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Nonstationary solutions to symmetric systems of second-order differential equations

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Abstract. We establish the existence of nonstationary solutions to a symmetric system of second-order autonomous differential equations. Our technique is based on the equivariant degree theory and involves a novel characterization of orbit types of maximal kind in the Burnside ring product of a finite number of basic degrees for the group $O(2) \times \Gamma \times \mathbb{Z}_2$.

1. Introduction. The classical forced pendulum equation $\ddot{x} = \boldsymbol{f}(x)$ serves as a paradigmatic problem for validating novel techniques in nonlinear analysis (see [19]–[21]). More generally, dynamical systems described by second-order differential equations strike a delicate balance between simplicity and a capacity for capturing rich and intricate nonlinear behavior, making them ideal candidates for the testing of analytical methods. On the other hand, autonomous differential equations with periodic boundary conditions always admit natural $SO(2)$-symmetries related to time shifting that make them particularly well-suited for a framework based on the equivariant degree theory. Second-order autonomous differential equations not involving any first derivative terms, specifically, admit $O(2)$-symmetries, where the reflection generator acts by time reversal. When these equations are subjected to an odd forcing function and arranged in symmetric networks characterized by a finite group Γ, their full symmetry group becomes

$$G := O(2) \times \Gamma \times \mathbb{Z}_2.$$

Our primary objective is to demonstrate how the G-equivariant Leray–Schauder degree can be used to establish the existence of nonstationary solutions to the following Γ-symmetric system of p-periodic, second-order,

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autonomous equations:

$$\begin{cases} \ddot{x}(t) = f(x(t)) + Ax(t), & t \in \mathbb{R},\ x(t) \in V, \\ x(t) = x(t+p),\ \dot{x}(t) = \dot{x}(t+p), \end{cases} \tag{1.1}$$

where $V := \mathbb{R}^N$ is an orthogonal Γ-representation, $A : V \to V$ is a symmetric Γ-equivariant matrix and $f : V \to V$ is a continuous function satisfying the following conditions:

(A_1) f is Γ-equivariant, i.e. $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, $t \in \mathbb{R}$, $x \in V$.
(A_2) f is odd, i.e. $f(-x) = -f(x)$ for all $x \in V$.
(A_3) $f(x)$ is $o(|x|)$ as x approaches 0, i.e.

$$\lim_{x \to 0} \frac{f(x)}{|x|} = 0.$$

(A_4) There exists a constant $M > 0$ such that for all $x \in V$ with $|x| \geq M$,

$$f(x) \cdot x > 0.$$

Conditions (A_1)–(A_2) ensure that the equations admit the symmetries of the product group $O(2) \times \Gamma \times \mathbb{Z}_2$, while conditions ($A_3$) and ($A_4$) imply the well-behavedness of our nonlinearity at solutions to (1.1) near zero and infinity, respectively. Specifically, (A_2) allows the equivariant degree to distinguish between stationary and nonstationary functions, (A_3) guarantees that the nonlinearity does not interact with the spectrum of A, and the Nagumo condition (A_4) is essential for establishing a priori bounds on the magnitude of solutions to (1.1).

System (1.1) is a natural model for the dynamics of a configuration of N identical pendula, subjected to nonlinear forcing f and with coupling relations specified by the matrix A and the group Γ. In this context, the vector $x(t) := (x_1(t), \dots, x_N(t))$ represents the angles of the pendula from the vertical position, $\dot{x}(t) := (\dot{x}_1(t), \dots, \dot{x}_N(t))$ the angular velocities and $\ddot{x}(t) := (\ddot{x}_1(t), \dots, \ddot{x}_N(t))$ the angular accelerations at any time $t > 0$.

Methods based on the Leray–Schauder degree have been used to solve a wide variety of second-order differential equations [2, 8, 13, 16, 17, 20, 21]. However, the inability of the Leray–Schauder degree to distinguish between periodic solutions that differ by a fixed time shift and differentiate constant solutions from nonconstant solutions have hindered its suitability for autonomous differential equations. The effectiveness of the G-equivariant Leray–Schauder degree in detecting nonconstant solutions to the system (1.1) for several explicit choices of the symmetry group Γ has already been demonstrated by García-Azpeitia et al. [12]. More generally, the G-equivariant Leray–Schauder degree has been applied to problems admitting appropriate spatial-temporal symmetries [3, 4, 10, 11, 14, 18]. Compared with other techniques to solve systems similar to (1.1), the equivariant degree approach requires relatively minimal regularity assumptions on the nonlinearity. For example, existence results were obtained in [1] by topological index argument and in [9] using Morse theory framework, under the assumption that the right-hand side interacts with only finitely many eigenvalues of the differential operator.

A prerequisite for the application of any G-equivariant degree-theory based argument to (1.1) is the reformulation of our system as a fixed point equation in an appropriate functional G-space with a nonlinear operator in the form of a G-equivariant compact perturbation of identity. In [12], García-Azpeitia et al. prove that the problem of finding p-periodic solutions to (1.1) is equivalent to the problem of finding zeros of the operator

$$\mathscr{F} : \mathscr{H} \to \mathscr{H}, \quad \mathscr{F}(u) := u - \mathscr{L}^{-1}(\beta^2 f(u) + \beta^2 Au - u),$$

where $\beta := \frac{p}{2\pi}$, $\mathscr{H} := H^2_{2\pi}([0, 2\pi]; V)$ and $\mathscr{L} : \mathscr{H} \to \mathscr{H}$ is the differential operator $\mathscr{L}u := \ddot{u} - u$. Under the conditions (A_1)—(A_4) and provided that the linearization $D\mathscr{F}(0) : \mathscr{H} \to \mathscr{H}$ is an isomorphism, García-Azpeitia et al. additionally demonstrate that

(i) $\mathscr{F}$ is a G-equivariant completely continuous field with respect to the natural isometric action of G on $\mathscr{H}$;
(ii) all of the nontrivial solutions to (1.1) can be confined to an annulus of the form $\Omega := B_R(\mathscr{H}) \setminus B_\varepsilon(\mathscr{H}) \subset \mathscr{H}$ for some sufficiently small $\varepsilon > 0$ and sufficiently large $R > 0$;
(iii) $(\mathscr{F}, \Omega)$ constitutes an admissible G-pair, such that the existence of nontrivial solutions to (1.1) is equivalent to nontriviality of the degree invariant

$$G\text{-deg}(\mathscr{F}, \Omega) = (G) - G\text{-deg}(D\mathscr{F}(0), B_1(\mathscr{H})),$$

where G-deg is the G-equivariant Leray–Schauder degree and $(G) \in A(G)$ is the unit element of the Burnside ring (see Appendix for the definition of the Burnside ring $A(G)$, the definition of G-admissibility/the set of all admissible G-pairs $\mathscr{M}^G$, and the construction of the degree $G\text{-deg} : \mathscr{M}^G \to A(G)$).

Together with a direct application of the existence property of the G-equivariant Leray–Schauder degree (cf. Appendix), the above considerations suggest the following sufficient condition for the existence of a nontrivial solution to (1.1):

LEMMA 1.1. *If for some orbit type* $(H) \in \Phi_0(G) \setminus \{(G)\}$ *one has*

$$\text{coeff}^H\big(G\text{-deg}(D\mathscr{F}(0), B(\mathscr{H}))\big) \neq 0,$$

(*see Appendix for the definition of the coefficient operator* $\text{coeff}^H : A(G) \to \mathbb{Z}$) *then there exists a function* $u \in \mathscr{H} \setminus \{0\}$ *satisfying* (2.1) *with an isotropy subgroup* $G_u \leq G$ *satisfying* $(G_u) \geq (H)$.

In this way, the problem of finding p-periodic solutions to (1.1) has been reformulated as the problem of computing the coefficients of nonunit orbit types (i.e. orbit types $(H) \in \Phi_0(G) \setminus \{(G)\}$) in the Burnside ring element $G\text{-deg}(D\mathscr{F}(0), B(\mathscr{H}))$. Moreover, since the G-equivariant Leray–Schauder degree provides a full equivariant classification to the solution set of $\mathscr{F}(u) = 0,\ u \in \mathscr{H}$, Lemma 1.1 represents a framework for identifying the exact spatio-temporal symmetries of any predicted solution. In order to take advantage of the full power of the G-equivariant Leray–Schauder degree for this purpose, we must first identify the irreducible representations of G and describe the G-isotypic decomposition of $\mathscr{H}$.

Without specifying the group Γ, we assume that a complete list of the irreducible Γ-representations $\{\mathcal{V}_j\}_{j=0}^r$ is available. As a Γ-representation, the space V is also a natural $\Gamma \times \mathbb{Z}_2$-representation with the $\Gamma \times \mathbb{Z}_2$-isotypic decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r,$$

where each $\Gamma \times \mathbb{Z}_2$-isotypic component V_j is modeled on the irreducible $\Gamma \times \mathbb{Z}_2$-representation $\mathcal{V}_j^-$ (here the superscript is meant to indicate that $\mathcal{V}_j$ has been equipped with the antipodal $\mathbb{Z}_2$-action) in the sense that V_j is equivalent to the direct sum of some finite number of copies of $\mathcal{V}_j^-$, i.e.

$$V_j \simeq \mathcal{V}_j^- \oplus \cdots \oplus \mathcal{V}_j^-.$$

The exact number of irreducible Γ-representations $\mathcal{V}_j^-$ 'contained' in the $\Gamma \times \mathbb{Z}_2$-isotypic component V_j is called the $\mathcal{V}_j$*-isotypic multiplicity* of V and is calculated according to the ratio

$$m_j := \dim V_j / \dim \mathcal{V}_j^-, \quad j \in \{0, 1, \dots, r\}. \tag{1.2}$$

Since $A : V \to V$ is Γ-equivariant, one has

$$\sigma(A) = \bigcup_{j=0}^{r} \sigma(A_j), \quad A_j := A|_{V_j} : V_j \to V_j.$$

For the sake of simplicity, we choose to sidestep a fixed point argument made by García-Azpeitia et al. [12] by enforcing the following nondegeneracy assumption:

(A_0) for each Γ-isotypic component V_j, there exists a number $\mu_j \in \mathbb{R}$ with $\mu_j \neq -m^2/\beta^2$ for all $m \in \mathbb{N}$ such that

$$A_j = \mu_j \operatorname{Id}|_{V_j} : V_j \to V_j.$$

On the other hand, for each $m \in \mathbb{N}$, we denote by $\mathcal{W}_m \simeq \mathbb{C}$ the irreducible $O(2)$-representation equipped with the m-folding $O(2)$-action

$$e^{i\theta} w := e^{im\theta} w, \quad \kappa w := \bar{w}, \quad e^{i\theta}, \kappa \in O(2),\ w \in \mathcal{W}_m,$$

(here, κ indicates the reflection generator in $O(2)$) and by $\mathcal{W}_0 \simeq \mathbb{R}$ the irreducible $O(2)$-representation on which $O(2)$ acts trivially such that the space $\mathscr{H}$ admits the $O(2)$-isotypic decomposition

$$\mathscr{H} := \overline{\bigoplus_{m=0}^{\infty} \mathscr{H}_m}, \quad \mathscr{H}_m := \{u \in \mathscr{H} : u(t) = \cos(mt)a + \sin(mt)b,\ a, b \in V\},$$

where

$$\mathscr{H}_m \simeq \mathcal{W}_m \otimes V, \quad m \in \mathbb{N} \cup \{0\}.$$

The G-isotypic decomposition of $\mathscr{H}$ can now be expressed in terms of G-isotypic components $\mathscr{H}_{m,j} \subset \mathscr{H}$ modeled on the irreducible G-representations

$$\mathcal{V}_{m,j} := \mathcal{W}_m \otimes \mathcal{V}_j^-, \quad m \in \mathbb{N} \cup \{0\},\ j \in \{0, 1, \dots, r\},$$

as follows:

$$\mathscr{H} = \bigoplus_{j=0}^{r} \overline{\bigoplus_{m=0}^{\infty} \mathscr{H}_{m,j}}, \tag{1.3}$$

where

$$\mathscr{H}_{m,j} := \{u \in \mathscr{H} : u(t) = \cos(mt)a + \sin(mt)b,\ a, b \in V_j\}.$$

We are now in a position to present our main existence results:

THEOREM 1.2. *Take $m > 0$ and let (H) be a maximal element in the isotropy lattice $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$. Under the assumptions (A_0)–(A_4), if the matrix $A : V \to V$ admits an* odd number *of eigenvalues $\mu_j \in \sigma(A)$ with odd isotypic multiplicity m_j (see (1.2)) satisfying*

$$\mu_j < -\left(\frac{m}{\beta}\right)^2 \quad \textit{and} \quad 2 \nmid \dim \mathcal{V}_{m,j}^H,$$

then there exists a nonstationary solution $u \in \mathscr{H} \setminus \{0\}$ to the system (1.1) with an isotropy subgroup $G_u \leq G$ satisfying $(G_u) \geq (H)$.

REMARK 1.3. In the field of equivariant bifurcation theory, the Ize Conjecture (IC) refers to the proposition that every irreducible representation of a compact Lie group admits at least one subgroup with an odd dimensional fixed point space. Although IC was shown to be false by Lauterbach et al. [22] via a counterexample, the conjecture holds in many cases. For example, in Section 7.1, we use G.A.P. to numerically verify that the irreducible representations of the group $G = O(2) \times \Gamma \times \mathbb{Z}_2$, with the assignment $\Gamma = D_8$, satisfy IC.

The remainder of this paper is organized as follows:

In Section 2, we prepare the problem (1.1) for application of the G-equivariant Leray–Schauder degree. This involves a functional reformulation in an appropriate Sobolev space, allowing us to derive a practical formula for a degree invariant, in terms of the Burnside ring product of a finite number of basic degrees defined on the irreducible G-representations, whose nontriviality is equivalent to the existence of nontrivial solutions to (1.1). Using this framework, we analyze the maximal elements in the isotropy lattices $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$, called *orbit types of maximal kind* in $\Phi_0(G)$. The significance of these orbit types is two-fold: first, the coefficients corresponding to orbit types of maximal kind in the degree invariant are computationally accessible, and second, the nontriviality of these coefficients guarantees the existence of nonstationary solutions to (1.1).

In Section 4, we present a classification for the generators of the Burnside ring $A(G)$ according to their lattice relations with respect to the s-folding homomorphism (3.2) and employ this classification, in Section 5, to derive a practical formula for characterizing the nontriviality of an orbit type of maximal kind in the product of a finite number of basic degrees. Specifically, we find that the contribution of each eigenvalue $\mu_{m,j} \in \sigma(D\mathscr{F}(0))$ to the appearance of an orbit type of maximal kind (H) in the degree invariant G-deg$(\mathscr{F}, \Omega)$ depends on (i) the sign of $\mu_{m,j}$, and (ii) the parity of the dimension of the corresponding H-fixed-point space $\mathcal{V}_{m,j}^H$.

Finally, in Section 7, we consider the implications of our main result for dihedral symmetry group $\Gamma = D_N$. For the special case of D_8 coupling symmetries, we use GAP to numerically verify (i) the pairwise disjointness of the maximal orbit type sets in each isotropy lattice $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$ and (ii) the odd-dimensionality of the fixed point sets $\mathcal{V}_{m,j}^H$ for every maximal element in $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$, allowing us to reformulate Theorem 1.2 in Section 7.1 as follows:

PROPOSITION 1.4. *Let $\Gamma = D_8$ and take any fixed $m > 0$. Under the assumptions (A_0)–(A_4), if the jth eigenvalue of the matrix A satisfies*

$$\mu_j < -\left(\frac{m}{\beta}\right)^2,$$

then, corresponding to every maximal element (H) in the isotropy lattice $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$, there exists a nonstationary solution $u \in \mathscr{H} \setminus \{0\}$ to the system (1.1) with an isotropy subgroup $G_u \leq G$ satisfying $(G_u) \geq (H)$.

For convenience, the Appendix includes an explanation of notations used, and a brief introduction to the G-equivariant Leray–Schauder degree. Readers who are interested in a deeper exposition of these topics are referred to [6, 7].

2. Setting for the Leray–Schauder G-equivariant degree. Notice that, with the substitutions $\beta := p/(2\pi)$ and $u(t) := x(\beta t)$, the system (1.1) becomes

$$\begin{cases} \ddot{u}(t) = \beta^2 f(u(t)) + \beta^2 A u(t), \quad t \in \mathbb{R},\, u(t) \in V, \\ u(t) = u(t+2\pi), \quad \dot{u}(t) = \dot{u}(t+2\pi). \end{cases} \tag{2.1}$$

Clearly, any 2π-periodic solution $u(t) : \mathbb{R} \to V$ for the system (2.1) corresponds to the p-periodic solution $x(t) = u(t/\beta)$ for our original equation.

Following [12], we denote by $\mathscr{H}$ the Sobolev space of 2π-periodic, V-valued functions

$$\mathscr{H} := \{u : \mathbb{R} \to V : u(0) = u(2\pi),\, u|_{[0,2\pi]} \in H^2(S^1; V)\},$$

equipped with the standard inner product

$$\langle u, v\rangle := \int_0^{2\pi} (\dot{u}(t) \bullet \dot{v}(t) + u(t) \bullet v(t))\, dt, \quad u, v \in \mathscr{H},$$

and the isometric G-action

$$(e^{i\theta}, \gamma, \pm 1)u(t) := \pm\gamma u(t+\theta), \quad (\kappa, \gamma, \pm 1)u(t) := \pm\gamma u(-t), \quad u \in \mathscr{H},\, t \geq 0,$$

where $(e^{i\theta}, \gamma, \pm 1), (\kappa, \gamma, \pm 1) \in O(2) \times \Gamma \times \mathbb{Z}_2$. We also consider the differential operator

$$\mathscr{L} : \mathscr{H} \to \mathscr{E}, \quad \mathscr{L}u := \ddot{u} - u,$$

the Nemytskiĭ operator

$$N : L^2(S^1; V) \to L^2(S^1; V), \quad (Nu)(t) := f(u(t)),$$

and the Banach embedding

$$j : \mathscr{H} \to L^2(S^1; V), \quad j(u)(t) := u(t).$$

Since $\mathscr{L} : \mathscr{H} \to L^2(S^1; V)$ is a linear isomorphism, $N : L^2(S^1; V) \to L^2(S^1; V)$ is continuous and $j : \mathscr{H} \to L^2(S^1; V)$ is compact, the operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ given by

$$\mathscr{F}(u) := u - \mathscr{L}^{-1}(\beta^2 N(j(u)) + \beta^2 A j(u) - j(u)) \tag{2.2}$$

is a compact perturbation of the identity on $\mathscr{H}$ and the system (2.1) is equivalent to the operator equation

$$\mathscr{F}(u) = 0, \tag{2.3}$$

in the sense that a function $u \in \mathscr{H}$ is a solution to (2.1) if and only if it satisfies (2.3). In what follows, (2.1) will be called the *operator equation* associated with the system (2.1).

García-Azpeitia et al. demonstrate the applicability of the G-equivariant Leray–Schauder degree to (2.3) by proving that the assumptions (A_1)–(A_4) imply that the nonlinear operator (2.2) is a completely continuous G-equivariant field, differentiable at the origin $0 \in \mathscr{H}$ with

$$\mathscr{A}u := \mathrm{Id} - \mathscr{L}^{-1}(\beta^2 A u - u), \quad \mathscr{A} := D\mathscr{F}(0) : \mathscr{H} \to \mathscr{H}.$$

2.1. Towards a computational formula for G**-deg**$(\mathscr{A}, B(\mathscr{H}))$**.** In order to effectively make use Lemma 1.1 to determine the existence of nontrivial solutions to (1.1), we must derive a practical formula for the computation of the local bifurcation invariant $G\text{-deg}(\mathscr{A}, B(\mathscr{H})) \in A(G)$. Our first step in this direction will be to take advantage of the G-isotypic decomposition (1.3) to collect spectral data related to the G-equivariant linear operator $\mathscr{A} : \mathscr{H} \to \mathscr{H}$.

By Schur's Lemma, one has

$$\mathscr{A}(\mathscr{H}_{m,j}) \subset \mathscr{H}_{m,j}, \quad m \in \mathbb{N} \cup \{0\},\ j \in \{0, 1, \dots, r\},$$

so that, adopting the notation

$$\mathscr{A}_{m,j} := \mathscr{A}|_{\mathscr{H}_{m,j}} : \mathscr{H}_{m,j} \to \mathscr{H}_{m,j},$$

the spectrum of $\mathscr{A}$ admits the decomposition

$$\sigma(\mathscr{A}) = \bigcup_{m=0}^{\infty} \bigcup_{j=0}^{r} \sigma(\mathscr{A}_{m,j}).$$

Under assumption (A_0), the spectrum of each matrix $\mathscr{A}_{m,j}$ consists of the nonzero eigenvalue

$$\mu_{m,j} := 1 + \frac{\beta^2 \mu_j - 1}{1 + m^2} = \frac{m^2 + \beta^2 \mu_j}{1 + m^2} \tag{2.4}$$

with multiplicity $m_j \in \mathbb{N}$ (see discussion of isotypic multiplicity (1.2)).

The product property of the G-equivariant Leray–Schauder degree (see Appendix 7.1) permits us to express the degree $G\text{-deg}(\mathscr{A}, B(\mathscr{H}))$ in terms of a Burnside ring product of the G-equivariant Leray–Schauder degrees of the various restrictions $\mathscr{A}_{m,j} : \mathscr{H}_{m,j} \to \mathscr{H}_{m,j}$ of the G-equivariant linear isomorphism $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ to the G-isotypic components $\mathscr{H}_{m,j}$ on their respective open unit balls $B(\mathscr{H}_{m,j}) := \{u \in \mathscr{H}_{m,j} : \|u\|_{\mathscr{H}} < 1\}$, as follows:

$$G\text{-deg}(\mathscr{A}, B(\mathscr{H})) = \prod_{j=0}^{r} \prod_{m=0}^{\infty} G\text{-deg}(\mathscr{A}_{m,j}, B(\mathscr{H}_{m,j})). \tag{2.5}$$

On the other hand, each degree $G\text{-deg}(\mathscr{A}_{m,j}, B(\mathscr{H}_{m,j}))$ is fully specified by the $\mathcal{V}_j$-isotypic multiplicities $\{m_j\}_{j=0}^{r}$ together with the real spectra of $\mathscr{A}_{m,j}$ according to the formula

$$G\text{-deg}(\mathscr{A}_{m,j}, B(\mathscr{H}_{m,j})) = \begin{cases} (\deg_{\mathcal{V}_{m,j}})^{m_j} & \text{if } \mu_{m,j} < 0, \\ (G) & \text{otherwise,} \end{cases} \tag{2.6}$$

where $\deg_{\mathcal{V}_{m,j}} \in A(G)$ is the basic degree associated with the irreducible G-representation $\mathcal{V}_{m,j}$ (see Appendix) and $(G) \in A(G)$ is the unit element of the Burnside ring. In addition, since each basic degree $\deg_{\mathcal{V}_{m,j}}$ is involutive in the Burnside ring (see Appendix), one has

$$(\deg_{\mathcal{V}_{m,j}})^{m_j} = \begin{cases} \deg_{\mathcal{V}_{m,j}} & \text{if } 2 \nmid m_j, \\ (G) & \text{otherwise.} \end{cases} \tag{2.7}$$

Putting together (2.5)–(2.7), we introduce some notations to keep track of the indices:

$$\Sigma := \big\{(m,j) : m \in \mathbb{N} \cup \{0\},\ j \in \{0,1,\dots,r\}\big\},$$

which contribute nontrivially to G-deg$(\mathscr{A}, B(\mathscr{H}))$. To begin, the *negative* spectrum of $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is accounted for with the index set

$$\Sigma_- := \{(m,j) \in \Sigma : \mu_{m,j} < 0\}.$$

Combining this with formulas (2.5) and (2.6) yields

$$G\text{-deg}(\mathscr{A}, B(\mathscr{H})) = \prod_{(m,j)\in\Sigma_-} (\deg_{\mathcal{V}_{m,j}})^{m_j}.$$

This computation can be further reduced by discarding the indices $(m,j) \in \Sigma$ associated with even $\mathcal{V}_j$-isotypic multiplicities m_j, whose corresponding basic degrees contribute trivially to the Burnside product (2.5). Specifically, we put

$$\Sigma_0 := \{(m,j) \in \Sigma_- : 2 \nmid m_j\},$$

which, together with (2.7), permits us to express G-deg$(\mathscr{A}, B(\mathscr{H}))$ as the Burnside ring product of a finite number of unique basic degrees, as follows:

$$G\text{-deg}(\mathscr{A}, B(\mathscr{H})) = \prod_{(m,j)\in\Sigma_0} \deg_{\mathcal{V}_{m,j}}. \tag{2.8}$$

3. Orbit types of maximal kind. It is always possible to study the nontriviality of the product of a finite number of Burnside ring elements (such as the product of basic degrees in the expression (2.8)) as a problem concerning the nontriviality of the product of a finite number of generator pairs $(K), (L) \in \Phi_0(G)$. Keeping in mind that the nontriviality of

$$(K) \cdot (L) \in A(G),$$

is equivalent to the existence of an orbit type $(H) \in \Phi_0(G) \setminus \{(G)\}$ with $\operatorname{coeff}^H((K) \cdot (L)) \neq 0$, notice that the recurrence formula for the Burnside ring product (see Appendix)

$$C^H(K,L) = \frac{n(H,L)|W(L)|n(H,K)|W(K)| - \sum_{(\tilde{H})>(H)} C^{\tilde{H}}(K,L)n(H,\tilde{H})|W(\tilde{H})|}{|W(H)|}$$

(here, we have adopted the ad hoc notation $C^H(K,L) := \operatorname{coeff}^H((K) \cdot (L))$ due to constraints of space), is only useful if one first identifies the set of all orbit types $(\tilde{H}) \in \Phi_0(G) \setminus \{(G)\}$ with $(\tilde{H}) > (H)$. Therefore, computation of the coefficients $\operatorname{coeff}^H((K)\cdot(L))$ is generally impractical except for those orbit types which are maximal in $\Phi_0(G)$, i.e. for orbit types $(H) \in \Phi_0(G)$ where $(\tilde{H}) > (H)$ implies $(\tilde{H}) = (G)$, in which case the recurrence formula simplifies to

$$\operatorname{coeff}^H((K)\cdot(L)) = \frac{n(H,L)|W(L)|n(H,K)|W(K)|}{|W(H)|}.$$

REMARK 3.1. One of the principal difficulties when working with a product group of the form $G = O(2) \times \Gamma \times \mathbb{Z}_2$ is that the conjugacy class of the subgroup $O(2) \times \{e_\Gamma\} \times \{1\} \le G$ is always a generator of the Burnside ring $A(G)$. Since any orbit type (H) which is maximal in the isotropy lattice $\Phi_0(G; \mathscr{H} \setminus \{0\})$ must also be maximal in the sublattice $\Phi_0(G; \mathscr{H}_0 \setminus \{0\})$, this implies that any nonzero function $u \in \mathscr{H}$ with an isotropy group $G_u \le G$ such that (G_u) is a maximal element in $\Phi_0(G; \mathscr{H} \setminus \{0\})$ must be $O(2)$-invariant. Therefore, any map $u \in \mathscr{H}$ which is both nontrivial and nonconstant has an isotropy group $G_u \le G$ with a conjugacy class that is maximal in $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ for some positive m.

Motivated by Remark 3.1, the conjugacy class of a subgroup $H \le G$ is called an *orbit type of maximal kind* if it is a maximal element in $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ for some $m > 0$. Moreover, we denote by $\mathfrak{M}$ the set of all orbit types of maximal kind, by $\mathfrak{M}_m$ the set of maximal elements in $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ and by $\mathfrak{M}_{m,j}$ the set of orbit types $\mathfrak{M}_m \cap \Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$. Since each orbit type of maximal kind $(H) \in \mathfrak{M}_m$ is contained in the sublattice of orbit types $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$ for at least one $j \in \{0, 1, \dots, r\}$, we have

$$\mathfrak{M} = \bigcup_{m>0} \mathfrak{M}_m, \qquad \mathfrak{M}_m = \bigcup_{j=0}^{r} \mathfrak{M}_{m,j},$$

corresponding to the $\Gamma \times \mathbb{Z}_2$-isotypic decomposition of the mth $O(2)$-isotypic component of $\mathscr{H}$:

$$\mathscr{H}_m = \mathscr{H}_{m,1} \oplus \cdots \oplus \mathscr{H}_{m,r}.$$

At this point, we can strengthen the sufficient condition for the existence of a nontrivial solution to (2.1) stated in Lemma 1.1 to a sufficient condition for the existence of a nonstationary solution:

LEMMA 3.2. *If for some orbit type of maximal kind* $(H) \in \mathfrak{M}$ *one has*

$$\operatorname{coeff}^H(G\text{-deg}(\mathscr{A}, B(\mathscr{H}))) \neq 0,$$

then there exists a nonstationary function $u \in \mathscr{H}$ *satisfying* (2.1).

3.1. The s-folding homomorphism. In this section, we introduce the concept of s-folding, which plays a key role in our study of orbit types of maximal kind. For each $s \in \mathbb{N}$, we define the *s-folding homomorphism* by

$$\phi_s : O(2) \to O(2)/\mathbb{Z}_s \simeq O(2), \quad \phi_s(e^{i\theta}) := e^{is\theta}, \quad \phi_s(\kappa e^{i\theta}) := \kappa e^{is\theta}.$$

There is a natural extension of ϕ_s to the Lie group homomorphism $\psi_s : O(2) \times \Gamma \times \mathbb{Z}_2 \to O(2) \times \Gamma \times \mathbb{Z}_2$ given by

$$\psi_s(e^{i\theta}, \gamma, \pm 1) := (\phi_s(e^{i\theta}), \gamma, \pm 1), \quad \psi_s(\kappa e^{i\theta}, \gamma, \pm 1) := (\phi_s(\kappa e^{i\theta}), \gamma, \pm 1). \tag{3.1}$$

In turn, (3.1) induces the Burnside ring homomorphism $\Psi_s : A(G) \to A(G)$ defined as follows:

$$\Psi_s(H) := ({}^sH), \quad {}^sH := \psi_s^{-1}(H). \tag{3.2}$$

Clearly, one has the following relation between the sets $\mathfrak{M}_m$ and $\mathfrak{M}_{sm}$ for all $s > 0$ and $m > 0$:

$$\Psi_s(\mathfrak{M}_m) = \mathfrak{M}_{sm}.$$

Balanov et al. hypothesize in [4]—where it serves as an essential component in the proof of their main result—that every orbit type $(H) \in \Phi_0(G; \mathscr{H}_m) \setminus \{(G)\}$ satisfies the relation

$$({}^sH) > (H) \quad \text{for all } s \in \mathbb{N}. \tag{3.3}$$

In this paper, we demonstrate that the relation (3.3) does not hold universally. Instead, we prove that the relation between (H) and $({}^sH)$ depends both on $(H) \in \Phi_0(G; \mathscr{H}_m)$ and $s \in \mathbb{N}$.

4. A classification of generators for the Burnside ring associated with the group $G = O(2) \times \Gamma \times \mathbb{Z}_2$. In order to proceed with our computations in the Burnside ring, we must employ the convention of *amalgamated notation*: a shorthand for the specification of subgroups in a product group, first considered by Balanov et al. [7]. It is a well known consequence of Goursat's Lemma [15] that any closed subgroup in the product group $G = O(2) \times \Gamma \times \mathbb{Z}_2$ can be identified, up to its conjugacy class in $\Phi_0(G)$, with a quintuple $(K_O, K_\Gamma, L, \varphi_O, \varphi_\Gamma)$ consisting of two subgroups $K_O \le O(2)$, $K_\Gamma \le \Gamma \times \mathbb{Z}_2$, a group L and a pair of epimorphisms $\varphi_O : K_O \to L$, $\varphi_\Gamma : K_\Gamma \to L$, as follows:

$$K_O {}^{\varphi_O}\times_L^{\varphi_\Gamma} K_\Gamma := \{(x, y) \in K_O \times K_\Gamma : \varphi_O(x) = \varphi_\Gamma(y)\}. \tag{4.1}$$

Let H be a closed subgroup in G with the *amalgamated decomposition* (4.1) and denote by $r(H) \in SO(2)$ the rotation generator in K_O. Since the order of $\varphi_O(r(H)) \in L$ is invariant under conjugation of $r(H)$ in K_O, the following map is well-defined:

$$\mathfrak{m} : \Phi_0(G) \to \mathbb{N}, \quad \mathfrak{m}(H) := |\varphi_O(r(H))|. \tag{4.2}$$

Given any orbit type $(H) \in \Phi_0(G)$ and $s_0 \in \mathbb{N}$, we will demonstrate how the map (4.2) can be used to describe the set

$$\{s \in \mathbb{N} : ({}^sH) \le ({}^{s_0}H)\}\,. \tag{4.3}$$

In turn, for any orbit type of maximal kind $(H) \in \mathfrak{M}$ and for any pair of generators $(K), (L) \in \Phi_0(G)$ we can use the set (4.3) to characterize the coefficients of $({}^sH)$ for all $s \in \mathbb{N}$ in the Burnside ring product

$$(K) \cdot (L) \in A(G).$$

REMARK 4.1. For any orbit type of maximal kind $(H) \in \mathfrak{M}$, the relation $(H) < (\tilde{H})$ implies $(\tilde{H}) = (G)$ or $(\tilde{H}) = ({}^sH)$ for some $s \in \mathbb{N}$. Therefore, since satisfaction of the relations $(H) < (K)$ and $(H) < (L)$ is a prerequisite for the nontriviality $\operatorname{coeff}^H((K) \cdot (L)) \neq 0$, we can restrict our focus to the coefficients of $({}^sH)$ for all $s \in \mathbb{N}$ in Burnside ring products of the form

$$({}^{s_0}H) \cdot ({}^{s_1}H) \in A(G), \qquad s_0, s_1 \in \mathbb{N}.$$

THEOREM 4.2. *For any orbit type $(H) \in \Phi_0(G)$ and any $s_0, s_1 \in \mathbb{N}$ with $s_1 \le s_0$, the corresponding pair of folded orbit types $({}^{s_1}H), ({}^{s_0}H) \in \Phi_0(G)$ satisfy $({}^{s_1}H) \le ({}^{s_0}H)$ if and only if*

$$\mathfrak{m}(H) \,|\, s_0 - s_1 \quad \textit{or} \quad \mathfrak{m}(H) \,|\, s_0 + s_1.$$

Proof. Assume that a subgroup representative $H \in (H)$ has the amalgamated decomposition $K_O \,{}^{\varphi_O}\!\times_L^{\varphi_\Gamma} K_\Gamma$. According to the natural ordering of $\Phi_0(G)$, the relation

$$({}^{s_1}H) \le ({}^{s_0}H) \tag{4.4}$$

is equivalent to the existence of an element $g \in G$ for which

$$g\,{}^{s_1}Hg^{-1} \le {}^{s_0}H. \tag{4.5}$$

Given any $s \in \mathbb{N}$, put ${}^sK_O := \phi_s^{-1}(K_O)$, $\varphi_O^s := \varphi_O \circ \phi_s : {}^sK_O \to L$ and notice that the subgroup ${}^sH \le G$ has the amalgamated decomposition

$$\begin{aligned}
\psi_s^{-1}(K_O \,{}^{\varphi_O}\!\times_L^{\varphi_\Gamma} K_\Gamma) &= \psi_s^{-1}(\{(x,y) \in K_O \times K_\Gamma : \varphi_O(x) = \varphi_\Gamma(y)\}) \\
&= \{(\phi_s^{-1}(x), y) \in \phi_s^{-1}(K_O) \times K_\Gamma : \varphi_O(x) = \varphi_\Gamma(y)\} \\
&= \{(x', y) \in {}^sK_O \times K_\Gamma : \varphi_O^s(x') = \varphi_\Gamma(y)\} \\
&= {}^sK_O \,{}^{\varphi_O^s}\!\times_L^{\varphi_\Gamma} K_\Gamma.
\end{aligned}$$

On the other hand, and without loss of generality, take $g := (a, e_\Gamma, 1) \in O(2) \times \Gamma \times \mathbb{Z}_2$, put $\tilde{K}_O := aK_Oa^{-1} \le O(2)$ and define the epimorphism $\tilde{\varphi}_O : \tilde{K}_O \to L$ by

$$\tilde{\varphi}_O(x) := \varphi_O(a^{-1}xa), \qquad x \in \tilde{K}_O,$$

so that the subgroup $gHg^{-1} \leq G$ can be expressed, using amalgamated notation, as follows:

$$\begin{aligned} gK_O{}^{\varphi_O}\times_L^{\varphi_\Gamma} K_\Gamma g^{-1} &= (a,e_\Gamma,1)\{(x,y)\in K_O\times K_\Gamma : \varphi_O(x)=\varphi_\Gamma(y)\}(a^{-1},e_\Gamma,1)\\ &= \{(axa^{-1},y)\in aK_Oa^{-1}\times K_\Gamma : \varphi_O(x)=\varphi_\Gamma(y)\}\\ &= \{(\tilde{x},y)\in \tilde{K}_O\times K_\Gamma : \tilde{\varphi}_O(\tilde{x})=\varphi_\Gamma(y)\}\\ &= \tilde{K}_O{}^{\tilde{\varphi}_O}\times_L^{\varphi_\Gamma} K_\Gamma. \end{aligned}$$

With these notations, it becomes clear that the subgroups ${}^{s_0}H, {}^{s_1}H \leq G$ satisfy (4.5) for some $g=(a,\gamma,\pm 1)\in O(2)\times\Gamma\times\mathbb{Z}_2$ if and only if $\varphi_O^{s_0}(axa^{-1}) = \varphi_O^{s_1}(x)$ for every $(x,y)\in K_O\times K_\Gamma$ with $\varphi_O(x)=\varphi_\Gamma(y)$ such that the relation (4.4) is equivalent to the existence of an element $a\in O(2)$ satisfying

$$\varphi_O^{s_1}(axa^{-1}) = \varphi_O^{s_0}(x) \quad \text{for all } x\in K_O.$$

Clearly, it is sufficient to consider the behavior of the rotation generator in $r\in K_0$ and the two cases $a\in\{\kappa, e_{O(2)}\}$. If $a=\kappa$, then from $\kappa r\kappa = r^{-1}$, one has

$$\begin{aligned} \kappa\,{}^{s_1}H\kappa \leq {}^{s_0}H &\iff \varphi_O^{s_1}(\kappa r\kappa)=\varphi_O^{s_0}(r)\\ &\iff \varphi_O(r)^{s_0+s_1}=e_L\\ &\iff \mathfrak{m}(H)\,|\,s_0+s_1. \end{aligned}$$

Supposing instead that $a=e_{O(2)}$, one has

$$\begin{aligned} {}^{s_1}H \leq {}^{s_0}H &\iff \varphi_O^{s_1}(r)=\varphi_O^{s_0}(r)\\ &\iff \varphi_O(r)^{s_0-s_1}=e_L\\ &\iff \mathfrak{m}(H)\,|\,s_0-s_1. \ \blacksquare \end{aligned}$$

To simplify the exposition throughout the remainder of this paper, we employ the *Iverson brackets* which map any logical predicate P to the set $\{0,1\}$ according to the rule

$$[P] := \begin{cases} 1 & \text{if } P \text{ is true},\\ 0 & \text{otherwise}, \end{cases} \tag{4.6}$$

and for any set of indices $I\subset\mathbb{N}$, we indicate by $\mathcal{B}_{\mathfrak{m}}(I)$ the Boolean expression

$$\mathcal{B}_{\mathfrak{m}}(I) \equiv \text{“Every pair of indices } x,y\in I \text{ satisfies either } \mathfrak{m}\,|\,(x-y) \text{ or } \mathfrak{m}\,|\,(x+y).\text{”} \tag{4.7}$$

THEOREM 4.3. *Let $(H)\in\mathfrak{M}$ be an orbit type of maximal kind. For any $s_0,s_1\in\mathbb{N}$, one has*

$$\operatorname{coeff}^{{}^sH}(({}^{s_0}H)\cdot({}^{s_1}H)) = \begin{cases} |W(H)| & \textit{if } s=\gcd(s_0,s_1) \textit{ and } [\mathcal{B}_{\mathfrak{m}(H)}(\{s_0,s_1\})],\\ 0 & \textit{otherwise}, \end{cases}$$

equivalently, for any $s \in \mathbb{N}$, *one has*

$$\operatorname{coeff}^{{}^sH}(({}^{s_0}H)\cdot({}^{s_1}H)) = |W(H)|\,[s=\gcd(s_0,s_1)]\,[\mathcal{B}_{\mathfrak{m}(H)}(\{s_0,s_1\})].$$

Proof. Recall that the coefficient $n_L \in \mathbb{Z}$ of an orbit type (L) in the Burnside ring product $({}^{s_0}H)\cdot({}^{s_1}H)$ is defined as the number of orbits of type (L) contained in the G-space $G/{}^{s_0}H \times G/{}^{s_1}H$. In particular, $(L) \in \Phi_0(G)$ is an orbit type in $G/{}^{s_0}H \times G/{}^{s_1}H$ if and only if there exists a group element $g \in G$ such that

$$L = {}^{s_0}H \cap g\,{}^{s_1}Hg^{-1}.$$

Without loss of generality, put $g = (a, e_\Gamma, 1)$ with an arbitrary $a \in O(2)$ and notice that the amalgamated decomposition

$$\begin{aligned}
&{}^{s_0}H \cap g\,{}^{s_1}Hg^{-1}\\
&\quad= \{(x,y) \in ({}^{s_0}K_O \cap a^{s_1}K_Oa^{-1}) \times K_\Gamma : \varphi_O^{s_0}(x) = \varphi_O^{s_1}(a^{-1}xa) = \varphi_\Gamma(y)\}\\
&\quad= ({}^{s_0}K_O \cap {}^{s_1}\tilde{K}_O)\,{}^{\psi}\times_L^{\varphi_\Gamma} K_\Gamma \qquad (\text{where } {}^{s_1}\tilde{K}_O := a^{s_1}K_Oa^{-1})
\end{aligned}$$

describes a subgroup belonging to the conjugacy class $({}^sH)$ if and only if there exists a group element $b \in O(2)$ with

(i) $\varphi_O(bxb^{-1}) = \varphi_O^{s_0}(x) = \varphi_O^{s_1}(axa^{-1})$ for all $x \in K_O$,
(ii) $b^sK_Ob^{-1} = {}^{s_0}K_O \cap a^{s_1}K_Oa^{-1}$.

The equivalence of condition (i) with the truthfulness of the Boolean expression $\mathcal{B}_{\mathfrak{m}(H)}(\{s,s_0,s_1\})$ has already been demonstrated in the proof of Theorem 4.2. On the other hand, the observation

$$K_O \in \{O(2), SO(2), D_n, \mathbb{Z}_n, n \in \mathbb{N}\},$$

together with the facts

$$\mathbb{Z}_{s_0n} \cap \mathbb{Z}_{s_1n} = \mathbb{Z}_{\gcd(s_1,s_0)n} \quad \text{and} \quad D_n = \mathbb{Z}_n \cup \kappa\mathbb{Z}_n,$$

implies that condition (ii) can be equated with $s = \gcd(s_0,s_1)$ (see [6, Example 3.12]).

For convenience, put $d := \gcd(s_0,s_1)$ and notice that, in the case one has $\mathcal{B}_{\mathfrak{m}(H)}(\{d,s_0,s_1\})$, the coefficient $n_d \in \mathbb{Z}$ standing next to the orbit type $({}^dH)$ in the Burnside ring product $({}^{s_0}H)\cdot({}^{s_1}H) \in A(G)$ can be obtained with the recurrence formula (see Appendix) as follows:

$$n_d = \frac{n({}^dH,{}^{s_0}H)|W({}^{s_0}H)|n({}^dH,{}^{s_1}H)|W({}^{s_1}H)|}{|W({}^dH)|},$$

and the result follows from the fact that $|W(H)| = |W({}^sH)|$ for any $s \in \mathbb{N}$ together with the observation that $\mathcal{B}_{\mathfrak{m}(H)}(\{d,s_0,s_1\})$ implies $n({}^dH,{}^{s_0}H) = n({}^dH,{}^{s_1}H) = 1$. ■

5. A characterization of the orbit types of maximal kind in the product of basic degrees. Since the classification put forward in the previous section fully characterizes the orbit types $(\tilde{H})$ with $(\tilde{H}) > (H)$ for any orbit type of maximal kind $(H) \in \mathfrak{M}$, we are now in a position to calculate their coefficients in Burnside ring products of the form (2.8).

Given a basic degree $\deg_{\mathcal{V}_{m,j}} \in A(G)$ corresponding to an index pair $(m,j) \in \Sigma$ with $m > 0$ and an orbit type of maximal kind $(H) \in \mathfrak{M}$, the recurrence formula for the G-equivariant Leray–Schauder degree (see Appendix) implies

$$\deg_{\mathcal{V}_{m,j}} = (G) - y_{m,j}(H) + a_{m,j}, \tag{5.1}$$

where $a_{m,j} \in A(G)$ is such that $\operatorname{coeff}^{{}^sH}(a_{m,j}) = 0$ for all $s \in \mathbb{N}$ and the coefficient $y_{m,j} \in \mathbb{Z}$ is determined by the rule

$$y_{m,j} = \begin{cases} 0 & \text{if } \dim \mathcal{V}^H_{m,j} \text{ is even,} \\ 1 & \text{if } \dim \mathcal{V}^H_{m,j} \text{ is odd and } |W(H)| = 2, \\ 2 & \text{if } \dim \mathcal{V}^H_{m,j} \text{ is odd and } |W(H)| = 1, \end{cases} \tag{5.2}$$

or equivalently (see discussion of the Iverson bracket notation (4.6))

$$y_{m,j} = x_0\, [2 \nmid \dim \mathcal{V}^H_{m,j}],$$

where

$$x_0 = \begin{cases} 1 & \text{if } |W(H)| = 2, \\ 2 & \text{if } |W(H)| = 1. \end{cases}$$

LEMMA 5.1. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any basic degree $\deg_{\mathcal{V}_{m,j}}$ with $(m,j) \in \Sigma$ and $m > 0$ one has the implication*

$$\operatorname{coeff}^H(\deg_{\mathcal{V}_{m,j}}) \neq 0 \implies (H) \in \mathfrak{M}_{m,j}.$$

Proof. Since $\mathcal{V}^H_{m,j} = \{0\}$ for *any* orbit type $(H) \notin \Phi_0(G, \mathscr{H}_{m,j})$, the assertion follows from formula (5.2) together with the assumption that (H) is of maximal kind. ■

For every $s \in \mathbb{N}$, the s-folding homomorphism (3.1) induces the following relation between the basic degrees $\deg_{\mathcal{V}_{m,j}}, \deg_{\mathcal{V}_{sm,j}} \in A(G)$:

$$\Psi_s(\deg_{\mathcal{V}_{m,j}}) = \deg_{\mathcal{V}_{sm,j}}, \tag{5.3}$$

so that, for any orbit type $(H) \in \Phi_0(G)$, one has

$$\operatorname{coeff}^H(\deg_{\mathcal{V}_{m,j}}) = \operatorname{coeff}^{{}^sH}(\deg_{\mathcal{V}_{sm,j}}). \tag{5.4}$$

Using the observations (5.1), (5.3), and (5.4), we will derive results which characterize the behavior of orbit types of maximal kind in the Burnside ring product of a finite collection of basic degrees of the form

$$\{\deg_{\mathcal{V}_{s_k m, j_k}} \in A(G) : s_k \in \mathbb{N}\}_{k=1,\dots,N}, \qquad m > 0,\ j_k \in \{0,1,\dots,r\}. \tag{5.5}$$

To begin, we recall that the coefficient of an orbit type of maximal kind $(H) \in \mathfrak{M}$ is 2-*nilpotent* with respect to the Burnside ring product of any pair of basic degrees $\deg_{\mathcal{V}_{m,j_1}}, \deg_{\mathcal{V}_{m,j_2}} \in A(G)$ with $m > 0$ and $j_1, j_2 \in \{0,1,\dots,r\}$, provided that both $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ are odd. Although this fact has been demonstrated elsewhere (see for example [6, 7, 4, 12]), the method of its proof is instructive for the proofs of subsequent, novel results, and for this reason we include our own version of the argument.

LEMMA 5.2. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any $m > 0$ and $j_1, j_2 \in \{0,1,\dots,r\}$, one has*

$$\operatorname{coeff}^H(\deg_{\mathcal{V}_{m,j_1}} \cdot \deg_{\mathcal{V}_{m,j_2}}) = \begin{cases} 0 & \text{if } \dim \mathcal{V}^H_{m,j_1} \text{ and } \dim \mathcal{V}^H_{m,j_2} \\ & \quad \text{are of the same parity,} \\ -x_0 & \text{else,} \end{cases}$$

or equivalently

$$\operatorname{coeff}^H(\deg_{\mathcal{V}_{m,j_1}} \cdot \deg_{\mathcal{V}_{m,j_2}}) = -x_0\,[2 \nmid \dim \mathcal{V}^H_{m,j_1} + \dim \mathcal{V}^H_{m,j_2}].$$

Proof. Consider the Burnside ring product of the relevant basic degrees,

$$\begin{aligned} \deg_{\mathcal{V}_{m,j_1}} \cdot \deg_{\mathcal{V}_{m,j_2}} &= ((G) - y_{m,j_1}(H) + a_{m,j_1}) \cdot ((G) - y_{m,j_2}(H) + a_{m,j_2}) \\ &= (G) - (y_{m,j_1} + y_{m,j_2} - y_{m,j_1} y_{m,j_2} |W(H)|)(H) + a, \end{aligned}$$

where $a \in A(G)$ is such that $\operatorname{coeff}^{^sH}(a) = 0$ for all $s \in \mathbb{N}$. Now, if $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ are both even, then $y_{m,j_1} = y_{m,j_2} = 0$ and the result follows. On the other hand, if $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ are both odd, then $y_{m,j_1} = y_{m,j_2} = x_0$, so that

$$y_{m,j_1} + y_{m,j_2} - y_{m,j_1} y_{m,j_2} |W(H)| = x_0(2 - x_0 |W(H)|),$$

where in either of the cases: $x_0 = 2$ and $|W(H)| = 1$ or $x_0 = 1$ and $|W(H)| = 2$, one has $2 - x_0|W(H)| = 0$. Suppose instead that $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ are of different parties and one has the corresponding two cases, $y_{m,j_1} = x_0$ and $y_{m,j_2} = 0$ *or* $y_{m,j_1} = 0$ and $y_{m,j_2} = x_0$. Then both imply $y_{m,j_1} + y_{m,j_2} - y_{m,j_1} y_{m,j_2} |W(H)| = x_0$. ∎

Since our computations in the Burnside ring often involve the product of a number of basic degrees, we consider a natural generalization of Lemma 5.2 to an orbit type of maximal kind $(H) \in \mathfrak{M}$ and the Burnside ring product of any finite collection $\{\deg_{\mathcal{V}_{m,j_k}}\}_{k=1}^N$ of basic degrees with $m > 0$ and $j_1, \dots, j_N \in \{0,1,\dots,r\}$.

COROLLARY 5.3. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any finite collection $\{\deg_{\mathcal{V}_{m,j_k}}\}_{k=1}^N$ of basic degrees with $m > 0$ and $j_1, \dots, j_N \in$*

$\{0,1,\dots,r\}$, *one has*

$$\operatorname{coeff}^H\Big(\prod_{k=1}^N \deg_{\mathcal{V}_{m,j_k}}\Big) = \begin{cases} 0 & \text{if } \dim \mathcal{V}^H_{m,j_k} \text{ is odd} \\ & \quad \text{for an even number of } j_k \in \{0,1,\dots,r\}, \\ -x_0 & \text{otherwise,} \end{cases}$$

or equivalently

$$\operatorname{coeff}^H\Big(\prod_{k=1}^N \deg_{\mathcal{V}_{m,j_k}}\Big) = -x_0 \Big[2 \nmid \sum_{k=1}^N \dim \mathcal{V}^H_{m,j_k}\Big].$$

The previous result only concerns the product of basic degrees associated with a fixed Fourier mode. We are also interested in characterizing the behavior of the coefficients of an orbit type of maximal kind $(H) \in \mathfrak{M}$ and its s-foldings $\{({}^sH)\}_{s\in\mathbb{N}} \subset \mathfrak{M}$ in the Burnside ring product of any pair of basic degrees $\deg_{\mathcal{V}_{s_1 m,j_1}}, \deg_{\mathcal{V}_{s_2 m,j_2}} \in A(G)$ with $m > 0$, $j_1, j_2 \in \{0,1,\dots,r\}$ and where $s_1, s_2 \in \mathbb{N}$ satisfy $s_1 \neq s_2$.

LEMMA 5.4. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any pair of basic degrees $\deg_{\mathcal{V}_{s_1 m,j_1}}, \deg_{\mathcal{V}_{s_2 m,j_2}} \in A(G)$ with $m > 0$, $j_1, j_2 \in \{0,1,\dots,r\}$ and $s_1, s_2 \in \mathbb{N}$ where $s_1 \neq s_2$, and for any $s_0 \in \mathbb{N}$, one has*

$$\begin{aligned}\operatorname{coeff}^{s_0 H}(\deg_{\mathcal{V}_{s_1 m,j_1}} \cdot \deg_{\mathcal{V}_{s_2 m,j_2}}) &= -x_0 \,[2 \nmid \dim \mathcal{V}^H_{m,j_0}]\,[s_0 \in \{s_1,s_2\}] \\ &\quad + 2x_0\,[2 \nmid \dim \mathcal{V}^H_{m,j_1} \dim \mathcal{V}^H_{m,j_2}]\,[\gcd(s_1,s_2) = s_0]\,[\mathcal{B}_{\mathfrak{m}(H)}(\{s_0,s_1,s_2\})],\end{aligned}$$

where

$$j_0 := \begin{cases} j_1 & \text{if } s_0 = s_1, \\ j_2 & \text{if } s_0 = s_2. \end{cases}$$

Proof. Consider the Burnside ring product of the relevant basic degrees,

$$\begin{aligned}(5.6)\qquad & \deg_{\mathcal{V}_{s_1 m,j_1}} \cdot \deg_{\mathcal{V}_{s_2 m,j_2}} \\ &= ((G) - y_{m,j_1}({}^{s_1}H) + a_{m,j_1}) \cdot ((G) - y_{m,j_2}({}^{s_2}H) + a_{m,j_2}) \\ &= (G) - y_{m,j_1}({}^{s_1}H) - y_{m,j_2}({}^{s_2}H) + y_{m,j_1} y_{m,j_2}({}^{s_1}H)\cdot({}^{s_2}H) + a \\ &= (G) - y_{m,j_1}({}^{s_1}H) - y_{m,j_2}({}^{s_2}H) \\ &\quad + y_{m,j_1} y_{m,j_2} |W(H)|({}^{d}H)[\mathcal{B}_{\mathfrak{m}(H)}(\{d,s_1,s_2\})] + a,\end{aligned}$$

where $d := \gcd(s_1,s_2)$ and $a \in A(G)$ satisfies $\operatorname{coeff}^{sH}(a) = 0$ for all $s \in \mathbb{N}$. In order to demonstrate each of the boolean conditions, we consider the following cases:

(i) Supposing that $s_0 \notin \{s_1,s_2\}$, if also both $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ are odd *and* $\gcd(s_1,s_2) = s_0$ *and* $\mathcal{B}_{\mathfrak{m}(H)}(\{s_0,s_1,s_2\})$, then

$$\operatorname{coeff}^{s_0 H}(\deg_{\mathcal{V}_{s_1 m,j_1}} \cdot \deg_{\mathcal{V}_{s_2 m,j_2}}) = x_0^2 |W(H)|({}^{s_0}H),$$

and the result follows from the fact that $x_0|W(H)| = 2$. On the other hand, if *either* of $\dim \mathcal{V}^H_{m,j_1}$ and $\dim \mathcal{V}^H_{m,j_2}$ is even *or* $\gcd(s_1, s_2) \neq s_0$, *or* $\mathcal{B}_{\mathfrak{m}(H)}(\{s_0, s_1, s_2\})$ does not hold, then the result follows immediately from the assumption that $s_0 \notin \{s_1, s_2\}$.

(ii) Supposing instead that $s_0 \in \{s_1, s_2\}$, if also $\dim \mathcal{V}^H_{m,j_0}$ is even, then $y_{m,j_0} = 0$ and the result follows from (5.6). If $s_0 \in \{s_0, s_1\}$ with both $\dim \mathcal{V}^H_{m,j_1}$, $\dim \mathcal{V}^H_{m,j_{s_2}}$ odd *and* $s_0 = \gcd(s_1, s_2)$ *and* $\mathcal{B}_{\mathfrak{m}(H)}(\{s_0, s_1, s_2\})$, then (5.6) becomes

$$\deg_{\mathcal{V}_{s_1 m,j_1}} \cdot \deg_{\mathcal{V}_{s_2 m,j_2}} = (G) - x_0(^{s_0}H) + 2x_0(^{s_0}H) + a',$$

where $a' \in A(G)$ is such that $\operatorname{coeff}^{^{sm}H}(a') = 0$, and the result follows. On the other hand, if either of $\dim \mathcal{V}^H_{s_1 m,j_{s_1}}, \dim \mathcal{V}^H_{s_1 m,j_{s_1}}$ is even (in which case $y_{m,j_1} y_{m,j_2} = 0$) or if $\gcd(t,l) \neq s_0$, or if $\mathcal{B}_{\mathfrak{m}(H)}(\{s_0, s_1, s_2\})$ does not hold, then the result follows from (5.6). ∎

What follows is the final generalization of the previous results to an orbit type of maximal kind $(H) \in \mathfrak{M}$ and its s-foldings $\{(^sH)\}_{s\in\mathbb{N}} \subset \mathfrak{M}$ in the Burnside ring product of any finite collection $\{\deg_{\mathcal{V}_{s_k m,j_{s_k}}}\}_{k=1}^N$ of basic degrees with $m > 0$, $j_{s_1}, \dots, j_{s_N} \in \{0, 1, \dots, r\}$ and where $s_1, \dots, s_N \in \mathbb{N}$ are distinct.

THEOREM 5.5. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any finite collection $\{\deg_{\mathcal{V}_{s_k m,j_{s_k}}}\}_{k=1}^N$ of basic degrees with $m > 0$, $j_{s_1}, \dots, j_{s_N} \in \{0, 1, \dots, r\}$ and where $s_1, \dots, s_N \in \mathbb{N}$ are distinct, and for any $s_0 \in \mathbb{N}$, one has*

$$\begin{aligned}\operatorname{coeff}^{^{s_0}H}\Big(\prod_{k=1}^N \deg_{\mathcal{V}_{s_k m,j_{s_k}}}\Big) &= -x_0\,[2 \nmid \dim \mathcal{V}^H_{m,j_{s_0}}]\,[s_0 \in \{s_1, \dots, s_N\}] \\ &\quad + 2x_0 \sum_{\substack{I \subset \{s_1,\dots,s_N\} \\ I \neq \emptyset, \{s_0\}}} (-2)^{|I|-2}\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,[s_0 = \gcd(I)]\,\Big[2 \nmid \prod_{s\in I} \dim \mathcal{V}^H_{m,j_s}\Big],\end{aligned}$$

where, if $s_0 = s_k$ for any $k = 1, \dots, N$, then $j_{s_0} := j_{s_k}$.

Proof. Consider the Burnside ring product of the relevant basic degrees,

$$\prod_{k=1}^N \deg_{\mathcal{V}_{s_k m,j_{s_k}}} = \prod_{k=1}^N \big((G) - y_{m,j_{s_k}}(^{s_k}H) + b_{m,j_{s_k}}\big), \tag{5.7}$$

and also the related Burnside ring product

$$\prod_{k=1}^N \big((G) - y_{m,j_{s_k}}(^{s_k}H)\big). \tag{5.8}$$

Notice that for any $s_0 \in \mathbb{N}$, one has

$$\operatorname{coeff}^{s_0 H}\Big(\prod_{k=1}^{N} \deg_{\mathcal{V}_{s_k m, j_{s_k}}}\Big) = \operatorname{coeff}^{s_0 H}\Big(\prod_{k=1}^{N} \big((G) - y_{m,j_{s_k}}({}^{s_k}H)\big)\Big),$$

and also that (5.8) has the expansion

$$\begin{aligned}\prod_{k=1}^{N} \big((G) - y_{m,j_{s_k}}({}^{s_k}H)\big) &= \sum_{I\subset\{s_1,\dots,s_N\}} \prod_{s\in I} -y_{m,j_s}({}^{s}H)\\ &= \sum_{I\subset\{s_1,\dots,s_N\}} |W(H)|^{|I|-1}\Big(\prod_{s\in I} -y_{m,j_s}\Big)\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,({}^{\gcd(I)}H),\end{aligned}$$

where the notation I is used to indicate a subset of $\{s_1,\dots,s_N\}$ and the expression $\sum_{I\subset\{s_1,\dots,s_N\}}$ describes a summation over all such subsets, including the empty set, in which case we put $\prod_{s\in\emptyset} -y_{m,j_s}({}^{s}H) := (G)$, and the full set. It follows that the coefficient of $({}^{s_0}H) \in \Phi_0(G;\mathscr{H})$ in the Burnside ring product (5.7), which for convenience of notation we denote by

$$n_{s_0} := \operatorname{coeff}^{s_0 H}\Big(\prod_{k=1}^{N} \deg_{\mathcal{V}_{s_k m, j_{s_k}}}\Big),$$

is specified by the formula

$$n_{s_0} = \sum_{\substack{I\subset\{s_1,\dots,s_N\}\\ I\neq\emptyset}} |W(H)|^{|I|-1}\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,[s_0 = \gcd(I)] \prod_{s\in I} -y_{m,s}. \tag{5.9}$$

Now, for any nonempty subset $I \subset \{s_1,\dots,s_N\}$, the product

$$\prod_{s\in I} -y_{m,j_s} = \prod_{s\in I} -x_0\,[2\nmid \dim \mathcal{V}^H_{m,j_s}] \tag{5.10}$$

is determined by the rule

$$\prod_{s\in I} -y_{m,j_s} = \begin{cases} (-x_0)^{|I|} & \text{if } \dim \mathcal{V}^H_{m,j_s} \text{ is odd for all } s\in I,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, (5.9) becomes

$$\begin{aligned}&\sum_{\substack{I\subset\{s_1,\dots,s_N\}\\ I\neq\emptyset}} |W(H)|^{|I|-1}(-x_0)^{|I|}\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,[s_0 = \gcd(I)]\Big[2\nmid \prod_{s\in I}\dim \mathcal{V}^H_{m,j_s}\Big]\\ &\quad= -x_0\,[2\nmid \dim \mathcal{V}^H_{m,j_0}]\,[s_0 \in \{s_1,\dots,s_N\}]\\ &\qquad+ 2x_0 \sum_{\substack{I\subset\{s_1,\dots,s_N\}\\ I\neq\emptyset,\{s_0\}}} (-2)^{|I|-2}\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,[s_0 = \gcd(I)]\Big[2\nmid \prod_{s\in I}\dim \mathcal{V}^H_{m,j_s}\Big],\end{aligned}$$

where we have used the fact that $x_0|W(H)| = 2$. ∎

The following practical corollary will prove useful in subsequent sections.

Corollary 5.6. *Let* $(H)\in\mathfrak{M}$ *be an orbit type of maximal kind. For any finite collection* $\{\deg_{\mathcal{V}_{s_k m, j_{s_k}}}\}_{k=1}^N$ *of basic degrees with* $m>0$, $j_{s_1},\dots,j_{s_N}\in\{0,1,\dots,r\}$ *and where* $s_1,\dots,s_N\in\mathbb{N}$ *are distinct and for any* $s\in\{s_1,\dots,s_N\}$ *with* $2\nmid\dim\mathcal{V}_{m,j_s}^H$,

$$\operatorname{coeff}^{{}^sH}\Big(\prod_{k=1}^N \deg_{\mathcal{V}_{s_k m, j_{s_k}}}\Big)\neq 0. \tag{5.11}$$

Proof. Indeed, notice that $-x_0+2x_0C\neq 0$ for any $C\in\mathbb{N}$. ∎

6. On the nontriviality of G-deg$(\mathscr{A},B(\mathscr{H}))$. Given an orbit type of maximal kind $(H)\in\mathfrak{M}$ and $s>0$, we put

$$\begin{cases}\Sigma^s(H):=\{(m,j)\in\Sigma_0 : 2\nmid\dim\mathcal{V}_{m,j}^{{}^sH}\}\\ \mathfrak{n}^s(H):=|\Sigma^s(H)|,\end{cases}$$

and

$$S(H):=\{s\in\mathbb{N} : \mathfrak{n}^s(H)\neq 0\}.$$

Remark 6.1. If $(H)\in\mathfrak{M}_{m_0}$ for some $m_0>0$, then $(m,j)\in\Sigma^s(H)$ only if $m=m_0 s$.

We are now in a position to present our main result.

Proposition 6.2. *Let* $(H)\in\mathfrak{M}$ *be an orbit type of maximal kind. For any* $s_0\in\mathbb{N}$, *one has*

$$\begin{aligned}\operatorname{coeff}^{{}^{s_0}H}(G\text{-deg}(\mathscr{A},B(\mathscr{H}))) &= -x_0\,[2\nmid\mathfrak{n}^{s_0}(H)]\,[s_0\in S(H)]\\ &\quad+2x_0\sum_{\substack{I\subset S(H)\\ I\neq\emptyset,\{s_0\}}}(-2)^{|I|-2}\,[\mathcal{B}_{\mathfrak{m}(H)}(I)]\,[s_0=\gcd(I)]\Big[2\nmid\prod_{s\in I}\mathfrak{n}^s(H)\Big].\end{aligned}$$

Proof. For convenience of notation, put

$$\Sigma^0(H):=\{(m,j)\in\Sigma_0 : 2\mid\dim\mathcal{V}_{m,j}^{{}^sH}\text{ for all }s>0\},$$

and define

$$\rho_G^s(H):=\prod_{(m,j)\in\Sigma^s(H)}\deg_{\mathcal{V}_{m,j}}.$$

Enumerating the set $S(H)=\{s_1,\dots,s_N\}$, one clearly has

$$\Sigma_0\setminus\Sigma^0(H)=\bigcup_{k=1}^N\Sigma^{s_k}(H),$$

so that the degree computation (2.8) becomes

$$G\text{-deg}(\mathscr{A}, B(\mathscr{H})) = \rho_G^0(H) \cdot \prod_{k=1}^{N} \rho_G^{s_k}(H).$$

From Lemma 5.2 and Corollary 5.3, one has

$$\rho_G^s(H) = \begin{cases} (G) + b_0 & \text{if } s = 0, \\ (G) - y_s({}^sH) + b_s & \text{if } s > 0, \end{cases}$$

where b_0, b_s satisfy $\operatorname{coeff}^{{}^sH}(b_0) = \operatorname{coeff}^{{}^sH}(b_s) = 0$ for all $s \in \mathbb{N}$ and

$$y_s = \begin{cases} 0 & \text{if } \mathfrak{n}^s(H) \text{ is even}, \\ x_0 & \text{if } \mathfrak{n}^s(H) \text{ is odd}. \end{cases}$$

Assume, without loss of generality, that $(H) \in \mathfrak{M}_{m_0}$ and notice that each of the elements $\rho_G^{s_k}(H)$ has the form of a basic degree corresponding to an index $(s_k m_0, j_{s_k}) \in \Sigma$ satisfying

$$2 \mid \dim \mathcal{V}_{m_0, j_{s_k}}^H + \mathfrak{n}^{s_k}(H),$$

i.e.,

$$\operatorname{coeff}^{{}^{s_k}H}(\deg_{\mathcal{V}_{s_k m_0, j_{s_k}}}) = \operatorname{coeff}^{{}^{s_k}H}(\rho_G^{s_k}(H)).$$

Therefore, the coefficient of any s_0-folding $({}^{s_0}H) \in \mathfrak{M}$ in the degree (2.8) can be computed as follows:

$$\operatorname{coeff}^{{}^{s_0}H}(G\text{-deg}(\mathscr{A}, B(\mathscr{H}))) = \operatorname{coeff}^{{}^{s_0}H}\Big(\prod_{k=1}^{N} \deg_{\mathcal{V}_{s_k m_0, j_{s_k}}}\Big),$$

and the result follows from Theorem 5.5. ■

Just as Corollary 5.6 naturally follows from Theorem 5.5, the following corollary is a direct implication of Proposition 6.2.

Corollary 6.3. *Let $(H) \in \mathfrak{M}$ be an orbit type of maximal kind. For any $s \in \mathbb{N}$ with $2 \nmid n^s(H)$, one has*

$$\operatorname{coeff}^{{}^sH}(G\text{-deg}(\mathscr{A}, B(\mathscr{H}))) \neq 0.$$

In turn, combining Lemma 1.1 with Corollary 6.3 leads immediately to Theorem 1.2.

7. Motivating example: an arrangement of N coupled pendula with dihedral symmetries and subject to nonlinear forcing. We can provide a more definitive prediction of the symmetries expressed by nonstationary solutions to (1.1) than that established in Theorem 1.2 by prescribing an exact coupling relation between the pendula in our configuration. Specifically, let Γ be the dihedral group of order $2N$ such that our full symmetry

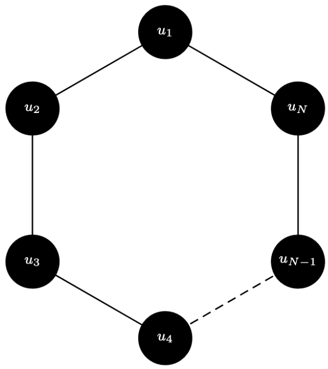

Fig. 1. Cycle of N oscillating pendula with $\Gamma = D_N$ symmetries

group becomes

$$G := O(2) \times D_N \times \mathbb{Z}_2.$$

Our model for this configuration is the system

$$\begin{cases} \ddot{u}(t) = |u|^q u - (L + \mathrm{Id})u, & t \in \mathbb{R},\ u(t) \in V, \\ u(t) = u(t+p),\ \dot{u}(t) = \dot{u}(t+p), \end{cases} \tag{7.1}$$

where $|u|^q u := (|u|^q u_1, \dots, |u|^q u_N)$ for any even $q > 1$ and $L : V \to V$ is the *graph Laplacian matrix* for an undirected graph $\mathbb{G}$ invariant under the permutation action of D_N, with N vertices representing a collection of the same number of oscillating pendula and with edges representing the coupling relations between vertices.

Since the natural permutation representation of Γ on V,

$$\rho_V : \Gamma \to \mathrm{GL}(V), \qquad \rho_V(\sigma)(u_1, \dots, u_N) := (u_{\sigma(1)}, \dots, u_{\sigma(N)}), \tag{7.2}$$

is induced by the permutation actions of the rotation and reflection generators $\gamma, \kappa \in \Gamma$ on the indices $i \in \{1, \dots, N\}$,

$$\gamma(i) := i + 1 \pmod N \quad \text{and} \quad \kappa(i) := N - i \pmod N,$$

its character can be determined according to the rule

$$\chi_V(\gamma) = 0 \quad \text{and} \quad \chi_V(\kappa) = \begin{cases} 1 & \text{if } N \text{ is odd}, \\ 0 & \text{if } N \text{ is even}. \end{cases}$$

The number and character of the irreducible Γ-representations also depend on the dihedral order N: there are always the *trivial representation* $\mathcal{V}_0 \simeq \mathbb{R}$, on which Γ acts trivially, $\lfloor (N+1)/2 \rfloor - 1$ *geometric representations* $\mathcal{V}_j$, each with an action induced by the corresponding matrix representation

$\rho_j : \Gamma \to \mathrm{GL}(\mathbb{C})$,

$$\begin{cases} \rho_j(\kappa) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \rho_j(\gamma) := \begin{pmatrix} \cos(\frac{2j\pi}{N}) & -\sin(\frac{2j\pi}{N}) \\ \sin(\frac{2j\pi}{N}) & \cos(\frac{2j\pi}{N}) \end{pmatrix}, \quad 1 \le j < \lfloor (N+1)/2 \rfloor, \end{cases}$$

the *sign representation* $\mathcal{V}_* \simeq \mathbb{R}$, with the action

$$\sigma \cdot v := \operatorname{sign}(\sigma) v, \qquad v \in \mathcal{V}_*,$$

and, in the case that N is even, two additional irreducible one-dimensional representations $\mathcal{V}_{\lfloor N+1/2 \rfloor}, \mathcal{V}_{**} \simeq \mathbb{R}$ equipped, respectively, with the actions

$$\sigma \cdot v := -\operatorname{sign}(\sigma) v, \qquad v \in \mathcal{V}_{\lfloor (N+1)/2 \rfloor},$$

and

$$\rho_{**}(\kappa) = \rho_{**}(\gamma) = -1.$$

Comparing characters for the irreducible representations of Γ with the character of V

Table 1. Character table for D_N

Conjugacy classes	e_Γ	κ	γ
χ_0	1	1	1
χ_1	2	0	$\cos(\frac{2\pi}{N}) + \cos(\frac{2\pi}{N})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_j	2	0	$\cos(\frac{2j\pi}{N}) + \cos(\frac{2j\pi}{N})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{\lfloor (N+1)/2 \rfloor}$	1	-1	1
χ_*	1	1	-1
χ_{**}	1	-1	-1
χ_V	N	$\frac{1}{2}(1-(-1)^N)$	0

one obtains the relation

$$\chi_V = \begin{cases} \chi_0 + \chi_1 + \cdots + \chi_{(N+1)/2-1} & \text{if } 2 \nmid N, \\ \chi_0 + \chi_1 + \cdots + \chi_{N/2-1} + \chi_{N/2} & \text{if } 2 \mid N, \end{cases} \tag{7.3}$$

implying that V has the Γ-isotypic decomposition

$$V = \begin{cases} V_0 \oplus V_1 \oplus \cdots \oplus V_{(N+1)/2-1} & \text{if } 2 \nmid N, \\ V_0 \oplus V_1 \oplus \cdots \oplus V_{N/2-1} \oplus V_{N/2} & \text{if } 2 \mid N. \end{cases} \tag{7.4}$$

For the sake of generality, we adopt the following notation for the set of Γ-isotypic indices relevant to the Γ-isotypic decomposition of V:

$$\mathfrak{J}(N) := \begin{cases} \{0,1,\dots,(N+1)/2-1\} & \text{if } 2 \nmid N, \\ \{0,1,\dots,N/2-1,N/2\} & \text{if } 2 \mid N. \end{cases}$$

The graph Laplacian associated with the undirected graph $\mathbb{G}$ with N vertices invariant under the permutation action of D_N has the form

$$L = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 1 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

One can verify that such a matrix has the eigenvectors

$$v_j := (1, \gamma^j, \gamma^{2j}, \dots, \gamma^{(N-1)j})^T, \quad j \in \{0,1,\dots,\lfloor (N+1)/2 \rfloor - 1\},\ \gamma := e^{\frac{2\pi i}{N}},$$

corresponding to the eigenvalues

$$-z_j := -2 + \gamma^j + \gamma^{-j} = -4\sin^2\left(\frac{\pi j}{N}\right).$$

And, for N even, one will find that there is also the eigenpair

$$v_{N/2} := (1,-1,1\dots,-1)^T, \quad z_{N/2} = -4.$$

Since each isotypic component of V has simple isotypic multiplicity, i.e.

$$m_j = 1, \quad j \in \mathfrak{J}(N),$$

the eigenvalues of $\mathscr{A}$ (see Definition 2.4) have the form

$$\mu_{m,j} = \frac{m^2 - \beta^2(z_j+1)}{1+m^2}.$$

Therefore, the condition $\mu_{m,j} < 0$ is equivalent to the condition

$$4\sin^2\left(\frac{\pi j}{N}\right) > \frac{m^2}{\beta^2} + 1$$

and Theorem 1.2 becomes:

THEOREM 7.1. *Take $m > 0$ and let (H) be a maximal element in the isotropy lattice $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$. For the symmetry group $\Gamma = D_N$ and under the assumptions (A_0)–(A_4), if the relations*

$$4\sin^2\left(\frac{\pi j}{N}\right) > \frac{m^2}{\beta^2} + 1 \quad \textit{and} \quad 2 \nmid \dim \mathcal{V}^H_{m,j}$$

are satisfied for an odd number of D_N isotypic indices $j \in \mathfrak{J}(N)$, then there exists a nonstationary solution $u \in \mathscr{H} \setminus \{0\}$ to the system (7.1) *with an isotropy subgroup $G_u \le G$ satisfying $(G_u) \ge (H)$.*

7.1. The special case of $\Gamma = D_8$. While the framework described in Section 7 applies to any number of pendula, we can explicitly identify spatio-temporal symmetries of the nonstationary solutions to (7.1) by choosing any particular dihedral order. Let us examine the special case of eight pendula, such that our full symmetry group becomes

$$G := O(2) \times D_8 \times \mathbb{Z}_2.$$

Since the nontriviality of the coefficient standing next to an orbit type of maximal kind $(H) \in \mathfrak{M}_m$ in the basic degree $\deg_{\mathcal{V}_{m,j}} \in A(G)$ is equivalent to the odd-dimensionality of the associated H-fixed point space $\mathcal{V}_{m,j}^H$, the full power of Theorem 7.1 can be demonstrated using a complete list of maximal elements in the isotropy lattice $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ for some fixed $m > 0$ together with computations of the relevant basic degrees $\deg_{\mathcal{V}_{m,j}}$ for $j \in \{0, 1, 2, 3, 4\}$. This can be achieved for $m = 1$ (and then extended to any $m \in \mathbb{N}$ via s-folding, see Section 3.1) by the following GAP code:

```
# A G.A.P. Program for the computation of Maximal Orbit Types and Basic Degrees associated with the G-isotypic decomposition
# V = V_0 \times V_1 \times V_2 \times V_3 \times V_4
LoadPackage("EquiDeg");
# create groups O(2)xD8xZ2
o2:=OrthogonalGroupOverReal(2);
d8:=pDihedralGroup(8);
z2:=pCyclicGroup(2);
# generate D8xZ2
g1:=DirectProduct(d8,z2);
# set names for subgroup conjugacy classes in D8xZ2
SetCCSsAbbrv(g1,["Z1","Z2","D1t","D1z","D1","Z1m","Z1p","D1zt","D1pt","D1p","D2","Z4","D2t","D2zt","D2d","Z4d","D2dt","D2z","Z2p","Z4p","D4dt","D2p","D4","D2pt","D4z","D4zt","Z8","Z8d","D4t","D4d","D4p","Z8p","D8","D4pt","D8d","D8z","D8dt","D8p"]);
# generate O(2)xD8xZ2
G:=DirectProduct(o2,g1);
ccs:=ConjugacyClassesSubgroups(G);
# find Maximal Orbit Types and Basic Degrees in first O(2)-isotypic component (these characterize the maximal orbit types in all O(2)-isotypic components via s-folding)
irrs := Irr(G);
# The D_8-isotypic components 0,1,2,3,4 are indexed in GAP as 1,9,11,13,7.
#Looping through these indices we obtain the desired data as
```

```
    follows:
for i in [1,9,11,13,7] do
    PrintFormatted( "\n Computing Maximal Orbit Types in M_1,{}
    \n", i );
   # The maximal orbit types in the isotypic component H_{m,j}
    and the corresponding basic degrees are obtained via the
    commands MaximalOrbitTypes( irrs[m,j] ); and BasicDegree(
    irrs[m,j] );
    max_orbtyps := MaximalOrbitTypes( irrs[1,i] );
    basic_deg := BasicDegree( irrs[1,i] );
    View(basic_deg);
    Print("\n");
    View(max_orbtyps);
od;
Print( "Done!\n" );
```

In the GAP package EquiDeg, developed for equivariant degree computations in the Burnside ring ([23]), the conjugacy class of an amalgamated subgroup (Section 4)

$$H = K_O {}^{\varphi_O}\times_L^{\varphi_\Gamma} K_\Gamma$$

is identified by the quadruple (K_1, K_2, Z_1, Z_2) where $Z_1 := \operatorname{Ker}\varphi_1$ and $Z_2 := \operatorname{Ker}\varphi_2$ with the notation

$$(H) =: (K_1 {}^{Z_1}\times^{Z_2} K_2).$$

Using this modified amalgamated notation, we can express the output of the above program, i.e. the maximal elements in the isotropy lattices $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\})$ for $j \in \{0,1,2,3,4\}$ and the corresponding basic degrees, as follows:

$$\begin{aligned}
\mathfrak{M}_{m,0} &= \{(D_m \times D_8^p)\},\\
\mathfrak{M}_{m,1} &= \{(D_{4m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_4^d} D_8^p),\ (D_{2m} {}^{D_m}\times^{\tilde{D}_4^d} \tilde{D}_4^p),\ (D_{2m} {}^{D_m}\times^{D_4^d} D_4^p)\},\\
\mathfrak{M}_{m,2} &= \{(D_{2m} {}^{D_m}\times^{D_2^d} D_2^p),\ (D_{2m} {}^{D_m}\times^{\tilde{D}_2^d} \tilde{D}_2^p),\ (D_{8m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_2^-} D_8^p)\},\\
\mathfrak{M}_{m,3} &= \{(D_{2m} {}^{D_m}\times^{D_1^p} D_2^p),\ (D_{2m} {}^{D_m}\times^{\tilde{D}_1^p} \tilde{D}_2^p),\ (D_{8m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_1^p} D_8^p)\},\\
\mathfrak{M}_{m,4} &= \{(D_{2m} {}^{D_m}\times^{\tilde{D}_4^p} D_8^p)\},
\end{aligned}$$

and

$$\begin{aligned}
\deg_{\mathcal{V}_{m,0}} &= (G) - (D_m \times D_8^p),\\
\deg_{\mathcal{V}_{m,1}} &= (G) + 2(D_{2m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_4^d} \tilde{D}_4^p) + 2(D_{2m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_4^d} D_4^p) + (D_{2m} {}^{D_m}\times^{\mathbb{Z}_4^d} \mathbb{Z}_4^p)\\
&\quad - 2(D_{4m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_4^d} D_8^p) - (D_{2m} {}^{D_m}\times^{\tilde{D}_4^d} \tilde{D}_4^p) - (D_{2m} {}^{D_m}\times^{D_4^d} D_4^p),\\
\deg_{\mathcal{V}_{m,2}} &= (G) + 2(D_{2m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_2^-} D_2^p) + 2(D_{2m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_2^-} \tilde{D}_2^p) + (D_{2m} {}^{D_m}\times^{\mathbb{Z}_2^-} \mathbb{Z}_2^p)\\
&\quad - (D_{2m} {}^{D_m}\times^{D_2^d} D_2^p) - (D_{2m} {}^{D_m}\times^{\tilde{D}_2^d} \tilde{D}_2^p) - 2(D_{8m} {}^{\mathbb{Z}_m}\times^{\mathbb{Z}_2^-} D_8^p),
\end{aligned}$$

$$\begin{aligned}\deg_{\mathcal{V}_{m,3}} &= (G) + 2(D_{2m}{}^{\mathbb{Z}_m}\times^{\mathbb{Z}_1^p} D_2^p) + 2(D_{2m}{}^{\mathbb{Z}_m}\times^{\mathbb{Z}_1^p} \tilde{D}_2^p) + (D_{2m}{}^{D_m}\times^{\mathbb{Z}_1^p} \mathbb{Z}_2^p)\\ &\quad - (D_{2m}{}^{D_m}\times^{D_1^p} D_2^p) - (D_{2m}{}^{D_m}\times^{\tilde{D}_1^d} \tilde{D}_2^p) - 2(D_{8m}{}^{\mathbb{Z}_m}\times^{\mathbb{Z}_1^p} D_8^p),\\ \deg_{\mathcal{V}_{m,4}} &= (G) - (D_{2m}{}^{D_m}\times^{\tilde{D}_4^p} D_8^p).\end{aligned}$$

In addition to the notation defined above, we have adopted the following ancillary shorthand for identification of subgroups in D_8 generated by the rotation $\gamma := e^{2\pi i/8}$ and the relection κ:

$$\begin{aligned}
&\mathbb{Z}_1 := \{1\},\\
&\mathbb{Z}_2 := \{1, \gamma^4\},\\
&\mathbb{Z}_4 := \{1, \gamma^2, \gamma^4, \gamma^6\},\\
&D_1 := \{1, \kappa\},\\
&\tilde{D}_1 := \{1, \kappa\gamma\},\\
&D_2 := \{1, \gamma^4, \kappa, \kappa\gamma^4\},\\
&\tilde{D}_2 := \{1, \gamma^4, \kappa\gamma, \kappa\gamma^5\},\\
&D_4 := \{1, \gamma^2, \gamma^4, \gamma^6, \kappa, \kappa\gamma^2, \kappa\gamma^4, \kappa\gamma^6\},\\
&\tilde{D}_4 := \{1, \gamma^2, \gamma^4, \gamma^6, \kappa\gamma, \kappa\gamma^3, \kappa\gamma^5, \kappa\gamma^7\},\\
&D_8 := \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6, \gamma^7, \kappa, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4, \kappa\gamma^5, \kappa\gamma^6, \kappa\gamma^7\}.
\end{aligned}$$

Likewise, the reader can consult the following list of shorthand to identify the subgroups in $D_8 \times \mathbb{Z}_2$:

$$\begin{aligned}
\mathbb{Z}_1^p &= \{(1,1), (1,-1)\}, \quad \mathbb{Z}_2^- = \{(1,1), (\gamma^4,-1)\},\\
\mathbb{Z}_4^d &:= \{(1,1), (\gamma^2,-1), (\gamma^4,1), (\gamma^6,-1)\},\\
D_1^p &:= \{(1,1), (\kappa,1), (1,-1), (\kappa,-1)\},\\
\tilde{D}_1^p &:= \{(1,1), (\kappa\gamma,1), (1,-1), (\kappa\gamma,-1)\},\\
D_2^d &:= \{(1,1), (\gamma^4,-1), (\kappa,1), (\kappa\gamma^4,-1)\},\\
\tilde{D}_2^d &:= \{(1,1), (\gamma^4,-1), (\kappa\gamma,1), (\kappa\gamma^5,-1)\},\\
D_2^p &:= \{(1,1), (\gamma^4,1), (\kappa,1), (\kappa\gamma^4,1), (1,-1), (\gamma^4,-1), (\kappa,-1), (\kappa\gamma^4,-1)\},\\
\tilde{D}_2^p &:= \{(1,1), (\gamma^4,1), (\kappa\gamma,1), (\kappa\gamma^5,1), (1,-1), (\gamma^4,-1), (\kappa\gamma,-1), (\kappa\gamma^5,-1)\},\\
D_4^d &:= \{(1,1), (\gamma^2,-1), (\gamma^4,1), (\gamma^6,-1), (\kappa,1), (\kappa\gamma^2,-1), (\kappa\gamma^4,1), (\kappa\gamma^6,-1)\},\\
\tilde{D}_4^d &:= \{(1,1), (\gamma^2,-1), (\gamma^4,1), (\gamma^6,-1), (\kappa\gamma,1), (\kappa\gamma^3,-1), (\kappa\gamma^5,1), (\kappa\gamma^7,-1)\},\\
\tilde{D}_4^p &:= \{(1,1), (\gamma^2,1), (\gamma^4,1), (\gamma^6,1), (\kappa\gamma,1), (\kappa\gamma^3,1), (\kappa\gamma^5,1),\\
&\qquad (\kappa\gamma^7,1), (1,-1), (\gamma^2,-1), (\gamma^4,-1), (\gamma^6,-1), (\kappa\gamma,-1),\\
&\qquad (\kappa\gamma^3,-1), (\kappa\gamma^5,-1), (\kappa\gamma^7,-1)\},
\end{aligned}$$

$$\begin{aligned} D_8^p := \{&(1,1),(\gamma,1),(\gamma^2,1),(\gamma^3,1),(\gamma^4,1),(\gamma^5,1),(\gamma^6,1),\\ &(\gamma^7,1),(\kappa,1),(\kappa\gamma,1),(\kappa\gamma^2,1),(\kappa\gamma^3,1),(\kappa\gamma^4,1),\\ &(\kappa\gamma^5,1),(\kappa\gamma^6,1),(\kappa\gamma^7,1),(1,-1),(\gamma,-1),(\gamma^2,-1),(\gamma^3,-1),\\ &(\gamma^6,-1),(\gamma^7,-1),(\kappa,-1),(\kappa\gamma,-1),(\kappa\gamma^2,-1),(\kappa\gamma^3,-1),\\ &(\kappa\gamma^4,-1),(\kappa\gamma^5,-1),(\kappa\gamma^6,-1),(\gamma^4,-1),(\gamma^5,-1),(\kappa\gamma^7,-1)\}.\end{aligned}$$

At this point, Proposition 1.4 follows directly from the observations that (i) the fixed point set $\mathcal{V}_{m,j}^H$ is odd-dimensional for any $(H) \in \mathfrak{M}_{m,j}$, and (ii) the sets $\mathfrak{M}_{m,j}$, $m > 0$, $j \in \{0,1,2,3,4\}$, are pairwise disjoint.

Appendix. The G-equivariant Leray–Schauder degree. Given an isometric Banach G-representation $\mathscr{H}$ of functions taking values in an orthogonal Γ-representation V, the G-equivariant Leray–Schauder degree is a topological tool used to study the solution sets associated with equations of the form

$$\mathscr{F}(u) = 0, \quad u \in \mathscr{H}, \tag{A.1}$$

where $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ is an operator satisfying the following conditions:

- (B_1) $\mathscr{F}$ is a G-equivariant completely continuous field;
- (B_2) there exists a sufficiently large $R > 0$ such that $\mathscr{F}$ is $B_R(0)$-admissibly G-homotopic to the identity operator $\mathrm{Id} : \mathscr{H} \to \mathscr{H}$;
- (B_3) $D\mathscr{F}(0) : \mathscr{H} \to \mathscr{H}$ exists and the operator $\mathrm{Id} - D\mathscr{F}(0) : \mathscr{H} \to \mathscr{H}$ is a G-equivariant compact field;
- (B_4) if $D\mathscr{F}(0) : \mathscr{H} \to \mathscr{H}$ is an isomorphism, there exists a sufficiently small $\epsilon > 0$ such that $\mathscr{F}$ is $B_\epsilon(0)$-admissibly G-homotopic to $D\mathscr{F}(0)$.

Condition (B_1) allows the problem of the existence of solutions for equation (A.1) to be reformulated as a question concerning the nontriviality of the degree $G\text{-deg}(\mathscr{F}, B(\mathscr{H}))$. In turn, conditions (B_2)–(B_4) reduce calculation of $G\text{-deg}(\mathscr{F}, B(\mathscr{H}))$ to a Burnside ring product involving a finite number of computationally simpler G-basic degrees.

Equivariant notation. Let G be a compact Lie group. For any subgroup $H \le G$, we denote by (H) its conjugacy class, by $N(H)$ its normalizer, and by $W(H) := N(H)/H$ its Weyl group in G. The set of all subgroup conjugacy classes in G is denoted by $\Phi(G) := \{(H) : H \le G\}$ and has a natural partial order defined as follows:

$$(H) \le (K) \iff \exists_{g \in G}\ gHg^{-1} \le K.$$

As is possible with any partially ordered set, we extend the natural order over $\Phi(G)$ to a total order, which we indicate by $<$ to differentiate the two relations. Moreover, we put $\Phi_0(G) := \{(H) \in \Phi(G) : W(H) \text{ is finite}\}$ and, for any $(H), (K) \in \Phi_0(G)$, we denote by $n(H,K)$ the number of subgroups $\tilde{K} \leq G$ with $\tilde{K} \in (K)$ and $H \leq \tilde{K}$.

Given a G-space X with an element $x \in X$, we denote by $G_x := \{g \in G : gx = x\}$ the *isotropy group* of x and we call $(G_x) \in \Phi(G)$ the *orbit type* of $x \in X$. Put $\Phi(G,X) := \{(H) \in \Phi_0(G) : (H) = (G_x) \text{ for some } x \in X\}$ and $\Phi_0(G,X) := \Phi(G,X) \cap \Phi_0(G)$. For a subgroup $H \leq G$, the subspace $X^H := \{x \in X : G_x \geq H\}$ is called the *H-fixed-point subspace* of X. If Y is another G-space, then a continuous map $f : X \to Y$ is said to be *G-equivariant* if $f(gx) = gf(x)$ for each $x \in X$ and $g \in G$.

The Burnside ring. The free $\mathbb{Z}$-module $A(G) := \mathbb{Z}[\Phi_0(G)]$ has a natural ring structure when equipped with the multiplicative operation defined, for any pair of generators $(H), (K) \in \Phi_0(G)$, as follows:

$$(H) \cdot (K) := \sum_{(L) \in \Phi_0(G)} n_L(L),$$

and where the coefficients $n_L \in \mathbb{Z}$ are given by the recurrence formula

$$n_L := \frac{n(L,H)|W(H)|n(L,K)|W(K)| - \sum_{(\tilde{L})>(L)} n_{\tilde{L}} n(L,\tilde{L})|W(\tilde{L})|}{|W(L)|}.$$

Any *Burnside ring* element $a \in A(G)$ can be expressed as a formal sum over some finite number of generator elements,

$$a = n_1(H_1) + n_2(H_2) + \cdots + n_N(H_N),$$

and we use the notation

$$\text{coeff}^H(a) = n_H$$

to specify the integer coefficient standing next to the generator element $(H) \in \Phi_0(G)$.

An axiomatic construction of the G-equivariant Leray–Schauder degree. Let $\mathscr{E}$ be any isometric Banach G-representation. A map $f : \mathscr{E} \to \mathscr{E}$ is said to be a *completely continuous G-equivariant field* if it can be expressed in the form

$$f(x) = x - F(x),$$

for some compact G-equivariant map $F : \mathscr{E} \to \mathscr{E}$. Moreover, given an open bounded G-invariant set $\Omega \subset \mathscr{E}$, a completely continuous G-equivariant field is said to be *Ω-admissible* and the pair (f, Ω) is called an *admissible G-pair* if $f(x) \neq 0$ for all $x \in \partial\Omega$. We denote by $\mathscr{M}^G(\mathscr{E})$ the set of all admissible G-pairs in $\mathscr{E}$ and by $\mathscr{M}^G$ the set of all admissible G-pairs defined by taking the union over all isometric Banach G-representations as follows:

$$\mathscr{M}^G := \bigcup_{\mathscr{E}} \mathscr{M}^G(\mathscr{E}).$$

The *G-equivariant Leray–Schauder degree* is defined as the unique map $G\text{-deg} : \mathscr{M}^G \to A(G)$ that assigns to every admissible G-pair (f,Ω) the Burnside ring element

$$G\text{-deg}(f,\Omega) = \sum_{(H)\in\Phi_0(G)} n_H(H), \tag{A.2}$$

satisfying the four degree axioms:

(Existence) If $n_H \neq 0$ for some $(H) \in \Phi_0(G)$ in (A.2), then there exists $x \in \Omega$ such that $f(x) = 0$ and $(G_x) \geq (H)$.

(Additivity) For any two disjoint open G-invariant subsets Ω_1 and Ω_2 with $f^{-1}(0)\cap\Omega \subset \Omega_1 \cup \Omega_2$, one has

$$G\text{-deg}(f,\Omega) = G\text{-deg}(f,\Omega_1) + G\text{-deg}(f,\Omega_2).$$

(Homotopy) For any Ω-admissible G-homotopy, $h : [0,1]\times V \to V$, one has

$$G\text{-deg}(h_t,\Omega) = \text{constant}.$$

(Normalization) For any open bounded neighborhood of the origin in an isometric Banach G-representation $\mathscr{E}$ with the identity operator $\text{Id} : \mathscr{E} \to \mathscr{E}$, one has

$$G\text{-deg}(\text{Id},\Omega) = (G).$$

The following are two additional properties of the map G-deg which can be derived from the four axiomatic properties defined above (see [7, 6]):

(Multiplicativity) For any $(f_1,\Omega_1),(f_2,\Omega_2) \in \mathcal{M}^G$,

$$G\text{-deg}(f_1\times f_2,\Omega_1\times\Omega_2) = G\text{-deg}(f_1,\Omega_1)\cdot G\text{-deg}(f_2,\Omega_2),$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$.

(Recurrence formula) For an admissible G-pair $(f,\Omega) \in \mathscr{M}^G(\mathscr{E})$, the G-degree (A.2) can be computed using the following recurrence formula:

$$n_H = \frac{\deg(f^H,\Omega^H) - \sum_{(K)>(H)} n_K\, n(H,K)\,|W(K)|}{|W(H)|}, \tag{A.3}$$

where $|X|$ stands for the number of elements in the set X and $\deg(f^H,\Omega^H)$ is the Brouwer degree of the map $f^H := f|_{\mathscr{E}^H}$ on the set $\Omega^H \subset \mathscr{E}^H$.

Computation of the G-equivariant Leray–Schauder degree. Let us denote by $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ the set of all irreducible G-representations and define the ith basic degree as follows:

$$\deg_{\mathcal{U}_i} := G\text{-deg}(-\text{Id}, B(\mathcal{U}_i)).$$

Given any isometric Banach G-representation with a G-isotypic decomposition

$$\mathscr{E} = \bigoplus_{i\in\mathbb{N}} \mathscr{E}_i,$$

and any bounded G-equivariant linear isomorphism $T : \mathscr{E} \to \mathscr{E}$, the Multiplicativity and Homotopy properties of the G-equivariant Leray–Schauder degree, together with Schur's Lemma, imply

$$G\text{-deg}(\mathrm{Id} - T, B(\mathscr{E})) = \prod_{i\in\mathbb{N}} G\text{-deg}(T_i, B(\mathscr{E}_i)) = \prod_{i\in\mathbb{N}} \prod_{\mu\in\sigma_-(T)} (\deg_{\mathcal{U}_i})^{m_i(\mu)},$$

where $T_i = \mathrm{Id} - T|_{\mathscr{E}_i} : \mathscr{E}_i \to \mathscr{E}_i$ and $\sigma_-(T)$ denotes the real negative spectrum of T.

Notice that each of the *basic degrees*,

$$\deg_{\mathcal{U}_i} = \sum_{(H)\in\Phi_0(G)} n_H(H),$$

can be practically computed, using the recurrence formula (A.3), as follows:

$$n_H = \frac{(-1)^{\dim \mathcal{U}_i^H} - \sum_{H<K} n_K n(H,K)|W(K)|}{|W(H)|}.$$

The following fact is well-known (see for example [5]).

LEMMA A.1. *For any irreducible G-representation $\mathcal{U}$, the basic degree $\deg_{\mathcal{U}} \in A(G)$ is an involutive element of the Burnside ring, i.e.*

$$(\deg_{\mathcal{U}})^2 = \deg_{\mathcal{U}} \cdot \deg_{\mathcal{U}} = (G).$$

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