4L_3^{-1}\Big(\sum_{k=1}^{m} M_k (Z_k \cdot dZ_k)\Big)^2 - \sum_{k=1}^{m} M_k |P(dZ_k)|^2 \\ &\quad + \frac{\mathsf{k}}{2} \sum_{j\neq k} M_j M_k \left[\frac{|dZ_j - dZ_k|^2}{|Z_k - Z_j|^3} - 3\frac{((dZ_j - dZ_k)\cdot(Z_k - Z_j))^2}{|Z_k - Z_j|^5}\right] \geq \eta_2 |dZ|^2.\end{aligned}$$

In preparation for Theorem 8 we describe the result we obtain from [4] in more detail. Let ρ_k^0 be the density of the solutions of the above problem with the masses M_k with $\Omega_k^0 = B_{R_k}$ without any rotation. By [4, Theorem 1.5] for $\sigma \geq \sigma_0$ there is a solution with m gas regions close to gas balls centered on the points σZ_k with masses M_k rotating with angular velocity $\omega = \sigma^{-3/2}$.

We denote these solutions by $(\rho^\sigma, \Omega^\sigma)$ with

$$\Omega^\sigma = \bigcup_{k=1}^{m} \Omega_k^\sigma$$

where $\sigma Z_k \in \Omega_k^\sigma$ for $\sigma \geq \sigma_0$, and indeed $|\sigma Z_k - X_k^\sigma| \to 0$ as $\sigma \to \infty$, where X_k^σ is the center of gravity of the gas in Ω_k^σ, and there are m one-parameter families $T_k^\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ converging to the identity in C^1 as $\sigma \to \infty$ such that $\Omega_k^\sigma = T_k^\sigma(B_{R_k}) + \sigma Z_k$, and $\rho^\sigma \circ (T^\sigma + \sigma Z_k)$ converges in $C^1(\overline{B}_{R_k})$ to ρ_k^0. These can also be chosen in such a way that

$$M_0 X_0^\sigma = \sum_{k=1}^{m} M_k X_k^\sigma = 0.$$

For the following two theorems we assume that $p(\rho)$ and $\mathsf{p}_0 > 0$ together fulfill Condition 6.

Theorem 8. *Assume Z fulfills Conditions* 5 *and* 7. *Then there exists a σ_1 such that for $\sigma \geq \sigma_1$ the equilibrium solutions given by the densities ρ^σ with the properties described above exist and fulfill Condition* 2.

This theorem will be proved in Section 3. In Section 4 we prove that Conditions 5 and 7 are always fulfilled for the restricted two-body problem, and therefore we also have the following theorem if Condition 6 is fulfilled.

Theorem 9. *For $m = 2$ there exists a σ_1 such that for $\sigma \geq \sigma_1$ the equilibrium solutions given by the densities ρ^σ with the properties described above exist and fulfill Condition* 2.

This has the consequence that if we begin with a configuration in which both balls carry out a motion close to a circular one fulfilling Condition 2, and if they are sufficiently close to the equilibrium, the solution will converge to a circular motion, with both gas balls together moving like rigid bodies in the limit.

1.1. Notation. The vectors used in this paper are ordinarily column vectors. For any matrix C we denote its transpose by $C^\dagger$. We define the vector product for column vectors v, w by $v \times w = (v^\dagger \times w^\dagger)^\dagger$. An exception to this are gradients of functions. Also, $'$ denotes the derivative of any single variable function with respect to that variable. We combine the mass points X_k forming a solution of a restricted m-body problem into a $3 \times m$ matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ X_{31} & X_{32} & \dots & X_{3m} \end{bmatrix} = [X_1 \quad X_2 \quad \dots \quad X_m].$$

Let $e_1 = [1,0,0]^\dagger, e_2 = [0,1,0]^\dagger, e_3 = [0,0,1]^\dagger$ be the canonical basis of $\mathbb{R}^3$.

The norm of any Banach space $\mathcal{B}$ is denoted by $\|\cdot\|_{\mathcal{B}}$. Let $S \subset \mathbb{R}^n$ be an arbitrary set. For $\delta \in [0,1)$ let $C^\delta(S)$ be the set of all continuous functions $f : S \to \mathbb{R}$ with

$$\|f\|_{C^\delta(S)} = \sup_{x,x' \in S,\, x \neq x'} |f(x) - f(x')|\,|x - x'|^{-\delta} + \sup_{x \in S} |f(x)| < \infty.$$

These are Banach spaces with the norms just defined. Now let S be an open set. Then we can define partial derivatives, and for $k = 0, 1, \ldots$ and $\delta \in [0,1)$ we can define $C^{k+\delta}(S)$ as the set of all functions f such that $\nabla^r f \in C^\delta(S)$ for all $r = 0, \ldots, k$. Then with the norm

$$\|f\|_{C^{k+\delta}(S)} = \sum_{r=0}^{k} \|\nabla^r f\|_{C^\delta(S)},$$

this is also a Banach space. If f is a function defined on $\overline{S}$, and all $\nabla^r(f|S)$ are functions that are continuously extendable to $\overline{S}$ and the extensions belong to $C^\delta(\overline{S})$, then we say $f \in C^{k+\delta}(\overline{S})$ and this set is a Banach space with the norm $\|f\|_{C^{k+\delta}(S)}$. If f is defined on a set larger than $\overline{S}$, then we say $f \in C^{k+\alpha}(\overline{\Omega})$ if $f|\overline{\Omega} \in C^{k+\alpha}(\overline{\Omega})$.

If Ω is a bounded open set with C^1 boundary $\partial\Omega$ then for $p \in [1, \infty]$ let $\widehat{L}^p = L^p(\Omega) \times L^p(\partial\Omega)$. While in some of the previous papers quoted here the equilibrium solutions are distinguished by an upper ^{e}, we will not do that here, as we are exclusively considering equilibria.

2. Estimates for equilibrium solutions and some of their consequences. We begin with an elementary lemma about solutions of the restricted m-body problem we need several times later. This lemma implies that $X_{3j} = 0$ for all j in general, and that $\omega = 0$ does not allow multiple mass points.

LEMMA 10. *Let $M_1, \ldots, M_m > 0$ and let X be a $3 \times m$ matrix with distinct columns such that*

$$M_k \omega^2 P X_k = \mathsf{k} \sum_{j=1,\, j \neq k}^{m} M_k M_j (X_k - X_j)|X_k - X_j|^{-3}$$

and $M_1 X_1 + \cdots + M_m X_m = 0$. If for a $\mu \in \{1,2,3\}$ we have $\omega(PX_k)_\mu = 0$ for $k = 1, \ldots, m$, then $X_{\mu k} = 0$ for $k = 1, \ldots, m$ as well.

Proof. Let k be such that $X_{\mu k} - X_{\mu j} \geq 0$ for all j. Then

$$0 = M_k \omega^2 (PX_k)_\mu = \mathsf{k} \sum_{j=1,\, j \neq k}^{m} M_k M_j (X_{\mu k} - X_{\mu j})|X_k - X_j|^{-3}.$$

All summands on the right are non-negative, thus the sum can only be zero if they are all equal to zero. Therefore $X_{\mu k} = X_{\mu j}$ for all j. As $M_1 X_{\mu 1} + \cdots + M_m X_{\mu m} = 0$, they are all zero. ∎

Now we proceed to the main subject of the section. As mentioned before, the aim is to prove Theorem 4. We begin with a series of a priori estimates. These concern solutions with densities ρ positive on a set $\overline{\Omega}$ of class C^4, $\rho \in C^4(\overline{\Omega})$, with $\overline{\Omega}$ consisting of m disjoint compact sets $\overline{\Omega}_k$. With angular velocity ω and constants C_k these satisfy equations (3) and (2). Also let

$$L = \omega \int_{\Omega} \rho |Px|^2 \, dx.$$

The following estimates are always true for each solution individually; the important point here is that we have universal constants for all solutions. The generic constants C in the following results only depend on the information mentioned before and in each lemma.

For the convenience of the reader we now give a proof of a well-known estimate for the Newton potential.

LEMMA 11. *For $f \in C_0^\infty(\mathbb{R}^3)$ $(1 < p < \infty)$ let*

$$\mathcal{V}(f)(x) = \int_{\mathbb{R}^3} |x-y|^{-1} f(y) \, dx.$$

There exists a constant C, only depending on p, such that for $p \in (1,\infty)\backslash\{3/2, 3\}$ we have

$$\|\nabla^2 \mathcal{V}(f)\|_{L_p} + \|\nabla \mathcal{V}(f)\|_{L_{q_1}} + \|\mathcal{V}(f)\|_{L_{q_2}} \le C(\|f\|_{L_p} + \|f\|_{L_1})$$

with $q_1 = 3p/(3-p)$ for $p < 3$, $q_1 = \infty$ for $p > 3$, $q_2 = 3p/(3-2p)$ for $p < 3/2$, and $q_2 = \infty$ for $p > 3/2$.

Proof. The estimates for $\nabla^2 \mathcal{V}(f)$ follow from [3, Theorem 9.9], as the constant C there only depends on n and p. For $p < 3/2$ the estimates follow from this by using [3, Theorem 7.10], which we can apply here as the constant there again is independent of the domain, and we can approximate the relevant functions by functions with compact support. For $p > 3/2$, as we also have $\|f\|_{L_1}$ on the left-hand side of the inequality, we can use the inequality with $p = 4/3$, combined with [3, (7.42) in Theorem 7.17], which does not depend on the boundary condition, to get the remainder of the claim. ∎

For Lemmas 12 to 15 below, the constants C only depend on $L, \beta, M, \varepsilon'(\vartheta)$, ϑ and η_1. (The assumptions on ε we need are stated in Theorem 4.) Note that

$$\vartheta \operatorname{Vol}(\Omega) = \int_{\Omega} \vartheta \, dx \le \int_{\Omega} \rho \, dx = M,$$

thus $\operatorname{Vol}(\Omega) \le \vartheta^{-1} M$.

LEMMA 12. *There is a constant C such that for solutions of equation* (3) *we have*

$$\|\rho\|_{L^{\beta+1}}^{\beta+1} + \int_\Omega \varepsilon'(\rho)\rho\,dx + \int_\Omega V\rho\,dx \le C.$$

Proof. Note that L and ω have the same sign and can be chosen nonnegative. Multiplying the second equation in (3) by ρ and integrating over Ω we obtain

$$\eta_1\|\rho\|_{L^{\beta+1}}^{\beta+1} - \int_\Omega \rho V\,dx \le \int_\Omega \rho\varepsilon'(\rho)\,dx - \int_\Omega \rho V\,dx = \frac{1}{2}L\omega + \sum_{k=1}^m C_k M_k.$$

For $x \in \partial\Omega_k$ we have

$$\varepsilon'(\vartheta) = V + \frac{1}{2}\omega^2|Px|^2 + C_k \ge \frac{1}{2}\omega^2|Px|^2 + C_k.$$

Now the function $|Px|^2$ cannot have an interior maximum in Ω_k, therefore also

$$\varepsilon'(\vartheta) \ge \frac{1}{2}\omega^2|Px|^2 + C_k \tag{8}$$

in $\overline{\Omega}_k$. Again, multiplying by ρ and integrating over Ω_k results in

$$\varepsilon'(\vartheta)M_k = \int_{\Omega_k} \rho\varepsilon'(\vartheta)\,dx \ge \frac{1}{2}\omega^2 \int_{\Omega_k} \rho|Px|^2\,dx + C_k M_k.$$

Adding over k, we get

$$M\varepsilon'(\vartheta) \ge \frac{1}{2}\omega^2 \int_\Omega \rho|Px|^2\,dx + \sum_{k=1}^m C_k M_k = \frac{1}{2}L\omega + \sum_{k=1}^m C_k M_k,$$

and therefore

$$\eta_1\|\rho\|_{L^{\beta+1}}^{\beta+1} - \int_\Omega \rho V\,dx \le \int_\Omega \rho\varepsilon'(\rho)\,dx - \int_\Omega \rho V\,dx \le M\varepsilon'(\vartheta).$$

Using [8, Lemma 17] and the fact that the L^1-norm of ρ equals M, we get $\|\rho\|_{L^{\beta+1}} \le C$, and even

$$\|\rho\|_{L^r} \le C$$

for all $r \in [1, 1+\beta]$, therefore in particular for $r = 4/3$. By Lemma 11 we obtain $\|V\|_{L^{12}} \le C$. Therefore

$$\|\rho V\|_{L^1} \le \|\rho\|_{L^{12/11}}\|V\|_{L^{12}} \le C$$

and

$$\int_\Omega \rho\varepsilon'(\rho)\,dx \le M\varepsilon'(\vartheta) + \int_\Omega \rho V\,dx \le C.$$

This completes the proof of our lemma. ■

LEMMA 13. *For $\beta \in (1/3, 1/2]$ there is a constant C such that $\|V\|_{C^0(\mathbb{R}^3)} \le C$.*

Proof. We differentiate equation (3) with respect to x_3 to obtain

$$\varepsilon''(\rho)\rho_{x_3} = V_{x_3}$$

and

$$\rho_{x_3} = (\varepsilon''(\rho))^{-1} V_{x_3}$$

inside Ω. Now by the assumption from Theorem 4,

$$|(\varepsilon''(\rho))^{-1}|^2 \le \eta_1^{-2}\rho^{2-2\beta},$$

therefore, using Lemma 12,

$$\|(\varepsilon''(\rho))^{-1}\|^2_{L_2(\Omega)} \le C\|\rho\|^{2-2\beta}_{L_{2-2\beta}} \le C,$$

as $1 \le 2-2\beta \le 4/3$. Now let $\rho_0(x) = \rho(x) - \vartheta$ for $x \in \overline{\Omega}$ and $\rho_0(x) = 0$ for other x. Note that $\rho_0 \in W^1_p(\mathbb{R}^3)$ for $p \in [1,\infty)$. Now $V = V_0 + V_1$ with

$$V_0(x) = \mathsf{k}\int\limits_{\mathbb{R}^3} \frac{1}{|x-y|}\rho_0\,dy, \qquad V_1(x) = \mathsf{k}\vartheta\int\limits_{\mathbb{R}^3} \frac{1}{|x-y|}\chi_\Omega\,dy.$$

Observe that $\|\chi_\Omega\|_{L_p} \le C$ with a C independent of p for $p \in [1,\infty]$. Therefore $\|V_{1x_3}\|_{L_\infty}$ is bounded. Also

$$V_{0x_3}(x_3) = \mathsf{k}\int\limits_{\Omega} \frac{1}{|x-y|}\rho_{0x_3}(y)\,dy.$$

If $2 \le \mu < \infty$ and $V_{0x_3} \in L_\mu(\Omega)$, with $s = \frac{2\mu}{\mu+2}$ we have

$$\begin{aligned}\|\rho_{x_3}\|_{L_s(\Omega)} &\le \|(\varepsilon''(\rho))^{-1}\|_{L_2(\Omega)}\|V_{x_3}\|_{L_\mu(\Omega)}\\ &\le \|(\varepsilon''(\rho))^{-1}\|_{L_2(\Omega)}(C + \|V_{0x_3}\|_{L_\mu}).\end{aligned}$$

As we already know from Lemma 12 that ρ is bounded in $L_{4/3}$, from Lemma 11 we get $\|V_{x_3}\|_{L_{12/5}} \le C$, and applying the above inequality with $\mu = 12/5$, we obtain $\|\rho_{x_3}\|_{L_{12/11}(\Omega)} = \|\rho_{0x_3}\|_{L_{12/11}} \le C$. Using Lemma 11 again, we get $\|V_{0x_3}\|_{L_4} \le C$ and with $\mu = 4$ we have $\|\rho_{0x_3}\|_{L_{4/3}} \le C$, as well as $\|V_{0x_3}\|_{L_{12}} \le C$, thus $\|\rho_{0x_3}\|_{L_{12/7}} \le C$, and $\|V_{0x_3}\|_{L_\infty} \le C$, thus finally

$$\|\rho_{0x_3}\|_{L_2} \le C.$$

Then

$$\begin{aligned}\frac{d}{dx_3}\int\limits_{\mathbb{R}^2} \rho_0^{5/3}\,dx_1\,dx_2 &= \frac{5}{3}\int\limits_{\mathbb{R}^2} \rho_0^{2/3}\rho_{0x_3}\,dx_1\,dx_2\\ &\le \frac{5}{3}\Big(\int\limits_{\mathbb{R}^2} \rho_0^{4/3}\,dx_1\,dx_2\Big)^{1/2}\Big(\int\limits_{\mathbb{R}^2} (\rho_{0x_3})^2\,dx_1\,dx_2\Big)^{1/2}\\ &\le \frac{5}{6}\Big(\int\limits_{\mathbb{R}^2} \rho_0^{4/3}\,dx_1\,dx_2 + \int\limits_{\mathbb{R}^2} (\rho_{0x_3})^2\,dx_1\,dx_2\Big).\end{aligned}$$

Integrating over the real axis we obtain

$$\max_{x_3} \int\limits_{\mathbb{R}^2} \rho_0^{5/3}\,dx_1\,dx_2 \le \frac{5}{6}(\|\rho_0\|_{L_{4/3}}^{4/3} + \|\rho_{0x_3}\|_{L_2}^2) \le C$$

and

$$\int\limits_{|x-y|<1} \rho_0^{5/3}\,dx \le C.$$

Then we easily see our claim, as we can write

$$V(x) = \mathsf{k} \int\limits_{\{y\in\Omega\,|\,|x-y|\ge 1\}} \frac{\rho(y)}{|x-y|}\,dy + \mathsf{k} \int\limits_{\{y\in\Omega\,|\,|x-y|<1\}} \frac{\rho(y)}{|x-y|}\,dy,$$

and so, using [3, Lemma 7.12] for the second integral, we have

$$|V(x)| \le \mathsf{k}\|\rho\|_{L_1} + C\|\rho\|_{L_{5/3}(\{y\in\Omega\,|\,|x-y|<1\})} \le C. \ \blacksquare$$

LEMMA 14. *Assume $\beta > 1/2$. Then there is a constant C such that $\|V\|_{C^0(\mathbb{R}^3)} \le C$.*

Proof. Let us first consider the case $1/2 < \beta \le 1$. Then by Lemmas 11 and 12, we have $\|\nabla V\|_{L^{3(1+\beta)/(2-\beta)}} \le C$ and $\|V\|_{L^\infty} \le C$. In case β is larger, the latter is obviously still true by Lemma 11. $\blacksquare$

LEMMA 15. *There exists a constant C such that*

$$|\omega| \le C, \qquad |C_k| \le C, \qquad \omega^2|Px|^2 \le C$$

for $x \in \overline{\Omega}_k$ $(k = 1,\dots,m)$ as well as $\|\rho\|_{L^\infty} \le C$.

Proof. Remember that from inequality (8) we know that for $x \in \overline{\Omega}_k$,

$$\frac{1}{2}\omega^2|Px|^2 + C_k \le \varepsilon'(\vartheta),$$

and in addition for $x \in \overline{\Omega}_k$ also

$$0 \le \varepsilon'(\rho(x)) = V(x) + \frac{1}{2}\omega^2|Px|^2 + C_k,$$

therefore

$$-\frac{1}{2}\omega^2|Px|^2 - C_k \le V(x) \le C.$$

Putting these two inequalities together, we obtain

$$\left|\frac{1}{2}\omega^2|Px|^2 + C_k\right| \le C \tag{9}$$

for $x \in \overline{\Omega}_k$. Now

$$\varepsilon'(\rho) = V + \frac{1}{2}\omega^2|Px|^2 + C_k,$$

and as the right-hand side is uniformly bounded, so too is the left-hand side, and, as one easily sees, also ρ. To see ω is bounded, let $x \in \overline{\Omega}$ be a point where ρ takes a local maximum. Then so does $\varepsilon'(\rho)$, and by [7, Theorem 5] we have $x \in \Omega$, therefore

$$0 \geq \Delta(\varepsilon'(\rho))(x) = \Delta\Big(V + \frac{1}{2}\omega^2|Px|^2 + C_k\Big) = -4\pi\mathsf{k}\rho(x) + 2\omega^2,$$

thus

$$2\pi\rho(x) \geq \omega^2.$$

As ρ is bounded, the boundedness of $|\omega|$ is proved. Now

$$\vartheta \int_{\Omega} \omega|Px|^2\,dx \leq \int_{\Omega} \omega\rho|Px|^2\,dx = L,$$

therefore, as ω is bounded, we have $\|\omega^2|Px|^2\|_{L^1} \leq C$. Also from inequality (9) we have

$$\Big\|\frac{1}{2}\omega^2|Px|^2 + C_k\Big\|_{L^1(\Omega_k)} \leq C\,\mathrm{Vol}(\Omega_k) \leq C\vartheta^{-1}M_k,$$

therefore also $\|C_k\|_{L^1(\Omega_k)} \leq C$ and

$$M_k(\|\rho\|_{C^0(\overline{\Omega}_k)})^{-1}|C_k| \leq \mathrm{Vol}(\Omega_k)|C_k| = \|C_k\|_{L^1(\Omega_k)} \leq C,$$

thus $|C_k|$ is bounded and so is $\frac{1}{2}\omega^2|Px|^2$. ∎

In the following lemma, the constants C depend on the numbers in the previous lemma, and the last one also on

$$\max_{\vartheta\leq\rho\leq\max_{x\in\overline{\Omega}}\rho(x)} |\varepsilon'''(\rho)|.$$

Lemma 16. *For every $\alpha \in (0,1)$ there is a C such that $\|V\|_{C^{1+\alpha}} \leq C$ as well as $\|\rho\|_{C^1(\Omega)} \leq C$ and $|\nabla\rho(x_1) - \nabla\rho(x_2)| \leq C|x_1 - x_2|^\alpha$ for $x_1, x_2 \in \overline{\Omega}$.*

Proof. The first inequality is easy to see as a result of Lemma 15. From equation (3) we have

$$\varepsilon''(\rho)\nabla\rho = \nabla V + \omega^2 Px.$$

Now by Lemma 15 we also have $|\omega|\,|Px| \leq C$ and therefore $\omega^2|Px| \leq C$, which implies $\|\rho\|_{C^1(\overline{\Omega})} \leq C$. Again, letting $\rho_0(x) = \rho(x) - \vartheta$ for $x \in \Omega$ and equal to zero outside, we get $\|\rho_0\|_{C^{0,1}} \leq C$. Also for $x \in \overline{\Omega}$,

$$\nabla\rho = (\varepsilon''(\rho_0 + \vartheta))^{-1}(\nabla V + \omega^2 Px).$$

Thus for arbitrary $x_1, x_2 \in \overline{\Omega}$ we have

$$|\nabla\rho(x_1) - \nabla\rho(x_2)| \leq C|x_1 - x_2|^\alpha,$$

using also the fact that $\nabla\rho$ is bounded. ∎

While the lemmas proved so far are independent of the stability of the solutions, it now comes into play. The constants depend on the numbers mentioned so far and the numbers listed in the statements.

LEMMA 17. *Given $\delta > 0$ there is a constant C such that* $\operatorname{diam}(\overline{\Omega}_k) \le C$ *for $k = 1, \ldots, m$ if $\lambda_\rho \ge \delta$.*

Proof. First let us confine ourselves to a fixed $\overline{\Omega}_k$. For $x, y \in \overline{\Omega}_k$ let

$$L(x,y) = \{\gamma : [0,1] \to \overline{\Omega}_k \mid \gamma \in C^1([0,1]),\ \gamma(0) = x,\ \gamma(1) = y\}.$$

Then we can define a metric

$$d(x,y) = \inf_{\gamma \in L(x,y)} l(\gamma)$$

in $\overline{\Omega}_k$, where $l(\gamma)$ is the length of the path given by γ. It generates the same topology as the Euclidean metric. Obviously

$$\max_{x,y \in \overline{\Omega}_k} d(x,y) \ge \operatorname{diam}(\overline{\Omega}_k).$$

Now for $x \in \overline{\Omega}_k$ let

$$\mathcal{B}_r(x) = \{y \in \overline{\Omega}_k \mid d(x,y) < r\}.$$

We want to estimate the volume of these "balls" from below. Let y_0 be the closest point to x in $\partial\Omega_k$. Then $x - y_0 = c\mathfrak{n}(y_0)$. After a rotation and translation we may assume $x = 0$ and $y_0 = (0,0,\xi)$ with a $\xi \ge 0$, and $\mathfrak{n}(y_0)=(0,0,1)$ in the transformed version. Let us denote the transformed ball by $\widetilde{\mathcal{B}}_r(0)$. As we have a lower bound for $\left|\frac{\partial\rho}{\partial\mathfrak{n}}\right|$ on $\partial\Omega_k$, there is an $r_0 > 0$ such that for $r \le r_0$ there is a function f such that $f(0,0) = \xi$ and

$$\widetilde{\mathcal{B}}_r(x) = \{y \in \mathbb{R}^3 \mid |y| < r,\ y_3 < f(y_1,y_2)\}.$$

Also $\nabla f(0,0) = 0$ and $|\nabla f(x_1,x_2)| \le C|(x_1,x_2)|^\alpha$. One easily sees there is a C such that $V(\mathcal{B}_r(x)) \ge \frac{2}{3}\pi r^3 - Cr^{3+\alpha}$, so we can ensure that $\operatorname{Vol}(\mathcal{B}_r(x)) \ge \frac{1}{3}\pi r^3$ for $r \le r_0$ for a sufficiently small new r_0, in particular for $r = r_0$. There is a minimizing rectifiable curve joining two points of maximal distance in $\overline{\Omega}_k$. Placing points along this curve with distance exactly $2r_0$, and looking at the balls $\mathcal{B}_{r_0}(x)$ around these points, which are disjoint, we can see that if the diameter of Ω_k goes to infinity, so must the volume. This proves our claim. ∎

LEMMA 18. *There exists a C such that* $\operatorname{diam}(\Omega) \le C$ *for all Ω if $\lambda_\rho \ge \delta > 0$.*

Proof. In case $m = 1$ this is already clear, so we may assume $m > 1$. We argue by contradiction. As all Ω_k contain points on $x_3 = 0$, we see that if $\operatorname{diam}(\Omega)$ is unbounded, so is $\operatorname{diam}(P(\Omega))$. This means we have sequences $\Omega^\nu, \rho^\nu, \omega^\nu$ corresponding to equilibrium solutions and $\operatorname{diam}(P(\Omega^\nu)) \to \infty$. By Lemma 14 then $\omega^\nu \to 0$. Let X_k^ν be the center of gravity of the gas in Ω_k^ν. With $\xi_\nu = \max_{k=1,\ldots,m} |PX_k^\nu|$ we have $\xi_\nu \to \infty$. Letting $Z_k^\nu = \xi_\nu^{-1} X_k^\nu$, we can select a subsequence such that $Z_k^\nu \to \overline{Z}_k$ for $k = 1, \ldots, m$. The $\overline{Z}_k$ cannot all be the same, as $|P(\overline{Z}_k)| = 1$ for one k, so if they were all equal, the center of gravity could not be at zero. Now let $\widetilde{Z}_k (k = 1, \ldots, \widetilde{m})$ be a complete listing of these limits without repetition, and let $\widetilde{\Omega}_k^\nu$ be the open sets obtained by joining together those with Z_k^ν converging to the same point. Now

$$0 = \int_{\widetilde{\Omega}_k^\nu} \rho^\nu \nabla(\varepsilon'(\rho^\nu))\,dx - \int_{\widetilde{\Omega}_k^\nu} \rho^\nu \nabla \widetilde{V}_k^\nu\,dx - \sum_{j=1,\,j\neq k}^{\widetilde{m}} \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx - \int_{\widetilde{\Omega}_k^\nu} \rho(\omega^\nu)^2 Px\,dx$$

and using [4, Lemmas 3.2, 3.3] we find that

$$\int_{\widetilde{\Omega}_k^\nu} \rho^\nu \nabla(\varepsilon'(\rho^\nu))\,dx = 0, \qquad \int_{\widetilde{\Omega}_k^\nu} \rho^\nu \nabla \widetilde{V}_k^\nu\,dx = 0,$$

thus

$$0 = \sum_{j=1,\,j\neq k}^{\widetilde{m}} \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx + (\omega^\nu)^2 \int_{\widetilde{\Omega}_k^\nu} \rho Px\,dx = \sum_{j=1,\,j\neq k}^{\widetilde{m}} \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx + (\omega^\nu)^2 \widetilde{M}_k P \widetilde{X}_k^\nu,$$

and also multiplying the equation by ξ_ν^2, we obtain

$$\sum_{j=1,j\neq k}^{\widetilde{m}} \xi_\nu^2 \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx + \xi_\nu^3 (\omega^\nu)^2 \widetilde{M}_k P \widetilde{Z}_k^\nu = 0.$$

Now for $k \neq j$,

$$\xi_\nu^2 \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx = -\mathsf{k} \int_{\widetilde{\Omega}_k^\nu} \int_{\widetilde{\Omega}_j^\nu} \rho^\nu(x)\rho^\nu(y)\xi_\nu^{-1}(x-y)|\xi_\nu^{-1}(x-y)|^{-3}\,dy\,dx.$$

Now

$$\max_{x\in\widetilde{\Omega}_k^\nu,\, y\in\widetilde{\Omega}_j^\nu} |\xi_\nu^{-1}(x-y) - (\widetilde{Z}_k - \widetilde{Z}_j)| \to 0$$

as $\nu \to \infty$, and therefore also

$$\xi_\nu^2 \int_{\widetilde{\Omega}_k^\nu} \rho \nabla \widetilde{V}_j^\nu\,dx \to \sum_{j=1,\,j\neq k}^{\widetilde{m}} \mathsf{k} \widetilde{M}_k \widetilde{M}_j |\widetilde{Z}_k - \widetilde{Z}_j|^{-3} (\widetilde{Z}_k - \widetilde{Z}_j).$$

so, using k with $|P\widetilde{Z}_k|$ maximal, we see that $\xi_\nu^3(\omega^\nu)^2$ is a bounded sequence, which we can assume to be convergent by passing to a subsequence. If the limit is zero, we get

$$\sum_{j=1, j\neq k}^{\widetilde{m}} \widetilde{M}_k \widetilde{M}_j |\widetilde{Z}_k - \widetilde{Z}_j|^{-3} (\widetilde{Z}_k - \widetilde{Z}_j) = 0,$$

and by Lemma 10 we have $\widetilde{Z}_k = 0$, so they are all equal. So $\xi_\nu^3(\omega^\nu)^2 \to A > 0$. Then for large ν we have

$$L \geq \frac{1}{2}\widetilde{M}_k \xi_\nu^2 \omega^\nu \geq \frac{1}{2}\widetilde{M}_k \left(\frac{1}{2}A\right)^{2/3} (\omega^\nu)^{-4/3}\omega_\nu = \frac{1}{2}\widetilde{M}_k \left(\frac{1}{2}A\right)^{2/3} (\omega^\nu)^{-1/3} \to \infty.$$

This is a contradiction. ■

Now we can prove Theorem 4. Using [4, Theorems 1.3, 5.2] it is easy to show there is a function as required on a, perhaps short, interval. Owing to local uniqueness, one can easily combine any two intervals I_1, I_2, obtaining a solution on $I_1 \cup I_2$. Now let us assume there is a sequence $L_k \in J$ with $L_k \to L_b$ such that $\lambda_{\rho(L_k)} \geq \delta > 0$. By [7, Theorem 4] we then have $\frac{\partial \rho(L_k)}{\partial \mathfrak{n}}(x) \leq -\sigma$ with a $\sigma > 0$. Combining our results we can select a subsequence converging to a limit with the same mass and angular momentum L_b. The limit Ω is in $C^{1+\alpha}$ as also are ρ and V. Following the ideas of [4, Section 7], we can easily obtain that these even belong to C^4, and it is easy to show that I is not maximal.

3. On the energy of systems of distant gas balls. The aim of this section is the analysis of the energy functional of distant gas balls with the aim of approximately splitting the energy up into that of mass points located at the centers of gravity of the gas balls and the thermodynamic and gravitational energies of each gas ball. As we are considering one solution at a time, we omit the index σ for most but not all of the section. Thus, in what follows Ω and ρ are the equilibrium solutions constructed using [4, Theorem 1.5] for a fixed σ. We consider a range of σ for which $X_k \in \Omega_k$ and $\operatorname{diam}(\Omega_k) \leq C$ for all k. Owing to the symmetry property proved in [7, Theorem 5], as we assume $X_0 = 0$, we have $X_{3k} = 0$ for $k = 1, \ldots, m$. In [7] we noted that by introducing a suitable coordinate system in $\mathbb{R}^3$ we can ensure that

$$\int_\Omega \rho x_i x_j \, dx = 0 \tag{10}$$

for $i \neq j$. We assume we have chosen our coordinate system so that this is true. Again, analogous to [7], we define

$$\mathsf{K}_i = \int_\Omega \rho(|x|^2 - x_i^2) \, dx \qquad (i = 1, 2, 3).$$

There we also proved the following lemma, except for the last inequality, which is, however, obvious.

Lemma 19. *For $w \in \mathbb{R}^3$ and $U = [w \times x, 0, 0]$ we have $Q_1^k(U) = 0$ for $k = 1, \ldots, m$, $Q_2(U) = 0$, $Q_3(U) = (\mathsf{K}_1 w_1, \mathsf{K}_2 w_2, \mathsf{K}_3 w_3)$, $Q_4(U) = 0$, and $E(U) = \sum_{i=1}^3 \mathsf{K}_i w_i^2 \geq \mathsf{K}_3 w_3^2$.*

We continue with an estimate:

Lemma 20. *There exists a constant C such that for σ sufficiently large,*

$$|\mathsf{K}_3 - \sigma^2 L_3| \leq C\sigma.$$

Proof. As $\mathsf{K}_3 = \sum_{\mu=1}^2 I_\mu$ with

$$I_\mu = \int_\Omega \rho x_\mu^2 \, dx = \sum_{k=1}^m \int_{\Omega_k} \rho x_\mu^2 \, dx$$

and

$$\int_{\Omega_k} \rho x_\mu^2 \, dx = M_k X_{\mu k}^2 + \int_{\Omega_k} \rho (x_\mu - X_{\mu k})(x_\mu + X_{\mu k}) \, dx,$$

we have

$$\Big| \int_{\Omega_k} \rho x_\mu^2 \, dx - M_k X_{\mu k}^2 \Big| \leq M_k \operatorname{diam}(\Omega_k)(2|X_k| + \operatorname{diam}(\Omega_k)).$$

Adding over μ and k, we obtain, also using $P X_k = X_k$,

$$\Big| \mathsf{K}_3 - \sum_{k=1}^m M_k |X_k|^2 \Big| \leq C \sum_{k=1}^m (|X_k| + 1).$$

These expressions are invariant under rotations around the x_3-axis, which are the transformations we need for equation (10), and, up to such rotations O_σ we have $|\sigma Z_k - O_\sigma X_k| \to 0$ as $\sigma \to \infty$. This easily implies our claim. ∎

Let us define an additional $3 \times m$ matrix dX by

$$dX_k = M_k^{-1} \Big[\int_\Omega x\alpha \, dy + \vartheta \int_{\partial\Omega} x\beta \, d\sigma \Big] = M_k^{-1} Q_4^k([0, \alpha, \beta]).$$

The columns of dX represent infinitesimal shifts of the centers of gravity of the different gas balls. Note that the condition $Q_4(U) = 0$ implies

$$\sum_{k=1}^m M_k dX_k = 0.$$

We want to restrict ourselves to $U = [u, \alpha, \beta]$ with $u = w \times x$. To enable this, we now state a condition we then show is equivalent to Condition 6.

CONDITION 21. For all real α, β, and $u = w \times x$ with a $w \in \mathbb{R}^3$ with $U = [u, \alpha, \beta] \in \mathfrak{H}_1$, and

$$\sum_{k=1}^{m} (e_3 \times X_k) \cdot dX_k = 0, \tag{11}$$

we have $E(U) \geq 0$, and $E(U) = 0$ implies $U = cU_r$ for some $c \in \mathbb{R}$.

Now we can prove the following. Note that we are considering Ω that are collections of slightly deformed round balls, so that the condition $\frac{\partial \rho}{\partial n} < 0$ on $\partial \Omega$ is always fulfilled here.

LEMMA 22. *If* $\frac{\partial \rho}{\partial n} < 0$ *on* $\partial \Omega$, *then Conditions* 2 *and* 21 *are equivalent.*

Proof. Let us first show that Condition 21′ obtained by removing equation 11 from Condition 21 is equivalent to Condition 2. To this end let $U = [u, \alpha, \beta] \in \mathfrak{H}_1$ with otherwise arbitrary $u \in L^2(\Omega)$. Let $[\widetilde{u}, 0, 0]$ be the projection of $[u, 0, 0]$ to the space consisting of all $[w \times x, 0, 0]$ in $\mathfrak{H}$ and $u^\perp = u - \widetilde{u}$. Finally let $\widetilde{U} = [\widetilde{u}, \alpha, \beta]$. Then obviously $Q_1^k(\widetilde{U}) = 0$ for $k = 1, \ldots, m$, $Q_4(\widetilde{U}) = 0$, $Q_2(\widetilde{U}) = Q_2([\widetilde{u}, 0, 0]) = 0$ by equation (4), and

$$\begin{aligned}(Q_3)_\nu(\widetilde{U}) &= (Q_3)_\nu(U) - (Q_3)_\nu([u^\perp, 0, 0]) = -(Q_3)_\nu([u^\perp, 0, 0]) \\ &= \int_\Omega \rho(e_\nu \times x) u^\perp \, dx = 0,\end{aligned}$$

as $u^\perp$ is orthogonal to all such velocities. Thus $\widetilde{U} \in \mathfrak{H}_1$. Finally, using the orthogonality properties of $u, \widetilde{u}, u^\perp$, we get

$$E(U) - E(\widetilde{U}) = \int_\Omega \rho(|u|^2 - |\widetilde{u}|^2) \, dx = \int_\Omega \rho |u^\perp|^2 \, dx.$$

By Condition 21′ we have $E(\widetilde{U}) \geq 0$ as $\widetilde{U} \in \mathfrak{H}_1$, therefore $E(U) \geq 0$ and $E(U) = 0$ implies $E(\widetilde{U}) = 0$ and $u^\perp = 0$, thus also $\widetilde{U} = U = cU_r$. Now we prove the equivalence of Conditions 21′ and 21.

Let $U = [w \times x, \alpha, \beta] \in \mathfrak{H}_1$. Using [5, Lemma 3.14], and an approximation argument, we can say that if $U \in \mathfrak{H}_1$, then for any real c also $U + cU_r \in \mathfrak{H}_1$ and $E(U + cU_r) = E(U)$. Furthermore

$$\begin{aligned}(Q_4^k)_j(U_r) &= \int_{\Omega_k} x_j \alpha_r \, dy + \vartheta \int_{\partial \Omega_k} x_j \beta_r \, d\sigma \\ &= -\int_{\Omega_k} x_j (e_3 \times x) \cdot \nabla \rho \, dx + \vartheta \int_{\partial \Omega_k} x_j (e_3 \times x) \cdot \mathfrak{n} \, dS \\ &= \int_{\Omega_k} \nabla x_j \cdot (e_3 \times x) \rho \, dx = \int_{\Omega_k} e_j \cdot (e_3 \times x) \rho \, dx \\ &= e_j \cdot (e_3 \times \int_{\Omega_k} x \rho \, dx) = M_k e_j \cdot (e_3 \times X_k),\end{aligned}$$

thus
$$Q_4^k(U_r) = M_k e_3 \times X_k,$$
and
$$Q_4^k(U + cU_r) = Q_4^k(U) + cM_k e_3 \times X_k,$$
therefore
$$\sum_{k=1}^{m} M_k^{-1}(e_3 \times X_k) \cdot [Q_4^k(U + cU_r)] = \sum_{k=1}^{m} M_k^{-1}(e_3 \times X_k) \cdot Q_4^k(U) + c\sum_{k=1}^{m} |e_3 \times X_k|^2.$$
This means that if we choose c properly and replace U with $U + cU_r$, then $U + cU_r$ fulfills (11). For this c we now have $E(U) = E(U + cU_r) \geq 0$ and $E(U + cU_r) = E(U) = 0$ only if $U + \widetilde{c}U_r = 0$, which is what we needed to prove. ■

To continue, we define different quantities we can see as constituent parts of the energy.

DEFINITION 23. For $u_j, \alpha_j \in L^2(\Omega)$, $\beta_j \in L^2(\partial\Omega)$ let $U_j = [u_j, \alpha_j, \beta_j] \in \mathfrak{H}$ $(j = 1, 2)$ and define
$$S_k^1(U_1, U_2) = \int_{\Omega_k} (\rho u_1 u_2 + \varepsilon''(\rho)\alpha_1\alpha_2)\, dx - \int_{\partial\Omega_k} \frac{\partial p}{\partial \mathfrak{n}} \beta_1\beta_2\, d\sigma,$$
$$S_k^2(U_1, U_2) = -\int_{\Omega_k} \alpha_1 I_{\Omega_k}(\alpha_2, \beta_2)\, dx - \vartheta \int_{\partial\Omega_k} \beta_1 I_{\Omega_k}(\alpha_2, \beta_2)\, d\sigma,$$
$$S_k^3(U_1, U_2) = -\sum_{j=1, j\neq k}^{m} \Big(\int_{\Omega_k} \alpha_1 I_{\Omega_j}(\alpha_2, \beta_2)\, dx + \vartheta \int_{\partial\Omega_k} \beta_1 I_{\Omega_j}(\alpha_2, \beta_2)\, d\sigma \Big),$$
$$S^\nu(U_1, U_2) = \sum_{k=1}^{m} S_k^\nu(U_1, U_2), S(U_1, U_2) = \sum_{\nu=1}^{3} S^\nu(U_1, U_2),$$
for $k = 1, \ldots, m$ and $\nu = 1, 2, 3$.

Note that $S_k^2(U, U)$ is the gravitational energy of the kth gas ball with respect to itself, and $E(U) = S(U, U)$.

As a result of Lemma 22 we just need to verify Condition 21 instead of Condition 6. To this end we split α and β into two functions each,
$$\alpha_0(x) = -dX_k \cdot \nabla\rho(x) \quad (x \in \Omega), \qquad \beta_0(x) = dX_k \cdot \mathfrak{n} \quad (x \in \partial\Omega)$$
and the differences $\alpha_1 = \alpha - \alpha_0$, $\beta_1 = \beta - \beta_0$. The functions α_0, β_0 correspond to variations generated by translating the equilibrium solutions by dX. It turns out that α_1, β_1 can be treated as perturbations in most contexts. Now

$Q_1^k([0,\alpha_0,\beta_0]) = 0$, and

$$(Q_4^k)_\mu([0,\alpha_0,\beta_0]) = -\int_{\Omega_k} x_\mu dX_k\cdot\nabla\rho(x)\,dx + \vartheta\int_{\partial\Omega_k} x_\mu dX_k\cdot\mathfrak{n}\,dS$$
$$= \int_{\Omega_k} \rho(x)dX_k\cdot\nabla(x_\mu)\,dx = M_k dX_k\cdot e_\mu,$$

therefore $Q_4^k([0,\alpha_0,\beta_0]) = M_k dX_k = Q_4^k([0,\alpha,\beta])$ and $Q_4^k([0,\alpha_1,\beta_1]) = 0$. In preparation for the next theorem we prove the following lemma.

LEMMA 24. *With $\alpha_1,\alpha_0,\beta_1,\beta_0$ as just defined we have*

$$Q_3^k([0,\alpha_0,\beta_0]) = \omega M_k(2dX_k\cdot X_k e_3 - dX_{3k}X_k)$$

and

$$|Q_3^k([0,\alpha_1,\beta_1])| \le C\omega\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}.$$

Proof. Using [5, Lemma 3.2] we get, as also $X_{3k} = 0$,

$$(Q_3^k)_\mu([0,\alpha_0,\beta_0]) = \omega\int_{\Omega_k^\sigma} \rho dX_k\cdot\nabla(\delta_{\mu 3}|x|^2 - x_3x_\mu)\,dx$$
$$= 2\delta_{\mu 3}\omega\int_{\Omega_k}\rho dX_k\cdot x\,dx - \omega\int_{\Omega_k}\rho dX_k\cdot(x_\mu e_3 + x_3e_\mu)\,dx$$
$$= 2\omega M_k\delta_{\mu 3}dX_k\cdot X_k - \omega\int_{\Omega_k^\sigma}\rho x_\mu dX_{3k}\,dx - \omega\int_{\Omega_k^\sigma} x_3\rho dX_{\mu k}\,dx$$
$$= 2\omega M_k\delta_{\mu 3}dX_k\cdot X_k - \omega M_k X_{\mu k}dX_{3k},$$

so we have proved our first equation.

To continue we temporarily use the linear form on $C^0(\overline{\Omega}^e)$ given by

$$F(g) = \int_{\Omega_k}\alpha_1 g\,dx + \vartheta\int_{\partial\Omega_k}\beta_1 g\,dS$$

for $g\in C^0(\overline{\Omega}_k)$. Then

$$|F(g)| \le C\|g\|_{C^0(\overline{\Omega}_k)}\|(\alpha_1,\beta_1)\|_{\widehat{L}_1}.$$

Note that the expressions in $Q_3^k([0,\alpha_1,\beta_1])$ are combinations of $F(x_ix_j)$. Also using $Q_1^k([0,\alpha_1,\beta_1]) = F(1) = 0$ and $(Q_4^k)_\mu([0,\alpha_1,\beta_1]) = F(x_\mu) = 0$, we find that for $i,j\in\{1,\dots,m\}$,

$$F((x_i - X_{ik})(x_j - X_{jk}))$$
$$= F(x_ix_j) - X_{ik}F(x_j) - X_{jk}F(x_i) + X_{ik}X_{jk}F(1) = F(x_ix_j).$$

Therefore

$$|F(x_ix_j)| = |F((x_i - X_{ik})(x_j - X_{jk}))| \le C(\mathrm{diam}(\Omega_k))^2\|(\alpha_1,\beta_1)\|_{\widehat{L}_1}$$

and

$$\Big|\omega \int\limits_{\Omega_k} \alpha_1 x_i x_j \, dx + \omega\vartheta \int\limits_{\partial\Omega_k} \beta_1 x_i x_j \, dS\Big| \le C\omega\|(\alpha_1,\beta_1)\|_{\widehat{L}^1},$$

which easily implies our claim. ∎

LEMMA 25. *There is a constant C independent of σ for large σ such that with $U = [u,\alpha,\beta]$ satisfying $Q_1^k(U) = 0$, $Q_3(U) = 0$, $Q_4(U) = 0$, and $u = w \times x$, we have*

$$\begin{aligned} &E([u,0,0]) \\ &\quad \ge 4L_3^{-1}\sigma^{-5}\Big(\sum_{k=1}^m M_k(X_k\cdot dX_k)\Big)^2 - C\sigma^{-4}(|dX|^2 + \|(\alpha_1,\beta_1)\|_{\widehat{L}^1}^2). \end{aligned}$$

Proof. For the velocities of the form $u = w \times x$ we know by Lemma 19 that $E([w\times x,0,0]) \ge \mathsf{K}_3 w_3^2$. Also, using Lemma 24 in addition, and the fact that $X_{3k} = 0$, we obtain

$$\begin{aligned} 0 &= (Q_3)_3([w\times x,\alpha,\beta]) \\ &= (Q_3)_3([w\times x,0,0]) + (Q_3)_3([0,\alpha_0,\beta_0]) + (Q_3)_3([0,\alpha_1,\beta_1]) \\ &\quad + \mathsf{K}_3 w_3 + 2\omega\sum_{k=1}^m M_k dX_k\cdot X_k + (Q_3)_3([0,\alpha_1,\beta_1]). \end{aligned}$$

Introducing the notation

$$c = \sum_{k=1}^m M_k dX_k \cdot X_k,$$

we obtain

$$|\mathsf{K}_3^{1/2} w_3| \ge 2\omega \mathsf{K}_3^{-1/2}|c| - \mathsf{K}_3^{-1/2}|(Q_3)_3([0,\alpha_1,\beta_1])|.$$

Remembering $\omega = \sigma^{-3/2}$ we have, again using Lemmas 20 and 24,

$$|\mathsf{K}_3^{1/2} w_3| \ge 2\sigma^{-3/2}\mathsf{K}_3^{-1/2}|c| - C\sigma^{-5/2}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}.$$

Therefore

$$\begin{aligned} E([u,0,0]) &\ge \mathsf{K}_3 w_3^2 \\ &\ge 4\sigma^{-3}\mathsf{K}_3^{-1}c^2 - C\sigma^{-5}|c|\|(\alpha_1,\beta_1)\|_{\widehat{L}^1} - C\sigma^{-5}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}^2 \\ &\ge 4\sigma^{-3}\mathsf{K}_3^{-1}c^2 - C(\sigma^{-6}|c|^2 + \sigma^{-4}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}^2) - C\sigma^{-5}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}^2 \\ &\ge 4\sigma^{-3}\mathsf{K}_3^{-1}c^2 - C\sigma^{-6}c^2 - C\sigma^{-4}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1}^2. \end{aligned}$$

Also for large σ,

$$\sigma^{-3}\mathsf{K}_3^{-1}c^2 = \frac{\sigma^{-3}}{\mathsf{K}_3}c^2 \ge \frac{\sigma^{-3}}{L_3\sigma^2 + C\sigma}c^2 = \frac{\sigma^{-5}}{L_3 + C\sigma^{-1}}c^2 \ge c^2 L_3^{-1}\sigma^{-5} - Cc^2\sigma^{-6},$$

thus

$$\begin{aligned}E([u,0,0]) &\geq 4c^2L_3^{-1}\sigma^{-5} - C\sigma^{-6}c^2 - C\sigma^{-4}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}\\ &\geq 4c^2L_3^{-1}\sigma^{-5} - C\sigma^{-4}|dX|^2 - C\sigma^{-4}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1},\end{aligned}$$

which is what we needed to prove. ∎

Now we reformulate $\alpha_0\varepsilon''(\rho^\sigma) - I(\alpha_0,\beta_0)$ in $\overline{\Omega}_k$ and $-\beta_0\frac{\partial p^\sigma}{\partial \mathfrak{n}} - I(\alpha_0,\beta_0)$ on $\partial\Omega_k$. Note that in the following lemma the upper index σ is omitted in the notation.

Lemma 26. *Defining the functions*

$$F_k = -\omega^2 dX_k\cdot Px + \sum_{j=1}^m \nabla V_j\cdot(dX_j - dX_k),$$

in $\overline{\Omega}_k$ we have $\varepsilon''(\rho)\alpha_0 - I(\alpha_0,\beta_0) = F_k$, while on $\partial\Omega_k$ we have $-\beta_0\frac{\partial p}{\partial \mathfrak{n}} - \vartheta I(\alpha_0,\beta_0) = \vartheta F_k$.

Proof. First we compute

$$\begin{aligned}I(\alpha_0,\beta_0) &= \mathsf{k}\sum_{k=1}^m\Big[-\int_{\Omega_k}\frac{dX_k\cdot\nabla\rho}{|x-y|}\,dy + \int_{\partial\Omega_k}\rho\frac{dX_k\cdot\mathfrak{n}}{|x-y|}\,dS\Big]\\ &= \mathsf{k}\sum_{k=1}^m\int_{\Omega_k}\rho dX_k\cdot\nabla_y\frac{1}{|x-y|}\,dy = -\mathsf{k}\sum_{k=1}^m\int_{\Omega_k}\rho dX_k\cdot\nabla_x\frac{1}{|x-y|}\,dy\\ &= -\sum_{k=1}^m dX_k\cdot\nabla V_k.\end{aligned}$$

As, by equation (3),

$$\varepsilon'(\rho) = \frac{1}{2}\omega^2|Px|^2 + C_k + V,$$

also

$$\varepsilon''(\rho)\nabla\rho = \omega^2 Px + \nabla V,$$

thus $\varepsilon''(\rho)\alpha_0 = -\omega^2 dX_k\cdot Px - dX_k\cdot\nabla V$ and

$$F_k = \varepsilon''(\rho)\alpha_0 - I(\alpha_0,\beta_0) = -\omega^2 dX_k\cdot Px + \sum_{j=1}^m \nabla V_j\cdot(dX_j - dX_k).$$

Further, on $\partial\Omega_k^\sigma$ we have

$$\begin{aligned}\vartheta\varepsilon''(\rho)\alpha_0 &= -\rho^e\varepsilon''(\rho)dX_k\cdot\nabla\rho = -p'(\rho)dX_k\cdot\nabla\rho\\ &= -dX_k\cdot\nabla(p(\rho)) = -dX_k\cdot\mathfrak{n}\frac{\partial p}{\partial n} = -\beta_0\frac{\partial p}{\partial \mathfrak{n}},\end{aligned}$$

thus $-\beta_0\frac{\partial p^e}{\partial \mathfrak{n}} - \vartheta I(\alpha_0,\beta_0) = \vartheta F_k$. ∎

Lemma 27. *We have*

$$E([0,\alpha_0,\beta_0]) = -\omega^2 \sum_{k=1}^{m} M_k |P(dX_k)|^2 + \sum_{j\neq k} \int_{\Omega_k} \rho dX_k \cdot \nabla^2 V_j(x) \cdot (dX_j - dX_k)\, dx.$$

Proof. The definition of E implies

$$\begin{aligned} E([0,\alpha_0,\beta_0]) &= \sum_{k=1}^{m} \Big[-\int_{\Omega_k} F_k dX_k \cdot \nabla\rho\, dx + \int_{\partial\Omega_k} \vartheta F_k dX_k \cdot \mathfrak{n}\, dS \Big] \\ &= \sum_{k=1}^{m} \int_{\Omega_k} \rho dX_k \cdot \nabla F_k\, dx \\ &= \sum_{k=1}^{m} \int_{\Omega_k} \rho dX_k \cdot \nabla \Big[-\omega^2 dX_k \cdot Px + \sum_{j=1}^{m} [\nabla V_j(x)\cdot(dX_j - dX_k)] \Big]\, dx \\ &= -\omega^2 \sum_{k=1}^{m} |P(dX_k)|^2 \int_{\Omega_k} \rho\, dx + \sum_{j\neq k} \int_{\Omega_k} \rho dX_k \cdot \nabla[\nabla V_j(x)\cdot(dX_j - dX_k)]\, dx \\ &= -\omega^2 \sum_{k=1}^{m} M_k |P(dX_k)|^2 + \sum_{j\neq k} \int_{\Omega_k} \rho dX_k \cdot \nabla^2 V_j(x) \cdot (dX_j - dX_k)\, dx. \end{aligned}$$

That proves our claim. ■

Lemma 28. *There exists a constant C such that for sufficiently large σ, for $\widehat{\alpha} \in L^1(\Omega)$ and $\widehat{\beta} \in L^1(\partial\Omega)$ we have*

$$|S([0,\widehat{\alpha},\widehat{\beta}],[0,\alpha_0,\beta_0])| \le C\sigma^{-2} |dX|\, \|(\widehat{\alpha},\widehat{\beta})\|_{\widehat{L}^1}.$$

Proof. Using Lemma 26, we deduce

$$\begin{aligned} S([0,\widehat{\alpha},\widehat{\beta}],[0,\alpha_0,\beta_0]) &= \sum_{k=1}^{m} \Big[-\int_{\Omega_k} \widehat{\alpha} F_k\, dx + \vartheta \int_{\partial\Omega_k} F_k \widehat{\beta}\, dS \Big] \\ &= \omega^2 \sum_{k=1}^{m} dX_k \cdot \Big[\int_{\Omega_k} \widehat{\alpha} Px\, dx - \vartheta \int_{\partial\Omega_k} \widehat{\beta} Px\, dS \Big] \\ &\quad + \sum_{j\neq k} \Big[-\int_{\Omega_k} \widehat{\alpha}[\nabla V_j \cdot (dX_j - dX_k)]\, dx + \vartheta \int_{\partial\Omega_k} [\nabla V_j \cdot (dX_j - dX_k)] \widehat{\beta}\, dS \Big], \end{aligned}$$

thus

$$|S([0,\widehat{\alpha},\widehat{\beta}],[0,\alpha_0,\beta_0])| \le C\omega^2|dX| \max_{x\in\overline{\Omega}} |Px| \, \|(\widehat{\alpha},\widehat{\beta})\|_{\widehat{L}^1} + \Big(\sum_{j\neq k=1}^{m} \max_{x\in\overline{\Omega}_k} |\nabla V_j(x)|\Big)|dX| \, \|(\widehat{\alpha},\widehat{\beta})\|_{\widehat{L}^1},$$

which easily implies our claim. ∎

LEMMA 29. *Letting* $\widetilde{E}(V) = S^1(V,V) + S^2(V,V)$ *for* $V = [0,\alpha_1,\beta_1]$, *for sufficiently large* σ *we have*

$$|E(V) - \widetilde{E}(V)| \le C\sigma^{-1}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}.$$

Proof. Clearly $E(V) - \widetilde{E}(V) = S^3(V,V)$ and

$$\begin{aligned} |S^3(V,V)| &\le \sum_{j\neq k}\Big|\Big[\int_{\Omega_k} \alpha_1 I_{\Omega_j}(\alpha_1,\beta_1)\,dx - \vartheta\int_{\partial\Omega_k} \beta_1 I_{\Omega_j}(\alpha_1,\beta_1)\,d\sigma\Big]\Big| \\ &\le C\|(\alpha_1,\beta_1)\|_{\widehat{L}^1} \sum_{j\neq k=1}^{m} \max_{\overline{\Omega}_k} |I_\Omega(\alpha_1,\beta_1)| \\ &\le C\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1} \sum_{j\neq k=1}^{m} \max_{x\in\overline{\Omega}_k,\, y\in\overline{\Omega}_j} |x-y|^{-1}, \end{aligned}$$

easily implying our claim. ∎

Now we put our calculations so far together.

LEMMA 30. *For sufficiently large* σ *and* $[u,\alpha,\beta]\in\mathfrak{H}_1$ *we have*

$$\begin{aligned} E([u,\alpha,\beta]) \ge{}& 4L_3^{-1}\sigma^{-5}\Big(\sum_{k=1}^{m} M_k(X_k\cdot dX_k)\Big)^2 + \widetilde{E}([0,\alpha_1,\beta_1]) \\ &- \omega^2\sum_{k=1}^{m} M_k|P(dX_k)|^2 + \sum_{j\neq k}\int_{\Omega_k^e} \rho dX_k^t \nabla^2 V_j^e(x)(dX_j - dX_k)\,dx \\ &- C\sigma^{-7/2}|dX|^2 - C\sigma^{-1/2}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}. \end{aligned}$$

Proof. We have

$$\begin{aligned} E([u,\alpha,\beta]) ={}& E([u,0,0]) + E([0,\alpha_1,\beta_1]) + E([0,\alpha_0,\beta_0]) \\ &+ 2S([0,\alpha_1,\beta_1],[0,\alpha_0,\beta_0]) \end{aligned}$$

and by Lemma 28,

$$\begin{aligned} |S([0,\alpha_1,\beta_1],[0,\alpha_0,\beta_0])| &\le C\sigma^{-7/4}|dX|\sigma^{-1/4}\|(\alpha_1,\beta_1)\|_{\widehat{L}^1} \\ &\le C\sigma^{-7/2}|dX|^2 + C\sigma^{-1/2}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}. \end{aligned}$$

By Lemma 25 also

$$E([u,0,0]) \geq 4\sigma^{-5}L_3^{-1}\Big(\sum_{k=1}^{m} M_k(X_k\cdot dX_k)\Big)^2 - C\sigma^{-4}(|dX|^2 + \|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}).$$

After using Lemma 27 to rewrite $E([0,\alpha_0,\beta_0])$, we only need to consider $E([0,\alpha_1,\beta_1])$, and by Lemma 29,

$$E([0,\alpha_1,\beta_1]) \geq \widetilde{E}([0,\alpha_1,\beta_1]) - C\sigma^{-1}\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}.$$

This immediately proves our claim. ∎

LEMMA 31. *There exists an $\eta_3 > 0$ such that for sufficiently large σ and $[w\times x,\alpha,\beta] \in \mathfrak{H}_1$ we have*

$$\begin{aligned}([w\times x,\alpha,\beta]) &\geq \eta_3\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^1}\\ &\quad + 4L_3^{-1}\sigma^{-5}\Big(\sum_{k=1}^{m} M_k(X_k\cdot dX_k)\Big)^2 - \sigma^{-3}\sum_{k=1}^{m} M_k|P(dX_k)|^2\\ &\quad + \frac{\mathsf{k}}{2}\sum_{j\neq k} M_jM_k\Big[\frac{|dX_j-dX_k|^2}{|X_k-X_j|^3} - 3\frac{((dX_j-dX_k)\cdot(X_k-X_j))^2}{|X_k-X_j|^5}\Big]\\ &\quad - C\sigma^{-7/2}|dX|^2.\end{aligned}$$

Proof. In analogy to [5, proof of Theorem 7.2], from Condition 6 we can prove that for sufficiently large σ there is an $\eta_4 > 0$ such that with $U_1 = [0,\alpha_1,\beta_1]$ we have

$$S_k^1(U_1,U_1) + S_k^2(U_1,U_1) \geq \eta_4\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^2}$$

for $k = 1,\ldots,m$, and therefore $\widetilde{E}(U_1) \geq \eta_4\|(\alpha_1,\beta_1)\|^2_{\widehat{L}^2}$, which, up to an application of the Hölder inequality, gives us what we need for this expression. For the remaining changes we first show that for $j \neq k$ we have, with $A = \mathsf{k}\nabla^2(|x|^{-1})$,

$$\Big|\int_{\Omega_k} \rho V_{jx_\mu x_\nu}(x)\,dx - M_jM_kA(X_k-X_j)\Big| \leq C\sigma^{-4}. \tag{12}$$

By Taylor's theorem, for $x \in \Omega_k$,

$$\begin{aligned}V_{jx_\mu x_\nu}(x) &= V_{jx_\mu x_\nu}(X_k) + \nabla V_{jx_\mu x_\nu}(X_k)\cdot(x-X^k)\\ &\quad + \frac{1}{2}(x-X_k)^t\nabla^2 V_{jx_\mu x_\nu}(sx+(1-s)X_k)(x-X_k)\end{aligned}$$

with an $s \in (0,1)$. As X_k is the center of gravity of the kth gas ball, this implies

$$\Big|\int_{\Omega_k^\sigma} \rho V_{jx_\mu x_\nu}(x)\,dx - M_kV_{jx_\mu x_\nu}(X_k)\Big| \leq C\sigma^{-5}.$$

As also

$$|\nabla(|x-y|^{-1})_{x_\mu x_\nu}| \le C|x-y|^{-4},$$

we get

$$|M_k V_{jx_\mu x_\nu}(X_k) - M_k M_j A(X_k - X_j)| \le C\sigma^{-4}.$$

Putting the two inequalities together, we get inequality (12). The remainder of the claim is obtained by splitting the double sum into two halves and exchanging the indices in one of the sums. ∎

LEMMA 32. *Assuming Condition 7, there exists an* $\varepsilon > 0$ *such that if*

$$|Z - \widetilde{Z}| < \varepsilon,$$

then for all $dZ_j \in \mathbb{R}^3$ *with*

$$\sum_{j=1}^m M_j dZ_j = 0, \qquad \sum_{j=1}^m (e_3 \times \widetilde{Z}_j)\cdot dZ_j = 0 \tag{13}$$

we have

$$4L_3^{-1}\Big(\sum_{k=1}^m M_k(\widetilde{Z}_k \cdot dZ_k)\Big)^2 - \sum_{k=1}^m M_k|P(dZ_k)|^2 + \frac{\mathsf{k}}{2}\sum_{j\ne k} M_j M_k\bigg[\frac{|dZ_j - dZ_k|^2}{|\widetilde{Z}_k - \widetilde{Z}_j|^3} - 3\frac{((dZ_j - dZ_k)\cdot(\widetilde{Z}_k - \widetilde{Z}_j))^2}{|\widetilde{Z}_k - \widetilde{Z}_j|^5}\bigg] \ge \frac{\eta_2}{2}|dZ|^2.$$

Proof. If we had Z_j instead of $\widetilde{Z}_j$ in the second condition of (13), the conclusion would follow directly from Condition 7, as we can confine ourselves to $|dZ| = 1$ owing to the homogeneity of the function to be estimated from below, and then this is clear by the continuity of this function. To be able to use Condition 7 we can make small adjustments to dZ to fulfill the condition $\sum_{k=1}^m e_3 \times Z_k \cdot dX_k = 0$ instead of $\sum_{k=1}^m e_3 \times \widetilde{Z}_k \cdot dX_k = 0$. To make this possible, we need to show that

$$\sum_{k=1}^m M_k dX_k = 0, \qquad \sum_{k=1}^m e_3 \times Z_k \cdot dX_k = b$$

is solvable for all $b \in \mathbb{R}$. To this end we prove that the matrix representing this system of equations has full rank. As the third component of each dX_k does not enter the second equation, we only need to consider the first two components, letting

$$d\widetilde{X}_k = [dX_k^1 \quad dX_k^2]^t.$$

Then the remaining three equations can be written in the form

$$\sum_{k=1}^m A_k d\widetilde{X}_k = 0$$

with

$$A_k = \begin{bmatrix} M_k & 0 \\ 0 & M_k \\ u_k & v_k \end{bmatrix}$$

with $(u_k, v_k, 0) = e_3^\dagger \times Z_k^\dagger$. The rank of the matrix $A = [A_1, \dots, A_m]$ is 3, as adding $-M_1^{-1}u_1$ times the first row to the third one and then $-M_1^{-1}v_1$ times the second row to the last one, we can only have rank 2 if the last row is now zero, and then $(u_k, v_k) = M_k M_1^{-1}(u_1, v_1)$. Then also $e_3 \times (u_k, v_k, 0) = M_k M_1^{-1} e_3 \times (u_1, v_1, 0)$, and therefore $Z_k = M_k M_1^{-1} Z_1$. Therefore all these points would be located on a ray emanating from 0, therefore the center of gravity can only be at zero if there is only one point, contrary to our assumption. ■

Now we can proceed to the proof of Theorem 8. Multiplying the inequality in Lemma 32 by σ^{-1}, replacing $\widetilde{Z}$ by $\sigma^{-1}X$, and dZ by $\sigma^{-1}dX$, together with our previous results, gives

$$\begin{aligned}
&E([u, \alpha, \beta]) \\
&\quad \geq 4\sigma^{-5} L_3^{-1} \Big(\sum_{k=1}^m M_k (X_k \cdot dX_k) \Big)^2 - \sigma^{-3} \sum_{k=1}^m M_k |P(dX_k)|^2 \\
&\qquad + \frac{\mathsf{k}}{2} \sum_{j \neq k} M_j M_k \left[\frac{|dX_j - dX_k|^2}{|X_k - X_j|^3} - 3 \frac{((dX_j - dX_k) \cdot (X_k - X_j))^2}{|X_k - X_j|^5} \right] \\
&\qquad - C\sigma^{-7/2} |dX|^2 + \eta_3 \|(\alpha_1, \beta_1)\|_{L^1(\Omega^\sigma)}^2 \\
&\quad \geq \sigma^{-3} \frac{\eta_2}{2} |dX|^2 - C\sigma^{-7/2} |dX|^2 + \eta_3 \|(\alpha_1, \beta_1)\|_{L^1(\Omega^\sigma)}^2 \\
&\quad \geq \sigma^{-3} \frac{\eta_2}{4} |dX|^2 + \eta_3 \|(\alpha_1, \beta_1)\|_{L^1(\Omega^\sigma)}^2 \geq 0
\end{aligned}$$

for sufficiently large σ. Thus $E(u, \alpha, \beta) \geq 0$ for the relevant vectors, and if it equals zero, we have $\alpha_1 = 0$, $\beta_1 = 0$, $dX = 0$, thus $\alpha = 0$, $\beta = 0$, and finally using Lemma 19, $u = 0$. This finishes our proof.

4. The case $m = 2$

LEMMA 33. *Every solution of the restricted two-body problem fulfills Condition* 5.

Proof. Let $Z = [Z_1, Z_2]$ be a solution of the two-body problem with masses M_1, M_2 and angular velocity $\omega = 1$, and choose our solution so that $Z_1^3 = Z_2^3 = 0$. As the center of gravity of the two masses together is at zero, and thus $M_1 Z_1 + M_2 Z_2 = 0$, we can write both vectors in terms of $Y = Z_1 - Z_2$, remembering $M_0 = M_1 + M_2$, and introducing the additional notation $\mu = M_1 M_2 M_0^{-1}$ in the form

$$Z_1 = M_1^{-1}\mu Y, \qquad Z_2 = -M_2^{-1}\mu Y.$$

It is then easy to see that the equation $Q_1(Z) = 0$ is equivalent to the equation

$$Y - \mathsf{k} M_0 Y |Y|^{-3} = 0$$

for Y. Therefore also

$$|Y|^3 = \mathsf{k} M_0. \tag{14}$$

Now one can partly write the differentials of $Q_{1,k}$ in terms of Y and dY, where $dY = dX_1 - dX_2$, as follows:

$$dQ_{1,1} = M_1 P dZ_1 + \mathsf{k} M_1 M_2 \big(-|Y|^{-3} dY + 3(dY \cdot Y)|Y|^{-5} Y\big),$$
$$dQ_{1,2} = M_2 P dZ_2 + \mathsf{k} M_1 M_2 \big(|Y|^{-3} dY - 3(dY \cdot Y)|Y|^{-5} Y\big).$$

Thus $dQ_{1,1} = dQ_{1,2} = 0$ implies, again adding both equations,

$$P(M_1 dZ_1 + M_2 dZ_2) = 0.$$

Therefore we get

$$P dZ_1 = M_1^{-1}\mu P dY, \qquad P dZ_2 = -M_2^{-1}\mu P dY,$$

and the equation

$$0 = M_0^{-1} P dY - \mathsf{k}|Y|^{-3} dY + 3\mathsf{k}|Y|^{-5}(dY \cdot Y) Y$$

for dY. Applying P to the equation we get, using (14),

$$0 = (M_0^{-1} - \mathsf{k}|Y|^{-3}) P dY + 3\mathsf{k}(dY \cdot Y)|Y|^{-5} Y = 3\mathsf{k}(dY \cdot Y)|Y|^{-5} Y,$$

so $dY \cdot Y = 0$. Therefore $M_0^{-1} P dY = \mathsf{k}|Y|^{-3} dY$, and thus $dY^3 = 0$. As $Y \neq 0$, $dY^3 = 0$ and $dY \cdot Y = 0$, we find that $dY = aAY$ for a number a, where A is the matrix from [4]. Therefore also

$$P dZ_k = aAZ_k$$

for $k = 1, 2$, and as $Y^3 = 0$ and therefore $Z_1^3 = Z_2^3$, we get

$$P dZ_k = aAZ_k + Z_1^3 e_3,$$

which we had to prove. ■

LEMMA 34. *Every solution of the restricted two-body problem fulfills Condition* 7.

Proof. In this case the energy $\mathcal{E}$ has the form

$$\begin{aligned}\mathcal{E} = {} & 4(M_1|Z_1|^2 + M_2|Z_2|^2)^{-1}[M_1(Z_1 \cdot dZ_1) + M_2(Z_2 \cdot dZ_2)]^2 \\ & - (M_1|P(dZ_1)|^2 + M_2|P(dZ_2)|^2) \\ & + \mathsf{k} M_2 M_1 \left[\frac{|dZ_1 - dZ_2|^2}{|Z_1 - Z_2|^3} - 3\frac{|(dZ_1 - dZ_2)\cdot(Z_1 - Z_2)|^2}{|Z_1 - Z_2|^5}\right].\end{aligned}$$

Also, using the notation and the relationship between Z_1, Z_2, Y as before, we have, using equation (14) as well,

$$|dZ_1 - dZ_2|^2|Z_1 - Z_2|^{-3} - 3|(dZ_1 - dZ_2)\cdot(Z_1 - Z_2)|^2|Z_1 - Z_2|^{-5}$$
$$= |Y|^{-3}|dY|^2 - 3|Y|^{-5}(Y\cdot dY)^2 = (\mathsf{k}M_0)^{-1}(|dY|^2 - 3|Y|^{-2}(Y\cdot dY)^2),$$

likewise

$$M_k(Z_k\cdot dZ_k) = M_k(\mu M_k^{-1})Y\cdot(\mu M_k^{-1}dY) = \mu^2(M_k^{-1})Y\cdot dY,$$

and therefore

$$[M_1(Z_1\cdot dZ_1) + M_2(Z_2\cdot dZ_2)]^2$$
$$= \mu^4(M_1^{-1} + M_2^{-1})^2(Y\cdot dY)^2 = \mu^2(Y\cdot dY)^2,$$

and

$$M_1|P(dZ_1)|^2 + M_2|P(dZ_2)|^2 = M_1\left(\frac{\mu}{M_1}\right)^2|P(dY)|^2 + M_2\left(\frac{\mu}{M_2}\right)^2|P(dY)|^2$$
$$= \mu^2\left(\frac{1}{M_1} + \frac{1}{M_2}\right)|P(dY)|^2 = \mu|P(dY)|^2.$$

Likewise

$$M_1|Z_1|^2 + M_2|Z_2|^2 = \mu|Y|^2,$$

resulting in

$$\mathcal{E} = 4(\mu|Y|^2)^{-1}\mu^2(Y\cdot dY)^2 - \mu|P(dY)|^2 + \mu[|dY|^2 - 3|Y|^{-2}(Y\cdot dY)^2]$$
$$= \mu[4|Y|^{-2}(Y\cdot dY)^2 - |P(dY)|^2 + |dY|^2 - 3|Y|^{-2}(Y\cdot dY)^2]$$
$$= \mu[|Y|^{-2}(Y\cdot PdY)^2 + |dY_3|^2].$$

Now, as we also assume

$$e_3\times Z_1\cdot dZ_1 + e_3\times Z_2\cdot dZ_2 = 0,$$

expressing Z_k and dZ_k in terms of Y and dY we can easily see that

$$e_3\times Y\cdot dY = e_3\times Y\cdot(PdY) = 0.$$

As Y and PdY are in the x_1, x_2 plane, this implies that Y and PdY are parallel and therefore $|Y\cdot PdY| = |Y|\,|PdY|$, and

$$\mathcal{E} = \mu[|PdY|^2 + |dY_3|^2] = \mu|dY|^2,$$

which implies our claim. ∎

References

[1] J. F. G. Auchmuty, *The global branching of rotating stars*, Arch. Ration. Mech. Anal. 114 (1991), 179–194.

[2] J. F. G. Auchmuty and R. Beals, *Variational solution of some non-linear free boundary problems*, Arch. Ration. Mech. Anal. 43 (1971), 255–271.

[3] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983.
[4] G. Ströhmer, *Existence and structural stability for certain configurations of rotating barotropic fluids*, J. Math. Fluid Mech. 16 (2014), 193–210.
[5] G. Ströhmer, *About linear stability for multiple gas balls*, J. Math. Fluid Mech. 18 (2016), 71–88.
[6] G. Ströhmer, *About nonlinear stability for multiple gas balls*, J. Math. Fluid Mech. 21 (2019), art. 7, 21 pp.
[7] G. Ströhmer, *Long-term existence for perturbed multiple gas balls and their asymptotic behavior*, J. Math. Fluid Mech. 27 (2025), art. 8, 19 pp.
[8] G. Ströhmer and W. Zajączkowski, *On the existence and properties of the rotationally symmetric equilibrium states of compressible barotropic self-gravitating fluids*, Indiana Univ. Math. J. 46 (1997), 1181–1220.

Gerhard Ströhmer
Professor emeritus
Department of Mathematics
University of Iowa
Iowa City, IA 52242, USA
E-mail: gerhard-strohmer@uiowa.edu