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On a Mattei–Salem theorem

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In memory of Arkadiusz Płoski

Abstract. We investigate the relationship between the valuations of a germ of a singular foliation $\mathcal{F}$ on the complex plane and those of a balanced equation of separatrices for $\mathcal{F}$, extending a theorem by Mattei–Salem. Under certain conditions, we also derive inequalities involving the valuation, tangency excess, and degree of a holomorphic foliation $\mathcal{F}$ on the complex projective plane.

1. Introduction. In [MS], J.-F. Mattei and E. Salem formulated a theorem that characterizes germs of non-dicritical second type foliations (possibly formal) on $(\mathbb{C}^2, 0)$ in terms of their algebraic multiplicity, valuation over components contained in the exceptional divisor of a reduction of singularities, and by an exact sequence of sheaves associated to these foliations. We recall that a foliation $\mathcal{F}$ is *non-dicritical* if the number of local separatrices – local irreducible invariant curves – is finite, otherwise it is called *dicritical*. A *second type foliation* is a foliation $\mathcal{F}$ that admits – at most – *non-tangent saddle-nodes*, meaning that no weak separatrix is contained in the exceptional divisor of a reduction of singularities of $\mathcal{F}$. In Section 2, we review some of the standard facts on foliations, singularities, and their separatrices.

In this paper, we study the *valuation* $\nu_D(\mathcal{F})$ of a germ of a singular foliation $\mathcal{F}$ at $p \in \mathbb{C}^2$ along a component D contained in the exceptional divisor $\mathcal{D}$ of a minimal reduction of singularities of $\mathcal{F}$ (see Definition 4.1). Our aim is to generalize [MS, Theorem 3.1.9(4)] to an arbitrary foliation (dicritical or not). It is worth pointing out that Y. Genzmer [G07, Lemma 3.2] extended [MS, Theorem 3.1.9(4)] to dicritical second type foliations (see also [GM18, Proposition 3.7]). Both, Mattei–Salem and Genzmer used this

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valuation to solve the *realization problem* for holomorphic foliations (see for example [G07, Theorems 1.1, 1.2]).

To state our main result, we introduce $\xi_D(\mathcal{F})$ in (4.1) and (4.2). This number is associated with the sum of the *tangency excess* of $\mathcal{F}$ at *infinitely near points* of D. Sections 3 and 4 provide a detailed exposition of these definitions.

We can now formulate our main result:

THEOREM 1. *Let $\mathcal{F}$ be a germ of a singular foliation at $p \in \mathbb{C}^2$ having $\hat{\Psi}$ as a balanced equation of separatrices. Let $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ be a minimal process of reduction of singularities for $\mathcal{F}$. Then, for every component $D \subset \mathcal{D}$, we have*

$$\nu_D(\hat{\Psi}) = \begin{cases} \nu_D(\mathcal{F}) + 1 - \xi_D(\mathcal{F}) & \text{if } D \text{ is non-dicritical,} \\ \nu_D(\mathcal{F}) - \xi_D(\mathcal{F}) & \text{if } D \text{ is dicritical.} \end{cases} \tag{1.1}$$

Since $\mathcal{F}$ is of second type if and only if $\xi_p(\mathcal{F}) = 0$ (see Definition 3.2), Theorem 1 generalizes [MS, Théorème 3.1.9(4)] and [G07, Lemma 3.2], as $\xi_D(\mathcal{F}) = 0$ by Definition 3.1. Finally, in Section 5, using [MS02, Theorem 1] for projective foliations, we will derive inequalities involving $\nu_D(\mathcal{F})$, $\xi_D(\mathcal{F})$, and the degree of a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$.

2. Basic tools. Let $\mathcal{F}$ be a germ of a singular foliation (possibly formal) at $p \in \mathbb{C}^2$. In local coordinates $(x, y) \in \mathbb{C}^2$ centered at p, the foliation is represented by a germ of a 1-form

$$\omega = P(x,y)dx + Q(x,y)dy, \tag{2.1}$$

or by its dual vector field

$$v = -Q(x,y)\frac{\partial}{\partial x} + P(x,y)\frac{\partial}{\partial y}, \tag{2.2}$$

where $P, Q \in \mathbb{C}[[x,y]]$ are relatively prime. The *algebraic multiplicity* $\nu_p(\mathcal{F})$ is the minimum of the orders $\nu_p(P)$, $\nu_p(Q)$ at p of the coefficients of a local generator of $\mathcal{F}$.

Let $f(x,y) \in \mathbb{C}[[x,y]]$. We say that $C : f(x,y) = 0$ is *invariant* by $\mathcal{F}$ if

$$\omega \wedge df = (f \cdot h)dx \wedge dy$$

for some $h \in \mathbb{C}[[x,y]]$. If C is irreducible, then we will say that C is a *separatrix* of $\mathcal{F}$. The separatrix C is *analytical* if f is convergent. We denote by $\mathrm{Sep}_p(\mathcal{F})$ the set of all separatrices of $\mathcal{F}$.

We say that $p \in \mathbb{C}^2$ is a *reduced* singularity for $\mathcal{F}$ if the linear part $\mathrm{D}v(p)$ of the vector field v in (2.2) is non-zero and has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ falling into one of the cases:

(i) $\lambda_1\lambda_2 \neq 0$ and $\lambda_1\lambda_2 \notin \mathbb{Q}^+$ (*non-degenerate*);
(ii) $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ (*saddle-node singularity*).

In case (i), there is a system of coordinates (x, y) in which $\mathcal{F}$ is defined by the equation

$$\omega = x(\lambda_1 + a(x, y))dy - y(\lambda_2 + b(x, y))dx, \tag{2.3}$$

where $a(x, y), b(x, y) \in \mathbb{C}[[x, y]]$ are non-units, so that $\mathrm{Sep}_p(\mathcal{F})$ is formed by two transversal analytic branches given by $\{x = 0\}$ and $\{y = 0\}$. In case (ii), up to a formal change of coordinates, the saddle-node singularity is given by a 1-form of the type

$$\omega = x^{k+1}dy - y(1 + \lambda x^k)dx, \tag{2.4}$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}^+$ are invariants after formal changes of coordinates (see [MR82, Proposition 4.3]). The curve $\{x = 0\}$ is an analytic separatrix, called *strong* separatrix, whereas $\{y = 0\}$ corresponds to a possibly formal separatrix, called *weak* separatrix. The integer $k + 1$ is called the *tangency index* of $\mathcal{F}$ with respect to the weak separatrix.

For a fixed minimal reduction process of singularities $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ of $\mathcal{F}$ (it always exists, as established by Seidenberg [S68]), and any component $D \subset \mathcal{D}$, two cases are possible:

- D is *non-dicritical* if D is $\tilde{\mathcal{F}}$-invariant. In this case, D contains a finite number of simple singularities. Each non-corner singularity carries a separatrix transversal to D, whose projection by π is a curve in $\mathrm{Sep}_p(\mathcal{F})$.
- D is *dicritical* if D is not $\tilde{\mathcal{F}}$-invariant. The definition of reduction of singularities gives that D may intersect only non-dicritical components and that $\tilde{\mathcal{F}}$ is everywhere transverse to D. The π-image of a local leaf of $\tilde{\mathcal{F}}$ at each non-corner point of D belongs to $\mathrm{Sep}_p(\mathcal{F})$.

Let σ be the blow-up of the reduction process π of $\mathcal{F}$ that generated the component $D \subset \mathcal{D}$. We will say that σ is *non-dicritical* (respectively *dicritical*) if D is non-dicritical (respectively dicritical).

Denote by $\mathrm{Sep}_p(D) \subset \mathrm{Sep}_p(\mathcal{F})$ the set of separatrices whose strict transforms by π intersect the component $D \subset \mathcal{D}$. If $B \in \mathrm{Sep}_p(D)$ with D non-dicritical, then B is said to be *isolated*. Otherwise, it is a *dicritical separatrix*. This yields the decomposition $\mathrm{Sep}_p(\mathcal{F}) = \mathrm{Iso}_p(\mathcal{F}) \cup \mathrm{Dic}_p(\mathcal{F})$, where notations are self-evident. The set $\mathrm{Iso}_p(\mathcal{F})$ is finite and contains all purely formal separatrices. It subdivides further into two classes: *weak* separatrices — those arising from weak separatrices of saddle-nodes — and *strong* separatrices — corresponding to strong separatrices of saddle-nodes and separatrices of non-degenerate singularities. On the other hand, if non-empty, $\mathrm{Dic}_p(\mathcal{F})$ is an infinite set of analytic separatrices. A foliation $\mathcal{F}$ is said to be *dicritical* when $\mathrm{Sep}_p(\mathcal{F})$ is infinite, which is equivalent to saying that $\mathrm{Dic}_p(\mathcal{F})$ is non-empty. Otherwise, $\mathcal{F}$ is *non-dicritical*.

Along the text, we would rather adopt the language of *divisors* of formal curves. More specifically, a *divisor of separatrices* for a foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$

is a formal sum

$$\mathcal{B} = \sum_{B \in \mathrm{Sep}_p(\mathcal{F})} a_B \cdot B$$

where the coefficients $a_B \in \mathbb{Z}$ are zero except for finitely many $B \in \mathrm{Sep}_p(\mathcal{F})$. We denote by $\mathrm{Div}_p(\mathcal{F})$ the set of all such divisors, which turns into a group with the canonical additive structure. We follow the usual terminology and notation:

- $\mathcal{B} \geq 0$ denotes an *effective* divisor, one whose coefficients are all non-negative;
- there is a unique decomposition $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$, where $\mathcal{B}_0, \mathcal{B}_\infty \geq 0$ are respectively the *zero* and *pole* divisors of $\mathcal{B}$;
- the *algebraic multiplicity* of $\mathcal{B}$ is $\nu_p(\mathcal{B}) = \sum_{B \in \mathrm{Sep}_p(\mathcal{F})} a_B \cdot \nu_p(B)$.

To a given formal meromorphic equation $\hat{\Psi}$, whose irreducible components define separatrices B_i with multiplicities ν_i, we associate the divisor $(\hat{\Psi}) = \sum_i \nu_i \cdot B_i$. A curve of separatrices $\hat{C}$, associated to a reduced equation $\hat{\Psi}$, is identified to the divisor $\hat{C} = (\hat{\Psi})$. Such an effective divisor is named *reduced*, that is, all coefficients are either 0 or 1. In general, $\mathcal{B} \in \mathrm{Div}_p(\mathcal{F})$ is reduced if both $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are reduced effective divisors. A divisor $\mathcal{B}$ is said to be *adapted* to a curve of separatrices $\hat{C}$ if $\mathcal{B}_0 - \hat{C} \geq 0$. Finally, the usual intersection number for formal curves $C_1 = \{g_1(x, y) = 0\}$ and $C_2 = \{g_2(x, y) = 0\}$ at $(\mathbb{C}^2, p)$, defined by $i_p(C_1, C_2) := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_1, g_2)}$, is canonically extended in a bilinear way to divisors of curves.

3. Tangency excess of a foliation. Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$ with a minimal reduction process $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ and let $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$ be the strict transform foliation of $\mathcal{F}$. A saddle-node singularity $q \in \mathsf{Sing}(\tilde{\mathcal{F}})$ is said to be a *tangent saddle-node* if its weak separatrix is contained in the exceptional divisor $\mathcal{D}$.

Let B be an irreducible curve invariant by $\mathcal{F}$ at p. Suppose that $\{y = 0\}$ is the tangent cone of B. Then we may choose a primitive Puiseux parametrization $\gamma(t) = (t^n, \phi(t))$ at $p = (0, 0)$ such that $n = \nu_p(B)$. The *tangency index of $\mathcal{F}$ along B at p* (or *weak index*, as in [FM19, p. 1114]) is

$$\mathrm{Ind}_p^{\omega}(\mathcal{F}) := \mathsf{ord}_t Q(\gamma(t)).$$

We have the following definition given by Mattei–Salem [MS] for the non-dicritical case and by Genzmer [G07] for arbitrary foliations:

Definition 3.1. A foliation is *in the second class* or is *of second type* if there are no tangent saddle-nodes in its reduction of singularities.

Given a component $D \subset \mathcal{D}$, we denote by $\rho(D)$ its multiplicity (following the notation of [GM18, p. 1424]), which coincides with the algebraic multiplicity of a curve γ at $(\mathbb{C}^2, p)$ whose strict transform $\pi^*\gamma$ meets D transversally outside a corner of $\mathcal{D}$. The following invariant is a measure of the existence of tangent saddle-nodes in the reduction of singularities of a foliation:

DEFINITION 3.2. The *tangency excess* of $\mathcal{F}$ is defined as $\xi_p(\mathcal{F}) = 0$ when p is a reduced singularity, and in the non-reduced case as the number

$$\xi_p(\mathcal{F}) = \sum_{q \in SN(\mathcal{F})} \rho(D_q)(\mathrm{Ind}_q^w(\tilde{\mathcal{F}}) - 1),$$

where $SN(\mathcal{F})$ stands for the set of tangent saddle-nodes on $\mathcal{D}$ and, if $q \in SN(\mathcal{F})$, we denote by D_q the component of E containing its weak separatrix and by $\mathrm{Ind}_q^w(\tilde{\mathcal{F}}) > 1$ its weak index.

Of course, $\xi_p(\mathcal{F}) \geq 0$ and, by definition, $\xi_p(\mathcal{F}) = 0$ if and only if $SN(\mathcal{F}) = \emptyset$, that is, if and only if $\mathcal{F}$ is of second type.

We recall the following object introduced in [G07, GM18]:

DEFINITION 3.3. A *balanced divisor of separatrices* for $\mathcal{F}$ is a divisor of the form

$$\mathcal{B} = \sum_{B \in \mathrm{Iso}_p(\mathcal{F})} B + \sum_{B \in \mathrm{Dic}_p(\mathcal{F})} a_B \cdot B,$$

where the coefficients $a_B \in \mathbb{Z}$ are non-zero except for finitely many $B \in \mathrm{Dic}_p(\mathcal{F})$, and for each dicritical component $D \subset \mathcal{D}$, the following equality holds:

$$\sum_{B \in \mathrm{Sep}_p(D)} a_B = 2 - \mathrm{Val}(D).$$

The integer $\mathrm{Val}(D)$ represents the valence of a component $D \subset \mathcal{D}$ in the reduction of singularities, that is, the number of components in $\mathcal{D}$ that intersect D other than D itself.

A balanced divisor $\mathcal{B}$ is called *primitive* if, for every dicritical component $D \in \mathcal{D}$ and every $B \in \mathrm{Sep}_p(D)$, we have $-1 \leq a_B \leq 1$. Recall that a balanced divisor $\mathcal{B}$ is *adapted* to a curve of separatrices C if $\mathcal{B}_0 - C \geq 0$. A *balanced equation of separatrices* is a formal meromorphic function $\hat{\Psi}$ whose associated divisor is a balanced divisor of separatrices. A balanced equation is *reduced*, *primitive* or *adapted* to a curve C if the underlying divisor has the corresponding property.

The tangency excess measures the extent to which a balanced divisor of separatrices computes the algebraic multiplicity, as expressed in the following result ([G07, Proposition 2.4] or [GM18, Proposition 3.3]):

PROPOSITION 3.4. *Let $\mathcal{F}$ be a germ of singular foliation at $(\mathbb{C}^2, p)$, and $\mathcal{B}$ is a balanced divisor of separatrices. Denote by $\nu_p(\mathcal{F})$ and $\nu_p(\mathcal{B})$ their* *algebraic multiplicities. Then*

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1 + \xi_p(\mathcal{F}). \tag{3.1}$$

Moreover,

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1$$

if and only if $\mathcal{F}$ is a second type foliation.

4. Valuation of a foliation along a component of the exceptional divisor. In this section, we introduce our primary object of study and establish the main result of this paper.

Definition 4.1. Let $\mathcal{F}$ be a germ of singular foliation at $(\mathbb{C}^2, p)$ and $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ be a minimal process of reduction of singularities of $\mathcal{F}$. If $D \subset \mathcal{D}$ is a component, the *valuation* of $\mathcal{F}$ along D, denoted by $\nu_D(\mathcal{F})$, is the order of vanishing of $\pi^*(\omega)$ along D, where ω is any 1-form defining $\mathcal{F}$. In the same way, if $\hat{\Psi}$ is a formal meromorphic function at $(\mathbb{C}^2, p)$, we define the *valuation* of $\hat{\Psi}$ along D, denoted by $\nu_D(\hat{\Psi})$, as the order of vanishing of $\pi^*\hat{\Psi} := \hat{\Psi} \circ \pi$ along D.

Now, we introduce $\xi_D(\mathcal{F})$ as follows: at height 1 of the blowing-up process π, i.e., when D is the exceptional projective line arising from the one blowing-up at p, we set

$$\xi_D(\mathcal{F}) := \xi_p(\mathcal{F}). \tag{4.1}$$

If $D = D_k$ is obtained at height $k \geq 2$ in the blowing-up process

$$\pi = \pi_1 \circ \pi_2 \circ \cdots : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p),$$

and $D = (\pi_k)^{-1}(q_{k-1})$, we define

$$\xi_D(\mathcal{F}) := \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + \sum_{D_{q_{k-1}} \in V(q_{k-1})} \xi_{D_{q_{k-1}}}(\mathcal{F}). \tag{4.2}$$

where $\sigma_{k-1}^*(\mathcal{F})$ denotes the strict transform of the foliation $\mathcal{F}$ by means of $\sigma_{k-1} := \pi_1 \circ \cdots \circ \pi_{k-1}$, and $V(q)$ refers to the set of irreducible components D_q of $\mathcal{D}$ that contain q. In this case, $V(q)$ contains at most two components and $\mathcal{D} = \sigma_{k-1}^{-1}(p)$.

Remark 4.2. If $\hat{\Psi} = f/g$ is a germ of a formal meromorphic function at $p \in \mathbb{C}^2$ and π is a blow-up at p with exceptional divisor D, then the strict transform of $\hat{\Psi}$ by means of π is

$$\hat{\Psi}_1 := \frac{\pi^*(\hat{\Psi})}{h^{\nu_p(f) - \nu_p(g)}},$$

where $\{h = 0\}$ is a local equation of D.

Now, we establish the main result of this paper, which is a generalization of [MS, Théorème 3.1.9(4)] and [G07, Lemma 3.2].

THEOREM 1. *Let $\mathcal{F}$ be a germ of a singular foliation at $p \in \mathbb{C}^2$ having $\hat{\Psi}$ as a balanced equation of separatrices. Let $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ be a minimal process of reduction of singularities for $\mathcal{F}$. Then, for every component $D \subset \mathcal{D}$, we have*

$$(4.3) \qquad \nu_D(\hat{\Psi}) = \begin{cases} \nu_D(\mathcal{F}) + 1 - \xi_D(\mathcal{F}) & \textit{if } D \textit{ is non-dicritical;} \\ \nu_D(\mathcal{F}) - \xi_D(\mathcal{F}) & \textit{if } D \textit{ is dicritical.} \end{cases}$$

Proof. The proof is by induction on the height k of the component D in the blowing-up process. If D is the exceptional projective line arising from the blowing-up at p, then $\nu_D(\hat{\Psi}) = \nu_p(\hat{\Psi})$ and $\nu_D(\mathcal{F}) = \nu_p(\mathcal{F}) + \epsilon(D)$, where

$$(4.4) \qquad \epsilon(D) = \begin{cases} 0 & \text{if } D \text{ is non-dicritical,} \\ 1 & \text{if } D \text{ is dicritical.} \end{cases}$$

It follows by (3.1) that

$$(4.5) \qquad \nu_D(\hat{\Psi}) = \nu_D(\mathcal{F}) + 1 - \epsilon(D) - \xi_p(\mathcal{F}).$$

Since $k = 1$, we have $\xi_D(\mathcal{F}) = \xi_p(\mathcal{F})$ by (4.1), and the theorem at height 1 follows from (4.5). Assume that (4.3) holds for height k and consider the component D obtained by the blowing-up π_k of the point q_{k-1}. Here, we suppose that $\pi = \pi_1 \circ \cdots \circ \pi_k \circ \cdots$. Let $\hat{\Psi}_{k-1}$ be the strict transform of $\hat{\Psi}$ by means of $\sigma_{k-1} = \pi_1 \circ \cdots \circ \pi_{k-1}$ (see Remark 4.2). Analyzing the behavior of the valuations of $\mathcal{F}$ and $\hat{\Psi}$ along D by blow-ups, we have

$$(4.6) \qquad \nu_D(\mathcal{F}) = \nu_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + \sum_{D_{q_{k-1}} \in V(q_{k-1})} \nu_{D_{q_{k-1}}}(\mathcal{F}) + \epsilon(D),$$

$$(4.7) \qquad \nu_D(\hat{\Psi}) = \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + \sum_{D_{q_{k-1}} \in V(q_{k-1})} \nu_{D_{q_{k-1}}}(\hat{\Psi}),$$

where $\sigma_{k-1}^*(\mathcal{F})$ denotes the strict transform of $\mathcal{F}$ by means of σ_{k-1}, and $V(q)$ refers to the set of irreducible components D_q of $\mathcal{D}$ that contain q and $\epsilon(D)$ as above.

We distinguish several cases:

PART 1: *$V(q_{k-1})$ consists of one component D_{k-1}.*

(1) *D_{k-1} is non-dicritical.* Let $\hat{\Psi}_{q_{k-1}}$ be the balanced equation of separatrices for $\sigma_{k-1}^*(\mathcal{F})$ at q_{k-1}. Since q_{k-1} is a smooth point of D_{k-1}, we find that $\hat{\Psi}_{q_{k-1}}$ is the germ at q_{k-1} given by the product of $\hat{\Psi}_{k-1}$ and of a germ of local equation of D_{k-1}, i.e., $\hat{\Psi}_{q_{k-1}} = \hat{\Psi}_{k-1} \cdot h_{k-1}$, where $D_{k-1} = \{h_{k-1} = 0\}$. Hence

$$(4.8) \qquad \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) = \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + 1.$$

On the other hand, applying Proposition 3.4 to $\sigma_{k-1}^*(\mathcal{F})$ at q_{k-1} we have

$$(4.9) \qquad \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) = \nu_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + 1 - \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})).$$

Then the relations (4.7), (4.8), (4.9), (4.6), and the induction hypothesis imply that

$$\begin{aligned}\nu_D(\hat{\Psi}) &= \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) - 1 + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) - \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) \underbrace{-\nu_{D_{k-1}}(\mathcal{F}) + \nu_{D_{k-1}}(\hat{\Psi})} -\xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F}))\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_{D_{k-1}}(\mathcal{F}) - \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})).\end{aligned}$$

We get $\xi_D(\mathcal{F}) = \xi_{D_{k-1}}(\mathcal{F}) + \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F}))$ by (4.2), and so $\nu_D(\hat{\Psi}) = \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_D(\mathcal{F})$, proving the result for the component D.

(2) D_{k-1} *is dicritical.* In this case, since D_{k-1} is not $\sigma_{k-1}^*(\mathcal{F})$-invariant, and q_{k-1} is a smooth point of D_{k-1}, the balanced equation of separatrices $\hat{\Psi}_{q_{k-1}}$ for $\sigma_{k-1}^*(\mathcal{F})$ at q_{k-1} is given by $\hat{\Psi}_{q_{k-1}} = \hat{\Psi}_{k-1}$. Hence

$$\nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) = \nu_{q_{k-1}}(\hat{\Psi}_{k-1}). \tag{4.10}$$

Then the relations (4.7), (4.10), (4.9), (4.6), the induction hypothesis, and (4.2) give the result for the component D:

$$\begin{aligned}\nu_D(\hat{\Psi}) &= \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + 1 - \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + \nu_{D_{k-1}}(\hat{\Psi})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) + \underbrace{\nu_{D_{k-1}}(\hat{\Psi}) - \nu_{D_{k-1}}(\mathcal{F})}\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 \underbrace{-\xi_{q_{k-1}}(\sigma_{k-1}^*(\mathcal{F})) - \xi_{D_{k-1}}(\mathcal{F})}\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_D(\mathcal{F}).\end{aligned}$$

PART 2: $V(q_{k-1})$ *consists of two components* D_{k-1} *and* D_{k-1}^1.

(1) D_{k-1} *and* D_{k-1}^1 *are non-dicritical.* In this case, $q_{k-1} \in D_{k-1} \cap D_{k-1}^1$ is a corner point, and so that the balanced equation of separatrices $\hat{\Psi}_{q_{k-1}}$ for $\sigma_{k-1}^*(\mathcal{F})$ at q_{k-1} is the product of $\hat{\Psi}_{k-1}$ and of a germ of local equation for $D_{k-1} \cup D_{k-1}^1$, i.e., $\hat{\Psi}_{q_{k-1}} = \hat{\Psi}_{k-1} \cdot h_{k-1} \cdot \phi_{k-1}$, where $D_{k-1} = \{h_{k-1} = 0\}$ and $D_{k-1}^1 = \{\phi_{k-1} = 0\}$. Hence,

$$\nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) = \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + 2. \tag{4.11}$$

Then, the relations (4.7), (4.11), (4.9), (4.6) and the induction hypothesis imply that

(4.12)

$$\begin{aligned}\nu_D(\hat{\Psi}) &= \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) - 2 + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) + 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) - 2 + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) - 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F}))\\ &\quad + \nu_{D_{k-1}}(\hat{\Psi}) - \nu_{D_{k-1}}(\mathcal{F}) + \nu_{D^1_{k-1}}(\hat{\Psi}) - \nu_{D^1_{k-1}}(\mathcal{F})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) - 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F}))\\ &\quad + (1 - \xi_{D_{k-1}}(\mathcal{F})) + (1 - \xi_{D^1_{k-1}}(\mathcal{F}))\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1\\ &\quad - \big(\xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) + \xi_{D_{k-1}}(\mathcal{F}) + \xi_{D^1_{k-1}}(\mathcal{F})\big).\end{aligned}$$

Thus, the result follows from (4.14) and (4.2):

$$\nu_D(\hat{\Psi}) = \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_D(\mathcal{F}).$$

(2) *D_{k-1} is dicritical and D^1_{k-1} is non-dicritical.* This case can be handled similarly.

(3) *D_{k-1} and D^1_{k-1} are dicritical.* In this case, $q_{k-1} \in D_{k-1} \cap D^1_{k-1}$ is a corner point, and since D_{k-1} and D^1_{k-1} are dicritical, we have the balanced equation of separatrices $\hat{\Psi}_{q_{k-1}}$ for $\sigma^*_{k-1}(\mathcal{F})$ at q_{k-1} is $\hat{\Psi}_{k-1}$. Hence,

$$\nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) = \nu_{q_{k-1}}(\hat{\Psi}_{k-1}). \tag{4.13}$$

Then the relations (4.7), (4.13), (4.9), (4.6) and the induction hypothesis imply that

(4.14)

$$\begin{aligned}\nu_D(\hat{\Psi}) &= \nu_{q_{k-1}}(\hat{\Psi}_{k-1}) + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\hat{\Psi}_{q_{k-1}}) + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) + 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) + \nu_{D_{k-1}}(\hat{\Psi}) + \nu_{D^1_{k-1}}(\hat{\Psi})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F}))\\ &\quad + \nu_{D_{k-1}}(\hat{\Psi}) - \nu_{D_{k-1}}(\mathcal{F}) + \nu_{D^1_{k-1}}(\hat{\Psi}) - \nu_{D^1_{k-1}}(\mathcal{F})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) - \xi_{D_{k-1}}(\mathcal{F}) - \xi_{D^1_{k-1}}(\mathcal{F})\\ &= \nu_D(\mathcal{F}) - \epsilon(D) + 1\\ &\quad - \big(\xi_{q_{k-1}}(\sigma^*_{k-1}(\mathcal{F})) + \xi_{D_{k-1}}(\mathcal{F}) + \xi_{D^1_{k-1}}(\mathcal{F})\big).\end{aligned}$$

Thus, the result follows from (4.14) and (4.2):

$$\nu_D(\hat{\Psi}) = \nu_D(\mathcal{F}) - \epsilon(D) + 1 - \xi_D(\mathcal{F}). \ \blacksquare$$

In order to illustrate Theorem 1 we consider the family of dicritical foliations given in [FGS, Example 6.5].

EXAMPLE 4.3. Let $\lambda \in \mathbb{C}$ and $k \geq 3$ integer. Let $\mathcal{F}_k$ be the singular foliation at $(\mathbb{C}^2, 0)$ defined by

$$\begin{aligned}\omega_k = y(2x^{2k-2} + 2(\lambda+1)x^2y^{k-2} - y^{k-1})dx \\ + x(y^{k-1} - (\lambda+1)x^2y^{k-2} - x^{2k-2})dy.\end{aligned}$$

The foliation $\mathcal{F}_k$ is dicritical and is not of second type. After one blow-up π_1 there appears a dicritical component $D_1 = \pi_1^{-1}(0)$, and the strict transform of $\mathcal{F}_k$ by means of π_1 has a unique non-reduced singularity $q \in D_1$. A further blow-up π_2 applied to q produces a non-dicritical component $D_2 = \pi_2^{-1}(q)$, and the reduction of singularities of $\mathcal{F}_k$ is achieved. A balanced equation of separatrices for $\mathcal{F}_k$ is $\hat{\Psi}(x,y) = xy$. Moreover, a straightforward calculation shows that

$$\begin{aligned}&\nu_{D_1}(\mathcal{F}_k) = k+1, \quad &&\xi_0(\mathcal{F}_k) = k-1, \quad &&\nu_{D_1}(\hat{\Psi}) = 2,\\ &\nu_{D_2}(\mathcal{F}_k) = 2k, &&\xi_q(\mathcal{F}_k) = k-1, &&\nu_{D_2}(\hat{\Psi}) = 3.\end{aligned}$$

Therefore, $\xi_{D_1}(\mathcal{F}_k) = k-1$ and $\xi_{D_2}(\mathcal{F}_k) = 2k-2$. Since

$$\begin{aligned}2 &= \nu_{D_1}(\hat{\Psi}) = \nu_{D_1}(\mathcal{F}_k) - \xi_{D_1}(\mathcal{F}_k) = k+1-(k-1),\\ 3 &= \nu_{D_2}(\hat{\Psi}) = \nu_{D_2}(\mathcal{F}_k) + 1 - \xi_{D_2}(\mathcal{F}_k) = 2k+1-(2k-2),\end{aligned}$$

we see that Theorem 1 holds.

As a consequence of [CLS84, Theorem 3], we have $\nu_p(\mathcal{F}) \geq \nu_p(\Gamma) - 1$ if the foliation $\mathcal{F}$ is non-dicritical at $p \in \mathbb{C}^2$, and Γ is the union of all separatrices through p. Then, since $\xi_D(\mathcal{F}) \geq 0$, we can obtain a similar inequality for $\nu_D(\mathcal{F})$ and $\nu_D(\hat{\Psi})$.

COROLLARY 4.4. *Let $\mathcal{F}$ be a germ of a singular foliation at $p \in \mathbb{C}^2$ having $\hat{\Psi}$ as a balanced equation of separatrices. Let $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ be a minimal process of reduction of singularities for $\mathcal{F}$. Then for every component $D \subset \mathcal{D}$, we have*

$$\nu_D(\hat{\Psi}) - 1 + \epsilon(D) \leq \nu_D(\mathcal{F}),$$

where $\epsilon(D)$ is as in (4.4).

5. A remark on projective foliations. In this section, under certain assumptions, we derive inequalities involving the valuation, tangency excess, and degree of a holomorphic foliation $\mathcal{F}$ on the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. Given that these invariants are related to the algebraic multiplicity $\nu_p(\mathcal{F})$, we believe they are of independent interest; see, for instance, [CFR21] and [G21, Section 3].

A holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 0$ is a foliation defined by a polynomial 1-form $\Omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$, where A, B, C are complex homogeneous polynomials of degree $d+1$, satisfying two conditions:

(1) the integrability condition $\Omega \wedge d\Omega = 0$,
(2) the Euler condition $Ax + By + Cz = 0$.

The singular set of $\mathcal{F}$ is by definition

$$\mathsf{Sing}(\mathcal{F}) = \{p \in \mathbb{P}^2_{\mathbb{C}} : A(p) = B(p) = C(p) = 0\}.$$

According to Mendes–Sad [MS02, p. 94], an analytic curve S contained in a complex surface M is said to be a *special invariant curve* for a holomorphic foliation $\mathcal{G}$ if S is smooth, isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, $\mathcal{G}$-invariant, $\mathsf{Sing}(\mathcal{G}) \cap S = \{q\}$ and there are local coordinates such that $q = (0,0)$ and $f(x,y) = x^m \cdot y$ ($m \in \mathbb{N}$) is a local holomorphic first integral for $\mathcal{G}$.

Now, we consider a holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 1$, and a minimal process of reduction of singularities $\pi : (\tilde{X}, \mathcal{D}) \to \mathbb{P}^2_{\mathbb{C}}$ for $\mathcal{F}$ such that the strict transform foliation of $\mathcal{F}$ by means of π is free of special invariant curves. Then it follows from Mendes–Sad's Theorem [MS02, Theorem 1] that

$$\sum_{p \in \mathsf{Sing}(\mathcal{F})} \sum_{D \subset \mathcal{D}} (\nu_D(\mathcal{F}_p) - 1)^2 \leq (d-1)^2, \tag{5.1}$$

where $\mathcal{F}_p$ is the germ of $\mathcal{F}$ at $p \in \mathsf{Sing}(\mathcal{F})$.

We have the following corollary.

Corollary 5.1. *Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 1$, and let $\pi : (\tilde{X}, \mathcal{D}) \to \mathbb{P}^2_{\mathbb{C}}$ be a minimal process of reduction of singularities for $\mathcal{F}$. For each $p \in \mathsf{Sing}(\mathcal{F})$, let $\hat{\Psi}_p$ be a balanced equation of separatrices for the germ $\mathcal{F}_p$ of $\mathcal{F}$ at p. Suppose that the strict transform foliation of $\mathcal{F}$ by means of π is free of special invariant curves. Then*

$$\sum_{p \in \mathsf{Sing}(\mathcal{F})} \sum_{D \subset \mathcal{D} \text{ non-dicritical}} (\nu_D(\hat{\Psi}_p) + \xi_D(\mathcal{F}_p) - 2)^2 + \sum_{p \in \mathsf{Sing}(\mathcal{F})} \sum_{D \subset \mathcal{D} \text{ dicritical}} (\nu_D(\hat{\Psi}_p) + \xi_D(\mathcal{F}_p) - 1)^2 \leq (d-1)^2. \tag{5.2}$$

Proof. Dividing the sum (5.1) into two parts (with D non-dicritical and dicritical) we get

$$\sum_{p \in \mathsf{Sing}(\mathcal{F})} \sum_{D \subset \mathcal{D} \text{ non-dicritical}} (\nu_D(\mathcal{F}_p) - 1)^2 + \sum_{p \in \mathsf{Sing}(\mathcal{F})} \sum_{D \subset \mathcal{D} \text{ dicritical}} (\nu_D(\mathcal{F}_p) - 1)^2 \leq (d-1)^2.$$

The inequality (5.2) follows by applying Theorem 1 to $\nu_D(\mathcal{F}_p)$. ∎

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