APPLICATIONES MATHEMATICAE
Online First version

Antoni Augustynowicz and Rafał Perczyński

ASYMPTOTIC EXPANSIONS FOR THE SOLUTION OF A LINEAR PDE WITH A MULTIFREQUENCY HIGHLY OSCILLATORY POTENTIAL

Abstract. Highly oscillatory differential equations present significant challenges in numerical treatments. The Modulated Fourier Expansion (MFE), used as an ansatz, is a commonly employed tool as a numerical approximation method. In this article, the Modulated Fourier Expansion is analytically derived for a linear partial differential equation with a multifrequency highly oscillatory potential. The solution of the equation is expressed as a convergent Neumann series in the appropriate Sobolev space. Our approach enables, firstly, to derive a general formula for the error associated with the approximation of the solution by MFE, and secondly, to determine the coefficients for this expansion – without the need to numerically solve the system of differential equations to find the coefficients of MFE. Numerical experiments illustrate the theoretical investigations.

1. Introduction. We consider the following highly oscillatory partial differential equation:

$$\begin{aligned}\partial_t u(x,t) &= \mathcal{L}u(x,t) + f(x,t)u(x,t), \qquad t \in [0,t^\star],\ x \in \Omega \subset \mathbb{R}^m,\\ u(x,0) &= u_0(x),\end{aligned} \tag{1.1}$$

with zero boundary conditions, where Ω is an open and bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\Omega$, $t^\star > 0$ and $\mathcal{L}$ is a linear differential

2020 *Mathematics Subject Classification*: Primary 65M38; Secondary 65M70.
Key words and phrases: modulated Fourier expansion, highly oscillatory PDEs, asymptotic expansion, Neumann series.
Received 14 December 2023; revised 3 June 2024.
Published online 28 February 2025.

DOI: 10.4064/am2510-10-2024

operator of degree $2p$, $p \in \mathbb{N}$, defined by the formula

$$\mathcal{L} = \sum_{|\boldsymbol{p}|\le 2p} a_{\boldsymbol{p}}(x) D^{\boldsymbol{p}}, \qquad D^{\boldsymbol{p}} = \frac{\partial^{p_1}}{\partial x_1^{p_1}} \cdots \frac{\partial^{p_m}}{\partial x_m^{p_m}}, \qquad x \in \Omega \subset \mathbb{R}^m. \tag{1.2}$$

The multi-index $\boldsymbol{p}$ is an m-tuple $(p_1, \ldots, p_m)$ of nonnegative integers and $a_{\boldsymbol{p}}(x)$ are smooth, complex-valued functions of $x \in \bar{\Omega}$. We assume that the function $f(x,t)$ in (1.1) is highly oscillatory of the type

$$f(x,t) = \sum_{n=1}^{N} \alpha_n(x) e^{in\omega t}, \qquad \omega \gg 1,\ N \in \mathbb{N}, \tag{1.3}$$

where α_n are complex-valued, sufficiently smooth functions. Our method can also be efficiently applied in the case of functions α_n depending on the time variable t. This will be demonstrated in numerical experiments. This paper aims to express the solution of (1.1) as the following partial sum of the asymptotic expansion:

$$\begin{aligned} u(x,t) = p_{0,0}(x,t) &+ \sum_{r=1}^{R} \frac{1}{\omega^r} \sum_{s=0}^{S} p_{r,s}(x,t) e^{is\omega t} \\ &+ \frac{1}{\omega^{R+1}} E_{R,S}(x,t), \qquad t \in [0, t^\star],\ x \in \Omega \subset \mathbb{R}^m, \end{aligned} \tag{1.4}$$

where the coefficients $p_{r,s}$ and the errors $E_{R,S}$ are independent of ω. Needless to say, the general form of (1.1) encompasses many important equations from both classical and quantum physics. The most important examples include the heat equation and the Schrödinger equation. We demonstrate that our method can also be applied to equations with second-order time derivatives, such as the wave equation and the Klein–Gordon equation. Highly oscillatory differential equations of type (1.1) arise in various fields, including electronic engineering [4, 6], computing scattering frequencies [7], and quantum mechanics [12, 8].

The asymptotic expansion of type (1.4), also called the Modulated Fourier Expansion or frequency expansion, is a key technique in computational mathematics. It is used to analyze highly oscillatory Hamiltonian systems over long times [10, 3] and to study heterogeneous multiscale methods for oscillatory ordinary differential equations [18]. A comprehensive and detailed description of the Modulated Fourier Expansion can be found in [11]. Furthermore, the Modulated Fourier expansion can be utilized in numerical-asymptotic approaches as an ansatz for solving linear or nonlinear highly oscillatory differential equations [2, 8, 5, 6]. This ansatz is incorporated into the equation, and subsequently the coefficients $p_{r,s}$ in the sum (1.4) are determined either recursively or numerically by solving non-oscillatory differential equations. This approach allows us to approximate highly oscillatory equations with great accuracy. It is known that the sum (1.4) approximates the solution with error $C\omega^{-R-1}$, but the constant C is unknown. We derive a formula for this constant. This may be important in practice since this constant can be large.

In this paper, instead of employing an ansatz, we approximate the solution to (1.1) by deriving a partial sum of the asymptotic expansion (1.4) purely analytically. This approach enables us to obtain formulas for the coefficients $p_{r,n}$ of (1.4), eliminating the need to determine them by solving a system of differential equations. Furthermore, this approach allows us to derive a formula for the error in the approximation by (1.4).

To express the solution of (1.1) as an asymptotic series, we intend to use computational methods applied for computing highly oscillatory integrals. To find the approximate solution of the partial differential equation, firstly, we show that the Neumann series – in other words, a series of multivariate integrals – converges to the solution of (1.1) in the Sobolev space $H^{2p}(\Omega)$, where $2p$ is the order of the differential operator $\mathcal{L}$ defined in (1.2). Then, by using integration by parts and the theory of semigroups, we expand asymptotically each of these integrals into a sum of known coefficients. This is the most complicated and technical part, since the domain of each integral is a d-dimensional simplex, $d = 1, 2, \ldots$, and as a result, the number of terms in the asymptotic expansion grows exponentially as d increases. Fortunately, our numerical experiments show that one can achieve a small enough error even for relatively small values of d. By organizing terms appropriately with respect to the magnitudes ω^{-r} and frequencies $\mathrm{e}^{\mathrm{i}s\omega t}$, we obtain the sum (1.4). Given that the Neumann series converges for any time t, we determine the long-time behavior of the highly oscillatory solution to the equation.

By considering the potential function f in the form (1.3), we avoid the occurrence of resonance points in the highly oscillatory integrals that appear in the Neumann series. In [14], the authors present the asymptotic expansion of the solution for a highly oscillatory equation with a single frequency, specifically, when f is given by $f(x,t) = \alpha(x)\mathrm{e}^{\mathrm{i}\omega t}$. The present paper is a continuation of the research commenced in [14] and is the next step towards approximating the solution of (1.1) with a more general potential of the form

$$f(x,t) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \alpha_n(x,t)e^{in\omega t}, \qquad \omega \gg 1,\ N \in \mathbb{N}. \tag{1.5}$$

For such a function, resonance points will appear repeatedly in the integrals which constitute the Neumann series. In this context, resonance occurs when the vector $\boldsymbol{n} \in \mathbb{Z}^d$ in the frequency term $\mathrm{e}^{\mathrm{i}\boldsymbol{n}^T\boldsymbol{\tau}}$, where $\boldsymbol{\tau} \in \mathbb{R}^d$, is orthogonal to the boundary of the integral $\int_{\sigma_d(t)} f(\boldsymbol{\tau})\mathrm{e}^{\mathrm{i}\boldsymbol{n}^T\boldsymbol{\tau}}\,\mathrm{d}\boldsymbol{\tau}$, with $\sigma_d(t)$ representing a d-dimensional simplex. The presence of resonance points renders the approximation of high oscillation integrals even more difficult. In this article, we consider a special case of such a situation and demonstrate that if the vector $\boldsymbol{n}$ is orthogonal to one edge of the simplex $\sigma_d(t)$, then resonance points vanish in the sum of integrals from the Neumann series.

Our results extend those in [14]. Specifically, we consider a potential function f with multifrequencies (1.3), rather than with a single frequency. Furthermore, we establish the convergence of the Neumann series in $H^{2p}(\Omega)$, instead of L^2. In addition, we show that our approach for computing highly oscillatory equation can be applied to equations featuring second-order time derivatives. The resonance case is also considered.

The paper is organized as follows. In Section 2 we show that the solution of (1.1) can be expressed as the Neumann series. In Section 3, we convert each term of the Neumann series into a sum of multivariate highly oscillatory integrals. Then we introduce necessary definitions which will be needed in Section 4 for the asymptotic expansion of each highly oscillatory integral. Section 5 is devoted to the error analysis of the approximation. Section 6 concerns highly oscillatory integrals with resonance points which form the Neumann series. In Section 7, we apply the asymptotic method to the wave equation. Numerical simulations are presented in Section 8, while the conclusion and ideas for future research are discussed in Section 9.

2. Representation of the solution as the Neumann series. In this section, we show that the solution of (1.1) can be represented as the Neumann series for any $t > 0$. We start by introducing the necessary notations and making a general assumption, which will be needed throughout the text.

NOTATIONS 1. By $H^{2p}(\Omega) = W^{2p,2}(\Omega)$, where p is a nonnegative integer, we understand the Sobolev space equipped with the standard norm $\| \ \|_{H^{2p}(\Omega)}$, and $H_0^p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^p(\Omega)$. The norm in $D(\mathcal{L}) := H_0^p(\Omega) \cap H^{2p}(\Omega)$ is denoted by $\| \ \| := \| \ \|_{H^{2p}(\Omega)}$. Additionally, we slightly abuse the notation and also denote by $u(t) := u(\cdot, t)$ an element of an appropriate Banach space.

The following assumption will be in force throughout the text.

ASSUMPTION 1. *Suppose that*

(1) *Ω is an open and bounded set in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$.*
(2) *The operator $-\mathcal{L} : D(\mathcal{L}) := H_0^p(\Omega) \cap H^{2p}(\Omega) \to L^2(\Omega)$, where $\mathcal{L}$ is of the form* (1.2)*, is strongly elliptic of order* $2p$ *and has smooth, complex-valued coefficients $a_{\boldsymbol{p}}(x)$ on $\bar{\Omega}$.*
(3) *$u_0 \in D(\mathcal{L})$ and $f \in C([0, t^\star]; H^{2p}(\Omega))$.*

We emphasize that $D(\mathcal{L})$ is a Banach space since it is a closed subspace of the Banach space $H^{2p}(\Omega)$.

Expressing the solution $u(t)$ of (1.1) with the highly oscillatory potential (1.3) as a Neumann series facilitates its numerical approximation. Namely, if we express $u(t)$ as the series

$$u(t) = \sum_{d=0}^{\infty} T^d e^{t\mathcal{L}} u_0$$

for a certain linear operator T, then each term $T^d e^{t\mathcal{L}} u_0$ is actually a sum of multivariate highly oscillatory integrals. In the following, we will show that if such an integral satisfies the nonresonance condition, then $T^d e^{t\mathcal{L}} u_0 \sim \mathcal{O}(\omega^{-d})$. Intuitively, for large $\omega \gg 1$, further terms of the Neumann series become less relevant in the numerical approximation.

We start by applying Duhamel's formula, and we write (1.1) in the integral form

$$u(t) = e^{t\mathcal{L}} u_0 + \int_0^t e^{(t-\tau)\mathcal{L}} f(\tau) u(\tau)\, \mathrm{d}\tau, \tag{2.1}$$

where u_0 and $f(\tau), u(\tau)$ for fixed τ are elements of appropriate Banach spaces. Assumption 1 guarantees that $\mathcal{L}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathrm{e}^{t\mathcal{L}}\}_{t\geq 0}$ on $L^2(\Omega)$ and therefore

$$\max_{t\in[0,t^\star]} \|\mathrm{e}^{t\mathcal{L}}\|_{L^2(\Omega)\leftarrow L^2(\Omega)} \leq C(t^\star),$$

where $C(t^\star)$ is some constant independent of t [16, 9]. Moreover, since the a priori estimate

$$\|u\|_{H^{2p}(\Omega)} \leq C(\|\mathcal{L}u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \tag{2.2}$$

holds for any $u \in D(\mathcal{L})$, where $C > 0$ is a constant [1, 16], the operator $\mathrm{e}^{t\mathcal{L}}$ is also bounded in the norm $\| \ \|$ of $D(\mathcal{L})$ for any $t \in [0, t^\star]$. Indeed, by using (2.2) we have

$$\begin{aligned}
\|\mathrm{e}^{t\mathcal{L}}\|_{D(\mathcal{L})\leftarrow D(\mathcal{L})} &= \sup_{\|u\|\leq 1} \|\mathrm{e}^{t\mathcal{L}} u\| \leq C \sup_{\|u\|\leq 1} (\|\mathcal{L}\mathrm{e}^{t\mathcal{L}} u\|_{L^2(\Omega)} + \|\mathrm{e}^{t\mathcal{L}} u\|_{L^2(\Omega)}) \\
&= C \sup_{\|u\|\leq 1} (\|\mathrm{e}^{t\mathcal{L}} \mathcal{L} u\|_{L^2(\Omega)} + \|\mathrm{e}^{t\mathcal{L}} u\|_{L^2(\Omega)}) \\
&\leq C \|\mathrm{e}^{t\mathcal{L}}\|_{L^2(\Omega)\leftarrow L^2(\Omega)} \sup_{\|u\|\leq 1} (\|\mathcal{L}u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\
&\leq C \|\mathrm{e}^{t\mathcal{L}}\|_{L^2(\Omega)\leftarrow L^2(\Omega)} \sup_{\|u\|\leq 1} (C_1 \|u\| + \|u\|_{L^2(\Omega)}) \\
&\leq C(t^\star)
\end{aligned}$$

and again the constant $C(t^\star)$ depends on the coefficients $a_{\boldsymbol{p}}(x)$ of $\mathcal{L}$, but is independent of t.

We define a sequence $\{u^{[n]}\}_{n=0}^{\infty} \subset C([0,t^\star];D(\mathcal{L})) =: V$ by

$$u^{[n]}(t) = \sum_{d=0}^{n} T^d \mathrm{e}^{t\mathcal{L}} u_0, \quad u^{[0]}(t) = \mathrm{e}^{t\mathcal{L}} u_0, \qquad t \in [0,t^\star], \tag{2.3}$$

and a linear operator $T : V \to V$ by

$$Tu(t) = \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}} f(\tau) u(\tau)\,\mathrm{d}\tau, \qquad t \in [0,t^\star], \tag{2.4}$$

where $\mathcal{L}$ is the infinitesimal generator of a strongly continuous semigroup, and $f \in C([0,t^\star];H^{2p}(\Omega))$. We show that the sequence (2.3) is convergent to the solution $u \in C^1([0,t^\star];L^2(\Omega)) \cap C([0,t^\star];D(\mathcal{L}))$ of (1.1). We use the following estimate for the Sobolev norm:

$$\|hg\|_{H^{2p}(\Omega)} \le M\|h\|_{H^{2p}(\Omega)}\|g\|_{H^{2p}(\Omega)}, \qquad \Omega \subset \mathbb{R}^m,\ 2p > m/2, \tag{2.5}$$

for $h, g \in H^{2p}(\Omega)$, where M depends only on p and m. First, we show that the expression

$$T^d u(t) = \int_0^t \mathrm{e}^{(t-\tau_1)\mathcal{L}} f(\tau_1) \int_0^{\tau_1} \mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \ldots \int_0^{\tau_{d-1}} \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\,\mathrm{d}\tau_d \ldots \mathrm{d}\tau_1$$

is uniformly bounded in the norm $\|\ \|$ of $D(\mathcal{L})$ by a constant independent of t.

Lemma 1. *Suppose that Assumption* 1 *is satisfied. Let $u \in V$ (V is the domain of operator T defined in* (2.4)) *and $2p > m/2$. Then there exists a constant M (depending only on p and m) such that*

$$\|T^d u(t)\| \le M^d C_1^d C_2^d C_3 \frac{(t^\star)^d}{d!}, \qquad t \in [0,t^\star], \tag{2.6}$$

where $C_1 := \max_{t\in[0,t^\star]} \|\mathrm{e}^{t\mathcal{L}}\|_{D(\mathcal{L})\leftarrow D(\mathcal{L})}$, $C_2 := \max_{t\in[0,t^\star]} \|f(t)\|$ *and* $C_3 := \max_{t\in[0,t^\star]} \|u(t)\|$.

Proof. By applying inequality (2.5), and using the basic properties of the operator norm, we can estimate

$$\begin{aligned}
&\|T^d u(t)\| \\
&\quad = \Big\| \int_0^t \mathrm{e}^{(t-\tau_1)\mathcal{L}} f(\tau_1) \int_0^{\tau_1} \mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \ldots \int_0^{\tau_{d-1}} \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\,\mathrm{d}\tau_d \ldots \mathrm{d}\tau_1 \Big\| \\
&\quad = \Big\| \int_0^t \ldots \int_0^{\tau_{d-1}} \mathrm{e}^{(t-\tau_1)\mathcal{L}} f(\tau_1) \mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \ldots \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\,\mathrm{d}\tau_d \ldots \mathrm{d}\tau_1 \Big\| \\
&\quad \le \int_0^t \ldots \int_0^{\tau_{d-1}} \|\mathrm{e}^{(t-\tau_1)\mathcal{L}} f(\tau_1) \mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \ldots \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\|\,\mathrm{d}\tau_d \ldots \mathrm{d}\tau_1
\end{aligned}$$

$$\leq \int_0^t \dots \int_0^{\tau_{d-1}} \|\mathrm{e}^{(t-\tau_1)\mathcal{L}}\| \, \|f(\tau_1)\mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \dots \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\| \, \mathrm{d}\tau_d \dots \mathrm{d}\tau_1$$

$$\leq C_1 C_2 M \int_0^t \dots \int_0^{\tau_{d-1}} \|\mathrm{e}^{(\tau_1-\tau_2)\mathcal{L}} f(\tau_2) \dots \mathrm{e}^{(\tau_{d-1}-\tau_d)\mathcal{L}} f(\tau_d) u(\tau_d)\| \, \mathrm{d}\tau_d \dots \mathrm{d}\tau_1 =: (\star).$$

The constant M comes from (2.5). Proceeding analogously and repeatedly we obtain

$$(\star) \leq M^d C_1^d C_2^d C_3 \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{d-1}} \mathrm{d}\tau_d \dots \mathrm{d}\tau_1 \leq M^d C_1^d C_2^d C_3 \frac{(t^\star)^d}{d!}, \ \forall t \in [0, t^\star],$$

and the right hand side is independent of t, which completes the proof. ■

THEOREM 1. *Suppose that Assumption* 1 *is satisfied,* $2p > m/2$ *and* $t \in [0, t^\star]$. *Then the Neumann series*

$$\sum_{d=0}^{\infty} T^d \mathrm{e}^{t\mathcal{L}} u_0, \quad t \in [0, t^\star],$$

converges absolutely and uniformly to a function $u^\star \in C^1([0, t^\star]; L^2(\Omega)) \cap C([0, t^\star]; D(\mathcal{L}))$ *and* $u^\star$ *is the solution of* (1.1).

Proof. Let $C_1 := \max_{t\in[0,t^\star]} \|\mathrm{e}^{t\mathcal{L}}\|_{D(\mathcal{L})\leftarrow D(\mathcal{L})}$ and $C_2 := \max_{t\in[0,t^\star]} \|f(t)\|$. By using (2.6), we estimate

$$\max_{t\in[0,t^\star]} \|T^d \mathrm{e}^{t\mathcal{L}} u_0\| \leq \underbrace{M^d C_1^{d+1} C_2^d \|u_0\| \frac{(t^\star)^d}{d!}}_{=:A_d}, \quad \forall d \in \mathbb{N}.$$

Since $\sum_{d=0}^{\infty} A_d = C_1 \|u_0\| \exp(t^\star M C_1 C_2)$ and $D(\mathcal{L})$ is a Banach space, the Weierstrass M-test ensures that $\{u^{[n]}\}_{n=0}^{\infty}$ converges absolutely and uniformly to some function $u^\star \in C([0, t^\star], D(\mathcal{L}))$. Thus, for each $t \in [0, t^\star]$ we have

$$u^\star(t) := \lim_{n\to\infty} u^{[n]}(t) = \sum_{d=0}^{\infty} T^d \mathrm{e}^{t\mathcal{L}} u_0, \tag{2.7}$$

where $u^{[n]}$ is defined by (2.3). The operator $I - T$ is a bijection and the function $u^\star$ is the solution of (2.1). Indeed, it is easy to observe that

$$(I - T) \sum_{d=0}^{n} T^d = \sum_{d=0}^{n} T^d (I - T) = I - T^{n+1}.$$

Since $\|T^{n+1}\| \to 0$ as $n \to \infty$, for each $t \in [0, t^\star]$ we obtain

$$(I - T) u^\star(t) = \mathrm{e}^{t\mathcal{L}} u_0,$$

which is the equivalent form of (2.1). Moreover, one can observe that the mapping $t \mapsto u^\star(t)$ is differentiable for each $t > 0$ when $u_0 \in D(\mathcal{L})$ and

$f \in C([0,t^\star], H^{2p}(\Omega))$. Indeed, consider the mild formulation of (1.1),

$$u^\star(t) = \mathrm{e}^{t\mathcal{L}} u_0 + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}} f(\tau) u^\star(\tau)\,\mathrm{d}\tau; \tag{2.8}$$

of course the summand $\mathrm{e}^{t\mathcal{L}}u_0$ is continuously differentiable, $\frac{d}{dt}\mathrm{e}^{t\mathcal{L}}u_0 = \mathcal{L}\mathrm{e}^{t\mathcal{L}}u_0$ for all $t > 0$, and since $u^\star(\tau)$ is continuous, also the last integral is continuously differentiable in t (see [17, Theorem 12.16]). Since $\mathcal{L}$ is a closed operator, we have

$$\frac{d}{dt} u^\star(t) = \mathrm{e}^{t\mathcal{L}} \mathcal{L} u_0 + f(t) u^\star(t) + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}} \mathcal{L} f(\tau) u^\star(\tau)\,\mathrm{d}\tau. \tag{2.9}$$

Since $f(t) \in H^{2p}(\Omega)$ and $u^\star(t) \in H^{2p}(\Omega) \cap H_0^p(\Omega)$ for all $t \in [0,t^\star]$, it follows that $f(t)u^\star(t) \in H^{2p}(\Omega) \cap H_0^p(\Omega) = D(\mathcal{L})$. Therefore, each expression on the right-hand side of (2.9) is well-defined. We have shown that the solution $u^\star$ of (2.8) belongs to $C^1([0,t^\star]; L^2(\Omega)) \cap C([0,t^\star]; D(\mathcal{L}))$. Now it is obvious that $u^\star$ is the solution of (1.1). ∎

The above convergence proof of the sequence (2.3) is similar to that presented in [14]. However, to show the convergence of (2.3) to the solution of (1.1), one needs to establish the convergence of (2.3) in the appropriate Sobolev norm.

Needless to say, the existence and uniqueness of solutions of linear evolution equations is a well-known and well-established theory. Theorem 1 is only needed to show that the solution to the equation can be represented as a series of multivariate integrals. We utilize this fact in the numerical approximation of a highly oscillatory solution. It is easy to notice that in the proof of Theorem 1, the operator T need not be a contraction mapping, but the Neumann series is still convergent. It can be shown that for arbitrary time $t > 0$, there exists s such that for any $d > s$, the operator T^d is a contraction mapping, i.e. $\|T^d\| < 1$.

Remark 1. The mild formulation of equation (1.1) allows us to relax the assumptions on the initial condition u_0. Instead of requiring $u_0 \in D(\mathcal{L})$, let us assume for a moment that $u_0 \in L^2(\Omega)$ and additionally that f is a bounded function, $\|f\|_\infty < \infty$. By following a similar approach to the proof of Theorem 1, one can also show the convergence of the sequence (2.3) in the norm of the space $C([0,t^\star]; L^2(\Omega))$ to the mild solution $u^\star$ of (1.1). In this case, the additional assumption $2p > m/2$ becomes redundant.

The series (2.7) is known as the *Neumann series*. Equation (2.1) can be expressed symbolically as

$$u(t) = \mathrm{e}^{t\mathcal{L}} u_0 + Tu(t).$$

It can be easy verified that for each n, the term $u^{[n]}(t)$ satisfies the relation

$$u^{[n]}(t) = \mathrm{e}^{t\mathcal{L}} u_0 + T u^{[n-1]}(t), \quad u^{[0]}(t) = \mathrm{e}^{t\mathcal{L}} u_0.$$

Therefore, in the remaining text, we will use the terms 'nth partial sum of the Neumann series' and 'nth iteration of the equation' interchangeably to refer to the expression $u^{[n]}(t)$.

3. Terms of the Neumann series as sums of multivariate integrals. According to Theorem 1, the solution to (1.1) can be expressed as the Neumann series, $u(t) = \sum_{d=0}^{\infty} T^d \mathrm{e}^{t\mathcal{L}} u_0$. In this section, we show that for the function f defined in (1.3), each term $T^d \mathrm{e}^{t\mathcal{L}}$, $d = 1, 2, \ldots,$ of the Neumann series is a sum of multivariate highly oscillatory integrals. Subsequently, we will prove that each $T^d \mathrm{e}^{t\mathcal{L}}$ can be approximated by a partial sum of the asymptotic expansion. By linearity of the semigroup operator $\{\mathrm{e}^{t\mathcal{L}}\}_{t\geq 0}$, we can convert expression $T^d \mathrm{e}^{t\mathcal{L}}$ into a more convenient form for f defined in (1.3):

$$\begin{aligned}
&T^d \mathrm{e}^{t\mathcal{L}} \\
&= \int_0^t \int_0^{\tau_d} \cdots \int_0^{\tau_2} \mathrm{e}^{(t-\tau_d)\mathcal{L}} f(\tau_d) \mathrm{e}^{(\tau_d-\tau_{d-1})\mathcal{L}} f(\tau_{d-1}) \ldots \mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}} f(\tau_1) \mathrm{e}^{\tau_1 \mathcal{L}} \, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_d \\
&= \sum_{1\leq n_1,\ldots n_d \leq N} \int_0^t \int_0^{\tau_d} \cdots \int_0^{\tau_2} \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} \mathrm{e}^{(\tau_d-\tau_{d-1})\mathcal{L}} \alpha_{n_{d-1}} \ldots \mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}} \alpha_{n_1} \mathrm{e}^{\tau_1\mathcal{L}} \\
&\qquad \times \mathrm{e}^{\mathrm{i}\omega(\tau_1 n_1 + \cdots + \tau_d n_d)} \, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_d \\
&= \sum_{\boldsymbol{n}\in\{1,\ldots,N\}^d} \int_{\sigma_d(t)} F_{\boldsymbol{n}}(t,\boldsymbol{\tau}) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{n}^T \boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{\tau} = \sum_{\boldsymbol{n}\in\{1,\ldots,N\}^d} I[F_{\boldsymbol{n}}, \sigma_d(t)],
\end{aligned}$$

where

$$\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d),$$

$$F_{\boldsymbol{n}}(t,\boldsymbol{\tau}) = \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} \mathrm{e}^{(\tau_d-\tau_{d-1})\mathcal{L}} \alpha_{n_{d-1}} \ldots \mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}} \alpha_{n_1} \mathrm{e}^{\tau_1\mathcal{L}}, \tag{3.1}$$

$$I[F_{\boldsymbol{n}}, \sigma_d(t)] = \int_{\sigma_d(t)} F_{\boldsymbol{n}}(t,\boldsymbol{\tau}) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{n}^T \boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{\tau} \tag{3.2}$$

and $\sigma_d(t)$ denotes the d-dimensional simplex

$$\sigma_d(t) = \{\boldsymbol{\tau} := (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d : t \geq \tau_d \geq \cdots \geq \tau_1 \geq 0\}.$$

Using the above notation, the solution $u(t)$ of (2.1) can be written as

$$u(t) = \mathrm{e}^{t\mathcal{L}} u_0 + \sum_{d=1}^{\infty} \sum_{\boldsymbol{n}\in\{1,\ldots,N\}^d} I[F_{\boldsymbol{n}}, \sigma_d(t)] u_0.$$

Each $F_{\boldsymbol{n}}(t,\boldsymbol{\tau})$ is a linear operator, $F_{\boldsymbol{n}}(t,\boldsymbol{\tau}) : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathcal{L})$ and $\boldsymbol{n} \in \mathbb{N}^d$ is a vector corresponding to $F_{\boldsymbol{n}}$ in the sense that $\boldsymbol{n}$ appears in the frequency exponent of the integral (3.2).

In our approach, to asymptotically expand a multivariate highly oscillatory integral of type (3.2), we will exploit repeatedly Fubini's theorem, the properties of the semigroup operator and integration by parts. Therefore one should be able to compute successive partial derivatives of the expression of type (3.1). One proves by induction the following:

LEMMA 2. *Assume* $\alpha \in D(\mathcal{L}^k)$, *where* $D(\mathcal{L}^k) := \{u \in D(\mathcal{L}^{k-1}) : \mathcal{L}^{k-1}u \in D(\mathcal{L})\}$, $k = 1,2,\ldots.$ *Then*

$$\partial_\tau^k(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha\mathrm{e}^{\tau\mathcal{L}}) = \mathrm{e}^{(t-\tau)\mathcal{L}}\underbrace{[\ldots[[\alpha,\mathcal{L}],\mathcal{L}],\ldots]}_{\mathcal{L}\text{ appears } k \text{ times}}\mathrm{e}^{\tau\mathcal{L}} = (-1)^k\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^k(\alpha)\mathrm{e}^{\tau\mathcal{L}},$$

where $\mathrm{ad}_{\mathcal{L}}^0(\alpha) = \alpha$, $\mathrm{ad}_{\mathcal{L}}^k(\alpha) = [\mathcal{L}, \mathrm{ad}_{\mathcal{L}}^{k-1}(\alpha)]$ *and* $[X,Y] \equiv XY - YX$ *is the commutator of* X *and* Y.

Unless otherwise stated, we assume that each expression of type $\mathcal{L}^k\alpha_n$ appearing below is well defined for any functions $\alpha_n, n = 1,\ldots,N$.

In the asymptotic expansion of the integral of type (3.2), the number of terms in a partial sum of the asymptotic series grows exponentially as the dimension d of the domain increases, therefore to simplify the notation we introduce the following definitions.

DEFINITION 1. We denote by $\boldsymbol{v}_\ell^d \in \{0,1\}^d$, $\ell = 0,1,\ldots,d$, the vertices of the simplex $\sigma_d(1)$:

$$\begin{aligned}\boldsymbol{v}_0^d &= (1,1,1,\ldots,1,1),\\ \boldsymbol{v}_1^d &= (0,1,1,\ldots,1,1),\\ &\ \ \vdots\\ \boldsymbol{v}_\ell^d &= (\underbrace{0,\ldots,0}_{\ell\text{ zeros}},1,\ldots,1).\end{aligned}$$

DEFINITION 2. Let Φ_ℓ^d, $\ell = 0,1,\ldots,d$, be a family of points such that

$$\begin{aligned}\Phi_0^d &= \{(1,1,\ldots,1)\},\\ \Phi_\ell^d &= \{\boldsymbol{\phi} \in \{0,1\}^d : \boldsymbol{\phi} = (\phi_1,\ldots,\phi_d),\ \phi_\ell = 0,\ \phi_j = 1 \text{ for } j > \ell\}.\end{aligned}$$

DEFINITION 3. For $\boldsymbol{\phi} = (\phi_1,\ldots,\phi_d) \in \{0,1\}^d$ and a multi-index $\boldsymbol{k} = (k_1,\ldots,k_d)$, we define the sequence of partial derivatives of $F(t,\tau_1,\ldots,\tau_d)$:

$$F^{\boldsymbol{k}}[\boldsymbol{\phi}](t) = \partial_{\tau_d}^{k_d}\big(\ldots\partial_{\tau_2}^{k_2}\big(\partial_{\tau_1}^{k_1}F(t,\tau_1,\ldots,\tau_d)|_{\tau_1=\phi_1\tau_2}\big)\big|_{\tau_2=\phi_2\tau_3}\cdots\big)\big|_{\tau_d=\phi_d t}.$$

We understand this definition as follows: First, we compute the k_1th partial derivative τ_1. Then in place of the τ_1 variable we substitute τ_2 or 0.

Subsequently, we compute the k_2th partial derivative τ_2 and then again we substitute $\tau_2 = \tau_3$ or $\tau_2 = 0$. We proceed in this manner until we compute the k_dth partial derivative with respect to τ_d and substitute $\tau_d = \phi_d t$, where $\phi_d = 0$ or $\phi_d = 1$.

EXAMPLE 1. To illustrate Definition 3 for $F_{\boldsymbol{n}}$ defined in (3.1), we compute the first few expressions $F_{\boldsymbol{n}}^{\boldsymbol{k}}[\boldsymbol{\phi}](t)$ for $d = 1, 2$. We assume that α_n, $n = 1, \dots, N$, are sufficiently smooth functions. Then

$$\begin{aligned}
F_{n_1}^{k_1}[1](t) &= \partial_{\tau_1}^{k_1}(\mathrm{e}^{(t-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}})|_{\tau_1=t} = (-1)^{k_1}\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1})\mathrm{e}^{t\mathcal{L}},\\
F_{n_1}^{k_1}[0](t) &= \partial_{\tau_1}^{k_1}(\mathrm{e}^{(t-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}})|_{\tau_1=0} = (-1)^{k_1}\mathrm{e}^{t\mathcal{L}}\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1}),\\
F_{\boldsymbol{n}}^{\boldsymbol{k}}[(1,1)](t) &= \partial_{\tau_2}^{k_2}\big(\partial_{\tau_1}^{k_1}\big(\mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_{n_2}\mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}}\big)\big|_{\tau_1=\tau_2}\big)\big|_{\tau_2=t}\\
&= (-1)^{|\boldsymbol{k}|}\operatorname{ad}_{\mathcal{L}}^{k_2}(\alpha_{n_2}\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1}))\mathrm{e}^{t\mathcal{L}},\\
F_{\boldsymbol{n}}^{\boldsymbol{k}}[(1,0)](t) &= \partial_{\tau_2}^{k_2}\big(\partial_{\tau_1}^{k_1}\big(\mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_{n_2}\mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}}\big)\big|_{\tau_1=\tau_2}\big)\big|_{\tau_2=0}\\
&= (-1)^{|\boldsymbol{k}|}\mathrm{e}^{t\mathcal{L}}\operatorname{ad}_{\mathcal{L}}^{k_2}(\alpha_{n_2}\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1})),\\
F_{\boldsymbol{n}}^{\boldsymbol{k}}[(0,1)](t) &= \partial_{\tau_2}^{k_2}\big(\partial_{\tau_1}^{k_1}\big(\mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_{n_2}\mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}}\big)\big|_{\tau_1=0}\big)\big|_{\tau_2=t}\\
&= (-1)^{|\boldsymbol{k}|}\operatorname{ad}_{\mathcal{L}}^{k_2}(\alpha_{n_2})\mathrm{e}^{t\mathcal{L}}\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1}),\\
F_{\boldsymbol{n}}^{\boldsymbol{k}}[(0,0)](t) &= \partial_{\tau_2}^{k_2}\big(\partial_{\tau_1}^{k_1}\big(\mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_{n_2}\mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}}\alpha_{n_1}\mathrm{e}^{\tau_1\mathcal{L}}\big)\big|_{\tau_1=0}\big)\big|_{\tau_2=0}\\
&= (-1)^{|\boldsymbol{k}|}\mathrm{e}^{t\mathcal{L}}\operatorname{ad}_{\mathcal{L}}^{k_2}(\alpha_{n_2})\operatorname{ad}_{\mathcal{L}}^{k_1}(\alpha_{n_1}).
\end{aligned}$$

It can be shown that for $F_{\boldsymbol{n}}$ defined in (3.1),

$$\begin{aligned}
&F_{\boldsymbol{n}}^{\boldsymbol{k}}[\boldsymbol{\phi}](t)\\
&= \sum_{\boldsymbol{m}\le\boldsymbol{k}}\binom{\boldsymbol{k}}{\boldsymbol{m}}(-1)^{|\boldsymbol{m}|+|\boldsymbol{k}|}\mathcal{L}^{r_{d+1}}\alpha_{n_d}\mathcal{L}^{r_d}\alpha_{n_{d-1}}\dots\mathcal{L}^{r_{\ell+1}}\alpha_{n_{\ell+1}}\mathrm{e}^{t\mathcal{L}}\mathcal{L}^{r_\ell}\alpha_{n_\ell}\dots\mathcal{L}^{r_2}\alpha_{n_1}\mathcal{L}^{r_1}
\end{aligned}$$

for $\boldsymbol{\phi} \in \Phi_\ell^d$, sufficiently smooth functions α_n, $n = 1, \dots, N$, multi-indices $\boldsymbol{k} = (k_1, \dots, k_d)$, $\boldsymbol{m} = (m_1, \dots, m_d)$ and $\binom{\boldsymbol{k}}{\boldsymbol{m}} = \frac{\boldsymbol{k}!}{\boldsymbol{m}!(\boldsymbol{k}-\boldsymbol{m})!}$. The numbers $r_1, \dots, r_{d+1}$ satisfy $r_1 + \dots + r_{d+1} = |\boldsymbol{k}|$ and $\boldsymbol{m} \le \boldsymbol{k}$ means $m_i \le k_i$ for $i = 1, 2, \dots, d$.

4. Asymptotic expansion of a highly oscillatory integral subject to the nonresonance condition. Let $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$. In this section, we assume the *nonresonance condition*

$$n_j + n_{j+1} + \dots + n_{r-1} + n_r \neq 0 \tag{4.1}$$

for each $1 \le j \le r \le d$. This condition implies that $\boldsymbol{n}$ is not orthogonal to the faces of the simplex $\sigma_d(t)$. This case involves a situation in which the highly oscillatory potential f takes the form (1.3). This section shows that the integral $I[F_{\boldsymbol{n}}, \sigma_d(t)]$ defined in (3.2) can be expressed as a partial sum of an asymptotic series.

Let us start with the first term of the Neumann series. By using integration by parts, the integral $[F_n, (0,t)]$ can be represented as

$$\begin{aligned}(4.2)\qquad &\int_0^t \mathrm{e}^{\mathrm{i}\omega\tau n}\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha_n \mathrm{e}^{\tau\mathcal{L}}u_0\,\mathrm{d}\tau \\ &= \sum_{k=0}^{r-1}\frac{1}{(\mathrm{i}\omega n)^{k+1}}[\mathrm{e}^{\mathrm{i}\omega t n}\operatorname{ad}^k_{\mathcal{L}}(\alpha_n)\mathrm{e}^{t\mathcal{L}} - \mathrm{e}^{t\mathcal{L}}\operatorname{ad}^k_{\mathcal{L}}(\alpha_n)]u_0 \\ &\quad + \frac{1}{(\mathrm{i}\omega n)^r}\int_0^t \mathrm{e}^{\mathrm{i}\omega\tau n}\mathrm{e}^{(t-\tau)\mathcal{L}}\operatorname{ad}^r_{\mathcal{L}}(\alpha_n)\mathrm{e}^{\tau\mathcal{L}}u_0\,\mathrm{d}\tau,\end{aligned}$$

and the above partial sum approximates the integral with error $\mathcal{O}(\omega^{-r-1})$. The general formula for the integral $I[F_{\boldsymbol{n}}, \sigma_d(t)]$ over a d-dimensional simplex is unfortunately much more complicated. However, (4.2) gives the main idea of our considerations. In the later derivation, to expand asymptotically a multivariate highly oscillatory integral of type (3.2), we will use simple tools, including induction, Fubini's theorem and the identity (4.2).

The following definition determines the coefficients that will appear as a result of integrating by parts.

Definition 4. For $\Phi^d_\ell \ni \boldsymbol{\phi} = (\phi_1, \ldots, \phi_d) = (\phi_1, \ldots, \phi_{\ell-1}, 0, 1, \ldots, 1)$, $\boldsymbol{k} = (k_1, \ldots, k_d)$, and $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ satisfying (4.1), we introduce the following rational numbers:

$$\begin{aligned}(4.3)\qquad A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n}) = {}&\frac{(-1)^{\phi_1+1}}{n_1^{k_1+1}}\,\frac{(-1)^{\phi_2+1}}{(n_1\phi_1+n_2)^{k_2+1}}\,\frac{(-1)^{\phi_3+1}}{((n_1\phi_1+n_2)\phi_2+n_3)^{k_3+1}} \\ &\times\cdots\times\frac{(-1)^{\phi_d+1}}{(\ldots((n_1\phi_1+n_2)\phi_2+n_3)\phi_3+\cdots+n_d)^{k_d+1}}\end{aligned}$$

Let us note that due to assumption (4.1), we never divide by zero in the above expressions.

One can observe that the numbers $A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n})$ satisfy the recursive relation

$$A_{k_1}[\phi_1](n_1) = \frac{(-1)^{\phi_1+1}}{n_1^{k_1+1}},$$

$$A_{\boldsymbol{k}}[(\boldsymbol{\phi}, \phi_{d+1})]((\boldsymbol{n}, n_{d+1})) = A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\boldsymbol{n})\frac{(-1)^{\phi_{d+1}+1}}{[(\boldsymbol{v}^d_\ell, 1)\cdot(\boldsymbol{n}, n_{d+1})]^{k_{d+1}+1}},$$

where $\boldsymbol{\phi} \in \Phi^d_\ell$, $\boldsymbol{v}^d_\ell = (0, \ldots, \overset{\ell}{0}, 1, \ldots, 1)$ is a vertex of the simplex $\sigma_d(1)$, $\boldsymbol{k} = (k_1, \ldots, k_{d+1})$, $\tilde{\boldsymbol{k}} = (k_1, \ldots, k_d)$, and $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ satisfies (4.1). This recursive definition will be needed later in the proof of Theorem 2.

EXAMPLE 2. For $d = 3$ we compute the coefficients $A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n})$ for different $\phi \in \Phi^3_\ell$, $\ell = 0, 1, 2, 3$:

$$\begin{aligned}
\ell = 0, &\quad A_{\boldsymbol{k}}[(1,1,1)](\boldsymbol{n}) = \frac{1}{n_1^{k_1+1}(n_1+n_2)^{k_2+1}(n_1+n_2+n_3)^{k_3+1}},\\
\ell = 1, &\quad A_{\boldsymbol{k}}[(0,1,1)](\boldsymbol{n}) = \frac{-1}{n_1^{k_1+1}(n_2)^{k_2+1}(n_2+n_3)^{k_3+1}},\\
\ell = 2, &\quad A_{\boldsymbol{k}}[(0,0,1)](\boldsymbol{n}) = \frac{1}{n_1^{k_1+1}n_2^{k_2+1}n_3^{k_3+1}},\\
\ell = 2, &\quad A_{\boldsymbol{k}}[(1,0,1)](\boldsymbol{n}) = \frac{-1}{n_1^{k_1+1}(n_1+n_2)^{k_2+1}n_3^{k_3+1}},\\
\ell = 3, &\quad A_{\boldsymbol{k}}[(0,0,0)](\boldsymbol{n}) = \frac{-1}{n_1^{k_1+1}n_2^{k_2+1}n_3^{k_3+1}},\\
\ell = 3, &\quad A_{\boldsymbol{k}}[(1,1,0)](\boldsymbol{n}) = \frac{-1}{n_1^{k_1+1}(n_1+n_2)^{k_2+1}(n_1+n_2+n_3)^{k_3+1}},\\
\ell = 3, &\quad A_{\boldsymbol{k}}[(0,1,0)](\boldsymbol{n}) = \frac{1}{n_1^{k_1+1}n_2^{k_2+1}(n_2+n_3)^{k_3+1}},\\
\ell = 3, &\quad A_{\boldsymbol{k}}[(1,0,0)](\boldsymbol{n}) = \frac{1}{n_1^{k_1+1}(n_1+n_2)^{k_2+1}n_3^{k_3+1}}.
\end{aligned}$$

REMARK 2. In the next section, we employ the following notation: if $\boldsymbol{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$ is an m-dimensional vector, then $\tilde{\boldsymbol{n}} = (n_1, \dots, n_{m-1}) \in \mathbb{Z}^{m-1}$.

To simplify notation in the proof Theorem 2 below, we will also write $F := F_{\boldsymbol{n}}$ for the operator defined in (3.1) if the vector $\boldsymbol{n}$ is clear from the context.

THEOREM 2. *Suppose that Assumption* 1 *is satisfied. In addition, suppose that* $u_0, \alpha_n \in H^{2p(r+1)}(\Omega)$, $n = 1, \dots, N$. *Let* F *be the operator defined in* (3.1), *and let* $\boldsymbol{n}$ *be the vector corresponding to* F *that satisfies the non-resonance condition* (4.1). *Then the integral* (3.2) *can be expressed as the* r*th partial sum* $\mathcal{S}_r^{(d)}(t)$ *of an asymptotic series with error* $E_r^{(d)}(t)$,

$$I[F, \sigma_d(t)] = \int_{\sigma_d(t)} F(t, \boldsymbol{\tau}) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{n}^T \boldsymbol{\tau}} \,\mathrm{d}\boldsymbol{\tau} = \mathcal{S}_r^{(d)}(t) + E_r^{(d)}(t), \tag{4.4}$$

where

$$\mathcal{S}_r^{(d)}(t) = \sum_{|\boldsymbol{k}|=0}^{r-d} \frac{(-1)^{|\boldsymbol{k}|}}{(\mathrm{i}\omega)^{d+|\boldsymbol{k}|}} \sum_{\ell=0}^{d} \mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_\ell^d} \sum_{\phi \in \Phi_\ell^d} A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n}) F^{\boldsymbol{k}}[\boldsymbol{\phi}](t), \qquad r \geq d, \tag{4.5}$$

and the error $E_r^{(d)}(t)$ *is in recursive form*

$$E_r^{(1)}(t)=\frac{(-1)^r}{(\mathrm{i}\omega)^r}\,\frac{1}{n_1^r}\int\limits_0^t \mathrm{e}^{\mathrm{i}\omega\tau_1 n_1}\partial_{\tau_1}^r F(t,\tau_1)\,\mathrm{d}\tau_1,$$

$$E_r^{(d)}(t)=\frac{(-1)^{r-d+1}}{(\mathrm{i}\omega)^r}\sum_{|\boldsymbol{k}|=r-d+1}\sum_{\ell=0}^{d-1}\sum_{\phi\in\Phi_\ell^{d-1}}\frac{1}{(\boldsymbol{n}^T\boldsymbol{v}_\ell^d)^{k_d}}A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}) \times\int\limits_0^t \mathrm{e}^{\mathrm{i}\omega\tau_d\boldsymbol{n}^T\boldsymbol{v}_\ell^d}\partial_{\tau_d}^{k_d}\big(\mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_d}F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d)\big)\,\mathrm{d}\tau_d + \int\limits_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_d}E_r^{(d-1)}(\tau_d)\mathrm{e}^{\mathrm{i}\omega n_d\tau_d}\,\mathrm{d}\tau_d \quad \textit{for } d\ge 2. \tag{4.6}$$

Proof. We use induction on d. For $d=1$ we integrate $I[F,(0,t)]$ by parts r times to obtain

$$\begin{aligned}
&I[F,(0,t)]\\
&\quad=\int\limits_0^t F(t,\tau_1)\mathrm{e}^{\mathrm{i}\omega n_1\tau_1}\,\mathrm{d}\tau_1\\
&\quad=\sum_{k_1=0}^{n-1}\frac{(-1)^{k_1}}{(\mathrm{i}\omega)^{1+k_1}}\bigg(\mathrm{e}^{\mathrm{i}\omega n_1}\frac{1}{n_1^{k_1+1}}\partial_{\tau_1}^{k_1}F(t,\tau_1)|_{\tau_1=t}+\frac{-1}{n_1^{k_1+1}}\partial_{\tau_1}^{k_1}F(t,\tau_1)|_{\tau_1=0}\bigg)\\
&\qquad+\frac{(-1)^r}{(\mathrm{i}\omega)^r}\frac{1}{n_1^r}\int\limits_0^t \mathrm{e}^{\mathrm{i}\omega\tau_1 n_1}\partial_{\tau_1}^r F(t,\tau_1)\,\mathrm{d}\tau_1\\
&\quad=\sum_{k_1=0}^{r-1}\frac{(-1)^{k_1}}{(\mathrm{i}\omega)^{1+k_1}}\sum_{\ell=0}^{1}\mathrm{e}^{\mathrm{i}\omega v_\ell^1 n_1}\sum_{\phi\in\Phi_\ell^1}A_{k_1}[\phi](n_1)F^{k_1}[\phi](t)+E_r^{(1)}(t)\\
&\quad=\mathcal{S}_r^{(1)}(t)+E_r^{(1)}(t).
\end{aligned}$$

Let $\tilde{\boldsymbol{n}}=(n_1,\dots,n_{d-1})$, $\tilde{\boldsymbol{k}}=(k_1,\dots,k_{d-1})\in\mathbb{N}^{d-1}$ and $\boldsymbol{n}=(n_1,\dots,n_d)$, $\boldsymbol{k}=(k_1,\dots,k_d)\in\mathbb{N}^d$. Suppose that (4.4) is true for

$$I[F,\sigma_{d-1}(t)]=\int\limits_{\sigma_{d-1}(t)}F(\boldsymbol{\tau})\mathrm{e}^{\mathrm{i}\omega\tilde{\boldsymbol{n}}^T\boldsymbol{\tau}}\,\mathrm{d}\boldsymbol{\tau}.$$

Then

$$\begin{aligned}
\int\limits_{S_d(t)}F(t,\boldsymbol{\tau})\mathrm{e}^{\mathrm{i}\omega\boldsymbol{n}^T\boldsymbol{\tau}}\,\mathrm{d}\boldsymbol{\tau}&\overset{(1)}{=}\int\limits_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_d}I[F,\sigma_{d-1}(\tau_d)]\mathrm{e}^{\mathrm{i}\omega\tau_d n_d}\,\mathrm{d}\tau_d\\
&\overset{(2)}{=}\int\limits_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_d}(\mathcal{S}_r^{(d-1)}(\tau_d)+E_r^{(d-1)}(\tau_d))\mathrm{e}^{\mathrm{i}\omega\tau_d n_d}\,\mathrm{d}\tau_d
\end{aligned}$$

$$\stackrel{(3)}{=} \sum_{|\tilde{\boldsymbol{k}}|=0}^{r-d+1} \frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}} \sum_{\ell=0}^{d-1} \sum_{\boldsymbol{\phi}\in\Phi_\ell^{d-1}} A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}) \int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d) \mathrm{e}^{\mathrm{i}\omega\tau_d \boldsymbol{n}^T \boldsymbol{v}_\ell^d}\, \mathrm{d}\tau_d$$

$$+ \int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} E_r^{(d-1)}(\tau_d) \mathrm{e}^{\mathrm{i}\omega\tau_d n_d}\, \mathrm{d}\tau_d$$

$$\stackrel{(4)}{=} \sum_{|\tilde{\boldsymbol{k}}|=0}^{r-d+1} \frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}} \sum_{\ell=0}^{d-1} \sum_{\boldsymbol{\phi}\in\Phi_\ell^{d-1}} A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}) \Bigg\{ \sum_{k_d=0}^{r-d-|\tilde{\boldsymbol{k}}|} \frac{(-1)^{k_d}}{(\mathrm{i}\omega)^{k_d+1}} \frac{1}{(\boldsymbol{n}^T \boldsymbol{v}_\ell^d)^{k_d+1}}$$

$$\times \left(\mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_\ell^d} \partial_{\tau_d}^{k_d} \left[\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d) \right] \Big|_{\tau_d=t} - \partial_{\tau_d}^{k_d} \left[\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d) \right] \Big|_{\tau_d=0} \right)$$

$$+ \frac{(-1)^{r-d-|\tilde{\boldsymbol{k}}|+1}}{(\mathrm{i}\omega)^{r-d-|\tilde{\boldsymbol{k}}|+1}} \frac{1}{(\boldsymbol{n}^T \boldsymbol{v}_\ell^d)^{r-d-|\tilde{\boldsymbol{k}}|+1}} \int_0^t \partial_{\tau_d}^{r-d-|\tilde{\boldsymbol{k}}|+1} (\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d)) \mathrm{e}^{\mathrm{i}\omega\tau_d \boldsymbol{n}^T \boldsymbol{v}_\ell^d}\, \mathrm{d}\tau_d \Bigg\}$$

$$+ \int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} E_r^{(d-1)}(\tau_d) \mathrm{e}^{\mathrm{i}\omega\tau_d n_d}\, \mathrm{d}\tau_d$$

$$\stackrel{(5)}{=} \sum_{|\boldsymbol{k}|=0}^{r-d} \frac{(-1)^{|\boldsymbol{k}|}}{(\mathrm{i}\omega)^{d+|\boldsymbol{k}|}} \sum_{\ell=0}^{d-1} \sum_{\boldsymbol{\phi}\in\Phi_\ell^{d-1}} A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}) \frac{1}{(\boldsymbol{n}^T \boldsymbol{v}_\ell^d)^{k_d+1}} \Big[\mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_\ell^d}$$

$$\times \partial_{\tau_d}^{k_d} \left[\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d) \right] \Big|_{\tau_d=1\cdot t} - \mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_d^d} \partial_{\tau_d}^{k_d} \left[\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d) \right] \Big|_{\tau_d=t\cdot 0} \Big]$$

$$+ \frac{(-1)^{r-d+1}}{(\mathrm{i}\omega)^r} \sum_{|\boldsymbol{k}|=r-d+1} \sum_{\ell=0}^{d-1} \sum_{\boldsymbol{\phi}\in\Phi_\ell^{d-1}} A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}) \frac{1}{(\boldsymbol{n}^T \boldsymbol{v}_\ell^d)^{k_d}}$$

$$\times \int_0^t \partial_{\tau_d}^{k_d} (\mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d)) \mathrm{e}^{\mathrm{i}\omega\tau_d \boldsymbol{n}^T \boldsymbol{v}_\ell^d}\, \mathrm{d}\tau_d + \int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}} \alpha_{n_d} E_r^{(d-1)}(\tau_d) \mathrm{e}^{\mathrm{i}\omega\tau_d n_d}\, \mathrm{d}\tau_d$$

$$\stackrel{(6)}{=} \sum_{|\boldsymbol{k}|=0}^{r-d} \frac{(-1)^{|\boldsymbol{k}|}}{(\mathrm{i}\omega)^{d+|\boldsymbol{k}|}} \sum_{\ell=0}^{d-1} \mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_\ell^d} \sum_{\boldsymbol{\phi}\in\Phi_\ell^d} A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n}) F^{\boldsymbol{k}}[\boldsymbol{\phi}](t)$$

$$+ A_{\boldsymbol{k}}[(\phi_1,\ldots,\phi_{d-1},0)](\boldsymbol{n}) F^{\boldsymbol{k}}[(\phi_1,\ldots,\phi_{d-1},0)](t) + E_r^{(d)}(t)$$

$$= \sum_{|\boldsymbol{k}|=0}^{r-d} \frac{(-1)^{|\boldsymbol{k}|}}{(\mathrm{i}\omega)^{d+|\boldsymbol{k}|}} \sum_{\ell=0}^{d} \mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}_\ell^d} \sum_{\boldsymbol{\phi}\in\Phi_\ell^d} A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n}) F^{\boldsymbol{k}}[\boldsymbol{\phi}](t) + E_r^{(d)}(t)$$

(we assume that $\sum_{k=0}^{-1} a_k = 0$). Throughout the above inductive proof we utilize the properties of the integral $I[F,\sigma_d(t)]$ and of the operator F, and simple summation identities. More precisely, where necessary, in the above identities we have used:

(1) the form (3.1) of F and Fubini's theorem,
(2) the induction hypothesis,
(3) the formula for the asymptotic expansion of $I[F,\sigma_{d-1}(t)]$ and the identity $\tilde{\boldsymbol{n}}^T \boldsymbol{v}_\ell^{d-1} + n_d = \boldsymbol{n}^T(\boldsymbol{v}_\ell^{d-1}, 1)$,
(4) integration by parts $r - d - |\tilde{\boldsymbol{k}}| + 1$ times and the nonresonance condition (4.1),

(5) the summation identities $\sum_{|\tilde{\boldsymbol{k}}|=0}^{r+1}\sum_{k_d=0}^{r-|\tilde{\boldsymbol{k}}|} a_{k_1,\dots,k_{d-1},k_d} = \sum_{|\boldsymbol{k}|=0}^{r} a_{k_1,\dots,k_d}$ and $\sum_{|\tilde{\boldsymbol{k}}|=0}^{r-d+1} a_{k_1,\dots,k_{d-1},r-d-|\tilde{\boldsymbol{k}}|+1} = \sum_{|\boldsymbol{k}|=r-d+1} a_{k_1,\dots,k_d}$,

(6) Definition 3, according to which

$$\partial_{\tau_d}\big[\mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_d}F^{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tau_d)\big]\big|_{\tau_d=\phi_d t} = F^{\boldsymbol{k}}[\boldsymbol{\phi}](t),$$

and Definition 4.

We have proved that the integral $I[F,\sigma_d(t)] \sim \mathcal{O}(\omega^{-d})$ can indeed be approximated by (4.5), with an error $\mathcal{O}(\omega^{-r-1})$ as given by (4.6). ∎

Let us note that we cannot consider an infinite expansion in (4.4) because $\mathcal{S}_r^{(d)}(t)$ may not converge to $I[F,\sigma_d(t)]$ as $r\to\infty$, even if $\omega \gg 1$.

A similar result was first obtained in [13], where the authors provide the asymptotic expansion of a multivariate highly oscillatory integral over a regular simplex. However, our result differs in that the integrand is a linear operator, instead of a real-valued function, which makes the formulas for the asymptotic expansion of highly oscillatory integrals more complicated. Furthermore, the coefficients $A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n})$ and the error of the expansion have to be derived explicitly.

5. Error analysis. First four terms of the Modulated Fourier Expansion. In this section, we analyze the error associated with approximating the solution of (1.1) using the sum (1.4). Additionally, we provide ready-to-use formulas for the first four terms of the partial sum (1.4).

By Theorem 2, each integral $I[F_{\boldsymbol{n}},\sigma_d(t)]$, for $\boldsymbol{n}$ satisfying the nonresonance condition (4.1), and for sufficiently smooth functions α_n, $n=1,\dots,N$, can be expressed as

$$I[F_{\boldsymbol{n}},\sigma_d(t)] = S_{r,\boldsymbol{n}}^{(d)}(t) + E_{r,\boldsymbol{n}}^{(d)}(t),$$

where $\mathcal{S}_{r,\boldsymbol{n}}^{(d)}(t)$ is the sum corresponding to $F_{\boldsymbol{n}}$ and $S_{r,\boldsymbol{n}}^{(d)}(t) \sim \mathcal{O}(\omega^{-d})$. Moreover, for the rth partial sum of the Neumann series we have

$$u^{[r]}(t) = \sum_{d=0}^{r} T^d \mathrm{e}^{t\mathcal{L}}u_0 = \mathrm{e}^{t\mathcal{L}}u_0 + \sum_{d=1}^{r}\sum_{\boldsymbol{n}\in\{1,\dots,N\}^d} I[F_{\boldsymbol{n}},\sigma_d(t)]u_0.$$

Therefore, one can approximate the solution $u(t)$ of (1.1) with a sum of type (1.4). Furthermore, since the Neumann series converges for any given time t, the asymptotic expansion is well-defined without requiring time steps.

We show that the sum consisting of terms of $\mathcal{S}_{r,\boldsymbol{n}}^{(d)}(t)u_0$ which are defined in (4.5),

$$U_{as}^{[r]}(t) = \mathrm{e}^{t\mathcal{L}}u_0 + \sum_{d=1}^{r}\sum_{\boldsymbol{n}\in\{1,\dots,N\}^d} \mathcal{S}_{r,\boldsymbol{n}}^{(d)}(t)u_0,$$

approximates the solution of (1.1) with error $\mathcal{O}(\omega^{-r-1})$. We need two lemmas.

LEMMA 3. *Suppose that Assumption* 1 *is satisfied. Let* $k \geq 1$ *and let* $D(\mathcal{L}^k) := H^{2pk}(\Omega) \cap H_0^p(\Omega)$ *be equipped with the Sobolev norm* $\| \ \|_{H^{2pk}(\Omega)}$. *Then the semigroup* $\{e^{t\mathcal{L}}\}_{t\geq 0}$ *is bounded on the space* $D(\mathcal{L}^k)$.

Proof. We denote $\| \ \|_{H^n(\Omega)} =: \| \ \|_n$. To show the boundedness of $\{e^{t\mathcal{L}}\}_{t\geq 0}$ with respect to the norm $\| \ \|_{2pk}$, we use k times the following estimate (see [1, Theorem 9.8]):

$$\|u\|_{2p+k} \leq C(\|\mathcal{L}u\|_k + \|u\|_{L^2(\Omega)})$$

for every strongly elliptic differential operator $\mathcal{L}$ of order $2p$ with smooth and bounded coefficients, and for every $u \in H^{2p+k}(\Omega) \cap H_0^p(\Omega)$, where $\Omega \subset \mathbb{R}^m$ is an open bounded with smooth boundary. We have

$$\begin{aligned}
\|e^{t\mathcal{L}}\|_{D(\mathcal{L}^k)\leftarrow D(\mathcal{L}^k)} &= \sup_{\|u\|_{2pk}\leq 1} \|e^{t\mathcal{L}}u\|_{2pk} \\
&\leq C_1 \sup_{\|u\|_{2pk}\leq 1} (\|\mathcal{L}e^{t\mathcal{L}}u\|_{2p(k-1)} + \|e^{t\mathcal{L}}u\|_{L^2(\Omega)}) \\
&\leq C_1 \sup_{\|u\|_{2pk}\leq 1} (\|\mathcal{L}e^{t\mathcal{L}}u\|_{2p(k-1)} + C_2) \\
&\leq C_3 \sup_{\|u\|_{2pk}\leq 1} (\|\mathcal{L}^2e^{t\mathcal{L}}u\|_{2p(k-2)} + \|\mathcal{L}e^{t\mathcal{L}}u\|_{L^2(\Omega)} + C_4) \\
&\leq C_5 \sup_{\|u\|_{2pk}\leq 1} (\|\mathcal{L}^2e^{t\mathcal{L}}u\|_{2p(k-2)} + C_6) \\
&\ \ \vdots \\
&\leq C_{m-1} \sup_{\|u\|_{2pk}\leq 1} (\|\mathcal{L}^k e^{t\mathcal{L}}u\|_{L^2(\Omega)} + C_m) \\
&\leq C_{m+1} \sup_{\|u\|_{2pk}\leq 1} (\|e^{t\mathcal{L}}\mathcal{L}^k u\|_{L^2(\Omega)} + C_{m+2}) \leq C(t^\star).
\end{aligned}$$

The constants $C_1, C_2, \ldots$ depend only on p, Ω, $t^\star$ and $\mathcal{L}$. We see that the operator $e^{t\mathcal{L}} : D(\mathcal{L}^k) \to D(\mathcal{L}^k)$ is bounded in the Sobolev norm $\| \ \|_{H^{2pk}(\Omega)}$ by a constant $C(t^\star)$ independent of t. ∎

LEMMA 4. *Suppose that Assumption* 1 *is satisfied. Further, suppose that* $u_0, \alpha_n \in H^{2p(r+1)}(\Omega)$, *for* $n = 1, \ldots, N$. *Let* $I[F_{\boldsymbol{n}}, \sigma_d(t)]$ *be defined by* (4.4), *and let* $\boldsymbol{n} = (n_1, \ldots, n_d)$ *satisfy* (4.1). *Let* $\mathcal{S}_{r,\boldsymbol{n}}^{(d)}(t)$ *and* $E_{r,\boldsymbol{n}}^{(d)}(t)$ *be given by* (4.5) *and* (4.6) *for* $F = F_n$. *Denote* $\| \ \|_{H^n(\Omega)} =: \| \ \|_n$ *and* $\| \ \|_{L^2(\Omega)} =: \| \ \|_{L^2}$. *Then there exists a constant* $C := C(\mathcal{L}, t^\star, \alpha, u_0)$, *independent of* ω *and* t, *such that*

$$\|\mathcal{S}_{r,\boldsymbol{n}}^{(d)}(t)u_0\|_{L^2} \leq C\omega^{-d}, \tag{5.1}$$

$$\|E_{r,\boldsymbol{n}}^{(d)}(t)u_0\|_{L^2} \leq C\omega^{-r-1}. \tag{5.2}$$

Proof. We prove (5.1); the proof of (5.2) is very similar. The expression $F_{\boldsymbol{n}}^{\boldsymbol{k}}[\boldsymbol{\phi}](t)$ from (4.5) can be written explicitly as

$$F_{\boldsymbol{n}}^{\boldsymbol{k}}[\boldsymbol{\phi}](t) = \sum_{\boldsymbol{m}\le\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{m}}(-1)^{|\boldsymbol{m}|+|\boldsymbol{k}|}\mathcal{L}^{r_{d+1}}\alpha_{n_d}\mathcal{L}^{r_d}\alpha_{n_{d-1}}\ldots\mathcal{L}^{r_{\ell+1}}\alpha_{n_{\ell+1}}\mathrm{e}^{t\mathcal{L}}\mathcal{L}^{r_\ell}\alpha_{n_\ell}\ldots\mathcal{L}^{r_2}\alpha_{n_1}\mathcal{L}^{r_1},$$

for $\boldsymbol{\phi} \in \Phi_\ell^d$, $\ell = 0,1,\ldots,d$, and $\boldsymbol{k} = (k_1,\ldots,k_d)$, $\boldsymbol{m} = (m_1,\ldots,m_d)$, where $\binom{\boldsymbol{k}}{\boldsymbol{m}} = \binom{k_1}{m_1}\ldots\binom{k_d}{m_d}$. The nonnegative numbers $r_1,\ldots,r_{d+1}$ satisfy $r_1+\cdots+r_{d+1} = |\boldsymbol{k}| \le r$ and $\boldsymbol{m}\le\boldsymbol{k}$ means $m_i \le k_i$ for $i = 1,\ldots,d$. Using (2.5) and $\|L^r v\|_{L^2} \le C\|v\|_{2pr}$ repeatedly, and Lemma 3, we obtain

$$\begin{aligned}\|\mathcal{L}^{r_{d+1}}\alpha_{n_d}\mathcal{L}^{r_d}\alpha_{n_{d-1}}\ldots\mathcal{L}^{r_{\ell+1}}\alpha_{n_{\ell+1}}\mathrm{e}^{t\mathcal{L}}\mathcal{L}^{r_\ell}\alpha_{n_\ell}\ldots\mathcal{L}^{r_2}\alpha_{n_1}\mathcal{L}^{r_1}u_0\|_{L^2(\Omega)}&\\ \le C_1\|\alpha_{n_d}\|_{2pr_{d+1}}\|\alpha_{n_{d-1}}\|_{2p(r_{d+1}+r_d)}\cdots\|\alpha_{n_1}\|_{2p(|\boldsymbol{k}|-r_1)}\|u_0\|_{2p|\boldsymbol{k}|} < C_2.&\end{aligned}$$

Therefore

$$\|F_{\boldsymbol{n}}^{\boldsymbol{k}}[\boldsymbol{\phi}](t)\|_{L^2} \le C_2\sum_{\boldsymbol{m}\le\boldsymbol{k}}\binom{\boldsymbol{k}}{\boldsymbol{m}} = C_2 2^{|\boldsymbol{k}|}.$$

We have $A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n}) \le 1$ and $|\mathrm{e}^{\mathrm{i}\omega t\boldsymbol{n}^T\boldsymbol{v}_\ell^d}| = 1$. We obtain the estimate

$$\begin{aligned}\|\mathcal{S}_r^{(d)}(t)\|_{L^2} &= \Big\|\sum_{|\boldsymbol{k}|=0}^{r-d}\frac{(-1)^{|\boldsymbol{k}|}}{(\mathrm{i}\omega)^{d+|\boldsymbol{k}|}}\sum_{\ell=0}^{d}\mathrm{e}^{\mathrm{i}\omega t\boldsymbol{n}^T\boldsymbol{v}_\ell^d}\sum_{\boldsymbol{\phi}\in\Phi_\ell^d}A_{\boldsymbol{k}}[\boldsymbol{\phi}](\boldsymbol{n})F^{\boldsymbol{k}}[\boldsymbol{\phi}](t)\Big\|_{L^2}\\ &\le \sum_{|\boldsymbol{k}|=0}^{r-d}\frac{1}{\omega^{d+|\boldsymbol{k}|}}C_2 2^{|\boldsymbol{k}|}\sum_{\ell=0}^{d}\sum_{\boldsymbol{\phi}\in\Phi_\ell^d}1 \le \sum_{|\boldsymbol{k}|=0}^{r-d}\frac{1}{\omega^{d+|\boldsymbol{k}|}}C_2 2^{|\boldsymbol{k}|}2^d \le C\omega^{-d},\end{aligned}$$

which completes the proof. ■

THEOREM 3. *Suppose that Assumption* 1 *is satisfied. In addition, suppose that* $u_0,\alpha_n \in H^{2p(r+1)}(\Omega)$, $n = 1,\ldots,N$. *Let* $u(t)$ *be the solution of* (1.1) *and* $u^{[r]}(t) = \sum_{d=0}^r T^d\mathrm{e}^{t\mathcal{L}}u_0$ *be the* r*th partial sum of the Neumann series. Denote* $\|\ \|_{L^2} := \|\ \|_{L^2(\Omega)}$. *Let* $\boldsymbol{T}$ *be the operator* $\boldsymbol{T} = \sum_{d=0}^\infty T^d$. *Then for* $t > 0$ *the following error estimations hold:*

(1) $\|u(t)-u^{[r]}(t)\|_{L^2} \le \sup_{\|v(t)\|_{L^2}\le 1}\|\mathbf{T}v(t)\|_{L^2}\|T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\|_{L^2} = \mathcal{O}(\omega^{-r-1})$,

(2) $\|u(t)-U_{as}^{[r]}(t)\|_{L^2} \le \sup_{\|v(t)\|_{L^2}\le 1}\|\mathbf{T}v(t)\|_{L^2}\|T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\|_{L^2} + \sum_{d=1}^{r}\sum_{\boldsymbol{n}\in\{1,\ldots,N\}^d}\|E_{r,\boldsymbol{n}}^{(d)}(t)u_0\|_{L^2} = \mathcal{O}(\omega^{-r-1})$.

Proof. Let $t > 0$. According to Theorem 1, the solution $u(t)$ of (1.1) can be expressed as the Neumann series $u(t) = \sum_{d=0}^\infty T^d\mathrm{e}^{t\mathcal{L}}u_0$. Therefore we have

$$\|u(t)-u^{[r]}(t)\|_{L^2} = \Big\|\sum_{d=r+1}^{\infty} T^d \mathrm{e}^{t\mathcal{L}}u_0\Big\|_{L^2} = \Big\|\sum_{d=0}^{\infty} T^d T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\Big\|_{L^2}$$
$$\le \sup_{\|v(t)\|_{L^2}\le 1} \|\mathbf{T}v(t)\|_{L^2}\|T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\|_{L^2}.$$

By Theorem 2 and Lemma 4,

$$\|T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\|_{L^2} \le \sum_{\boldsymbol{n}\in\{1,\dots,N\}^d} \big(\|\mathcal{S}^{(r+1)}_{r+1,\boldsymbol{n}}(t)u_0\|_{L^2} + \|E^{(r+1)}_{r+1,\boldsymbol{n}}(t)u_0\|_{L^2}\big) = \mathcal{O}(\omega^{-r-1}).$$

To show (2) we use the triangle inequality:

$$\|u(t)-U^{[r]}_{as}(t)\|_{L^2} \le \|u(t)-u^{[r]}(t)\|_{L^2} + \|u^{[r]}(t)-U^{[r]}_{as}(t)\|_{L^2}$$
$$\le \sup_{\|v(t)\|_{L^2}\le 1} \|\mathbf{T}v(t)\|_{L^2}\|T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\|_{L^2}$$
$$+ \sum_{d=1}^{r}\sum_{\boldsymbol{n}\in\{1,\dots,N\}^d} \|E^{(d)}_{r,\boldsymbol{n}}(t)u_0\|_{L^2},$$

and for each $\boldsymbol{n} \in \{1,\dots,N\}^d$, by Lemma 4, we have $\|E^{(d)}_{r,\boldsymbol{n}}(t)u_0\|_{L^2} = \mathcal{O}(\omega^{-r-1})$. It remains to show that $\sup_{\|v(t)\|_{L^2}\le 1}\|\|\mathbf{T}v(t)\|_{L^2}\|$ is bounded. The function $\boldsymbol{T}v(t)$, where $\|v(t)\|_{L^2}\le 1$, is the solution of the integral equation

$$\psi(t) = v(t) + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}} f(\tau)\psi(\tau)\,\mathrm{d}\tau. \tag{5.3}$$

Let $C_1 := \max_{t\in[0,t^\star]}\|\mathrm{e}^{t\mathcal{L}}\|_{L^2(\Omega)\leftarrow L^2(\Omega)}$. By Grönwall's inequality, the solution of (5.3) can be estimated as follows:

$$\|\mathbf{T}v(t)\|_{L^2} = \|\psi(t)\|_{L^2} \le \exp(tC_1\|f\|_\infty), \tag{5.4}$$

which concludes the proof. ∎

Remark 3. The upper bound (5.4) of $\|\mathbf{T}v(t)\|_{L^2}$ may be large, especially for $\|f\|_\infty$ large. Therefore, let us also consider a different approach. In the proof of inequality (1) of Theorem 3, we estimate the truncation of the Neumann series,

$$\|u(t)-u^{[r]}(t)\|_{L^2} = \Big\|\sum_{d=0}^{\infty} T^d T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0\Big\|_{L^2}. \tag{5.5}$$

Let now $v(t) = T^{r+1}\mathrm{e}^{t\mathcal{L}}u_0$. It is straightforward to verify that $\psi(t) = \sum_{d=0}^{\infty} T^d v(t)$ is the solution of the nonhomogeneous equation

$$\begin{aligned} \psi'(t) &= \mathcal{L}\psi(t) + f(t)\psi(t) + v'(t) - \mathcal{L}v(t),\\ \psi(0) &= v(0). \end{aligned} \tag{5.6}$$

From the formula for $v(t)$ we have

$$v'(t) - \mathcal{L}v(t) = f(t)T^r \mathrm{e}^{t\mathcal{L}} u_0 + \mathcal{L}T^{r+1}\mathrm{e}^{t\mathcal{L}} u_0 - \mathcal{L}T^{r+1}\mathrm{e}^{t\mathcal{L}} u_0 = f(t)T^r \mathrm{e}^{t\mathcal{L}} u_0$$

and $v(0) \equiv 0$. Therefore, (5.6) reads

$$\begin{aligned} \psi'(t) &= (\mathcal{L} + f(t))\psi(t) + f(t)T^r \mathrm{e}^{t\mathcal{L}} u_0, \\ \psi(0) &= 0, \end{aligned} \tag{5.7}$$

and $f(t)T^r\mathrm{e}^{t\mathcal{L}}u_0 = \mathcal{O}(\omega^{-r})$. Moreover, for t sufficiently small, the solution of (5.7) can be written as

$$\psi(t) = \int_0^t \Phi(t,s) f(s) T^r \mathrm{e}^{s\mathcal{L}} u_0 \, \mathrm{d}s,$$

where $\Phi(t,s)$ is the solution of the homogeneous problem

$$\Phi'(t,s) = (\mathcal{L} + f(s))\Phi(t,s), \qquad \Phi(s,s) = 1,$$

and $\Phi(t,s) = \exp(\Omega(t,s))$, where $\Omega(t,s)$ is the Magnus expansion. If, for example, (1.1) is the Schrödinger equation with a time-dependent potential, then $\|\Phi(t,s)\|_{L^2} \equiv 1$ and therefore the truncation (5.5) can be estimated in a different manner:

$$\Big\| \sum_{d=0}^{\infty} T^d T^{r+1} \mathrm{e}^{t\mathcal{L}} u_0 \Big\|_{L^2} = \|\psi(t)\|_{L^2} \le t \|f\|_\infty \max_{s\in[0,t]} \|T^r \mathrm{e}^{s\mathcal{L}} u_0\|_{L^2}.$$

This provides a different estimate of the error constant $\|\mathbf{T}v(t)\|_{L^2}$.

Now we provide a ready-made formula for the sum (1.4) for $R = 3$, which approximates the solution $u(t)$ of problem (1.1) with error $\mathcal{O}(\omega^{-4})$. The solution $u(t)$ can be written as

$$u(t) = \mathrm{e}^{t\mathcal{L}} u_0 + T^1 \mathrm{e}^{t\mathcal{L}} u_0 + T^2 \mathrm{e}^{t\mathcal{L}} u_0 + T^3 \mathrm{e}^{t\mathcal{L}} u_0 + \boldsymbol{T} T^4 \mathrm{e}^{t\mathcal{L}} u_0,$$

where

$$\boldsymbol{T} = \sum_{d=0}^{\infty} T^d \quad \text{and} \quad \|\boldsymbol{T} T^4 \mathrm{e}^{t\mathcal{L}} u_0\|_2 = \mathcal{O}(\omega^{-4}).$$

We expand each term $T^1\mathrm{e}^{t\mathcal{L}}u_0$, $T^2\mathrm{e}^{t\mathcal{L}}u_0$ and $T^3\mathrm{e}^{t\mathcal{L}}u_0$ as follows:

$$T^1 \mathrm{e}^{t\mathcal{L}} u_0 = \sum_{n_1 \in \{1,\dots,N\}} (\mathcal{S}^{(1)}_{3,n_1}(t) + E^{(1)}_{3,n_1}(t)) u_0,$$

$$T^2 \mathrm{e}^{t\mathcal{L}} u_0 = \sum_{\boldsymbol{n} \in \{1,\dots,N\}^2} (\mathcal{S}^{(2)}_{3,\boldsymbol{n}}(t) + E^{(2)}_{3,\boldsymbol{n}}(t)) u_0,$$

$$T^3 \mathrm{e}^{t\mathcal{L}} u_0 = \sum_{\boldsymbol{n} \in \{1,\dots,N\}^3} (\mathcal{S}^{(3)}_{3,\boldsymbol{n}}(t) + E^{(3)}_{3,\boldsymbol{n}}(t)) u_0.$$

The partial sum that approximates $u(t)$ is

$$U_{as}^{[3]}(t) = \mathrm{e}^{t\mathcal{L}}u_0 + \sum_{n_1\in\{1,\dots,N\}} \mathcal{S}_{3,n_1}^{(1)}(t)u_0 + \sum_{\boldsymbol{n}\in\{1,\dots,N\}^2} \mathcal{S}_{3,\boldsymbol{n}}^{(2)}(t)u_0 + \sum_{\boldsymbol{n}\in\{1,\dots,N\}^3} \mathcal{S}_{3,\boldsymbol{n}}^{(3)}(t)u_0,$$

where, by (4.5), $\mathcal{S}_{3,n_1}^{(1)}(t)u_0$, $\mathcal{S}_{3,\boldsymbol{n}}^{(2)}(t)u_0$ and $\mathcal{S}_{3,\boldsymbol{n}}^{(3)}(t)u_0$ are

$$\begin{aligned}\mathcal{S}_{3,n_1}^{(1)}(t)u_0 = {}& \frac{1}{\mathrm{i}n_1\omega}(\mathrm{e}^{\mathrm{i}n_1\omega t}\alpha_{n_1}\mathrm{e}^{t\mathcal{L}} - \mathrm{e}^{t\mathcal{L}}\alpha_{n_1})u_0\\ &+ \frac{1}{(\mathrm{i}n_1\omega)^2}(\mathrm{e}^{\mathrm{i}n_1\omega t}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_1})\mathrm{e}^{t\mathcal{L}} - \mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_1}))u_0\\ &+ \frac{1}{(\mathrm{i}n_1\omega)^3}(\mathrm{e}^{\mathrm{i}n_1\omega t}\,\mathrm{ad}_{\mathcal{L}}^2(\alpha_{n_1})\mathrm{e}^{t\mathcal{L}} - \mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^2(\alpha_{n_1}))u_0,\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_{3,\boldsymbol{n}}^{(2)}(t)u_0\\ &= \frac{1}{(\mathrm{i}\omega)^2}\Bigg(\mathrm{e}^{\mathrm{i}\omega t(n_1+n_2)}\frac{1}{n_1(n_1+n_2)}\alpha_{n_2}\alpha_{n_1}\mathrm{e}^{t\mathcal{L}}\\ &\qquad - \mathrm{e}^{\mathrm{i}\omega t n_2}\frac{1}{n_1n_2}\alpha_{n_2}\mathrm{e}^{t\mathcal{L}}\alpha_{n_1} + \frac{1}{n_2(n_1+n_2)}\mathrm{e}^{t\mathcal{L}}\alpha_{n_2}\alpha_{n_1}\Bigg)u_0\\ &\quad + \frac{1}{(\mathrm{i}\omega)^3}\Bigg(\mathrm{e}^{\mathrm{i}\omega t(n_1+n_2)}\bigg(\frac{1}{n_1^2(n_1+n_2)}\alpha_{n_2}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_1})\mathrm{e}^{t\mathcal{L}}\\ &\qquad + \frac{1}{n_1(n_1+n_2)^2}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_2}\alpha_{n_1})\mathrm{e}^{t\mathcal{L}}\bigg)\\ &\quad + \mathrm{e}^{\mathrm{i}\omega t n_2}\bigg(\frac{-1}{n_1^2n_2}\alpha_{n_2}\mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_1}) - \frac{1}{n_1n_2^2}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_2})\mathrm{e}^{t\mathcal{L}}\alpha_{n_1}\bigg)\\ &\quad + \frac{1}{n_1n_2(n_1+n_2)}\mathrm{e}^{t\mathcal{L}}\alpha_{n_2}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_1})\\ &\quad + \frac{-1}{n_1(n_1+n_2)^2}\mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_2}\alpha_{n_1}) + \frac{1}{n_1n_2^2}\mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_2})\alpha_{n_1}\Bigg)u_0,\end{aligned}$$

$$\begin{aligned}S_{3,\boldsymbol{n}}^{(3)}(t)u_0 = {}& \frac{1}{(\mathrm{i}\omega)^3}\Bigg(\frac{\mathrm{e}^{\mathrm{i}\omega t(n_1+n_2+n_3)}}{n_1(n_1+n_2)(n_1+n_2+n_3)}\alpha_{n_3}\alpha_{n_2}\alpha_{n_1}\mathrm{e}^{t\mathcal{L}}\\ &- \frac{\mathrm{e}^{\mathrm{i}\omega t(n_2+n_3)}}{n_1n_2(n_2+n_3)}\alpha_{n_3}\alpha_{n_2}\mathrm{e}^{t\mathcal{L}}\alpha_{n_1} + \frac{\mathrm{e}^{\mathrm{i}\omega t n_3}}{n_2n_3(n_1+n_2)}\alpha_{n_3}\mathrm{e}^{t\mathcal{L}}\alpha_{n_2}\alpha_{n_1}\\ &- \frac{1}{(n_1+n_2+n_3)(n_2+n_3)n_3}\mathrm{e}^{t\mathcal{L}}\alpha_{n_3}\alpha_{n_2}\alpha_{n_1}\Bigg)u_0.\end{aligned}$$

Expressions of type $\mathrm{e}^{t\mathcal{L}}u_0$, $\mathrm{e}^{t\mathcal{L}}\alpha_{n_j}u_0$, $\mathrm{e}^{t\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha_{n_j})u_0$, etc. can be computed either explicitly, or very efficiently and accurately by using the spectral methods [19] and/or the splitting methods [15].

It is convenient to present the idea of the asymptotic expansion in the following form:

$$\begin{aligned}
T^1 \mathrm{e}^{t\mathcal{L}} u_0 &= \frac{1}{\omega}\sum_{s=0}^{N} \mathrm{e}^{\mathrm{i}\omega s} P^1_{1,s} + \frac{1}{\omega^2}\sum_{s=0}^{N} \mathrm{e}^{\mathrm{i}\omega s} P^1_{2,s} + \ldots + \frac{1}{\omega^r}\sum_{s=0}^{N} \mathrm{e}^{\mathrm{i}\omega s} P^1_{r,s} + \frac{1}{\omega^{r+1}} \boldsymbol{E}^1_r,\\
T^2 \mathrm{e}^{t\mathcal{L}} u_0 &= \qquad\qquad \frac{1}{\omega^2}\sum_{s=0}^{2N} \mathrm{e}^{\mathrm{i}\omega s} P^2_{2,s} + \ldots + \frac{1}{\omega^r}\sum_{s=0}^{2N} \mathrm{e}^{\mathrm{i}\omega s} P^2_{r,s} + \frac{1}{\omega^{r+1}} \boldsymbol{E}^2_r,\\
&\vdots \qquad\qquad\qquad\qquad \ddots \qquad \vdots\\
T^r \mathrm{e}^{t\mathcal{L}} u_0 &= \qquad\qquad\qquad\qquad \frac{1}{\omega^r}\sum_{s=0}^{rN} \mathrm{e}^{\mathrm{i}\omega s} P^r_{r,s} + \frac{1}{\omega^{r+1}} \boldsymbol{E}^r_r.
\end{aligned}$$

Here $P^n_{r,s}$ is a term of expansion of the nth term of the Neumann series $T^n \mathrm{e}^{t\mathcal{L}} u_0$, which is independent of ω, next to frequency $\mathrm{e}^{\mathrm{i}\omega s}$ and magnitude ω^{-r}. The error of the method consists of a truncation of the Neumann series and a truncation of the asymptotic expansion of each term $T^n \mathrm{e}^{t\mathcal{L}} u_0$.

6. Asymptotic expansion – integrals with resonance points. Theorem 2 allows for the asymptotic expansion of the integral $I[F_{\boldsymbol{n}}, \sigma_d(t)]$ provided the vector $\boldsymbol{n}$ satisfies the nonresonance condition (4.1). However, given the potential function of a more general form (1.5) with negative frequencies, the solution of the problem (1.1) is

$$u(t) = \mathrm{e}^{t\mathcal{L}} u_0 + \sum_{d=1}^{\infty} \sum_{\boldsymbol{n}\in \boldsymbol{N}^d} I[F_{\boldsymbol{n}}, \sigma_d(t)] u_0,$$

where $\boldsymbol{N}^d$ is the set of all possible vectors $\boldsymbol{n}$,

$$\boldsymbol{N}^d := \{-N, \ldots, -2, -1, 1, 2, \ldots, N\}^d. \tag{6.1}$$

$\boldsymbol{N}^d$ contains vectors $\boldsymbol{n}$ which are orthogonal to the simplex $\sigma_d(t)$ or its boundary. The coordinates of such $\boldsymbol{n}$ satisfy

$$n_j + n_{j+1} + \cdots + n_{r-1} + n_r = 0$$

for some $1 \le j < r \le d$. For those vectors we cannot apply Theorem 2 since we would divide by zero in (4.3). However, instead of one integral with resonance points, we are to deal with a sum of such integrals. In Theorem 2, an increase in the dimension of integral results in an increase in the rate of decay – while $I[F, \sigma_{d-1}(t)] \sim \mathcal{O}(\omega^{-d+1})$, we have $I[F, \sigma_d(t)] \sim \mathcal{O}(\omega^{-d})$. This is because at any step of the proof we could integrate by parts in each integral since, due to the non-resonance assumption, the argument of the frequency exponent is $\boldsymbol{n}^T \boldsymbol{v}^d_\ell \neq 0$ and so $\mathrm{e}^{\mathrm{i}\omega t \boldsymbol{n}^T \boldsymbol{v}^d_\ell}$ never disappears.

At this stage we consider the integral $I[F_{\boldsymbol{n}}, \sigma_d(t)]$ with

$$\boldsymbol{n} = (n_1, \dots, n_d) \in \boldsymbol{N}^d$$

whose coordinates satisfy

$$n_1 + \dots + n_d = 0 \quad \text{and} \quad n_j + n_{j+1} + \dots + n_{j+r} \neq 0 \quad \text{for } j = 1, \dots, d,\ 1 \leq j + r \leq d. \tag{6.2}$$

In other words, $\boldsymbol{n}$ is orthogonal only to one edge of the simplex $\sigma_d(t)$ contained in the line $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = \dots = x_d\}$. In such a situation, we have $\boldsymbol{n}^T \boldsymbol{v}_0^d = n_1 + \dots + n_d = 0$ and therefore in the proof of Theorem 2 we cannot integrate by parts the last, outer integral with $\mathrm{e}^{\mathrm{i}\omega\tau_d \boldsymbol{n}^T \boldsymbol{v}_0^d}$. The general case involving the whole set (6.1) is a matter of further research.

Let $\boldsymbol{n}_j \in \mathbb{N}^d$, $j = 1, \dots, d$, be vectors satisfying (6.2) such that

$$\begin{aligned}
\boldsymbol{n}_1 &= (n_1, n_2, n_3, \dots, n_{d-1}, n_d),\\
\boldsymbol{n}_2 &= (n_2, n_3, n_4, \dots, n_d, n_1),\\
&\ \ \vdots\\
\boldsymbol{n}_j &= (n_j, n_{j+1}, \dots, n_d, n_1, \dots, n_{j-1}),\\
&\ \ \vdots\\
\boldsymbol{n}_d &= (n_d, n_1, n_2, \dots, n_{d-2}, n_{d-1}).
\end{aligned}$$

If $\boldsymbol{n}_1 \in \boldsymbol{N}^d$, then the vectors $\boldsymbol{n}_j$, $j = 2, \dots, d$, are in $\boldsymbol{N}^d$ as well. Because of assumption (6.2), for each $j = 1, \dots, d$ we have

$$\sum_{j=1}^{d} \boldsymbol{n}_j = 0.$$

From the coordinates of $\boldsymbol{n}_j$ we form the following fractions:

$$\begin{aligned}
A[\tilde{\boldsymbol{n}}_1] &= \frac{1}{n_1(n_1+n_2)(n_1+n_2+n_3)\dots(n_1+n_2+n_3+\dots+n_{d-1})},\\
A[\tilde{\boldsymbol{n}}_2] &= \frac{1}{n_2(n_2+n_3)(n_2+n_3+n_4)\dots(n_2+n_3+n_4+\dots+n_d)},\\
&\ \ \vdots\\
A[\tilde{\boldsymbol{n}}_{k+1}] &= \frac{1}{n_{k+1}(n_{k+1}+n_{k+2})\dots(n_{k+1}+n_{k+2}+\dots+n_d+n_1+\dots+n_{k-1})},\\
&\ \ \vdots\\
A[\tilde{\boldsymbol{n}}_d] &= \frac{1}{n_d(n_d+n_1)(n_d+n_1+n_2)\dots(n_d+n_1+n_2\dots+n_{d-3}+n_{d-2})}.
\end{aligned}$$

(We use cyclic notation $n_s = n_{s+d}$ for $s \in \mathbb{Z}$.)

Lemma 5. *Let $\boldsymbol{n}_j$, $j = 1, \dots, d$, be vectors which satisfy condition* (6.2). *Then*

$$\sum_{j=1}^{d} A[\tilde{\boldsymbol{n}}_j] = 0.$$

Proof. It is sufficient to apply the partial fraction decomposition to $A[\tilde{\boldsymbol{n}}_1]$, by treating n_1 as a variable and the other n_r, $r \neq 1$, as constants. We aim to express $A[\tilde{\boldsymbol{n}}_1]$ as

$$A[\tilde{\boldsymbol{n}}_1] = \sum_{j=1}^{d-1} \frac{N_j}{n_1 + \dots + n_j} \tag{6.3}$$

for some N_j, $j = 1, \dots, d-1$. Fix j. To determine N_j, we use the Heaviside cover-up method. Substituting $n_1 = -(n_2 + \dots + n_j)$ we have

$$\begin{aligned} N_j &= \frac{n_1 + \dots + n_j}{n_1(n_1 + n_2)\dots(n_1 + \dots + n_{d-1})}\bigg|_{n_1 = -(n_2 + \dots + n_j)} \\ &= \frac{(-1)^{j-1}}{(n_2 + \dots + n_j)\dots n_j n_{j+1}(n_{j+1} + n_{j+2})\dots(n_{j+1} + \dots + n_{d-1})} \\ &= \frac{1}{(n_{j+1} + \dots + n_1)\dots(n_{j+1} + \dots + n_{j-1}) n_{j+1}(n_{j+1} + n_{j+2})\dots(n_{j+1} + \dots n_{d-1})} \\ &= \frac{n_{j+1} + \dots + n_d}{n_{j+1}(n_{j+1} + n_{j+2})\dots(n_{j+1} + \dots + n_d)(n_{j+1} + \dots + n_1)\dots(n_{j+1} + \dots + n_{j-1})} \end{aligned}$$

(in the penultimate equality we have used assumption (6.2)). Now since $n_{j+1} + \dots + n_d = -(n_1 + \dots + n_j)$, substituting N_j into (6.3) we obtain

$$A[\tilde{\boldsymbol{n}}_1] = \sum_{j=1}^{d-1} \frac{-1}{n_{j+1}(n_{j+1} + n_{j+2})\dots(n_{j+1} + \dots + n_{j-1})} = -\sum_{j=1}^{d-1} A[\tilde{\boldsymbol{n}}_{j+1}],$$

which completes the proof. ∎

Now notice, that if $\tilde{\boldsymbol{k}} = (k_1, \dots, k_{d-1})$ satisfies $|\tilde{\boldsymbol{k}}| = 0$ and $\boldsymbol{\phi} = (1, \dots, 1) \in \Phi_0^{d-1}$, then $A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}_j) = A[\tilde{\boldsymbol{n}}_j]$, where $A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}_j)$ are the coefficients from Definition 4. In other words, the numbers $A[\tilde{\boldsymbol{n}}_j]$, $j = 1, \dots, d$, appear with the first term $\frac{1}{(\mathrm{i}\omega)^{d-1}}$ of the asymptotic expansion of the integral $I[F_{\tilde{\boldsymbol{n}}_j}, \sigma_{d-1}(t)]$.

Theorem 4. *Let $\boldsymbol{n}_j \in \boldsymbol{N}^d$, $j = 1, \dots, d$, satisfy* (6.2) *and let $F_{\boldsymbol{n}_j}$ be the corresponding operator defined in* (3.1). *Then*

$$\sum_{j=1}^{d} \int_{\sigma_d(t)} F_{\boldsymbol{n}_j}(t, \boldsymbol{\tau}) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{n}_j^T \boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{\tau} \sim \mathcal{O}(\omega^{-d}).$$

In other words, the sum of these integrals with resonance points over a simplex $\sigma_d(t)$ decays in the same manner as the integral over the same domain without resonance points.

Proof. Let

$$F_{\boldsymbol{n}_j}(t,\boldsymbol{\tau}) = \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}}\mathrm{e}^{(\tau_d-\tau_{d-1})\mathcal{L}}\alpha_{n_{d-1,j}}\dots\mathrm{e}^{(\tau_2-\tau_1)\mathcal{L}}\alpha_{n_{1,j}}\mathrm{e}^{\tau_1\mathcal{L}}$$

with $\boldsymbol{n}_j = (n_{1,j}, n_{2,j}, \dots, n_{d,j})$, and let $\mathcal{S}^{(d-1)}_{r,\tilde{\boldsymbol{n}}_j}, E^{(d-1)}_{r,\tilde{\boldsymbol{n}}_j}$ be the partial sum of the asymptotic series and the error term for $I[F_{\tilde{\boldsymbol{n}}_j}, \sigma_{d-1}(t)]$. We use Fubini's theorem and then we apply Theorem 2 to expand $I[F_{\tilde{\boldsymbol{n}}_j}, \sigma_{d-1}(t)]$ asymptotically. This is possible since the nonresonance condition is violated only for the vectors $\boldsymbol{n}_j \in \mathbb{N}^d$, $j = 1,\dots,d$. We get

$$\begin{aligned}
&\sum_{j=1}^{d}\int_{\sigma_d(t)} F_{\boldsymbol{n}_j}(t,\boldsymbol{\tau})\mathrm{e}^{\mathrm{i}\omega\boldsymbol{n}_j^T\boldsymbol{\tau}}\,\mathrm{d}\boldsymbol{\tau}\\
&\quad= \sum_{j=1}^{d}\int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} I[F_{\tilde{\boldsymbol{n}}_j}, \sigma_{d-1}(\tau_d)]\mathrm{e}^{\mathrm{i}\omega\tau_d n_{d,j}}\,\mathrm{d}\tau_d\\
&\quad= \sum_{j=1}^{d}\int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} (\mathcal{S}^{(d-1)}_{r,\tilde{\boldsymbol{n}}_j}(\tau_d) + E^{(d-1)}_{r,\tilde{\boldsymbol{n}}_j}(\tau_d))\mathrm{e}^{\mathrm{i}\omega\tau_d n_{d,j}}\,\mathrm{d}\tau_d\\
&\quad= \sum_{j=1}^{d}\sum_{|\tilde{\boldsymbol{k}}|=0}^{r-d+1}\frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}}\sum_{\ell=1}^{d-1}\sum_{\boldsymbol{\phi}\in\Phi_\ell^{d-1}} A_{\tilde{\boldsymbol{k}}}[\boldsymbol{\phi}](\tilde{\boldsymbol{n}}_j)\\
&\qquad\qquad\times \underbrace{\int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} F^{\tilde{\boldsymbol{k}}}_{\tilde{\boldsymbol{n}}_j}[\boldsymbol{\phi}](\tau_d)\mathrm{e}^{\mathrm{i}\omega\tau_d\boldsymbol{n}_j^T\boldsymbol{v}_\ell^d}\,\mathrm{d}\tau_d}_{=:P_1(t)}\\
&\qquad+ \sum_{j=1}^{d}\sum_{|\tilde{\boldsymbol{k}}|=0}^{r-d+1}\frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}} A_{\tilde{\boldsymbol{k}}}[(1,\dots,1)](\tilde{\boldsymbol{n}}_j)\\
&\qquad\qquad\times \underbrace{\int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} \boldsymbol{F}^{\tilde{\boldsymbol{k}}}_{\tilde{\boldsymbol{n}}_j}[(1,\dots,1)](\tau_d)\mathrm{e}^{\mathrm{i}\omega\tau_d\boldsymbol{n}_j^T\boldsymbol{v}_0^d}\,\mathrm{d}\tau_d}_{=:P_2(t)}\\
&\qquad+ \sum_{j=1}^{d}\int_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} E^{(d-1)}_{r,j}(\tau_d)\mathrm{e}^{\mathrm{i}\omega\tau_d n_{d,j}}\,\mathrm{d}\tau_d.
\end{aligned}$$

Now since the vectors $\boldsymbol{n}_j$, $j = 1,\dots,d$, satisfy (6.2), we have $\boldsymbol{n}_j^T\boldsymbol{v}_0^d = 0$, so $\mathrm{e}^{\mathrm{i}\omega\tau_d\boldsymbol{n}_j^T\boldsymbol{v}_0^d} = 1$ and therefore we cannot expand $P_2(t)$ asymptotically. However, if $|\tilde{\boldsymbol{k}}| = 0$ then for each j we have $\mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} F^{\tilde{\boldsymbol{k}}}_{\tilde{\boldsymbol{n}}_j}[(1,\dots,1)](\tau_d) = \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_1\dots\alpha_d \mathrm{e}^{\tau_d\mathcal{L}}$. As a consequence, in $P_2(t)$, by Lemma 5, the terms with $|\tilde{\boldsymbol{k}}| = 0$ vanish, and therefore

$$\sum_{j=1}^{d}\sum_{|\tilde{\boldsymbol{k}}|=0}^{n-d+1} \frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}} A_{\tilde{\boldsymbol{k}}}[(1,\dots,1)](\tilde{\boldsymbol{n}}_j)P_2(t)$$
$$= \sum_{j=1}^{d}\sum_{|\tilde{\boldsymbol{k}}|=1}^{n-d+1} \frac{(-1)^{|\tilde{\boldsymbol{k}}|}}{(\mathrm{i}\omega)^{d-1+|\tilde{\boldsymbol{k}}|}} A_{\tilde{\boldsymbol{k}}}[(1,\dots,1)](\tilde{\boldsymbol{n}}_j)P_2(t) \sim \mathcal{O}(\omega^{-d}).$$

We integrate $P_1(t)$ by parts according to Theorem 2 since no integral in $P_1(t)$ has resonance points; therefore $P_1(t) \sim \mathcal{O}(\omega^{-1})$, and consequently

$$\sum_{j=1}^{d} \int\limits_{\sigma_d(t)} F_{\boldsymbol{n}_j}(t,\boldsymbol{\tau})\mathrm{e}^{\mathrm{i}\omega \boldsymbol{n}_j^T\boldsymbol{\tau}}\,\mathrm{d}\boldsymbol{\tau} \sim \mathcal{O}(\omega^{-d}). \quad \blacksquare$$

In the asymptotic series of $\sum_{j=0}^{d} I[F_{n_j},\sigma_d(t)]$, the expressions denoted by $P_2(t)$ in the proof of Theorem 4 contain terms with integrals

$$\int\limits_0^t \mathrm{e}^{(t-\tau_d)\mathcal{L}}\alpha_{n_{d,j}} F^{\tilde{\boldsymbol{k}}}_{\tilde{\boldsymbol{n}}_j}[(1,\dots,1)](\tau_d)\,\mathrm{d}\tau_d,$$

but they are not highly oscillatory, so we expect we can approximate them effortlessly and effectively, for example by using Gauss–Legendre quadrature.

EXAMPLE 3. Consider the set $\boldsymbol{N}^2 = \{\boldsymbol{n}_1 = (-1,1), \boldsymbol{n}_2 = (1,-1)\}$ and the two integrals

$$I[F_{\boldsymbol{n}_1},\sigma_2(t)] \quad \text{and} \quad I[F_{\boldsymbol{n}_2},\sigma_2(t)].$$

Then $I[F_{\boldsymbol{n}_1},\sigma_2(t)]+I[F_{\boldsymbol{n}_2},\sigma_2(t)] \sim \mathcal{O}(\omega^{-2})$ and the first term of the asymptotic expansion of $I[F_{\boldsymbol{n}_1},\sigma_2(t)] + I[F_{\boldsymbol{n}_2},\sigma_2(t)]$ is equal to

$$\frac{1}{(\mathrm{i}\omega)^2}\Big(\alpha_1\mathrm{e}^{t\mathcal{L}}\alpha_{-1}\mathrm{e}^{\mathrm{i}\omega t} + \alpha_{-1}\mathrm{e}^{t\mathcal{L}}\alpha_1\mathrm{e}^{-\mathrm{i}\omega t} - 2\mathrm{e}^{t\mathcal{L}}\alpha_1\alpha_{-1}$$
$$+\int\limits_0^t \mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_{-1}\,\mathrm{ad}^1_{\mathcal{L}}(\alpha_1)\mathrm{e}^{\tau_2\mathcal{L}}\,\mathrm{d}\tau_2 + \int\limits_0^t \mathrm{e}^{(t-\tau_2)\mathcal{L}}\alpha_1\,\mathrm{ad}^1_{\mathcal{L}}(\alpha_{-1})\mathrm{e}^{\tau_2\mathcal{L}}\,\mathrm{d}\tau_2\Big).$$

The integrals here are not highly oscillatory and can be computed, for example, by Gauss–Legendre quadrature with high accuracy.

To summarize this section, it is much more difficult to provide formulas for the coefficients of the asymptotic expansion for integrals with resonance points. However, Theorem 4 describes the asymptotic behaviour of terms from the Neumann series, and it seems possible to use this fact to construct quadrature rules based on the Filon method.

7. A highly oscillatory wave equation. Our method can also be successfully applied to the approximation of highly oscillatory equations with a second time derivative. Consider the second-order PDE

$$\begin{aligned}
&\partial_{tt}u = \mathcal{L}u(x,t) + f(x,t)u(x,t), \qquad t \in [0,t^\star],\ x \in \Omega \subset \mathbb{R}^m, \\
&u(x,0) = u_1(x), \qquad \partial_t u(x,0) = u_2(x), \\
&u = 0 \text{ on } \partial\Omega \times [0,t^\star],
\end{aligned} \tag{7.1}$$

with f given in (1.3). We write (7.1) as a first-order system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{I} \\ \mathcal{L} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + f \begin{bmatrix} 0 & 0 \\ \mathcal{I} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{7.2}$$

where $v = \partial_t u$. Thus

$$\partial_t \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\varphi} = \underbrace{\begin{bmatrix} 0 & \mathcal{I} \\ \mathcal{L} & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} + \sum_{n=1}^{N} \mathrm{e}^{in\omega t} \underbrace{\begin{bmatrix} 0 & 0 \\ \alpha_n & 0 \end{bmatrix}}_{\beta_n} \begin{bmatrix} u \\ v \end{bmatrix},$$

and therefore

$$\begin{aligned}
&\partial_t \varphi = A\varphi + h\varphi, \quad \varphi(x,0) = [u_1(x), u_2(x)], \\
&A[u,v]^T = [v, \mathcal{L}u]^T, \quad \beta_n[u,v]^T = [0, \alpha_n u]^T,
\end{aligned} \tag{7.3}$$

where $h(x,t) = \sum_{n=1}^{N} \mathrm{e}^{in\omega t}\beta_n(x)$ is a highly oscillatory function and φ is a vector-valued function. Let $\mathcal{L}$ be a second-order differential operator

$$\mathcal{L}u = \sum_{i,j=0}^{m} \partial_{x_j}(a_{ij}\partial_{x_i}u) - cu,$$

where $a_{ij} = a_{ji}$, $i,j = 1,\dots,m$, and $c \geq 0$. For simplicity, assume that $\|\alpha_n\|_\infty < \infty$ for all $n = 1,\dots,N$. By applying Duhamel's formula, we can write (7.3) as

$$\varphi(t) = \mathrm{e}^{tA}\varphi_0 + \int_0^t \mathrm{e}^{(t-\tau)A} h(\tau)\varphi(\tau)\,\mathrm{d}\tau. \tag{7.4}$$

The operator $A : D(A) := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is the infinitesimal generator of a C_0-semigroup $\{\mathrm{e}^{tA}\}$ on $H_0^1(\Omega) \times L^2(\Omega)$ (see [9]). Using the same arguments as in the proof of Theorem 1, one can show that the Neumann series converges absolutely and uniformly in the norm of $H_0^1(\Omega) \times L^2(\Omega)$ to the solution of (7.4).

8. Numerical examples. In this section, we apply our method to equations of type (1.1) and (7.1). For each of the equations, one can find an analytical solution to compare them accurately with a numerical approximation. The L^2 norm of the error is considered in every example below. For each equation, the solution can be approximated by a partial sum of the asymptotic expansion

$$u(x,t) \approx p_{0,0}(x,t) + \sum_{r=1}^{R} \frac{1}{\omega^r} \sum_{s=0}^{S} p_{r,s}(x,t) e^{is\omega t}, \tag{8.1}$$

for different R and ω.

Example 4. We first consider the equation

$$\begin{aligned} &\partial_t u = (1-x^2)^4 \partial_{xx}^2 u + f(x,t)u(x,t), \qquad t \in [0,3],\ x \in (-1,1),\\ &u(x,0) = u_0(x),\\ &u(-1,t) = 0 = u(1,t), \end{aligned} \tag{8.2}$$

where the initial condition u_0 and the highly oscillatory potential f are

$$u_0(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-x^2}} & \text{if } x \in (-1,1),\\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} f(x,t) = {} & \underbrace{\frac{1}{\omega^2}\big((1-x^2)^4 + 4\mathrm{i}\omega x(1-x^2)^2 + 2\omega^2(1-3x^4)\big)}_{\alpha_0(x)} \\ & + \mathrm{e}^{\mathrm{i}\omega t} \underbrace{\frac{1}{\omega^2}\big(-2(1-x^2)^4 - 4\mathrm{i}x\omega(1-x^2)^2 + x\omega^2\big)}_{\alpha_1(x)} \\ & + \mathrm{e}^{2\mathrm{i}\omega t} \underbrace{\frac{1}{\omega^2}(1-x^2)^4}_{\alpha_2(x)}. \end{aligned}$$

The solution of (8.2) is

$$u(x,t) = \mathrm{e}^{-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}x}{\omega} + \frac{\mathrm{i}x}{\omega}} u_0(x)$$

and the differential operator $\mathcal{L}$ is

$$\mathcal{L} = (1-x^2)^4 \partial_{xx}^2 + \alpha_0.$$

To approximate the solution we take the first four terms of the Neumann series

$$u^{[3]}(t) = \mathrm{e}^{t\mathcal{L}} u_0(x) + T^1 \mathrm{e}^{t\mathcal{L}} u_0 + T^2 \mathrm{e}^{t\mathcal{L}} u_0 + T^3 \mathrm{e}^{t\mathcal{L}} u_0.$$

Subsequently, we expand $T^1\mathrm{e}^{t\mathcal{L}}u_0, T^2\mathrm{e}^{t\mathcal{L}}u_0$ and $T^3\mathrm{e}^{t\mathcal{L}}u_0$ asymptotically with error $\mathcal{O}(\omega^{-4})$. In other words, we approximate the solution of (8.2) by the partial sum of (8.1) with the first four terms. Figure 1 and Table 1 show the approximation error of the solution $u(x,3)$ for different values of ω and different numbers R.

Example 5. As already mentioned, the method can be applied to the potential f with time–dependent functions $\alpha_n(x,t)$. Indeed, consider the

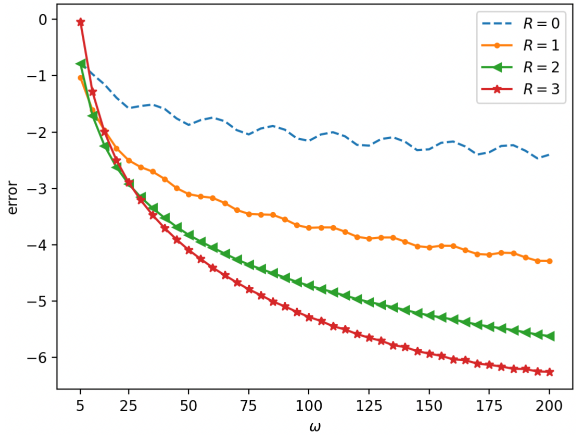

Fig. 1. L^2 norm of the error of the method for (8.2) for $t = 3$. We use a base-10 log scale.

Table 1. Error of the method – equation (8.2)

	$R = 0$	$R = 1$	$R = 2$	$R = 3$
$\omega = 10$	1.12e-01	3.17e-02	2.75e-02	7.38e-02
$\omega = 100$	1.23e-02	3.06e-04	2.68e-05	7.36e-06
$\omega = 1000$	1.27e-03	3.15e-06	2.68e-08	1.76e-09

equation

$$
\begin{aligned}
&\partial_t u = \partial_{xx}^2 u + \mathrm{e}^{\mathrm{i}\omega t}\sin(t)u(x,t), \qquad t \in [0,5],\ x \in (0,2\pi),\\
&u(x,0) = \sin(x), \qquad\qquad (8.3)\\
&u(0,t) = 0 = u(2\pi,t).
\end{aligned}
$$

The solution of (8.3) is

$$u(x,t) = \mathrm{e}^{-t-1/(\omega^2-1)+\mathrm{e}^{\mathrm{i}\omega t}\cos(t)/(\omega^2-1)-\mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}\omega\sin(t)/(\omega^2-1)}\sin(x).$$

We have $\mathcal{L} = \partial_{xx}^2$ and $f(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\alpha(x,t)$, where $\alpha(x,t) = \sin(t)$ is a time-dependent function. We approximate the solution by taking the first three terms of the Neumann series,

$$u^{[2]}(t) = \mathrm{e}^{t\mathcal{L}}u_0 + T^1\mathrm{e}^{t\mathcal{L}}u_0 + T^2\mathrm{e}^{t\mathcal{L}}u_0.$$

To expand the integrals $T^1\mathrm{e}^{t\mathcal{L}}u_0$ and $T^2\mathrm{e}^{t\mathcal{L}}u_0$ asymptotically we utilize the following generalization of Lemma 2.

LEMMA 6. *Let $\alpha(\tau) \in C^k([0,t^\star], D(\mathcal{L}^k))$. Then the kth time derivative of $\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}$ is*

$$\partial_\tau^k(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}) = \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}.$$

The proof can be found in the Appendix. Table 2 presents the error of the method up to the second iteration.

Table 2. Error of the method – equation (8.3)

	$R=0$	$R=1$	$R=2$
$\omega=10$	9.33e-03	5.28e-03	5.55e-05
$\omega=100$	5.33e-04	4.87e-05	4.94e-08
$\omega=1000$	4.96e-05	7.64e-08	4.86e-11

Example 6. Consider now the wave equation with potential with negative frequencies

$$\begin{aligned}
&\partial_{tt}u = \partial_{xx}u + f(x,t)u(x,t), \qquad t\in[0,1],\ x\in(-L,L),\ L=10,\\
&u(x,0) = \mathrm{e}^{-x^2(1/2+1/\omega^2)}, \qquad \partial_t u(x,0)=0,\\
&u(-L,t)=u(L,t),\\
&\partial_t u(-L,t) = \partial_t u(L,t),
\end{aligned} \tag{8.4}$$

where

$$f(x,t) = \left(1-x^2+\frac{(2+x^2\omega^2-4x^2)\cos(\omega t)}{\omega^2} - \frac{4x^2\cos^2(\omega t)}{\omega^4} + \frac{x^4\sin^2(\omega t)}{\omega^2}\right).$$

The solution of (8.4) is

$$u(x,t) = \mathrm{e}^{-\cos(\omega t)x^2/\omega^2}\mathrm{e}^{-x^2/2}.$$

Due to the presence of resonance points, we do not have a general formula for the asymptotic expansion of the integrals that appear in the Neumann series in this case. Nevertheless, we can employ Theorem 2 and Theorem 4 to approximate just $T^1\mathrm{e}^{tA}$ and $T^2\mathrm{e}^{tA}$, where A is the linear operator (7.3). Table 3 presents the error of the method for ω and various R.

Table 3. Error of the method – equation (8.4)

	$R=0$	$R=1$	$R=2$
$\omega=10$	3.17e-02	3.17e-02	1.71e-03
$\omega=100$	5.45e-04	5.45e-04	2.00e-07
$\omega=1000$	5.54e-06	5.54e-06	1.98e-11

Example 7. In the last example, we consider the equation with the biharmonic operator

$$\begin{aligned}
&\partial_t u = \partial^4_{xxxx}u + \mathrm{e}^{\mathrm{i}\omega t}u(x,t), \qquad x\in(0,\pi),\ t\in[0,1],\\
&u(x,0) = u_0(x) = \sin(x)\exp\left(-\frac{\mathrm{i}}{\omega}\right),\\
&u(0,t) = u(\pi,t) = 0,
\end{aligned} \tag{8.5}$$

with periodic boundary conditions. The solution is

$$u(x,t) = \mathrm{e}^{t-\mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}/\omega}\sin(x).$$

Table 4 presents the error of the method for various ω and R.

Table 4. Error of the method – equation (8.5)

	$R=0$	$R=1$	$R=2$	$R=3$
$\omega=10$	3.58e+00	3.47e-01	2.22e-02	1.07e-03
$\omega=100$	1.00e-01	2.64e-04	4.62e-07	6.07e-10
$\omega=1000$	1.80e-02	8.41e-06	2.62e-09	6.14e-13

REMARK 4. One can notice that equations (8.2), (8.4) and (8.5) do not fully satisfy Assumption 1. Indeed, the linear operator $\mathcal{L}$ from (8.2) is not strongly elliptic for x near -1 or 1, and solutions $u(x,t)$ of (8.4) and (8.5) do not satisfy the zero boundary conditions. These are just examples to illustrate the proposed methodology and suggest that the method can be applied to a wider class of differential equations.

9. Concluding remarks. There are several ways of extending the proposed approach. The primary concern is to establish rigorous estimations for the error formulas that arise in the asymptotic expansion of highly oscillatory integrals. This task is particularly challenging, especially when dealing with a general differential operator $\mathcal{L}$ of the form (1.2). It should be noted that the partial sum $\mathcal{S}_r^{(d)}(t)$, used to approximate the integral $I[F_{\boldsymbol{n}},\sigma_d(t)]$, may be divergent as $r\to\infty$. A significant advancement in the approximation of highly oscillatory PDEs would involve determining the maximum value of $r^\star$, beyond which the method's error begins to increase.

To approximate each integral of the form (3.2), we have used the asymptotic method, which represents a simple form of quadrature rules tailored to highly oscillatory integrals. If adding successive terms of the sum (1.4) does not improve the expected accuracy, one could modify the method by incorporating more sophisticated formulas, such as Filon-type methods. This approach should further reduce the approximation error.

The next step would be to provide the asymptotic expansion of the solution for the potential (1.5). Furthermore, it would be interesting to investigate whether it is possible to represent the solution as an asymptotic series for an even more general highly oscillatory potential given by

$$f(x,t) = \sum_{n=1}^{N}\alpha_n(x,t)\mathrm{e}^{\mathrm{i}\omega g_n(t)}, \qquad \omega\gg 1,\ N\in\mathbb{N},$$

where α_n and g_n are sufficiently smooth functions. If such an expansion is possible, the method could be applicable to a wide range of highly oscillatory differential equations.

Appendix

LEMMA 6. *The kth time derivative of* $\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}$ *is equal to*

$$\partial_\tau^k(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}) = \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}. \tag{A.1}$$

Proof. We use induction on k. For $k=1$,

$$\begin{aligned}
\partial_\tau^1(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}) &= \partial_\tau^1(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau))\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathcal{L}\mathrm{e}^{\tau\mathcal{L}}\\
&= (-1)\mathcal{L}\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha'(\tau)\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathcal{L}\mathrm{e}^{\tau\mathcal{L}}\\
&= (-1)\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^1(\alpha(\tau))\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha'(\tau)\mathrm{e}^{\tau\mathcal{L}}\\
&= \sum_{\ell=0}^{1}(-1)^\ell\binom{1}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(1-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}.
\end{aligned}$$

Suppose now (A.1) is valid for $k-1, k>1$. Then

$$\begin{aligned}
&\partial_\tau^k(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}) = \partial_\tau^1(\partial_\tau^{k-1}(\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha(\tau)\mathrm{e}^{\tau\mathcal{L}}))\\
&= \sum_{\ell=0}^{k-1}(-1)^\ell\binom{k-1}{\ell}\partial_\tau^1\big(\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-1-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}\big)\\
&= \sum_{\ell=0}^{k-1}(-1)^\ell\binom{k-1}{\ell}\big((-1)\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^{\ell+1}(\alpha^{(k-1-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}\big)\\
&= \sum_{\ell=0}^{k-2}(-1)^{\ell+1}\binom{k-1}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^{\ell+1}(\alpha^{(k-(\ell+1))}(\tau))\mathrm{e}^{\tau\mathcal{L}} + (-1)^k\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^k(\alpha(\tau))\mathrm{e}^{\tau\mathcal{L}}\\
&\quad+\sum_{\ell=1}^{k-1}(-1)^\ell\binom{k-1}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha^{(k)}(\tau)\mathrm{e}^{\tau\mathcal{L}}\\
&= \sum_{\ell=1}^{k-1}(-1)^\ell\binom{k-1}{\ell-1}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}} + (-1)^k\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^k(\alpha(\tau))\mathrm{e}^{\tau\mathcal{L}}\\
&\quad+\sum_{\ell=1}^{k-1}(-1)^\ell\binom{k-1}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}} + \mathrm{e}^{(t-\tau)\mathcal{L}}\alpha^{(k)}(\tau)\mathrm{e}^{\tau\mathcal{L}}\\
&= \sum_{\ell=1}^{k-1}(-1)^\ell\binom{k}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}\\
&\quad+\mathrm{e}^{(t-\tau)\mathcal{L}}\alpha^{(k)}(\tau)\mathrm{e}^{\tau\mathcal{L}} + (-1)^k\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^k(\alpha(\tau))\mathrm{e}^{\tau\mathcal{L}}\\
&= \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\mathrm{e}^{(t-\tau)\mathcal{L}}\,\mathrm{ad}_{\mathcal{L}}^\ell(\alpha^{(k-\ell)}(\tau))\mathrm{e}^{\tau\mathcal{L}}.
\end{aligned}$$

In the penultimate equality we used the identity $\binom{k}{\ell} = \binom{k-1}{\ell} + \binom{k-1}{\ell-1}$. ■

References

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, NJ, 1965.

[2] D. Cohen, *Analysis and numerical treatment of highly oscillatory differential equations*, PhD thesis, 2004.

[3] D. Cohen, E. Hairer and C. Lubich, *Modulated Fourier expansions of highly oscillatory differential equations*, Found. Comput. Math. 3 (2003), 327–345.

[4] M. Condon, A. Deaño and A. Iserles, *On highly oscillatory problems arising in electronic engineering*, Math. Model. Numer. Anal. 43 (2009), 785–804.

[5] M. Condon, A. Deaño and A. Iserles, *On second-order differential equations with highly oscillatory forcing terms*, Proc. Roy. Soc. London Ser. A Math. Phys. Engrg. Sci. 466 (2010), 1809–1828.

[6] M. Condon, A. Deaño and A. Iserles, *Effcient computation of delay differential equations with highly oscillatory terms*, ESAIM Math. Model. Numer. Anal. 46 (2012), 1407–1420.

[7] M. Condon, A. Iserles, K. Kropielnicka and P. Singh, *Solving the wave equation with multifrequency oscillations*, J. Comput. Dynam. 6 (2019), 239–249.

[8] M. Condon, K. Kropielnicka, K. Lademann and R. Perczyński, *Asymptotic numerical solver for the linear Klein–Gordon equation with space- and time-dependent mass*, Appl. Math. Lett. 115 (2021), art. 106935, 7 pp.

[9] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, Amer. Math. Soc., 1998.

[10] E. Hairer and C. Lubich, *Long-time energy conservation of numerical methods for oscillatory differential equations*, SIAM J. Numer. Anal. 38 (2000), 414–441.

[11] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, 2nd ed., Springer Ser. Comput. Math. 31, Springer, Berlin, 2006.

[12] A. Iserles, K. Kropielnicka and P. Singh, *Solving Schrödinger equation in semiclassical regime with highly oscillatory time-dependent potentials*, J. Comput. Phys. 376 (2019), 564–584.

[13] A. Iserles and S. Norsett, *Quadrature methods for multivariate highly oscillatory integrals using derivatives*, Math. Comput. 75 (2006), 1233–1258.

[14] K. Kropielnicka and R. Perczyński, *Asymptotic expansions for the linear PDEs with oscillatory input terms; Analytical form and error analysis*, Comput. Math. Appl. 156 (2024), 16–27.

[15] R. I. McLachlan and G. R. W. Quispel, *Splitting methods*, Acta Numer. 11 (2002), 341–434.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., Springer, 1983.

[17] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, Springer, 2004.

[18] J. M. Sanz-Serna, *Modulated Fourier expansions and heterogeneous multiscale methods*, J. Numer. Anal. 29 (2009), 595–605.

[19] L. N. Trefethen, *Spectral methods in MATLAB*, SIAM, Philadelphia, PA, 2000.

Antoni Augustynowicz, Rafał Perczyński
Institute of Mathematics
Faculty of Mathematics, Physics and Informatics
University of Gdańsk
80-308 Gdańsk, Poland
E-mail: antoni.augustynowicz@ug.edu.pl
rafal.perczynski@phdstud.ug.edu.pl