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On the Zariski invariant of plane branches

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Dedicated to the memory of Professor Arkadiusz Płoski

Abstract. We show how to obtain the Zariski invariant of a plane branch employing the contact order or the intersection multiplicity with elements in a particular family of curves, and we present some consequences of this result.

1. Introduction. Let $C_f : \{f = 0\}$ be an irreducible singular plane curve (briefly, a *plane branch*) defined by an irreducible convergent power series $f \in \mathbb{C}\{x, y\}$. We denote by $\mathrm{mult}(h)$ the multiplicity of $h \in \mathbb{C}\{x, y\} \setminus \{0\}$, that is, the smallest $s \in \mathbb{N}$ such that $h \in \mathcal{M}^s \setminus \mathcal{M}^{s+1}$, where $\mathcal{M}$ stands for the maximal ideal of $\mathbb{C}\{x, y\}$. Up to a change of coordinates, we may assume that $\{x = 0\}$ is transversal to C_f and $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial, that is, $f = y^n + \sum_{i=1}^{n} c_i(x) y^{n-i}$ where $c_i(x) \in \mathbb{C}\{x\}$ with $\mathrm{mult}(c_i(x)) > i$ and $n = \mathrm{mult}(f) = \deg_y(f)$ is the multiplicity of f. We denote $\mathrm{mult}(C_f) := \mathrm{mult}(f)$.

By the Newton–Puiseux theorem, f admits a root given by

$$\alpha(x) := \sum_{i>n} a_i x^{i/n} \in \mathbb{C}\{x^{1/n}\}.$$

In addition, the zero set of f is $\{\alpha_j(x) := \sum_{i>n} a_i \epsilon_j^i x^{i/n} : \epsilon_j \in \mathbb{U}_n\}$, where $\mathbb{U}_n$ is the multiplicative group of complex nth roots of unity. In this way, we get

$$f(x, y) = \prod_{j=1}^{n} (y - \alpha_j(x)).$$

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Setting $t = x^{1/n}$ we have $\alpha(t^n) = \sum_{i>n} a_i t^i \in \mathbb{C}\{t\}$. The pair

$$\left(t^n, \sum_{i>n} a_i t^i\right) \tag{1.1}$$

is called a *Puiseux parametrization* of C_f. Notice that $f(t^n, \sum_{i>n} a_i t^i) = 0$. In addition, by the Newton–Puiseux theorem, (1.1) is a *primitive* parametrization, that is, n and the elements of $\{i : a_i \neq 0\}$ do not admit a non-trivial common divisor.

Given a Puiseux parametrization (1.1) we define two sequences of integers:

$$\begin{aligned} &\beta_0 := n, && e_0 := n, \\ &\beta_j := \min\{i : a_i \neq 0 \text{ and } i \notin e_{j-1}\mathbb{N}\}, && e_j := \gcd(e_{j-1}, \beta_j), \end{aligned} \tag{1.2}$$

for $j > 0$.

In what follows we denote $m := \beta_1$. Since the parametrization is primitive, there exists an integer $g \geq 1$ such that $e_g = 1$ and the sequences (1.2) are finite.

The sequence $(\beta_i)_{i=0}^g$ is called the *characteristic sequence* of C_f and it determines the topological type of the curve C_f (see [Z32, p. 465]). The set of all irreducible plane curves with the same characteristic sequence $(\beta_i)_{i=0}^g$ or equivalently with the same topological type is denoted by $K(n, m, \beta_2, \ldots, \beta_g)$.

Given $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ we consider the set

$$\Gamma_f := \{\mathrm{I}(f, h) : f \text{ does not divide } h \text{ in } \mathbb{C}\{x, y\}\},$$

where

$$\mathrm{I}(f, h) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\langle f, h\rangle}$$

is the *intersection multiplicity* of C_f and C_h at the origin. We also denote $\mathrm{I}(f, h)$ by $\mathrm{I}(C_f, C_h)$. It follows from the properties of codimension of ideals that Γ_f is an additive semigroup of $\mathbb{N}$, called the *values semigroup* of C_f. Moreover, Γ_f admits a conductor

$$\mu_f = \min\{\gamma \in \Gamma_f : \gamma - 1 \notin \Gamma_f \text{ and } \gamma + k \in \Gamma_f \text{ for any } k \in \mathbb{N}\}$$

and it coincides with the Milnor number of C_f, that is, $\mu_f = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle f_x, f_y\rangle}$.

If $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ then the semigroup Γ_f is finitely generated by $g + 1$ natural numbers $v_0 < v_1 < \cdots < v_g$ and there is a relationship between the sequences $(\beta_i)_{i=0}^g$ and $(v_i)_{i=0}^g$ as follows (see [Z86, Theorem 3.9] for instance):

$$\begin{aligned} &v_0 = \beta_0 = n, \qquad v_1 = \beta_1 = m, \\ &v_j = n_{j-1} v_{j-1} + \beta_j - \beta_{j-1} \text{ for } 2 \leq j \leq g, \quad \text{where} \quad n_{j-1} := \frac{e_{j-2}}{e_{j-1}}. \end{aligned} \tag{1.3}$$

In what follows we write $\Gamma_f = \langle v_0, v_1, \ldots, v_g\rangle := \mathbb{N}v_0 + \mathbb{N}v_1 + \cdots + \mathbb{N}v_g$.

In addition, according to [R09, Proposition 9.15], the conductor of Γ_f can be expressed as $\mu_f = \sum_{i=1}^{g}(n_i - 1)v_i - (v_0 - 1)$.

Let $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ be a plane branch with Puiseux parametrization $(t^n, \sum_{i>n} a_i t^i)$. Zariski [Z66, pp. 785–786] proved that if $a_j \neq 0$, $n < j \neq m$ and $j + n \in \Gamma_f$ then there exists a change of coordinates such that C_f is analytically equivalent to a plane branch with Puiseux parametrization

$$\Big(t^n, \sum_{n<i<j} a_i t^i + \sum_{i>j} a_i' t^i\Big).$$

Moreover, he showed that C_f is analytically equivalent to a plane branch with Puiseux parametrization (t^n, t^m) or

$$\Big(t^n, t^m + b_{\lambda_f} t^{\lambda_f} + \sum_{i>\lambda_f} b_i t^i\Big) \quad \text{with } b_{\lambda_f} \neq 0 \text{ and } \lambda_f + n \notin \Gamma_f. \tag{1.4}$$

The integer λ_f is an analytical invariant (see [Z66, p. 785]) called the *Zariski invariant* of C_f. If C_f is analytically equivalent to (t^n, t^m) we put $\lambda_f = \infty$.

In general, it is not immediate to identify the Zariski invariant directly from any Puiseux parametrization, as shown by the following example.

Example 1.1. Let $C_f \in K(4, 7)$ be given by the Puiseux parametrization

$$\varphi(t) := (t^4, t^7 + t^{10} + t^{12} + bt^{13}).$$

Notice that $10 + 4, 12 + 4 \in \Gamma_f$ and $13 + 4 \notin \Gamma_f$. But we cannot conclude that $\lambda_f = 13$ for any $b \neq 0$.

In fact, taking the change of coordinates

$$\sigma(x, y) = \left(x + \frac{4}{7}y, y - x^3\right)$$

and parameter

$$\rho(t) = t - \frac{1}{7}t^4 - \frac{3}{98}t^7 - \frac{1}{7}t^9$$

we get

$$\psi(t) := \sigma \circ \varphi \circ \rho(t) = \left(t^4 + \left(\frac{4}{7}b - \frac{32}{49}\right)t^{13} + A(t), t^7 + \left(b - \frac{17}{14}\right)t^{13} + B(t)\right),$$

where $A(t), B(t) \in \mathbb{C}\{t\}$ have order greater than 13.

Now after the change of parameter

$$t_1 := t \cdot \left(1 + \left(\frac{4}{7}b - \frac{32}{49}\right)t^9 + \frac{A(t)}{t^4}\right)^{1/4}$$

we find that C_f is analytically equivalent to the plane branch with parametrization

$$\left(t_1^4, t_1^7 + \left(b - \frac{17}{14}\right)t_1^{13} + S(t_1)\right), \tag{1.5}$$

where $S(t_1) \in \mathbb{C}\{t_1\}$ has order greater than 13.

Since any integer $z > 13$ satisfies $z + 4 \in \Gamma_f$, by a change of coordinates and parameter (see [Z66, p. 784]) any term in (1.5) with order greater than 13 can be eliminated, and consequently C_f is analytically equivalent to a plane branch with Puiseux parametrization

$$\left(t^4, t^7 + \left(b - \frac{17}{14}\right)t^{13}\right).$$

In this way, we get $\lambda_f = 13$ if and only if $b \neq \frac{17}{14}$, that is, C_f is analytically equivalent to the plane branch defined by $y^4 - x^7 = 0$ if and only if $b = \frac{17}{14}$.

In this paper we characterize the Zariski invariant by means of the contact order and the intersection multiplicity with a particular family of plane branches and we present some consequences of this result.

2. The Zariski invariant, contact and intersection multiplicity of plane branches. As before, we consider $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ with Puiseux parametrization $(t^n, \sum_{i>n} a_i t^i)$, where $\Gamma_f = \langle n, m, v_2, \ldots, v_g\rangle$ is its values semigroup and λ_f is the Zariski invariant of C_f as introduced in (1.4).

Remark 2.1. If $\Gamma_f = \langle n, m, v_2, \ldots, v_g\rangle$ and $n_i = e_{i-1}/e_i$ for $i = 1, \ldots, g$ in (1.3) then any $z \in \mathbb{Z}$ can be uniquely represented (see [R09, Lemma 9.14]) as

$$z = \sum_{i=0}^{g} s_i v_i \quad \text{with } 0 \leq s_i < n_i \text{ for } 1 \leq i \leq g \text{ and } s_0 \in \mathbb{Z}. \tag{2.1}$$

In particular, an integer $z = \sum_{i=0}^{g} s_i v_i$ as in (2.1) belongs to Γ_f if and only if $s_0 \geq 0$.

Let $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ be a plane branch with $g \geq 2$ and Puiseux parametrization $(t^n, \sum_{i>n} a_i t^i)$. By (1.3), we get $\beta_2 = v_2 + \beta_1 - n_1 v_1 = v_2 + v_1 - m_1 v_0$ with $m_1 := m/e_1 > 2$. In particular, $\beta_2 + v_0 = v_2 + v_1 - (m_1 - 1)v_0$, so $\beta_2 + v_0$ is an integer as in (2.1), with $-(m_1 - 1) = s_0 < 0$, hence $\beta_2 + v_0 \notin \Gamma_f$. By (1.2), we get $a_{\beta_2} \neq 0$ and for $g \geq 2$ we have

$$m < \lambda_f \leq \beta_2.$$

Let us recall the notion of contact order between two branches.

Let C_f and C_h be two plane branches defined by Weierstrass polynomials $f, h \in \mathbb{C}\{x\}[y]$ with $n = \mathrm{mult}(f) = \deg_y(f)$ and $n' = \mathrm{mult}(h) = \deg_y(h)$. If $\{\alpha_i(x) : 1 \leq i \leq n\}$ and $\{\delta_j(x) : 1 \leq j \leq n'\}$ denote the zero set of f and h respectively, then the *contact order* of C_f with C_h is defined as

$$\mathrm{cont}(C_f, C_h) = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n'}} \mathrm{mult}(\alpha_i(x) - \delta_j(x)). \tag{2.2}$$

The following proposition relates the contact order and the intersection multiplicity of two branches:

Proposition 2.2 ([M77, Proposition 2.4]). *Let $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$, let $\Gamma_f = \langle n, m, v_2, \ldots, v_g \rangle$ be its values semigroup and C_h be any plane branch. The following statements are equivalent:*

(i) $\mathrm{cont}(C_f, C_h) = \theta$ *with* $\theta \in \mathbb{Q}$ *and* $\beta_q/n \le \theta < \beta_{q+1}/n$ *for some* $1 \le q \le g$ *(by convention* $\beta_{g+1} = \infty$*).*

(ii) $\dfrac{\mathrm{I}(C_f, C_h)}{\mathrm{mult}(C_h)} = \dfrac{n_q v_q + n\theta - \beta_q}{n_0 n_1 \cdots n_q}$ *(where* $n_0 = 1$*).*

Since the intersection multiplicity and the multiplicity of plane branches are invariant under analytical changes of coordinates it follows that the contact order between two branches is also an invariant of analytical equivalence.

Remark 2.3. A direct application of the contact formula (2.2) shows that for three plane branches C_1, C_2 and C_3, at least two of the three values

$$\mathrm{cont}(C_1, C_2),\ \mathrm{cont}(C_1, C_3),\ \mathrm{cont}(C_2, C_3)$$

are equal and the third one is no smaller than the other two. In addition, according to Płoski [Pl85, Théorème 1.2], at least two of the three values

$$\frac{\mathrm{I}(C_1, C_2)}{\mathrm{mult}(C_1)\mathrm{mult}(C_2)},\ \frac{\mathrm{I}(C_1, C_3)}{\mathrm{mult}(C_1)\mathrm{mult}(C_3)},\ \frac{\mathrm{I}(C_2, C_3)}{\mathrm{mult}(C_2)\mathrm{mult}(C_3)}$$

are equal and the third one is no smaller than the other two. This property is known in the literature as the *triangular inequality.*

Notice that the integers $\beta_0, \beta_1, \ldots, \beta_g$ and $v_0, v_1, \ldots, v_g$ associated to a plane branch $C_f \in K(\beta_0, \beta_1, \ldots, \beta_g)$ are geometrically characterized by

$$\begin{aligned} &\beta_0 = \min\{\mathrm{cont}(C_f, C) : C \text{ is a regular curve}\},\\ (2.3)\qquad &\beta_1 = \max\{\mathrm{cont}(C_f, C) : C \text{ is a regular curve}\},\\ &\beta_i = \max\Big\{\mathrm{cont}(C_f, C) : C \in K\big(\tfrac{\beta_0}{e_{i-1}}, \ldots, \tfrac{\beta_{i-1}}{e_{i-1}}\big)\Big\} \quad \text{for } 2 \le i \le g, \end{aligned}$$

and

$$\begin{aligned} &v_0 = \min\{\mathrm{I}(C_f, C) : C \text{ is a regular curve}\},\\ (2.4)\qquad &v_1 = \max\{\mathrm{I}(C_f, C) : C \text{ is a regular curve}\},\\ &v_i = \max\Big\{\mathrm{I}(C_f, C) : C \in K\big(\tfrac{\beta_0}{e_{i-1}}, \ldots, \tfrac{\beta_{i-1}}{e_{i-1}}\big)\Big\} \quad \text{for } 2 \le i \le g. \end{aligned}$$

In what follows, similar to (2.3) and (2.4), we present a geometric interpretation for the Zariski invariant of a plane branch using the contact order or the intersection multiplicity with elements in a family $\mathcal{B}$ of curves in $K(n_1, m_1)$.

THEOREM 2.4. *Let $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ be a plane branch defined by a Weierstrass polynomial $f \in \mathbb{C}\{x\}[y]$. Then*

$$\lambda_f = n \cdot \max_{C \in \mathcal{B}} \operatorname{cont}(C_f, C) = \max_{C \in \mathcal{B}} \mathrm{I}(C_f, C) - (n_1 - 1)m,$$

where $\mathcal{B} \subset K(n_1, m_1)$ is the set of branches which are analytically equivalent to $y^{n_1} - x^{m_1} = 0$ with $n_1 = n/e_1$ and $m_1 = m/e_1$.

Proof. If the Zariski invariant of C_f is $\lambda_f = \infty$ then, by [Z66, p. 784], we get $e_1 = gcd(n, m) = 1$, that is, $n = n_1$, $m = m_1$ and C_f is analytically equivalent to $y^{n_1} - x^{m_1} = 0$, so $C_f \in \mathcal{B}$ and the theorem follows since $\mathrm{I}(C_f, C_f) = \infty = \operatorname{cont}(C_f, C_f)$.

Suppose that C_f has a finite Zariski invariant λ_f. Then there exists an analytic change of coordinates $\varPhi$ such that $\varPhi(C_f)$ has a Puiseux parametrization (1.4), that is,

$$\Big(t^n, t^m + b_{\lambda_f} t^{\lambda_f} + \sum_{i > \lambda_f} b_i t^i\Big) \quad \text{with } b_{\lambda_f} \neq 0.$$

After Proposition 2.2 and in order to compute $\max_{C \in \mathcal{B}} \mathrm{I}(C_f, C)$, it is enough to determine $\max_{C \in \mathcal{B}} \operatorname{cont}(C_f, C)$.

Notice that $C_h \in \mathcal{B} \subset K(n_1, m_1)$ defined by $h = y^{n_1} - x^{m_1}$ whose Puiseux parametrization is (t^{n_1}, t^{m_1}) is such that $\operatorname{cont}(\varPhi(C_f), C_h) = \lambda_f/n$.

In addition, given $C \in \mathcal{B} \subset K(n_1, m_1)$, if $\operatorname{cont}(\varPhi(C_f), C) > \lambda_f/n = \operatorname{cont}(\varPhi(C_f), C_h)$ then by Remark 2.3 we get $\operatorname{cont}(C_h, C) = \lambda_f/n$. By definition of contact order (see (2.2)), C admits a Puiseux parametrization $(t^{n_1}, t^{m_1} + c_k t^k + \sum_{i>k} c_i t^i)$ with $\operatorname{cont}(C_h, C) = \lambda_f/n = k/n_1$ for some $k > m_1$ and $c_k \neq 0$, that is, $k = \lambda_f/e_1$. Since $C \in \mathcal{B} \subset K(n_1, m_1)$, we must have $k + n_1 \in \langle n_1, m_1 \rangle$, as otherwise k would be the Zariski invariant of C, contradicting $C \in \mathcal{B}$. But in this way, $\lambda_f + n = e_1 k + e_1 n_1 \in \langle n, m \rangle \subseteq \varGamma_f$, which is absurd, because λ_f is the Zariski invariant of C_f. Hence, $\max_{C \in \mathcal{B}} \operatorname{cont}(\varPhi(C_f), C) = \lambda_f/n$.

Notice that for any change of coordinates $\varPhi$ and for every $C \in \mathcal{B}$ we get $\varPhi(C) \in \mathcal{B}$. In particular, $\varPhi(\mathcal{B}) \subseteq \mathcal{B}$ and $\varPhi^{-1}(\mathcal{B}) \subseteq \mathcal{B}$, and consequently $\varPhi(\mathcal{B}) = \mathcal{B}$. Since the contact order is invariant under change of coordinates, we get

$$\begin{aligned}\lambda_f/n &= \max_{C \in \mathcal{B}} \operatorname{cont}(\varPhi(C_f), C)\\ &= \max_{\varPhi^{-1}(C) \in \varPhi^{-1}(\mathcal{B})} \operatorname{cont}(C_f, \varPhi^{-1}(C)) = \max_{C \in \mathcal{B}} \operatorname{cont}(C_f, C).\end{aligned}$$

This finishes the proof of the first equality of the statement.

Since $m < \lambda_f \leq \beta_2$ and $\mathrm{mult}(C) = n_1$ for every $C \in \mathcal{B}$, again by Proposition 2.2 we get

$$\max_{C\in\mathcal{B}} \mathrm{I}(C_f, C) = \begin{cases} (n_1-1)m + \lambda_f & \text{if } \lambda_f < \beta_2, \\ v_2 = (n_1-1)m + \beta_2 & \text{if } \lambda_f = \beta_2, \end{cases}$$

and the theorem follows. ∎

Example 2.5. If C_f is the plane branch with Puiseux parametrization $(t^4, t^7 + t^{10} + t^{12})$ then, according to Example 1.1, we get $\lambda_f = 13$ and the plane branch C_h with parametrization $\left(t^4, t^7 + t^{10} + t^{12} + \frac{17}{14}t^{13}\right)$ is an element in $\mathcal{B} \subset K(4,7)$ such that $\mathrm{cont}(C_f, C_h) = 13/4 = \lambda_f/n$. Consequently, the branch C_h satisfies

$$\mathrm{cont}(C_f, C_h) = \max_{C\in\mathcal{B}} \mathrm{cont}(C_f, C) \quad \text{and} \quad \mathrm{I}(C_f, C_h) = \max_{C\in\mathcal{B}} \mathrm{I}(C_f, C).$$

In [C82, pp. 62–63] Casas-Alvero studied a similar property to Theorem 2.4 using the theory of infinitely near points, although no formula is presented in this context.

Let $C_f \in K(\beta_0, \beta_1, \ldots, \beta_g)$ and $C_h \in K(\beta'_0, \beta'_1, \ldots, \beta'_{g'})$ be two plane branches with values semigroups $\Gamma_f = \langle v_0, v_1, \ldots, v_g\rangle$ and $\Gamma_h = \langle v'_0, v'_1, \ldots, v'_{g'}\rangle$ respectively. Using the definition of contact order and a simple computation, it follows that if $\mathrm{cont}(C_f, C_h) = \theta > \beta_k/\beta_0$ then

$$\frac{e_i}{e'_i} = \frac{\beta_i}{\beta'_i} = \frac{v_i}{v'_i} \quad \text{for } 0 \leq i \leq k, \tag{2.5}$$

where $e'_i = \gcd(\beta'_0, \ldots, \beta'_i) = \gcd(v'_0, \ldots, v'_i)$.

As an application of Theorem 2.4 we will see that (2.5) is also valid for the Zariski invariant.

Proposition 2.6. *Let $f, h \in \mathbb{C}\{x\}[y]$ be two irreducible Weierstrass polynomials defining $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ and $C_h \in K(n', m', \beta'_2, \ldots, \beta'_{g'})$ with λ and λ' their respective Zariski invariants.*

(i) *If $\mathrm{cont}(C_f, C_h) > \lambda/n$ then $\lambda/n = \lambda'/n'$.*
(ii) *If $\mathrm{I}(C_f, C_h) > n' \cdot (n_1-1)m + \lambda/n_1$ then $\lambda/n = \lambda'/n'$.*

Proof. (i) Since $\mathrm{cont}(C_f, C_h) > \lambda/n > m/n$ it follows by (2.5) that

$$n_1 = \frac{n}{e_1} = \frac{n'}{e'_1} \quad \text{and} \quad m_1 = \frac{m}{e_1} = \frac{m'}{e'_1}.$$

By Theorem 2.4 we have

$$\frac{\lambda}{n} = \max_{C\in\mathcal{B}} \mathrm{cont}(C_f, C) \quad \text{and} \quad \frac{\lambda'}{n'} = \max_{C\in\mathcal{B}} \mathrm{cont}(C_h, C),$$

where $\mathcal{B}$ is the set of plane branches which are analytically equivalent to $y^{n_1} - x^{m_1} = 0$.

Since $\mathrm{cont}(C_f, C_h) > \lambda/n$, by Remark 2.3 we get

$$\mathrm{cont}(C_f, C_h) > \mathrm{cont}(C_f, C) = \mathrm{cont}(C_h, C)$$

for any $C \in \mathcal{B}$. So, $\max_{C\in\mathcal{B}} \mathrm{cont}(C_f, C) = \max_{C\in\mathcal{B}} \mathrm{cont}(C_h, C)$, and consequently $\lambda/n = \lambda'/n'$.

(ii) Let us denote $\mathrm{cont}(C_f, C_h) = \theta$. Since $x = 0$ is transversal to C_f and C_h, we get $\theta \geq 1$. We will show that $\theta > \lambda/n$.

By hypothesis we get

$$\mathrm{I}(C_f, C_h) > n' \cdot \frac{(n_1-1)m+\lambda}{n_1}. \tag{2.6}$$

Suppose towards a contradiction that $\theta \leq \lambda/n$. We have the following possibilities:

(a) If $n/n \leq \theta < m/n$ then, by Proposition 2.2, we get $\mathrm{I}(C_f, C_h)/n' = n\theta$. Since $m < \lambda$, we have

$$\mathrm{I}(C_f, C_h) = n'n\theta < n'm = n'\frac{n_1 m}{n_1} < n' \cdot \frac{(n_1-1)m+\lambda}{n_1},$$

which contradicts (2.6).

(b) If $m/n \leq \theta \leq \lambda/n$ and $\lambda < \beta_2$, Proposition 2.2 gives

$$\mathrm{I}(C_f, C_h) = n' \cdot \frac{(n_1-1)m+n\theta}{n_1} \leq n' \cdot \frac{(n_1-1)m+\lambda}{n_1},$$

which is absurd since we have (2.6).

(c) Finally, if $\theta = \lambda/n$ and $\lambda = \beta_2$, by Proposition 2.2 we get

$$\mathrm{I}(C_f, C_h) = n' \cdot \frac{n_2 v_2 + n\theta - \beta_2}{n_1 n_2} = n' \cdot \frac{v_2}{n_1} = n' \cdot \frac{(n_1-1)m+\lambda}{n_1},$$

which is not possible according to (2.6).

So, $\mathrm{cont}(C_f, C_h) = \theta > \lambda/n$ and the result follows by (i). ∎

Example 2.7. Consider the plane branches $C_1 \in K(3,7)$ defined by the Puiseux parametrization (t^3, t^7+t^8) and $C_2 \in K(6,14,17)$ given by

$$f_2 = y^6 - 6x^5y^4 - 2x^7(1+4x)y^3 + 9x^{10}(1-x)y^2 + 6x^{12}(1+x-x^2)y + x^{14}(1-x+10x^2-x^3).$$

As $8+3 \notin \langle 3,7\rangle$, the parametrization of C_1 is given by (1.4) and, by definition, the Zariski invariant of C_1 is $\lambda_1 = 8$.

Since

$$\mathrm{I}(C_1, C_2) = \mathrm{mult}(f_2(t^3, t^7+t^8)) = 45 > 44 = 6 \cdot \frac{(3-1)\cdot 7+8}{3},$$

Proposition 2.6(ii) implies that the Zariski invariant λ_2 of C_2 satisfies $\lambda_2/6 = 8/3$, so $\lambda_2 = 16$.

Up to a change of coordinates we can assume that C_f is given by a Weierstrass polynomial $f = y^n + \sum_{i=1}^{n} c_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ such that

$$n = v_0 = \mathrm{I}(f,x) = \min\{\mathrm{I}(C_f, C) : C \text{ is a regular curve}\},$$
$$m = v_1 = \mathrm{I}(f,y) = \max\{\mathrm{I}(C_f, C) : C \text{ is a regular curve}\}.$$

Let $C_h \in \mathcal{B} \subset K(n_1, m_1)$ be a plane branch analytically equivalent to $y^{n_1} - x^{m_1} = 0$ and such that $\mathrm{I}(C_f, C_h) = (n_1 - 1)m + \lambda_f$, or equivalently $\mathrm{cont}(C_f, C_h) = \lambda_f/n$. Since C_f is given by a Weierstrass polynomial, we can consider C_h defined by a monic polynomial $h \in \mathbb{C}\{x\}[y]$ with degree (and multiplicity) equal to $n_1 = n/e_1$. In addition, systematically applying Euclidean division by h, that is, considering the h-expansion of f, we obtain $A_k \in \mathbb{C}\{x\}[y]$ with $\deg_y(A_k) < n_1$ such that

$$f = h^{e_1} + \sum_{k=0}^{e_1-1} A_k h^k \quad \text{and} \quad \mathrm{I}(f,h) = \mathrm{I}(A_0, h) = (n_1-1)m + \lambda_f.$$

Notice that the conductor μ_h of $\langle n_1, m_1\rangle$ is $\mu_h = (n_1 - 1)(m_1 - 1) < (n_1 - 1)m + \lambda_f$, so any integer $z \geq (n_1-1)m+\lambda_f$ belongs to $\langle n_1, m_1\rangle$. Thus, by Remark 2.1, there exist $p, q \in \mathbb{N}$ with $0 \leq q < n_1$ such that

$$pn_1 + qm_1 = (n_1 - 1)m + \lambda_f = \mathrm{I}(A_0, h).$$

Since $\mathrm{I}(f,h) = (n_1 - 1)m + \lambda_f$, $\mathrm{I}(f,x) = n$ and $\mathrm{I}(f,y) = m$, it follows, by Remark 2.3, that $\mathrm{I}(h,x) = n_1$ and $\mathrm{I}(h,y) = m_1$. In this way, we get $\mathrm{I}(h, x^p y^q) = (n_1 - 1)m + \lambda_f$, and consequently there exist unique $0 \neq c \in \mathbb{C}$ and $h_1 \in \mathbb{C}\{x\}[y]$ with $\deg_y(h_1) < n_1$ such that $A_0 = cx^p y^q + h_1$ and $\mathrm{I}(h, h_1) > (n_1 - 1)m + \lambda_f = pn_1 + qm_1$.

Similarly, we can write $h_1 = \sum_{in_1+jm_1>pn_1+qm_1} a_{i,j}x^i y^j$ with $j < n_1$. We have proved the following result.

Proposition 2.8. *Let $C_f \in K(n, m, \beta_2, \ldots, \beta_g)$ be a plane branch defined by a Weierstrass polynomial $f \in \mathbb{C}\{x\}[y]$ with Zariski invariant λ_f, $\mathrm{I}(f,x) = n$ and $\mathrm{I}(f,y) = m$. There exist $C_h \in \mathcal{B} \subset K(n_1, m_1)$ with $\mathrm{I}(C_f, C_h) = (n_1 - 1)m + \lambda_f = pn_1 + qm_1$ and a unique $0 \neq c \in \mathbb{C}$ such that*

$$f = h^{e_1} + \sum_{k=1}^{e_1-1} A_k h^k + cx^p y^q + \sum_{\substack{in_1+jm_1>pn_1+qm_1\\ j<n_1}} a_{ij}x^i y^j,$$

where $A_k \in \mathbb{C}\{x\}[y]$ with $\deg_y(A_k) < n_1$.

Example 2.9. Consider the plane branch $C_2 \in K(6, 14, 17)$ given by

$$f_2 = y^6 - 6x^5y^4 - 2x^7(1+4x)y^3 + 9x^{10}(1-x)y^2 + \\ + 6x^{12}(1 + x - x^2)y + x^{14}(1 - x + 10x^2 - x^3).$$

By Example 2.7, the Zariski invariant of C_2 is $\lambda_2 = 16$.

Considering the plane branch $C_h \in \mathcal{B} \subset K(3,7)$ given by $h = y^3 - x^7$ with Puiseux parametrization (t^3, t^7), we get

$$f_2(t^3, t^7) = 9t^{44} - 9t^{45} + 6t^{46} - 9t^{47} + 10t^{48} - 6t^{49} - t^{51}.$$

In this way, we obtain

$$\mathrm{I}(C_2, C_h) = \mathrm{mult}(f_2(t^3, t^7)) = 44 = (3-1)\cdot 14 + 16.$$

According to Proposition 2.8, we can write $f_2 = h^{e_1} + \sum_{k=1}^{e_1-1} A_k h^k + cx^p y^q + h_1$, where $A_k \in \mathbb{C}\{x\}[y]$ with $\deg_y(A_k) < n_1$, $pn_1 + qm_1 = 44$, $c \neq 0$ and $h_1 = \sum_{(i,j)} a_{i,j} x^i y^j$, where each (i,j) satisfies $in_1 + jm_1 > 44$ and $j < n_1$. In fact, since $n_1 = 3$, $m_1 = 7$ and $e_1 = 2$, we get $44 = 10 \cdot 3 + 2 \cdot 7$, that is, $p = 10$ and $q = 2$, which yields

$$f_2 = h^2 - (6x^5 y + 8x^8)h + 9x^{10}y^2 + h_1$$

with $h_1 = -9x^{11}y^2 + (6x^{13} - 6x^{14})y - 9x^{15} + 10x^{16} - x^{17}$.

Notice that the plane branch $C_h \in \mathcal{B} \in K(n_1, m_1)$ in Proposition 2.8 is not unique. In fact, consider $C_{h'}$ given by $h' = y^3 - 3x^3y^2 + 3x^6y - x^7 - x^9$ that admits Puiseux parametrization $\varphi(t) = (t^3, t^7 + t^9)$. Since the change of coordinates $\sigma(x,y) = (x, y - x^2)$ is such that $\sigma \circ \varphi(t) = (t^3, t^7)$, it follows that $C_{h'} \in \mathcal{B} \subset K(3,7)$. Moreover, $\mathrm{I}(C_2, C_{h'}) = \mathrm{mult}(f_2(t^3, t^7 + t^9)) = 44 = (3-1)\cdot 14 + 16 = (n_1 - 1)m + \lambda_2$. So, applying Proposition 2.8 we obtain

$$f_2 = (h')^2 + (6x^3y^2 + 3x^6y - 6x^5 - 26x^8 + 11x^9)h' + 9x^{10}y^2 + h_1'$$

with $h_1' = (-69x^{11} + 15x^{12})y^2 + (15x^{13} + 66x^{14} - 24x^{15})y - 27x^{15} + 19x^{16} - 27x^{17} + 10x^{18}$.

In particular, if $C_f \in K(n,m)$ and, considering a change of coordinates, the Puiseux parametrization of C_f is given by (1.4), then $e_1 = 1$, $h = y^n - x^m$ and Proposition 2.8 gives

$$f = y^n - x^m + cx^p y^q + \sum_{\substack{in+jm>pn+qm \\ j<n}} a_{ij} x^i y^j,$$

which is a similar expression to the one considered by Peraire [Pe98].

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