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On the containment problem and sporadic simplicial line arrangements

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In memory of Arkadiusz Płoski

Abstract. We present two examples of inductively free sporadic simplicial arrangements of 31 lines that have the same weak combinatorics, which allows us to answer negatively the questions regarding the containment problem formulated recently by Drabkin and Seceleanu.

1. Introduction. The study of relations between symbolic and ordinary powers of homogeneous ideals in the polynomial ring over a given field $\mathbb{K}$ has a long history and is derived from many different problems in mathematics. In 1995, Eisenbud and Mazur [19], referring to the proof of Fermat's Last Theorem, were investigating the so-called "fitting ideals" and some symbolic powers of certain associated ideals. They proved, among other things, that $I^{(2)} \subset \mathfrak{m}I$ in the case of perfect ideals of codimension 2 in the ring of polynomials over a field of characteristic 0. They also showed that this kind of containment holds for several other classes of ideals. In 2013, Harbourne and Huneke [24] proposed a certain generalization and began to study the relations $I^{(m)} \subset \mathfrak{m}^k I^r$; their work was continued in [2, 8].

Another direction of investigations on symbolic and ordinary powers of ideals was proposed by Skoda [31] and Waldschmidt [34]. They focused on some estimates of the degree of hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$ passing through fixed points with prescribed multiplicities. A paper by Chudnovsky [7] fits into this context. Using some complex analysis tools, Chudnovsky improved the results of Skoda and Waldschmidt in $\mathbb{P}^2$ and formulated a still open conjecture for the case of $\mathbb{P}^N$. A generalization of this conjecture was also given by

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Demailly [11], and the combination of these conjectures has been the subject of intense research [2, 3, 4, 5, 15, 17, 21, 29].

Using the Nagata–Zariski theorem makes it possible to relate geometric questions to algebra. Therefore, the study of containment relations between ordinary and symbolic powers of homogeneous ideals of points provides a connection between conjectures formulated by Chudnovsky and Demailly. This perspective led to an increased interest in the so-called containment problem, i.e., the determination of the exponents (m, r) for which the mth symbolic power of a homogeneous ideal is contained in the rth ordinary power of that ideal. Work in this topic was begun by Hochster in 1973 [25], but the groundbreaking result was published only in 2001 by Ein, Lazarsfeld, and Smith [18], who gave a lower bound on the exponent of the symbolic power in characteristic zero; Hochster and Huneke considered the case of positive characteristic in [26]. Since then, the cases unsolved by the main theorem in [18] have become the subject of intensive study. This regards in particular the case of smallest powers, namely the containment $I^{(3)} \subset I^2$ for ideals of reduced points in $\mathbb{P}^2$. While at the beginning researchers tried to prove that this particular containment holds for all homogeneous ideals, after [16], where the first counterexample defined over $\mathbb{C}$ was presented, a lot of counterexamples defined over different fields have been published (see [6, 28]). Despite a growing number of counterexamples, the true nature of the relation between $I^{(3)}$ and I^2 is still unknown. In [14], Drabkin and Seceleanu studied reflection arrangements given by (irreducible) complex pseudoreflection groups. As a result, they gave a complete description of the relation between the third symbolic power and the second ordinary power of a radical ideal $J(\mathcal{A})$, which defines the singular locus of the complex reflection arrangement $\mathcal{A}$. The work on this problem motivated them to state some open questions, including the following:

Question 4.2 ([14, Question 6.7]). Are the containments $(J(\mathcal{A}))^{(2r-1)} \subseteq (J(\mathcal{A}))^r$ always satisfied for any $r \geq 2$ and any hyperplane arrangement that is inductively free?

In the present paper we give a negative answer to this question.

Theorem 4.4. *There are two non-isomorphic inductively free simplicial arrangements consisting of* 31 *lines each, with the same weak combinatorics, and such that for one arrangement the containment* $(J(\mathcal{A}))^{(3)} \subseteq (J(\mathcal{A}))^2$ *holds, and for the other arrangement it does not.*

The structure of the paper is the following. In Section 2, we recall some basic definitions and tools concerning line arrangements and symbolic powers of homogeneous ideals. In Section 3, we give detailed information about a family of line arrangement known as $\mathcal{A}(12k + 7)$, giving line equations and proving that some line arrangements from this family are inductively free. This result is used in Section 4, where we prove Theorem 4.4 of this paper. At the end, we provide our SINGULAR [10] code to let interested readers check the containment between $(J(\mathcal{A}))^{(3)}$ and $(J(\mathcal{A}))^2$.

2. Preliminaries. In this section we recall all necessary definitions regarding hyperplane arrangements that we will exploit in the paper. For more information, see [13, 30].

Let $\mathbb{K}$ be a field of characteristic zero and let V be a fixed vector space of dimension ℓ over $\mathbb{K}$. Let $\{x_1, \ldots, x_\ell\}$ be the basis of the dual space V^*. Then the symmetric algebra $S(V^*)$ is isomorphic to the ring of polynomials $S = \mathbb{K}[x_1, \ldots, x_\ell]$.

A *hyperplane* H in V is defined as a vector subspace of $\operatorname{codim}_{\mathbb{K}}(H) = 1$. A finite set $\mathcal{A} = \{H_1, \ldots, H_d\}$ of pairwise distinct hyperplanes is called an *arrangement of hyperplanes* in V. The pair $(\mathcal{A}, V)$ is then called an ℓ-*arrangement* of hyperplanes. The symbol Φ_ℓ denotes the empty ℓ-arrangement. If the dimension is clear from the context, we use the name arrangement for short. Each hyperplane $H \in \mathcal{A}$ is the kernel of a (unique up to a constant) linear form $l_H \in V^*$. The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} l_H$$

is called the *defining polynomial* of $\mathcal{A}$. In the case of empty arrangements, we put $Q(\Phi_l) = 1$.

We denote by $L(\mathcal{A})$ the *intersection lattice* of $\mathcal{A}$, i.e., the set of all nonempty intersections $\bigcap \mathcal{B}$ for $\mathcal{B} \subset \mathcal{A}$. Take $X \in L(\mathcal{A})$. The subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ defined as

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\},$$

is called the *localization* of $\mathcal{A}$ to X, and the subarrangement

$$\mathcal{A}^X = \{X \cap H \ : X \nsubseteq H \text{ and } X \cap H \neq \emptyset\}$$

is the *restriction* of $\mathcal{A}$ to X.

Definition 2.1. A *simplicial arrangement* is a finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of (central, i.e., containing the origin) hyperplanes in $\mathbb{R}^\ell$ such that all connected components of the complement

$$M(\mathcal{A}) := \mathbb{R}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$$

are simplicial cones.

Remark 2.2. There are three infinite families (in the sense of the number of lines) of simplicial arrangements in the real projective plane which were described in [23]. Besides these three series there are a number of simplicial arrangements of at most 37 lines which do not fit into the three families, and these arrangements are called sporadic simplicial line arrangements; see [23] for further details.

Denote by $\mathrm{Der}_{\mathbb{K}}(S)$ the set of all $\mathbb{K}$-linear maps (derivations) $\theta : S \to S$ such that for all $f, g \in S$ one has

$$\theta(fg) = f\theta(g) + g\theta(f).$$

It is known that the set $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{\ell}$ forms a (canonical) basis for $\mathrm{Der}_{\mathbb{K}}(S)$, i.e.,

$$\mathrm{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S \cdot \frac{\partial}{\partial x_i}.$$

Any non-zero homogeneous element $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ can be expressed as $\theta = \sum_{i=1}^{\ell} g_i \cdot \frac{\partial}{\partial x_i}$, where $g_i \in S$ are homogeneous polynomials of degree d. For such θ we denote by $\operatorname{pdeg}\theta = d$ its polynomial degree.

For any homogeneous $f \in S$, we define an S-submodule of $\mathrm{Der}_{\mathbb{K}}(S)$ by

$$D(f) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) : \theta(f) \in f \cdot S\}.$$

If f is the defining polynomial $Q(\mathcal{A})$ of $\mathcal{A}$, we use the notation $D(\mathcal{A})$ instead of $D(Q(\mathcal{A}))$.

Definition 2.3. If $D(\mathcal{A})$ is a free S-module, then we say that $\mathcal{A}$ is a *free arrangement.*

Let $\mathcal{A}$ be a free arrangement for which $\{\theta_1, \ldots, \theta_\ell\}$ is a homogeneous basis of $D(\mathcal{A})$. We say that $\exp(\mathcal{A}) = \{\operatorname{pdeg}\theta_1, \ldots, \operatorname{pdeg}\theta_\ell\}$ is the *set of exponents* of $\mathcal{A}$.

If we denote by $\theta_E \in \mathrm{Der}_{\mathbb{K}}(S)$ the Euler derivation, then we have the decomposition

$$D(\mathcal{A}) = S \cdot \theta_E \oplus D_0(\mathcal{A}).$$

Now, for an arrangement $\mathcal{A}$ and fixed $H \in \mathcal{A}$, it is convenient to study triples of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, where $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$. The next theorem will be useful.

Theorem 2.4 (Addition-Deletion, [30]). *Suppose* $\mathcal{A} \neq \Phi_\ell$. *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \textit{ is free with } \exp(\mathcal{A}) &= \{b_1, \ldots, b_{\ell-1}, b_\ell\}, \\ \mathcal{A}' \textit{ is free with } \exp(\mathcal{A}') &= \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}, \\ \mathcal{A}'' \textit{ is free with } \exp(\mathcal{A}'') &= \{b_1, \ldots, b_{\ell-1}\}. \end{aligned}$$

In this paper we deal with line arrangements $\mathcal{A}$ defined over the complex numbers, therefore we will use the following reformulation of Theorem 2.4.

THEOREM 2.5. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ and $H \in \mathcal{A}$. Let $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. If*

(1) *$\mathcal{A}'$ is free and* $\exp(\mathcal{A}') = \{1, a, b\}$,
(2) $|\mathrm{Sing}(\mathcal{A}) \cap H| = b + 1$ (*or* $a + 1$, *respectively*),

then $\mathcal{A}$ is free with $\exp(\mathcal{A}) = \{1, a + 1, b\}$ (*or* $\exp(\mathcal{A}) = \{1, a, b + 1\}$, *respectively*).

We will need the following definition.

DEFINITION 2.6 ([30, Definition 4.53]). The class $\mathcal{IF}$ of *inductively free arrangements* is the smallest class of arrangements such that

(1) $\Phi_\ell \in \mathcal{IF}$ for $\ell \geq 0$,
(2) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A} \in \mathcal{IF}$.

3. Examples of inductively free arrangements. The main object of our considerations is a special family of line arrangements, denoted in the literature by $\mathcal{A}(12k + 7)$. This infinite family was originally described by Grünbaum [1]. We recall here its construction and the basic properties.

For fixed k, each element of the family consists of exactly $12k + 7$ lines, including the line at infinity $z = 0$. The equations of these lines are given explicitly in Table 1, where $e^2 - 3 = 0$.

Table 1. Equations of lines of $\mathcal{A}(12k + 7)$

$\mathcal{A}(12k+7)$	
$2x - eiz = 0$, $x - ey + iez = 0$, $x + ey - iez = 0$,	for $i \in \{-(k+1), -k, \ldots, -1, 0, 1, \ldots, k, k+1\}$
$2y - jz = 0$, $ex - y + jz = 0$, $ex + y - jz = 0$,	for $j \in \{-(k-1), -(k-2), \ldots, -1, 0, 1, \ldots, k-2, k-1\}$
$z = 0$	

The arrangements $\mathcal{A}(19)$ and $\mathcal{A}(31)$ are exactly the sporadic simplicial arrangements $A(19, 1)$ and $A(31, 2)$ listed in Grünbaum's catalogue [22, 23]. From [12], we know that these are free arrangements with $\exp(\mathcal{A}(19, 1)) = \{1, 7, 11\}$ and $\exp(\mathcal{A}(31, 2)) = \{1, 13, 17\}$. We start with the following combinatorial observation that is crucial for our further considerations. To our best knowledge, it has not been recorded in the literature.

THEOREM 3.1. *The arrangements $\mathcal{A}(19, 1)$ and $\mathcal{A}(31, 2)$ are inductively free.*

Proof. We first show that $\mathcal{A}(19, 1)$ is inductively free. For this purpose, we present Table 2 below, where we give the sequences of the exponents for $\mathcal{A}'$, the equation of each line that we add to the arrangement starting from Φ_3, and then the exponents of $\exp(\mathcal{A}'')$. Each subsequent row of the table allows us to verify the conditions of Theorem 2.5 and the condition $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$ from Definition 2.6.

Table 2. List of exponents for arrangements building $\mathcal{A}(19,1)$

$\exp \mathcal{A}'$	ℓ_i	$\exp \mathcal{A}''$
$\{0,0,0\}$	$\ell_1 : z = 0$	$\{0,0\}$
$\{0,0,1\}$	$\ell_2 : \sqrt{3}x + y = 0$	$\{0,1\}$
$\{0,1,1\}$	$\ell_3 : \sqrt{3}x - y = 0$	$\{1,1\}$
$\{1,1,1\}$	$\ell_4 : y = 0$	$\{1,1\}$
$\{1,1,2\}$	$\ell_5 : x = 0$	$\{1,1\}$
$\{1,1,3\}$	$\ell_6 : x + \sqrt{3}y = 0$	$\{1,1\}$
$\{1,1,4\}$	$\ell_7 : x - \sqrt{3}y = 0$	$\{1,1\}$
$\{1,1,5\}$	$\ell_8 : 2x - \sqrt{3}z = 0$	$\{1,5\}$
$\{1,2,5\}$	$\ell_9 : 2x + \sqrt{3}z = 0$	$\{1,5\}$
$\{1,3,5\}$	$\ell_{10} : x + \sqrt{3}y + \sqrt{3}z = 0$	$\{1,5\}$
$\{1,4,5\}$	$\ell_{11} : x - \sqrt{3}y - \sqrt{3}z = 0$	$\{1,5\}$
$\{1,5,5\}$	$\ell_{12} : x + \sqrt{3}y - \sqrt{3}z = 0$	$\{1,5\}$
$\{1,5,6\}$	$\ell_{13} : x - \sqrt{3}y + \sqrt{3}z = 0$	$\{1,5\}$
$\{1,5,7\}$	$\ell_{14} : 2x - 2\sqrt{3}z = 0$	$\{1,7\}$
$\{1,6,7\}$	$\ell_{15} : x + \sqrt{3}z = 0$	$\{1,7\}$
$\{1,7,7\}$	$\ell_{16} : x + \sqrt{3}y + 2\sqrt{3}z = 0$	$\{1,7\}$
$\{1,7,8\}$	$\ell_{17} : x - \sqrt{3}y + 2\sqrt{3}z = 0$	$\{1,7\}$
$\{1,7,9\}$	$\ell_{18} : x - \sqrt{3}y - 2\sqrt{3}z = 0$	$\{1,7\}$
$\{1,7,10\}$	$\ell_{19} : x + \sqrt{3}y - 2\sqrt{3}z = 0$	$\{1,7\}$
$\{1,7,11\}$		

Based on the last row in Table 2, we see that the arrangement $\mathcal{A}(19,1)$ is free with $\exp(\mathcal{A}(19,1)) = \{1,7,11\}$.

For the second part of the proof, we apply Theorem 2.5 to the arrangement $\mathcal{A}(19,1)$ by adding suitably chosen lines as indicated in Table 3.

Table 3. Equations of lines ℓ_{19+i} with $i \in \{1,...,12\}$

ℓ_{19+i}	Equations of lines	ℓ_{19+i}	Equations of lines
ℓ_{20} :	$\sqrt{3}x + y - z = 0$	ℓ_{26} :	$x + \sqrt{3}y + 3\sqrt{3}z = 0$
ℓ_{21} :	$\sqrt{3}x + y + z = 0$	ℓ_{27} :	$x + \sqrt{3}y - 3\sqrt{3}z = 0$
ℓ_{22} :	$\sqrt{3}x - y + z = 0$	ℓ_{28} :	$x - \sqrt{3}y + 3\sqrt{3}z = 0$
ℓ_{23} :	$\sqrt{3}x - y - z = 0$	ℓ_{29} :	$x - \sqrt{3}y - 3\sqrt{3}z = 0$
ℓ_{24} :	$2y - z = 0$	ℓ_{30} :	$2x + 3\sqrt{3}z = 0$
ℓ_{25} :	$2y + z = 0$	ℓ_{31} :	$2x - 3\sqrt{3}z = 0$

Attaching successively lines $\ell_{20}, \ldots, \ell_{31}$ to $\mathcal{A}(19,1)$, and using each time Theorem 2.5 and Definition 2.6, we conclude that the resulting arrangements are free. The details are presented in Table 4.

Table 4. Transition from $\mathcal{A}(19,1)$ to $\mathcal{A}(31,2)$

exp $\mathcal{A}'$	ℓ_i	exp $\mathcal{A}''$	exp $\mathcal{A}'$	ℓ_i	exp $\mathcal{A}''$
$\{1,7,11\}$	ℓ_{20}	$\{1,11\}$	$\{1,12,13\}$	ℓ_{27}	$\{1,13\}$
$\{1,8,11\}$	ℓ_{21}	$\{1,11\}$	$\{1,13,13\}$	ℓ_{28}	$\{1,13\}$
$\{1,9,11\}$	ℓ_{22}	$\{1,11\}$	$\{1,13,14\}$	ℓ_{29}	$\{1,13\}$
$\{1,10,11\}$	ℓ_{23}	$\{1,11\}$	$\{1,13,15\}$	ℓ_{30}	$\{1,13\}$
$\{1,11,11\}$	ℓ_{24}	$\{1,11\}$	$\{1,13,16\}$	ℓ_{31}	$\{1,13\}$
$\{1,11,12\}$	ℓ_{25}	$\{1,11\}$	$\{1,13,17\}$		
$\{1,11,13\}$	ℓ_{26}	$\{1,13\}$			

Thus we conclude that the arrangement

$$\mathcal{A}(31,2) := \mathcal{A}(19,1) \cup \{\ell_{20},\ldots,\ell_{31}\}$$

is inductively free with $\exp(\mathcal{A}(31,2)) = \{1,13,17\}$, which completes the proof. ■

4. Inductively free arrangements and a counterexample to the containment problem. The line arrangement $\mathcal{A}(31,2)$ from the previous section turns out to be important in the context of the so-called containment problem for symbolic powers of homogeneous ideals. Let us recall here the definition of symbolic powers.

Definition 4.1. Let $I \subseteq S = \mathbb{K}[x,y,z]$ be a homogeneous ideal. For a fixed positive integer m, we define the mth *symbolic power* of I as

$$I^{(m)} = S \cap \bigcap_{Q \in \operatorname{Ass}(I)} I_Q^m,$$

where $\operatorname{Ass}(I)$ denotes the set of all prime ideals associated with I and I_Q denotes the location of I at Q.

The reader unfamiliar with the topic of symbolic powers and the containment problem is referred to [32].

In [14], Drabkin and Seceleanu study arrangements of hyperplanes that come from irreducible complex reflection groups, proving that for some cases we have the failure of containment:

$$(J(\mathcal{A}))^{(3)} \not\subseteq (J(\mathcal{A}))^2, \tag{4.1}$$

where $J(\mathcal{A})$ denotes the radical ideal associated with the configuration of all intersection points of a given arrangement $\mathcal{A}$ and $(J(\mathcal{A}))^{(3)}$ denotes the third symbolic power of $J(\mathcal{A})$. In the light of the results obtained in [14], it is natural to ask the following.

Question 4.2 ([14, Question 6.7]). Are the containments $(J(\mathcal{A}))^{(2r-1)} \subseteq (J(\mathcal{A}))^r$ always satisfied for any $r \geq 2$ and any hyperplane arrangement that is inductively free?

Here we answer this question in the negative for $r = 2$ by taking the whole singular locus of the arrangement $\mathcal{A}(31, 2)$ described in the previous section. In fact, we will show even more: we produce two inductively free arrangements of 31 lines such that for one arrangement condition (4.1) holds, but for the other it does not. In order to do so, let us present briefly our construction of the second arrangement of 31 lines. Surprisingly, this is a simplicial line arrangement which is denoted by $\mathcal{A}(31, 3)$ in Grünbaum's catalogue [23]. Here are the details.

Consider the ten lines given by

$$x \pm \sqrt{3}az = 0, \quad 2y \pm az = 0, \quad 2x \pm \sqrt{3}bz = 0,$$

where $a \in \{0, 1\}$ and $b \in \{0, 1, 3\}$. The visualization in the affine part of the projective plane of these ten lines is presented in Figure 1.

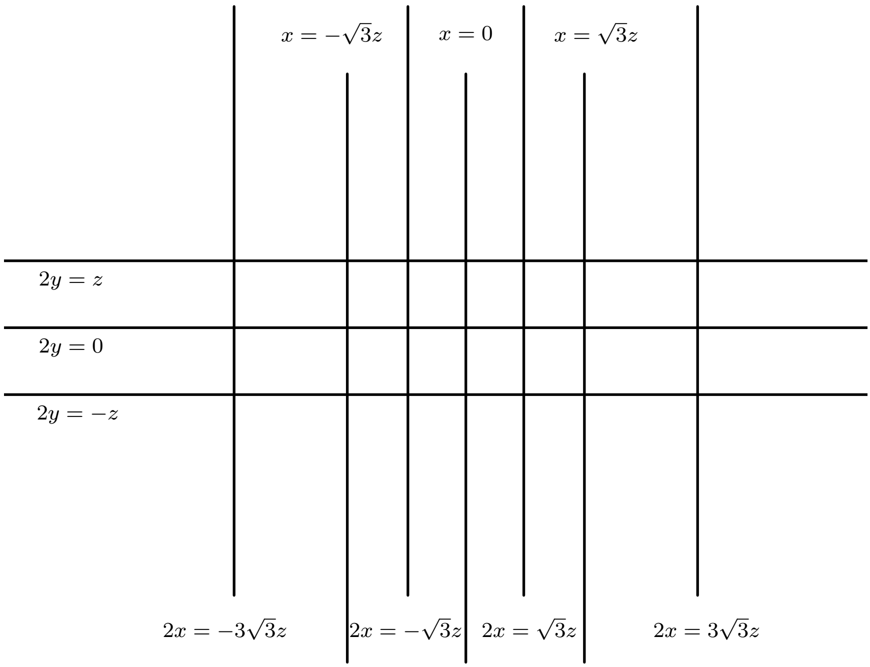

Fig. 1. A set of 10 lines inducing realizations of $\mathcal{A}(31, 3)$

Now, we perform two rotations of these lines (affinely), firstly by 60°, then by 120°, around the point $(0, 0)$. In this way we obtain 30 lines. Finally, to obtain 31 lines, we add to our arrangement the line at infinity $z = 0$. The resulting configuration is shown in Figure 2.

The following statement shows that the answer to Question 4.2 is negative.

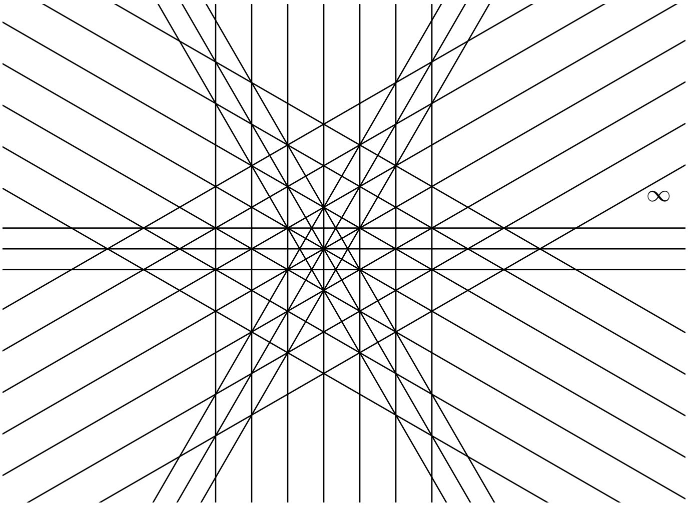

Fig. 2. An affine realization of the arrangement $\mathcal{A}(31,3)$. The symbol ∞ denotes the line at infinity $z=0$.

THEOREM 4.3. *The arrangement* $\mathcal{A}(31,3)$ *is inductively free and*

$$(J(\mathcal{A}(31,3)))^{(3)} \not\subseteq (J(\mathcal{A}(31,3)))^2.$$

Proof. To prove that $\mathcal{A}(31,3)$ is inductively free, we will produce a table containing the exponents of $\exp(\mathcal{A}')$, the equations of the lines that we add to $\mathcal{A}(19,1)$, and then the exponents of $\exp(\mathcal{A}'')$.

Table 5. Transition from $\mathcal{A}(19,1)$ to $\mathcal{A}(31,3)$

exp $\mathcal{A}'$	ℓ_i	exp $\mathcal{A}''$
$\{1,7,11\}$	$\ell_{20}: \sqrt{3}x+y+z=0$	$\{1,11\}$
$\{1,8,11\}$	$\ell_{21}: \sqrt{3}x+y-z=0$	$\{1,11\}$
$\{1,9,11\}$	$\ell_{22}: \sqrt{3}x-y+z=0$	$\{1,11\}$
$\{1,10,11\}$	$\ell_{23}: \sqrt{3}x-y-z=0$	$\{1,11\}$
$\{1,11,11\}$	$\ell_{24}: 2y-z=0$	$\{1,11\}$
$\{1,11,12\}$	$\ell_{25}: 2y+z=0$	$\{1,11\}$
$\{1,11,13\}$	$\ell_{26}: 4x+\sqrt{3}z=0$	$\{1,13\}$
$\{1,12,13\}$	$\ell_{27}: 4x-\sqrt{3}z=0$	$\{1,13\}$
$\{1,13,13\}$	$\ell_{28}: 2x-2\sqrt{3}y+\sqrt{3}z=0$	$\{1,13\}$
$\{1,13,14\}$	$\ell_{29}: 2x-2\sqrt{3}y-\sqrt{3}z=0$	$\{1,13\}$
$\{1,13,15\}$	$\ell_{30}: 2x+2\sqrt{3}y-\sqrt{3}z=0$	$\{1,13\}$
$\{1,13,16\}$	$\ell_{31}: 2x+2\sqrt{3}y+\sqrt{3}z=0$	$\{1,13\}$
$\{1,13,17\}$		

Observe that each row in Table 5 allows us to verify condition (2) in Definition 2.6. Evidence for non-containment $(J(\mathcal{A}(31,3)))^{(3)} \not\subseteq (J(\mathcal{A}(31,3)))^2$ has been provided in [27]. The verification was performed using `SINGULAR`.

The affine part ($z = 1$) of the element $F \in (J(\mathcal{A}(31,3)))^{(3)} \setminus (J(\mathcal{A}(31,3)))^2$ is shown in Figure 3. ∎

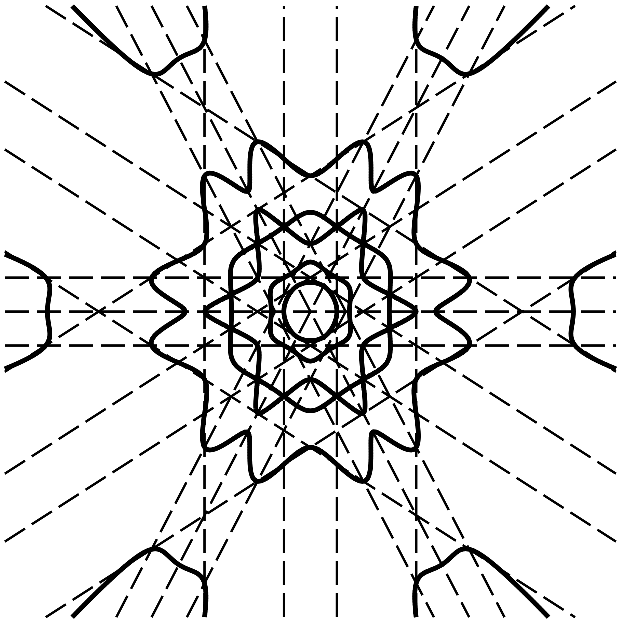

Fig. 3. The element $F \in (J(\mathcal{A}(31,3)))^{(3)} \setminus (J(\mathcal{A}(31,3)))^2$ consists of 21 lines and a curve of degree 12.

Theorem 4.4 illustrates the fact an arrangement $\mathcal{A}$ that being inductively free does not directly transfer into lack of containment $(J(\mathcal{A}))^{(3)} \subseteq (J(\mathcal{A}))^2$.

Theorem 4.4. *There are two non-isomorphic inductively free simplicial arrangements consisting of* 31 *lines, with the same weak combinatorics, and such that for one arrangement the containment* $(J(\mathcal{A}))^{(3)} \subseteq (J(\mathcal{A}))^2$ *holds, and for the other does not.*

In other words, the weak combinatorics of line arrangements does not determine whether $(J(\mathcal{A}))^{(3)} \subseteq (J(\mathcal{A}))^2$ or not. For clarity, let us recall that for an arrangement $\mathcal{A}$ of d lines in the plane, the weak combinatorics is the vector of the form $(d; t_2, \ldots, t_d)$ with t_i being the number of i-fold intersection points of lines in $\mathcal{A}$.

It is worth noticing that Theorem 4.4 should be compared with a result from [20], where the authors observed a similar phenomenon, but in a different setting, namely in the case of real line arrangements possessing the maximal possible number of triple intersection points.

Proof of Theorem 4.4. The arrangement $\mathcal{A}(31,3)$ is constructed from $\mathcal{A}(19,1)$ by adding 12 appropriately chosen lines, according to Table 5. The arrangements $\mathcal{A}(31,2)$ and $\mathcal{A}(31,3)$ are non-isomorphic simplicial arrangements of lines [23], even though they have the same weak combinatorics:

$$t_2 = 54,\ t_3 = 42,\ t_4 = 21,\ t_5 = 6,\ t_6 = 1,\ t_8 = 3,$$

and $t_i = 0$ for $i > 8$.

Let us pass to the containment question $(J(\mathcal{A}))^{(3)} \subseteq (J(\mathcal{A}))^2$. In Theorem 4.3, we explained the non-containment for the singular locus of $\mathcal{A}(31,3)$, and this check was done using `SINGULAR`. In the case of $\mathcal{A}(31,2)$, we can perform exactly the same computations showing that the containment

$$(J(\mathcal{A}(31,2))^{(3)} \subseteq (J(\mathcal{A}(31,2))^2$$

does hold, which completes the proof. ∎

In Appendix, the reader can find our script in `SINGULAR` that allows verifying $(J(\mathcal{A}(31,2))^{(3)} \subseteq (J(\mathcal{A}(31,2))^2$.

We have the following relations (see [9, 33, 35]):

$$\text{inductively free} \subsetneq \text{recursively free} \subsetneq \text{free}.$$

It turns out that our example of a pair of arrangements $\mathcal{A}(31,2)$ and $\mathcal{A}(31,3)$ allows us to answer in the negative a question posed by Drabkin and Seceleanu [14, Question 6.8]. More precisely, the example shows that $J(\mathcal{A})^{(3)} \subset J(\mathcal{A})^2$ need not hold for recursively free line arrangements $\mathcal{A}$. It is still an open question if there is a configuration of lines which is recursively free, but not inductively free for which $(J(\mathcal{A}))^{(3)} \subset (J(\mathcal{A}))^2$ holds.

Appendix

```
proc PtsIdeal(poly p, poly q, poly r) {
matrix M[2][3]=p,q,r,x,y,z;
ideal @I=minor(M,2);
return(std(@I));
}
option(redSB);
ring R=(0,e),(x,y,z),dp;
minpoly=e2-3;

/* The list L contains the coordinates of the singular points
of the arrangement A(31,2). */

"loading arrangement A(31,2)...";
list L=
(-7/2e),-1/2,1,(7/2e),-1/2,1,(7/2e),1/2,1,(-7/2e),1/2,1,
(-3/2e),-11/2,1,(2e),5,1,(3/2e),11/2,1,(-2e),-5,1,
(3/2e),-11/2,1,(-2e),5,1,(-3/2e),11/2,1,(2e),-5,1,
(3/4e),5/4,1,(1/4e),7/4,1,(1/4e),-7/4,1,(3/4e),-5/4,1,
(-3/4e),5/4,1,(-1/4e),7/4,1,(-1/4e),-7/4,1,(-3/4e),-5/4,1,
(-e),-1/2,1,(-e),1/2,1,(e),-1/2,1,(e),1/2,1,
(5/2e),-1/2,1,(-3/2e),7/2,1,(5/2e),1/2,1,(-3/2e),-7/2,1,
(-5/2e),-1/2,1,(3/2e),7/2,1,(-5/2e),1/2,1,(3/2e),-7/2,1,
(-e),-4,1,(-e),4,1,(e),4,1,(e),-4,1,
```

```
(3/2e),5/2,1,(-2e),-1,1,(-3/2e),5/2,1,(2e),-1,1,
(-3/2e),-5/2,1,(2e),1,1,(3/2e),-5/2,1,(-2e),1,1,
(-1/2e),-7/2,1,(-1/2e),7/2,1,(1/2e),7/2,1,(1/2e),-7/2,1,
(-3/2e),-1/2,1,(e),2,1,(3/2e),-1/2,1,(-e),2,1,
(3/2e),1/2,1,(-e),-2,1,(-3/2e),1/2,1,(e),-2,1,
(-1/2e),5/2,1,(-1/2e),-5/2,1,(1/2e),5/2,1,(1/2e),-5/2,1,
(3/2e),3/2,1,(-3/2e),-3/2,1,(3/2e),-3/2,1,(-3/2e),3/2,1,
0,3,1,0,-3,1,(-1/4e),-1/4,1,(1/4e),1/4,1,
(1/4e),-1/4,1,(-1/4e),1/4,1,0,-1/2,1,0,1/2,1,
(-e),-1,1,(e),1,1,(-e),1,1,(e),-1,1,
0,2,1,0,-2,1,(-1/2e),-1/2,1,(1/2e),1/2,1,
(-1/2e),1/2,1,(1/2e),-1/2,1,0,1,1,0,-1,1,
(3/2e),0,1,(-3/2e),0,1,(3/4e),9/4,1,(-3/4e),-9/4,1,
(3/4e),-9/4,1,(-3/4e),9/4,1,(3e),0,1,(-3e),0,1,
(-3/2e),-9/2,1,(3/2e),9/2,1,(-3/2e),9/2,1,(3/2e),-9/2,1,
(-1/3e),0,1,(1/3e),0,1,(-1/6e),-1/2,1,(1/6e),1/2,1,
(1/6e),-1/2,1,(-1/6e),1/2,1,(2e),0,1,(-2e),0,1,
(-e),-3,1,(e),3,1,(-e),3,1,(e),-3,1,
(-1/2e),0,1,(1/2e),0,1,(1/4e),3/4,1,(-1/4e),-3/4,1,
(-1/4e),3/4,1,(1/4e),-3/4,1,(e),0,1,(-e),0,1,
(-1/2e),-3/2,1,(1/2e),3/2,1,(-1/2e),3/2,1,(1/2e),-3/2,1,
0,0,1,(e),1,0,(-e),1,0,0,1,0,-1,0,0,1,(e),0,-1,(e),0;

"generating ideals I^(3) and I^2...";
ideal I=1; ideal I3=1;
for(int i=1;i<=(size(L) div 3);i++){
I=intersect(I,PtsIdeal(L[3*i-2],L[3*i-1],L[3*i]));
I3=intersect(I3,(PtsIdeal(L[3*i-2],L[3*i-1],L[3*i]))^3);
if((i mod 10) == 0){ string(i)+" points of "
        +string(size(L) div 3)+" in total used";}
}
I=std(I^2); I3=std(I3);

"number of generators of I^(3) not in I^2: "+string(size(NF(I3,I)));
```

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