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L^p estimates for discrete Schrödinger equations on $\mathbb{Z}^d$

by

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Abstract. Let Δ_d defined be the discrete Laplacian on $\mathbb{Z}^d, d \geq 1$, defined by

$$\Delta_d f(x) = \sum_{j=1}^{d} -[f(x+e_j) + f(x-e_j) - 2f(x)], \qquad x \in \mathbb{Z}^d,$$

where $\{e_j : j = 1, \ldots, d\}$ is the standard basis for $\mathbb{R}^d$.

The main aim of this paper is to prove the $\ell^p(\mathbb{Z}^d)$ estimates for the solutions to the Schrödinger equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta_{\mathrm{d}} u = 0, \\ u(x,0) = f(x). \end{cases}$$

Our approach is inspired by harmonic analysis methods such as the estimates of joint spectral multipliers and the theory of Hardy spaces associated with the discrete Laplacian.

1. Introduction and the statement of main results. Consider the (continuous) Schrödinger equation on $\mathbb{R}^d$,

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = 0, \\ u(x,0) = f(x), \end{cases} \tag{1.1}$$

where Δ is the Laplacian defined by

$$\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

The solution to (1.1) is given by $u(x,t) = e^{it\Delta} f(x)$. The family $\{e^{it\Delta}\}$ of operators is referred to as the Schrödinger group due to its connection with the Schrödinger equation (1.1). It is well-known that the Schrödinger group $e^{it\Delta}$

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is bounded on $L^2(\mathbb{R}^d)$ and unbounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$. See [17]. However, Lanconelli [22] (see also [25, 26]) showed that $e^{it\Delta}$ maps continuously the Sobolev space $W^s_p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and $s \geq d|1/p - 1/2|$. That is, the operator $(I + \Delta)^{-s}e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and $s \geq d|1/p - 1/2|$.

The main aim of this paper is to prove a similar result for the discrete Laplacian on $\mathbb{Z}^d$. Before coming to the main results, we introduce the framework of the discrete Laplacian. The discrete Laplacian Δ_{d} on $\mathbb{Z}^d$ is defined by

$$\begin{aligned}\Delta_{\mathrm{d}} f(x) &= \sum_{j=1}^{d} -[f(x+e_j) + f(x-e_j) - 2f(x)] \\ &=: \sum_{j=1}^{d} \Delta^j_{\mathrm{d}} f(x), \qquad x \in \mathbb{Z}^d,\end{aligned}$$

where $\Delta^j_{\mathrm{d}} f(x) = -[f(x+e_j) + f(x-e_j) - 2f(x)]$ and $\{e_j : j = 1, \ldots, d\}$ is the standard basis for $\mathbb{R}^d$.

For $0 < p < \infty$, the function space $\ell^p(\mathbb{Z}^d)$ consists of all complex-valued functions on $\mathbb{Z}^d$ satisfying

$$\|f\|_{\ell^p(\mathbb{Z}^d)} = \Big[\sum_{x\in\mathbb{Z}^d} |f(x)|^p\Big]^{1/p} < \infty.$$

For $1 < p < \infty$ and $s \in \mathbb{R}$, we define the Sobolev space $W^s_p(\mathbb{Z}^d)$ to be the set of all functions $f \in \ell^p(\mathbb{Z}^d)$ such that

$$\|f\|_{W^s_p(\mathbb{Z}^d)} = \|(I + \Delta_{\mathrm{d}})^{s/2} f\|_{\ell^p(\mathbb{Z}^d)} < \infty.$$

It was proved in [16] that

$$\|f\|_{W^1_p(\mathbb{Z}^d)} \sim \|f\|_{\ell^p(\mathbb{Z}^d)} + \sum_{j=1}^{d} \|D_j f\|_{\ell^p(\mathbb{Z}^d)},$$

where $D_j f(x) = f(x + e_j) - f(x)$, $j = 1, \ldots, d$.

Consider the discrete Schrödinger equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta_{\mathrm{d}} u = 0, \\ u(x, 0) = f(x). \end{cases} \tag{1.2}$$

The study of discrete Schrödinger equations is an interesting topic in the theory of nonlinear partial differential equations and has many applications ranging from optical lattices in physics to the flow of a chemical in an infinite tank. See for example [6, 7, 2, 18, 21, 20, 29, 31, 27] and the references therein. Our main result is the following theorem.

THEOREM 1.1. *For* $p \in (1, \infty)$ *and* $s \geq d|1/p - 1/2|$, *we have*

$$\|e^{it\Delta_{\mathrm{d}}} f\|_{\ell^p(\mathbb{Z}^d)} \leq C(s,p)(1+|t|)^s \|f\|_{W_p^s(\mathbb{Z}^d)}, \qquad t \in \mathbb{R},$$

where the constant $C(s,p)$ *depends on* s *and* p *only.*

As a consequence of Theorem 1.1, we have the following.

COROLLARY 1.2. *Let* $u(x,t)$ *be the solution to* (1.2). *Then for* $p \in (1,\infty)$ *and* $s \geq d|1/p - 1/2|$,

$$\|u(\cdot,t)\|_{\ell^p(\mathbb{Z}^d)} \leq C(s,p)(1+|t|)^s \|f\|_{W_p^s(\mathbb{Z}^d)}, \qquad t \in \mathbb{R},$$

where the constant $C(s,p)$ *depends on* s *and* p *only.*

Theorem 1.1 is a consequence of the boundedness of the Schrödinger group $e^{it\Delta_{\mathrm{d}}}$ on $\ell^2(\mathbb{Z}^d)$ and the following theorem which gives the estimates on Hardy space for $0 < p \leq 1$.

THEOREM 1.3. *For* $p \in (0,1]$ *and* $s \geq d(1/p - 1/2)$,

$$\|(I+\Delta_{\mathrm{d}})^{-s} e^{it\Delta_{\mathrm{d}}} f\|_{H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)} \leq C(s,p)(1+|t|)^s \|f\|_{H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)}, \qquad t \in \mathbb{R},$$

where $H^p_{\Delta_{\mathrm{d}}}$ *is the Hardy space associated to the discrete Laplacian* Δ_{d} *and the constant* $C(s,p)$ *depends on* s *and* p *only. See Section* 3.

As a consequence, by interpolation, for $p \in (1,\infty)$ *and* $s \geq d|1/p - 1/2|$, *we have*

$$\|(I+\Delta_{\mathrm{d}})^{-s} e^{it\Delta_{\mathrm{d}}}\|_{\ell^p(\mathbb{Z}^d)} \leq C(s,p)(1+|t|)^s \|f\|_{\ell^p(\mathbb{Z}^d)}, \qquad t \in \mathbb{R},$$

where the constant $C(s,p)$ *depends on* s *and* p *only.*

Before going into details, we explain why our main results cannot be obtained by using the Fourier multiplier approach as in the classical case [25, 26]. Recall that for each $j = 1, \dots, d$,

$$\Delta_{\mathrm{d}}^j f(x) = f(x+e_j) + f(x-e_j) - 2f(x), \qquad x \in \mathbb{Z}^d.$$

Obviously, $\|\Delta_{\mathrm{d}}^j\|_{\ell^p(\mathbb{Z}^d)\to\ell^p(\mathbb{Z}^d)} \leq 4$ for each j and $1 \leq p \leq \infty$. Consequently, the spectrum of Δ_{d}^j is contained in $[0,4]$ for each $j = 1, \dots, d$. Since $\{\Delta_{\mathrm{d}}^j : j = 1, \dots, d\}$ is pairwise commutative, the spectral projections of Δ_{d}^j, $j = 1, \dots, d$, commute pairwise. In this case one can define the joint spectral measure $E(\lambda)$ on $[0,\infty)^d$ such that

$$\Delta_{\mathrm{d}}^j = \int\limits_{[0,\infty)^d} \lambda_j \, dE(\lambda) = \int\limits_{[0,\infty)} \lambda_j \, dE_j(\lambda_j), \qquad j = 1, \dots, d,$$

where $E_j(\cdot)$ is the spectral projection of Δ_{d}^j for each $j = 1, \dots, d$.

Let $m : [0,\infty)^d \to \mathbb{C}$ be a Borel measurable function. We define the multivariate spectral multiplier $m(\Delta_{\rm d}^1,\dots,\Delta_{\rm d}^d)$ by

$$m(\Delta_{\rm d}^1,\dots,\Delta_{\rm d}^d) = \int_{[0,\infty)^d} m(\lambda_1,\dots,\lambda_d)\,dE(\lambda).$$

From now on, we will denote

$$\overrightarrow{\Delta}_{\rm d} = (\Delta_{\rm d}^1,\dots,\Delta_{\rm d}^d), \quad t\overrightarrow{\Delta}_{\rm d} = (t\Delta_{\rm d}^1,\dots,t\Delta_{\rm d}^d), \quad t>0.$$

For $f \in \ell^1(\mathbb{Z}^d)$, the discrete Fourier transform $\mathcal{F}_{\rm d}(f)$ of f is defined by

$$\mathcal{F}_{\rm d}(f)(\theta) = \sum_{x\in\mathbb{Z}^d} f(x)e^{ix\cdot\theta}, \quad \theta \in [-\pi,\pi]^d.$$

The inverse operator $\mathcal{F}_{\rm d}^{-1}$ of $\mathcal{F}_{\rm d}$ is determined by

$$\mathcal{F}_{\rm d}^{-1}(\varphi)(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \varphi(\theta)e^{-ix\cdot\theta}\,d\theta, \quad \varphi \in L^2([-\pi,\pi]^d).$$

Let $m \in L^\infty([0,\infty)^d)$. We define

$$\begin{aligned} T_m f &= \mathcal{F}_{\rm d}^{-1}[m(2(1-\cos\theta_1),\dots,2(1-\cos\theta_d))\mathcal{F}_{\rm d}(f)] \\ &= \mathcal{F}_{\rm d}^{-1}[m(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))\mathcal{F}_{\rm d}(f)]. \end{aligned}$$

Recall that in the particular case $d=1$, it is well-known that for every $\theta\in[-\pi,\pi]$,

$$\Delta_{\rm d} e_\theta = 2(1-\cos\theta)e_\theta, \tag{1.3}$$

where $e_\theta(n) = e^{in\theta}$.

Therefore, from (1.3) and the spectral theory, it is straightforward that for $m \in L^\infty([0,\infty)^d)$,

$$m(\overrightarrow{\Delta}_{\rm d})f = T_{\widetilde{m}} f, \tag{1.4}$$

where $\widetilde{m}(\theta) = m(2(1-\cos\theta_1),\dots,2(1-\cos\theta_d))$. Consequently, for $s>0$ and $t\in\mathbb{R}$,

$$(I+\Delta_{\rm d})^{-s}e^{it\Delta_{\rm d}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \Psi(\theta)e^{-ix\cdot\theta}\,d\theta,$$

where for $\theta = (\theta_1,\dots,\theta_d)$,

$$\Psi(\theta) = (1+4\sin^2(\theta_1/2)+\cdots+4\sin^2(\theta_d/2))^{-s}e^{it(4\sin^2(\theta_1/2)+\cdots+4\sin^2(\theta_d/2))}.$$

This symbol is different from $\widetilde{\Psi}(\theta) = (1+|\theta|^2)^{-s}e^{it|\theta|^2}$ corresponding to the Schrödinger groups $(I+\Delta)^{-s}e^{it\Delta}$ of the Laplacian Δ on $\mathbb{R}^d$. Because of this difference, the method to transfer the results regarding the Schrödinger groups $(I+\Delta)^{-s}e^{it\Delta}$ in the continuous setting to the discrete setting as outlined in [33, Theorem 3.8] is not applicable, hence it does not facilitate the proofs of the main results in this paper.

Let us remark that the boundedness of the Schrödinger groups associated with an abstract operator L has been studied intensively in [8, 9, 5]. In these papers, the underlying operator L is assumed to satisfy the Gaussian upper bound or the Davies–Gaffney estimate. In contrast to the approach in [8, 9, 5, 12], our approach relies on the theory of joint spectral multipliers (or multivariate spectral multipliers). See [28, 30]. Note that the joint spectral multiplier theorem was proved in [30] under the Gaussian upper bound conditions for the main operators. However, this is not the case in our setting since it is not clear if the heat kernel of the discrete Laplacian Δ_{d} satisfies the Gaussian upper bound or the Davies–Gaffney estimate even in the case $d = 1$. See for example [10]. Moreover, one of the key elements in the proof of the main results in this paper is the inequality

$$\sum_{x\in\mathbb{Z}^d} |F(\Delta_{\mathrm{d}}^1,\dots,\Delta_{\mathrm{d}}^d)(x,y)|^2(1+R|x-y|)^{2k} \le C(k)R^d\|\delta_{R^2}F\|^2_{W^k_\infty(\mathbb{R}^d)},$$

where $F(\Delta_{\mathrm{d}}^1,\dots,\Delta_{\mathrm{d}}^d)(x,y)$ is the kernel of $F(\Delta_{\mathrm{d}}^1,\dots,\Delta_{\mathrm{d}}^d)(x,y)$ and

$$\|f\|_{W^k_\infty(\mathbb{R}^d)} := \sum_{0<\ell\le k} \|\partial^\ell f\|_{L^\infty(\mathbb{R}^d)}$$

for $k \in \mathbb{N}$. (See the inequality (2.2) in Proposition 2.1 for the precise statement.)

Let us emphasize that the approach in [30] will give the estimate

$$\sum_{x\in\mathbb{Z}^d} |F(\Delta_{\mathrm{d}}^1,\dots,\Delta_{\mathrm{d}}^d)(x,y)|^2(1+R|x-y|)^{2k} \le C(k,\epsilon)R^d\|\delta_{R^2}F\|^2_{\mathcal{W}^{k+\epsilon}_\infty(\mathbb{R}^d)}$$

for any $\epsilon > 0$, and this estimate is not enough to obtain the sharp estimate in Theorem 1.3. Here $\|f\|_{\mathcal{W}^s_\infty(\mathbb{R}^d)} := \|(I-\Delta)^{s/2}f\|_{L^\infty}$ with Δ being the Laplacian on $\mathbb{R}^d$.

The proof of the main result (Theorem 1.3) proceeds in the following steps:

- We first establish general spectral multiplier theorems on $\ell^p(\mathbb{Z}^d)$ and on the Hardy spaces $H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$; see Theorems 2.5 and 4.1. These results, however, do not provide the sharp estimates stated in Theorem 1.3, and thus cannot be applied directly.
- Using Theorem 4.1, we prove Theorem 1.3 for $0 < p \le 1$.
- Theorem 2.5 is then used to establish the coincidence between the Hardy spaces associated with Δ_{d} and the classical spaces $\ell^p(\mathbb{Z}^d)$ for $1 < p < \infty$ (see Theorem 3.5). This identification, together with the complex interpolation argument from [26], yields Theorem 1.3 for $1 < p < \infty$, thereby completing the proof.

The organization of the paper is as follows. In Section 2 we prove some kernel estimates and a joint spectral multiplier theorem. In Section 3, we consider the Hardy spaces associated with the discrete Laplacian Δ_{d}. We also prove the discrete square function characterization and an interpolation theorem of these function spaces. The proofs of the main results will be given in Section 4.

Throughout the paper, we usually use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write $A \lesssim B$ if there is a universal constant C such that $A \leq CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Some kernel estimates and a joint spectral theorem. We begin this section with some notation we will use in the sequel.

Let $E \subset \mathbb{Z}^d$. Denote by $|E|$ the counting measure of E, i.e., $|E| = \sharp E$. For $x \in \mathbb{Z}^d$ and $r > 0$, denote by

$$B(x,r) := \{y \in \mathbb{Z}^d : |x-y| < r\}$$

the ball centered x with radius r. Then we have

$$|B(x,r)| \sim (1+r)^d.$$

We will often use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset X$ we set

$$S_0(B) = B \quad \text{and} \quad S_j(B) = 2^j B \setminus 2^{j-1} B \text{ for } j \geq 1.$$

For each $r > 0$ the Hardy–Littlewood maximal function $\mathcal{M}_r$ is defined by setting

$$\mathcal{M}_r f(x) = \sup_{B \ni x} \bigg(\frac{1}{|B|} \sum_{y \in B} |f(y)|^r \bigg)^{1/r},$$

where the supremum is taken over all balls B containing x.

Since in this case the space $\mathbb{Z}^d$ is a space of homogeneous type in the sense of Coifman and Weiss [11], it is well-known that for $p > r$,

$$\|\mathcal{M}_r f\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^d)} \tag{2.1}$$

for all $f \in \ell^p(\mathbb{Z}^d)$.

2.1. Some kernel estimates. We first prove the following kernel estimate for the joint spectral multiplier which plays a key role in the sequel.

PROPOSITION 2.1. *Let $R \in (0, 2\sqrt{d}]$ and $k \in \mathbb{N}$. Then there exists a constant $C = C(k)$ such that*

(2.2) $$\sum_{x\in\mathbb{Z}^d} |F(\overrightarrow{\Delta}_{\mathrm{d}})(x,y)|^2(1+R|x-y|)^{2k} \le CR^d\|\delta_{R^2}F\|^2_{W^k_\infty(\mathbb{R}^d)}$$

for all $x,y\in\mathbb{Z}^d$ *and all Borel functions* F *such that* $\operatorname{supp} F\subset[0,R^2]^d$, *where* $\delta_t F = F(t\,\cdot)$ *and* $F(\overrightarrow{\Delta}_{\mathrm{d}})(x,y)$ *is the kernel of* $F(\overrightarrow{\Delta}_{\mathrm{d}})$.

In order to prove the proposition, we need the following technical lemma.

LEMMA 2.2. *Let* $R\in(0,2\sqrt{d}]$ *and let* f *be a function supported in* $[0,R^2]$. *Then for each* $k\in\mathbb{N}$, *there exists* $C=C(k)$ *such that*

$$\left|R^k\frac{\partial^k}{\partial\theta^k}f(4\sin^2(\theta/2))\right| \le C\sum_{j=1}^k R^{2j}|f^{(j)}(4\sin^2(\theta/2))|$$

for all $\theta\in[-\pi,\pi]$.

Proof. For $\vec{m}=(m_1,\dots,m_k)\in\mathbb{N}^k$, we denote

$$\begin{aligned}|\vec{m}| &= m_1+\cdots+m_k,\\ |\vec{m}_{\mathrm{odd}}| &= m_1+m_3+\cdots+m_{2\lfloor(k+1)/2\rfloor-1},\\ |\vec{m}_{\mathrm{even}}| &= m_2+m_4+\cdots+m_{2\lfloor k/2\rfloor}.\end{aligned}$$

By Faà di Bruno's formula,

$$\begin{aligned}&\frac{\partial^k}{\partial\theta^k}f(4\sin^2(\theta/2))\\ &\quad=\sum c_{k,m_1,\dots,m_k}f^{(m_1+\cdots+m_k)}(4\sin^2(\theta/2))\prod_{i=1}^k\left(\frac{\partial^i(4\sin^2(\theta/2))}{i!}\right)^{m_i}\\ &\quad=\sum c_{k,m_1,\dots,m_k}f^{(m_1+\cdots+m_k)}(4\sin^2(\theta/2))\prod_{i=1}^k\left(\frac{\partial^i(2(1-\cos\theta))}{i!}\right)^{m_i},\end{aligned}$$

where $c_{k,m_1,\dots,m_k}$ are constants and the sum is taken over all k-tuples $(m_1,\dots,m_k)$ of nonnegative integers satisfying the constraint

$$m_1+2m_2+\cdots+km_k=k.$$

Note that there exists c_0 such that $|\theta|\le c_0R$ whenever $\theta\in[-\pi,\pi]$ and $4\sin^2(\theta/2)\le R^2$. This, along with the fact that $|\sin\theta|\le|\theta|$ and $|\cos\theta|\le 1$, implies that

$$\prod_{i=1}^k\left(\frac{\partial^i(2(1-\cos\theta))}{i!}\right)^{m_i}\lesssim|\theta|^{|\vec{m}_{\mathrm{odd}}|}\lesssim R^{|\vec{m}_{\mathrm{odd}}|}.$$

Therefore,

$$
\begin{aligned}
&\left|R^k\frac{\partial^k}{\partial\theta^k}f(4\sin^2(\theta/2))\right|\\
&\qquad\lesssim \sum_{m_1+2m_2+\cdots+km_k=k} R^{k+|\vec{m}_{\rm odd}|}|f^{(m_1+\cdots+m_k)}(4\sin^2(\theta/2))|\\
&\qquad\lesssim \sum_{m_1+2m_2+\cdots+km_k=k} R^{2(m_1+\cdots+m_k)}|f^{(m_1+\cdots+m_k)}(4\sin^2(\theta/2))|\\
&\qquad\lesssim \sum_{j=1}^k R^{2j}|f^{(j)}(4\sin^2(\theta/2))|,
\end{aligned}
$$

where in the second inequality we have used the fact that $R\in(0,2\sqrt{d}]$ and the implicit constant depends on k. ∎

We are ready to give the proof of Proposition 2.1.

Proof of Proposition 2.1. From (1.4),

$$
F(\overrightarrow{\Delta}_{\rm d})(x,y)=\frac{1}{(2\pi)^d}\int\limits_{[-\pi,\pi]^d} F(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))e^{-i(x-y)\cdot\theta}\,d\theta
$$

for $x,y\in\mathbb{Z}^d$.

For each $j\in\{1,\dots,d\}$, by integration by parts we have, for $x,y\in\mathbb{Z}^d$,

$$
\begin{aligned}
(2.3)\qquad &F(\Delta^1_{\rm d},\dots,\Delta^d_{\rm d})(x,y)(x_j-y_j)^k\\
&=\frac{1}{(2\pi)^d}\int\limits_{[-\pi,\pi]^d}(-i)^k\frac{\partial^k}{\partial\theta_j^k}F(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))e^{-i(x-y)\cdot\theta}\,d\theta\\
&=\mathcal{F}_{\rm d}^{-1}\bigg((-i)^k\frac{\partial^k}{\partial\theta_j^k}F(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))\bigg)(x-y).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}|F(\overrightarrow{\Delta}_{\rm d})(x,y)|^2|x_j-y_j|^{2k}\\
&\qquad=\left\|(-i)^k\frac{\partial^k}{\partial\theta_j^k}F(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))\right\|^2_{L^2([-\pi,\pi]^d)}.
\end{aligned}
$$

Recall that there exists c_0 such that $|\theta_j|\le c_0R$ for all $j=1,\dots,d$ whenever $\theta_j\in[-\pi,\pi]$ and $4\sin^2(\theta_j/2)\le R^2$. This, along with Lemma 2.2, yields

$$
\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}|F(\overrightarrow{\Delta}_{\rm d})(x,y)|^2|x_j-y_j|^{2k}\\
&\qquad\le\left\|(-i)^k\frac{\partial^k}{\partial\theta_j^k}F(4\sin^2(\theta_1/2),\dots,4\sin^2(\theta_d/2))\right\|^2_{L^2([-\pi,\pi]^d)}\\
&\qquad\lesssim R^{2d}\sum_{\ell=1}^k[R^{2\ell}\|\partial_j^\ell F\|_\infty]^2\lesssim R^{2d}\|\delta_{R^2}F\|^2_{W^k_\infty(\mathbb{R}^d)}.
\end{aligned}
$$

This completes our proof. ∎

As a byproduct, we obtain the following kernel estimate.

PROPOSITION 2.3. *Let $R \in (0, 2\sqrt{d}]$ and $k \in \mathbb{N}$. Then there exists a constant $C = C(k)$ such that*

$$|F(\overrightarrow{\Delta}_{\mathrm{d}})(x,y)| \le CR^d(1 + R|x-y|)^{-k}\|\delta_{R^2}F\|_{W^k_\infty(\mathbb{R}^d)} \tag{2.4}$$

for all $x, y \in \mathbb{Z}^d$ and all Borel functions F such that $\operatorname{supp} F \subset [0, R^2]^d$, *where $\delta_t F = F(t\cdot)$.*

Proof. From (2.3) in the proof of Proposition 2.1, we obtain

$$|F(\overrightarrow{\Delta}_{\mathrm{d}})(x,y)|\,|x_j - y_j|^k \le \left\|(-i)^k \frac{\partial^k}{\partial\theta_j^k} F(4\sin^2(\theta_1/2), \dots, 4\sin^2(\theta_d/2))\right\|_{\ell^1([-\pi,\pi]^d)}$$

for every $j = 1, \dots, d$ and $k \in \mathbb{N}$.

At this stage, proceeding similarly to the rest of the proof of Proposition 2.1, we come up with (2.4). ■

In what follows, for a function ψ defined on $[0,\infty)$, we define

$$m_\psi(\lambda_1, \dots, \lambda_d) = \psi(\lambda_1 + \dots + \lambda_d). \tag{2.5}$$

From Proposition 2.3, we have the following result.

LEMMA 2.4. *Let $\psi \in \mathcal{S}(\mathbb{R})$ with* $\operatorname{supp}\psi \subset [1/4, 4]$ *and $\psi \ne 0$. Then for each $N > 0$ there exists $C_N > 0$ such that*

$$|m_\psi(t^2\overrightarrow{\Delta}_{\mathrm{d}})(x,y)| \le \frac{C_N}{t^d}\left(1 + \frac{|x-y|}{t}\right)^{-N}\|m_\psi\|_{W^N_\infty(\mathbb{R}^d)}$$

for all $t > 0$ and $x, y \in \mathbb{Z}^d$.

Proof. Note that

$$m_\psi(t^2\lambda_1, \dots, t^2\lambda_d) = \psi(t^2(\lambda_1 + \dots + \lambda_d)).$$

Since the spectrum of Δ_{d}^j is contained in $[0,4]$ for each j, we have $m_\psi(t^2\overrightarrow{\Delta}_{\mathrm{d}}) = 0$ whenever $t < 1/(4\sqrt{d})$. Hence, we need only take care of the case when $t \ge 1/(4\sqrt{d})$. In this case, $\operatorname{supp} m_\psi(t^2\cdot, \dots, t^2\cdot) \subset [0, R^2]^d$ with $R = 2/t \le 8\sqrt{d}$. ■

2.2. A spectral multiplier theorem. In this section, we will prove the following result.

THEOREM 2.5. *Let $s \in \mathbb{N}$, $s > d/2$ and $\eta \in C_c^\infty(0,\infty)$ be a fixed function, not identically zero. If F is a smooth function defined on $[0,\infty)^d$ such that*

$$\sup_{j\in\mathbb{Z}} \|m_\eta\, \delta_{2^{2j}}F\|_{W^s_\infty(\mathbb{R}^d)} < \infty \tag{2.6}$$

where $m_\eta(\lambda_1,\dots,\lambda_d)=\eta(\lambda_1+\dots+\lambda_d)$ *and* $\delta_t F(\lambda_1,\dots,\lambda_d)=F(t\lambda_1,\dots,t\lambda_d)$ *for* $t>0$*, then the spectral multiplier* $F(\overrightarrow{\Delta}_{\rm d})$ *is bounded on* $\ell^p(\mathbb{Z}^d)$ *for all* $1<p<\infty$*.*

REMARK 2.6. (a) Since the spectrum of $\Delta_{\rm d}^j$ is contained in the interval $[0,4]$ for each j, condition (2.6) is equivalent to

$$\sup_{j\le \log_2(4\sqrt{d})} \|m_\eta\, \delta_{2^{2j}} F\|_{W^s_\infty(\mathbb{R}^d)}<\infty.$$

(b) Note that a similar result was obtained in [30, Theorem 2.1] under the Gaussian upper bound conditions imposed on the main operators. We would like to emphasize that since it is unclear if the discrete Laplacian satisfies the Gaussian upper bound, Theorem 2.5 does not follow from [30, Theorem 2.1].

In order to prove Theorem 2.5, we need the following sharp maximal function which was introduced by Martell [24]. Assume that $\mathcal{A}:=\{\mathcal{A}_t\}_{t\ge 0}$ is a family of operators defined by

$$\mathcal{A}_t f(x)=\sum_{y\in\mathbb{Z}^d}\mathcal{A}_t(x,y)f(y),$$

where $\mathcal{A}_t(x,y)$ is a function defined on $\mathbb{Z}^d\times\mathbb{Z}^d$ such that there exist $C>0$ and $N>d$ with

$$|\mathcal{A}_t(x,y)|\le \frac{C}{t^d}\Big(1+\frac{|x-y|}{t}\Big)^{-N} \tag{2.7}$$

for all $x,y\in\mathbb{Z}^d$ and $t>0$. Associated to the family of $\mathcal{A}:=\{\mathcal{A}_t\}_{t\ge 0}$ we define the new maximal function $\mathcal{M}^\sharp_{\mathcal{A}}$ by setting

$$\mathcal{M}^\sharp_{\mathcal{A}} f(x)=\sup_{\substack{x\ni B\\ B\text{ balls}}}\frac{1}{|B|}\sum_{y\in B}|f(y)-\mathcal{A}_{r_B}f|.$$

We have the following theorem.

THEOREM 2.7 ([24]). *Let* $\mathcal{A}:=\{\mathcal{A}_t\}_{t\ge 0}$ *be a family of operators satisfying* (2.7)*. Then for* $1<p<\infty$*, we have*

$$\|f\|_{\ell^p(\mathbb{Z}^d)}\lesssim \|\mathcal{M}^\sharp_{\mathcal{A}} f\|_{\ell^p(\mathbb{Z}^d)}.$$

REMARK 2.8. Since in our setting $B(x,1)=B(x,r)$ for all $x\in\mathbb{Z}^d$ and $r\in(0,1)$, the statement of the above theorem still holds true if the supremum in the definition of $\mathcal{M}^\sharp_{\mathcal{A}}$ is taken over the balls whose radii are greater than or equal to 1 and the condition (2.7) is assumed to be true for $t\ge 1$ only.

We are now ready to give the proof of Theorem 2.5.

Proof of Theorem 2.5. Due to spectral theory, the multiplier $F(\overrightarrow{\Delta}_{\rm d})$ is bounded on $\ell^2(\mathbb{Z}^d)$.

We now fix $s \in \mathbb{N}, s > d/2$. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\psi \subset [1/4, 4]$ be such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-2j}\lambda) = 1, \qquad \lambda > 0.$$

Then we have

$$F(\lambda_1, \ldots, \lambda_d) = \sum_{j\in\mathbb{Z}} \psi(2^{-2j}(\lambda_1 + \cdots + \lambda_d))F(\lambda_1, \ldots, \lambda_d) =: \sum_{j\in\mathbb{Z}} F_j(\lambda).$$

Let B be a ball in $\mathbb{Z}^d$. Since in our setting $B(x, 1) = B(x, r)$ for all $x \in \mathbb{Z}^d$ and $r \in (0, 1)$, we may assume that $r_B \ge 1$. We then define

$$\mathcal{A}_{r_B} = \varphi_1(\Delta^1_{\rm d}, \ldots, \Delta^d_{\rm d}),$$

where $\varphi_1(\lambda_1, \ldots, \lambda_d) = \sum_{j<0} \psi(2^{-2j}r_B^2(\lambda_1 + \cdots + \lambda_d))$.

It follows that

$$I - \mathcal{A}_{r_B} = \varphi_2(\Delta^1_{\rm d}, \ldots, \Delta^d_{\rm d}),$$

where $\varphi_2(\lambda_1, \ldots, \lambda_d) = \sum_{j\ge0} \psi(2^{-2j}r_B^2(\lambda_1 + \cdots + \lambda_d))$.

Since $\operatorname{supp}\psi \subset [1/4, 4]$ and $\operatorname{supp}\varphi_1 \subset [0, 4/r_B^2]^d$, applying Proposition 2.3 we have, for $N > 0$, $t > 0$ and $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} |\mathcal{A}_{r_B}(x, y)| &= |\varphi_1(\Delta^1_{\rm d}, \ldots, \Delta^d_{\rm d})(x, y)| \\ &\le \frac{C_N}{r_B^d}\left(1 + \frac{|x-y|}{r_B}\right)^{-N} \|\delta_{4r_B^{-2}}\varphi_1\|_{W^N_\infty(\mathbb{R}^d)} \\ &\le \frac{C_N}{r_B^d}\left(1 + \frac{|x-y|}{r_B}\right)^{-N}, \end{aligned}$$

which confirms (2.7).

We now claim that

$$\mathcal{M}^\sharp_{\mathcal{A}}[F(\overrightarrow{\Delta}_{\rm d})f] \le \mathcal{M}_2 f.$$

To see this, for $x \in \mathbb{Z}^d$, and any ball $x \in B$ with $r_B \ge 1$, we write

$$\frac{1}{|B|}\sum_{y\in B} |(I - \mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})f(y)| = \sum_{\ell=0}^{\infty} \frac{1}{|B|}\sum_{y\in B} |(I - \mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})f_\ell(y)|,$$

where $f_\ell = f \cdot 1_{\chi_{S_\ell(B)}}$, and $S_\ell(B)$ is defined at the beginning of Section 2.

For $\ell = 0, 1, 2$, by the ℓ^2-boundedness of $\mathcal{A}_{r_B}$ and $F(\overrightarrow{\Delta}_{\rm d})$ and Hölder's inequality,

$$\frac{1}{|B|}\sum_{y\in B} |(I - \mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})f_\ell(y)| \lesssim \left[\frac{1}{|B|}\sum_{y\in B} |f_\ell(y)|^2\right]^{1/2} \lesssim \mathcal{M}_2 f(x).$$

For $\ell \geq 3$, we have

$$\begin{aligned}\frac{1}{|B|}\sum_{x\in B}&|(I-\mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})f_\ell(y)|\\&\lesssim \frac{1}{|B|}\sum_{x\in B}\sum_{y\in S_\ell(B)}|(I-\mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})(x,y)|\,|f_\ell(y)|\\&\lesssim \sup_{x\in B}\Big[\sum_{y\in S_\ell(B)}|(I-\mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})(x,y)|^2\Big]^{1/2}\Big[\sum_{y\in S_\ell(B)}|f_\ell(y)|^2\Big]^{1/2}.\end{aligned}$$

Set

$$F_{\mathcal{A}_{r_B}}(\overrightarrow{\Delta}_{\rm d}) := (I-\mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d}) = \sum_{k\geq 0}F(\lambda_1,\ldots,\lambda_d)\psi(2^{-2k}r_B^2(\lambda_1+\cdots+\lambda_d)).$$

Then we have

$$F_{\mathcal{A}_{r_B}}(\lambda_1,\ldots,\lambda_d) = \sum_{j\in\mathbb{Z}}\psi(2^{-2j}(\lambda_1+\cdots+\lambda_d))F_{\mathcal{A}_{r_B}}(\lambda_1,\ldots,\lambda_d).$$

Then by Proposition 2.1 with $\operatorname{supp}\psi(2^{-2j}(\lambda_1+\cdots+\lambda_d))F_{\mathcal{A}_{r_B}}(\lambda_1,\ldots,\lambda_d) \subset [0,2^{2(j+1)}]^d$ we have, for $x\in B$ and $s>d/2$,

$$\begin{aligned}\Big[\sum_{y\in S_\ell(B)}&|F(\overrightarrow{\Delta}_{\rm d})(I-\mathcal{A}_{r_B})(x,y)|^2\Big]^{1/2}\\&\lesssim \sum_{j\in\mathbb{Z}}\Big[\sum_{y\in S_\ell(B)}|\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d})F_{\mathcal{A}_{r_B}}(\overrightarrow{\Delta}_{\rm d})(x,y)|^2\Big]^{1/2}\\&\lesssim \sum_{j\in\mathbb{Z}}2^{jd/2}(1+2^{j+\ell}r_B)^{-s}\|\delta_{2^{2(j+1)}}F_{\mathcal{A}_{r_B}}\|_{W^s_\infty(\mathbb{R}^d)}.\end{aligned}$$

Using (2.6), by a straightforward calculation we have

$$\|\delta_{2^{2j+2}}F_{\mathcal{A}_{r_B}}\|_{W^s_\infty(\mathbb{R}^d)} \lesssim \min\{1,(2^jr_B)^{2s}\}.$$

Consequently,

$$\begin{aligned}\frac{1}{|B|}&\sum_{y\in B}|(I-\mathcal{A}_{r_B})F(\overrightarrow{\Delta}_{\rm d})f_\ell(y)|\\&\lesssim \sum_{j\in\mathbb{Z}}2^{jd/2}(1+2^{j+\ell}r_B)^{-s}\min\{1,(2^jr_B)^{2s}\}\Big[\sum_{y\in S_\ell(B)}|f_\ell(y)|^2\Big]^{1/2}\\&\lesssim \sum_{j\in\mathbb{Z}}(2^{j+\ell}r_B)^{d/2}(1+2^{j+\ell}r_B)^{-s}\min\{1,(2^jr_B)^{2s}\}\Big[\frac{1}{|2^\ell B|}\sum_{y\in S_\ell(B)}|f_\ell(y)|^2\Big]^{1/2}\end{aligned}$$

$$\lesssim \sum_{j\colon 2^j r_B \ge 2^{-\ell}} 2^{-\ell(s-d/2)}(2^j r_B)^{d/2} \min\{1,(2^j r_B)^{2s}\}\bigg[\frac{1}{|2^\ell B|}\sum_{y\in S_\ell(B)} |f_\ell(y)|^2\bigg]^{1/2}$$
$$+ \sum_{j\colon 2^j r_B < 2^{-\ell}} (2^{j+\ell} r_B)^{d/2}(2^j r_B)^{2s}\bigg[\frac{1}{|2^\ell B|}\sum_{y\in S_\ell(B)} |f_\ell(y)|^2\bigg]^{1/2}$$
$$\lesssim 2^{-\ell(s-d/2)}\bigg[\frac{1}{|2^\ell B|}\sum_{y\in S_\ell(B)} |f_\ell(y)|^2\bigg]^{1/2}$$
$$\lesssim 2^{-\ell(s-d/2)}\mathcal{M}_2 f(x)$$

for every ball $B\subset\mathbb{Z}^d$ and every $x\in B$ as long as $s>d/2$.

It follows that

$$\mathcal{M}^\sharp_{\mathcal{A}}(F(\overrightarrow{\Delta}_{\mathrm{d}})f)(x)\lesssim \mathcal{M}_2 f(x).$$

By Remark 2.8, Theorem 2.7 and the boundedness of the maximal function $\mathcal{M}_2$ we have, for $p>2$,

$$\|F(\overrightarrow{\Delta}_{\mathrm{d}}))f\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|\mathcal{M}^\sharp(F(\overrightarrow{\Delta}_{\mathrm{d}})f)\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|\mathcal{M}_2 f\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

This means that $F(\overrightarrow{\Delta}_{\mathrm{d}})$ is bounded on $\ell^p(\mathbb{Z}^d)$ for $p>2$. Then, by using duality, $F(\overrightarrow{\Delta}_{\mathrm{d}})$ is bounded on $\ell^p(\mathbb{Z}^d)$ for $1<p<2$. This completes our proof. ∎

3. Hardy spaces associated with the discrete Laplacian Δ_{d}. Let $\psi\in\mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\psi\subset[1/4,4]$ and $\psi\ne 0$, and let m_ψ be defined as in (2.5). Consider the area square function $\mathcal{A}_{\Delta_{\mathrm{d}},\psi}$ defined by

$$\mathcal{A}_{\Delta_{\mathrm{d}},\psi} f(x) = \bigg(\int_0^\infty \sum_{y\colon |x-y|<t} |m_\psi(t^2\overrightarrow{\Delta}_{\mathrm{d}})f(y)|^2\,\frac{dt}{t(t+1)^d}\bigg)^{1/2}. \tag{3.1}$$

For $0<p<\infty$, we define the Hardy space $H^p_{\Delta_{\mathrm{d}},\psi}(\mathbb{Z}^d)$ to be the completion of the set

$$\{f\in\ell^2(\mathbb{Z}^d): \mathcal{A}_{\Delta_{\mathrm{d}},\psi}\in\ell^p(\mathbb{Z}^d)\}$$

under the norm

$$\|f\|_{H^p_{\Delta_{\mathrm{d}},\psi}(\mathbb{Z}^d)} := \|\mathcal{A}_{\Delta_{\mathrm{d}},\psi} f\|_{\ell^p(\mathbb{Z}^d)} < \infty.$$

We would like to emphasize that the theory of Hardy spaces adapted to differential operators was developed in [1, 13, 14, 15, 19]. In these papers, the Hardy spaces are also defined via the area square functions associated to the semigroup generated by the main operator. However, in our setting it is not clear whether or not the semigroup $e^{-t\Delta_{\mathrm{d}}}$ satisfies either the Poisson upper bound or the Davies–Gaffney estimates as in [13, 14, 15, 19]. This is the reason why we approach the Hardy spaces by using the functional calculus $m_\psi(t^2\overrightarrow{\Delta}_{\rm d})$.We also note that in the case $d=1$, the theory of Besov and Triebel–Lizorkin spaces has been studied recently by the authors in [4].

We now adapt ideas from [14, 19] to define the molecules associated to the discrete Laplacian $\Delta_{\rm d}$.

DEFINITION 3.1. Let $\epsilon>0$, $0<p\le 1$ and $M\in\mathbb{N}$. A function a is called a $(p,M,\epsilon)_{\Delta_{\rm d}}$-*molecule* associated to a ball $B\subset\mathbb{Z}^d$ if there exists a function b such that

(i) $a=\Delta_{\rm d}^M b$;
(ii) $\|\Delta_{\rm d}^k b\|_{\ell^2(S_j(B))}\le 2^{-j\epsilon}r_B^{2(M-k)}|2^jB|^{1/2-1/p}$ for all $k=0,1,\dots,M$ and $j=0,1,\dots$.

DEFINITION 3.2. Given $\epsilon>0$, $0<p\le 1$ and $M\in\mathbb{N}$, we say that $f=\sum\lambda_j a_j$ is a *molecular* $(p,M,\epsilon)_{\Delta_{\rm d}}$*-representation* if $\{\lambda_j\}_{j=0}^\infty\in\ell^p$, each a_j is a $(p,M,\epsilon)_{\Delta_{\rm d}}$-molecule, and the sum converges in $\ell^2(\mathbb{Z}^d)$. The space $H^p_{\Delta_{\rm d},{\rm mol},M,\epsilon}(\mathbb{Z}^d)$ is then defined to be the completion of the set

$$\{f\in\ell^2(\mathbb{Z}^d): f \text{ has a molecular } (p,M,\epsilon)_{\Delta_{\rm d}}\text{-representation}\}$$

under the norm given by

$$\|f\|^p_{H^p_{\Delta_{\rm d},{\rm mol},M,\epsilon}(\mathbb{Z})} = \inf\Big\{\sum|\lambda_j|^p : f=\sum\lambda_j a_j \text{ is a molecular } (p,M,\epsilon)_{\Delta_{\rm d}}\text{-representation}\Big\}.$$

Since the kernel of $m_\psi(t^2\overrightarrow{\Delta}_{\rm d})$ satisfies the estimate in Lemma 2.4, by the standard argument as in the proofs of [15, Theorem 4.2] (see also [19, Propositions 4.1 & 4.3]) we have the following:

THEOREM 3.3. *Let $\epsilon>0$, $p\in(0,1]$ and $M>\frac{d}{2}(1/p-1/2)$. Then the Hardy spaces $H^p_{\Delta_{\rm d},{\rm mol},M,\epsilon}(\mathbb{Z}^d)$ and $H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)$ coincide and have equivalent norms.*

Let $\psi\in\mathcal{S}(\mathbb{R})$ such that ${\rm supp}\,\psi\subset[1/4,4]$ and $\psi\ne 0$ and m_ψ be defined in (2.5). We define the discrete square function $S_{\Delta_{\rm d},\psi}$ by

$$S_{\Delta_{\rm d},\psi}f=\Big(\sum_{j\in\mathbb{Z}}|m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d})f|^2\Big)^{1/2},$$

By a similar proof, mutatis mutandis, to that of [3, Proposition 3.13], we have the discrete square function characterization for the Hardy spaces $H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)$ for $0<p<\infty$.

THEOREM 3.4. *Let $\psi\in\mathcal{S}(\mathbb{R})$ with ${\rm supp}\,\psi\subset[1/4,4]$ and $\psi\ne0$, and let m_ψ be defined in (2.5). Then for each $0<p<\infty$, we have*

$$\|f\|_{H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)}\sim\|S_{\Delta_{\rm d},\psi}f\|_{\ell^p(\mathbb{Z}^d)}$$

for all $f\in H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)$.

We next prove the following identification between the Hardy space $H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)$ and $\ell^p(\mathbb{Z}^d)$ for $1<p<\infty$.

THEOREM 3.5. *Let $\psi \in \mathcal{S}(\mathbb{R})$ with* $\operatorname{supp}\psi \subset [1/4,4]$ *and $\psi \neq 0$, and let m_ψ be defined in* (2.5). *Then for $1<p<\infty$ we have*

$$H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d) \equiv \ell^p(\mathbb{Z}^d).$$

Proof. By Theorem 3.4 it suffices to prove that

$$\|S_{\Delta_{\rm d},\psi} f\|_{\ell^p(\mathbb{Z}^d)} \sim \|f\|_{\ell^p(\mathbb{Z}^d)}$$

for $1<p<\infty$.

Let $\epsilon := \{\epsilon_j\}_{j\in\mathbb{Z}}$ be an arbitrary sequence with $\epsilon = \pm 1$, we define

$$T_\epsilon f = \sum_{j\in\mathbb{Z}} \epsilon_j m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d}) f.$$

By a straightforward calculation we can show that the function

$$F_\epsilon(\lambda_1,\dots,\lambda_d) = \sum_{j\in\mathbb{Z}} \epsilon_j m_\psi(2^{-2j}\lambda_1,\dots,2^{-2j}\lambda_d)$$

satisfies the condition (2.6) in Proposition 2.5. As a consequence we have, for $p\in(1,\infty)$,

$$\|T_\epsilon f\|_{\ell^p(\mathbb{Z}^d)} \le C\|f\|_{\ell^p(\mathbb{Z}^d)},$$

where the constant C is independent of ϵ.

This, in combination with the Khinchin inequality and the Fubini theorem, implies that

$$\|S_{\Delta_{\rm d},\psi} f\|^p_{\ell^p(\mathbb{Z}^d)} \lesssim \sum_{x\in\mathbb{Z}^d} \mathbb{E}\{|T_\epsilon f(x)|^p\} = \mathbb{E}(\|T_\epsilon f\|^p_{\ell^p(\mathbb{Z}^d)}) \lesssim \|f\|_p^p,$$

which yields $\|S_{\Delta_{\rm d},\psi} f\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^d)}$.

For the lower bound, it is well-known that we can find $\widetilde{\psi} \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\widetilde{\psi} \subset [1/4,4]$ and

$$1 = \sum_{j\in\mathbb{Z}} \widetilde{\psi}(2^{-2j}\lambda)\psi(2^{-2j}\lambda).$$

This implies that

$$f = \sum_{j\in\mathbb{Z}} m_{\widetilde{\psi}}(2^{-2j}\overrightarrow{\Delta}_{\rm d}) m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d}) f,$$

where $m_{\widetilde{\psi}}$ is defined as in (2.5).

Hence, for $g \in \ell^{p'}(\mathbb{Z}^d)$, by Hölder's inequality we have

$$
\begin{aligned}
(3.2)\qquad \sum_{x\in\mathbb{Z}^d} f(x)g(x) &= \sum_{x\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} m_{\widetilde{\psi}}(2^{-2j}\overrightarrow{\Delta}_{\rm d})m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d})f(x)g(x)\\
&= \sum_{x\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d})f(x)m_{\widetilde{\psi}}(2^{-2j}\overrightarrow{\Delta}_{\rm d})g(x)\\
&\le \sum_x\Big(\sum_{j\in\mathbb{Z}}|m_\psi(2^{-2j}\overrightarrow{\Delta}_{\rm d})f(x)|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}|m_{\widetilde{\psi}}(2^{-2j}\overrightarrow{\Delta}_{\rm d})g(x)|^2\Big)^{1/2}\\
&= \sum_x S_{\Delta_{\rm d},\psi}f(x)S_{\Delta_{\rm d},\widetilde{\psi}}g(x)\\
&\lesssim \|S_{\Delta_{\rm d},\psi}f\|_{\ell^p(\mathbb{Z}^d)}\|S_{\Delta_{\rm d},\widetilde{\psi}}g\|_{\ell^{p'}(\mathbb{Z}^d)}.
\end{aligned}
$$

We have proved that

$$\|S_{\Delta_{\rm d},\widetilde{\psi}}g\|_{\ell^{p'}(\mathbb{Z}^d)} \lesssim \|g\|_{\ell^{p'}(\mathbb{Z}^d)}.$$

This, along with (3.2), yields

$$\sum_x f(x)g(x) \lesssim \|S_\Delta f\|_{\ell^p(\mathbb{Z}^d)}\|g\|_{\ell^{p'}(\mathbb{Z}^d)}$$

for all $g \in \ell^{p'}(\mathbb{Z}^d)$. Therefore,

$$\|f\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|S_\Delta g\|_{\ell^p(\mathbb{Z}^d)}.$$

This completes our proof. ∎

From Theorems 3.3 and 3.5, for $0 < p < \infty$ we define the Hardy space $H^p_{\Delta_{\rm d}}(\mathbb{Z}^d)$ to be any function space $H^p_{\Delta_{\rm d},\psi}(\mathbb{Z}^d)$ with $\psi \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp}\psi \subset [1/4, 4]$ and $\psi \neq 0$.

The following interpolation property for our new Hardy spaces can be obtained by using the argument used in [14, proof of Lemma 9.1].

Proposition 3.6. *Let* $0 < p_0, p_1 < \infty$, *and* $0 < \theta < 1$. *If*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

then

$$[H^{p_0}_{\Delta_{\rm d}}(\mathbb{Z}^d), H^{p_1}_{\Delta_{\rm d}}(\mathbb{Z}^d)]_\theta = H^p_{\Delta_{\rm d}}(\mathbb{Z}^d),$$

where $[\cdot,\cdot]_\theta$ *stands for the complex interpolation brackets.*

4. Boundedness of Schrödinger groups. We first prove the following theorem which gives the bounds of the spectral multipliers on the Hardy spaces $H^p_{\Delta_{\rm d}}$. This can be viewed as a counterpart of Theorem 2.5 for the Hardy spaces scale.

THEOREM 4.1. *Let* $0 < p \le 1$ *and* $s \in \mathbb{N}$, $s > d(1/p - 1/2)$. *Let* $\eta \in C_c^\infty(0,\infty)$ *be a fixed function, not identically zero. If* F *is a smooth function defined on* $[0,\infty)^d$ *satisfying the condition* (2.6) *as in Theorem* 2.5, *then the spectral multiplier* $F(\overrightarrow{\Delta}_{\rm d})$ *is bounded on* $H^p_{\Delta_{\rm d}}(\mathbb{Z}^d)$.

Proof. Fix $p \in (0,1]$ and $s \in \mathbb{N}, s > d(1/p - 1/2)$. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\psi \subset [1/4, 4]$ and $\psi \neq 0$. It suffices to prove that

$$\|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})a]\|_{\ell^p(\mathbb{Z}^d)} \lesssim 1$$

for every $(p, M, \epsilon)_{\Delta_{\rm d}}$-molecule a associated to a ball B.

Using the identity

$$I = (I - e^{-r_B^2\Delta_d})^M a + \sum_{i=1}^M c_i e^{-ir_B^2\Delta_d},$$

where c_i are constants, we obtain

$$\begin{aligned}(4.1)\qquad \|S_{\Delta_{\rm d},\psi}a\|^p_{\ell^p(\mathbb{Z}^d)} &\lesssim \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I - e^{-r_B^2\Delta_d})^M a]\|^p_{\ell^p(\mathbb{Z}^d)} \\ &\quad + \sum_{i=1}^M \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})\Delta_d^M e^{-ir_B^2\Delta_d}b]\|^p_{\ell^p(\mathbb{Z}^d)} \\ &=: J + K.\end{aligned}$$

We only need to show that $J \lesssim 1$, since the inequality $K \lesssim 1$ can be proved similarly.

In order to estimate the term J, we write

$$\begin{aligned} J &\le \sum_{j\ge 0} \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I - e^{-r_B^2\Delta_d})^M (a\cdot 1_{S_j(B)})]\|^p_{\ell^p(\mathbb{Z}^d)} \\ &\le \sum_{\ell\ge 0}\sum_{j\ge 0} \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I - e^{-r_B^2\Delta_d})^M (a\cdot 1_{S_j(B)})]\|^p_{\ell^p(S_\ell(2^jB))} \\ &=: \sum_{\ell\ge 0}\sum_{j\ge 0} J_{j\ell}. \end{aligned}$$

It follows from Hölder's inequality that

$$J_{j\ell} \le |2^{j+\ell}B|^{\frac{2-p}{2}} \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I - e^{-r_B^2\Delta_d})^M (a\cdot 1_{S_j(B)})]\|^p_{\ell^2(S_\ell(2^jB))}.$$

For $\ell = 0,1,2,3$ and $j \ge 0$, by Hölder's inequality and the L^2-boundedness of $S_{\Delta_{\rm d},\psi}$ and $(I - e^{-r_B^2\Delta_d})^M$,

$$\begin{aligned} J_{j\ell} &\le |2^jB|^{\frac{2-p}{2}} \|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I - e^{-r_B^2\Delta_d})^M (a\cdot 1_{S_j(B)})]\|^p_{\ell^2(S_\ell(2^jB))} \\ &\lesssim |2^jB|^{\frac{2-p}{2}} \|a\|^p_{\ell^2(S_j(B))} \\ &\lesssim 2^{-j\epsilon p} \sim 2^{-(j+\ell)\epsilon p}. \end{aligned}$$

For $\ell \geq 4$, we have

$$
\begin{aligned}
&\|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M(a\cdot 1_{S_j(B)})]\|^2_{\ell^2(S_\ell(2^jB))}\\
&\quad=\sum_{k\in\mathbb{Z}}\|m_\psi(2^{-2k}\overrightarrow{\Delta}_{\rm d})F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M(a\cdot 1_{S_j(B)})\|^2_{\ell^2(S_\ell(2^jB))}\\
&\quad=\sum_{k\in\mathbb{Z}}\Big\|\sum_{y\in S_j(B)}[m_\psi(2^{-2k}\overrightarrow{\Delta}_{\rm d})F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M](x,y)a(y)\Big\|^2_{\ell^2(S_\ell(2^jB))}\\
&\quad\lesssim\sum_{k\in\mathbb{Z}}\Big[\sum_{y\in S_j(B)}\|[m_\psi(2^{-2k}\overrightarrow{\Delta}_{\rm d})F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M](x,y)\|_{\ell^2(S_\ell(2^jB))}a(y)\Big]^2,
\end{aligned}
$$

where in the last inequality we have used Minkowski's inequality and the fact that

$$[m_\psi(2^{-2k}\overrightarrow{\Delta}_{\rm d})F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M](x,y)$$

is the kernel of $m_\psi(2^{-2k}\overrightarrow{\Delta}_{\rm d})F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M$.

Note that in this situation $|x-y|\sim 2^{j+\ell}r_B$. We thus apply Proposition 2.1 to the function

$$m_\psi(2^{-2k}(\lambda_1,\ldots,\lambda_d))F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2(\lambda_1,\ldots,\lambda_d)})^M$$

supported in $[0,2^{2(k+1)}]^d$ to deduce that

$$
\begin{aligned}
&\|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M(a\cdot 1_{S_j(B)})]\|^2_{\ell^2(S_\ell(2^jB))}\\
&\lesssim\sum_{k\in\mathbb{Z}}2^{kd}\|\delta_{2^{2k+2}}[m_\psi(2^{-2k}(\lambda_1,\ldots,\lambda_d))F(\lambda_1+\cdots+\lambda_d)(I-e^{-r_B^2(\lambda_1,\ldots,\lambda_d)})^M]\|^2_{W^s_\infty(\mathbb{R}^d)}\\
&\qquad\qquad\times(1+2^{k+j+\ell}r_B)^{-2s}\|a\|^2_{\ell^1(S_j(B))}.
\end{aligned}
$$

Using (2.6) we can verify that

$$
\begin{aligned}
&\|\delta_{2^{2k+2}}[m_\psi(2^{-2k}\lambda_1,\ldots,2^{-2k}\lambda_d)F(\lambda_1,\ldots,\lambda_d)(I-e^{-r_B^2(\lambda_1+\cdots+\lambda_d)})^M]\|^2_{W^s_\infty(\mathbb{R}^d)}\\
&\qquad\qquad\lesssim\min\{1,(2^kr_B)^{4M}\},
\end{aligned}
$$

and

$$\|a\|^2_{\ell^1(S_j(B))}\leq|2^jB|\,\|a\|^2_{\ell^2(S_j(B))}\lesssim 2^{-2j\epsilon}|2^jB|^{2-2/p}.$$

Therefore,

$$
\begin{aligned}
&\|S_{\Delta_{\rm d},\psi}[F(\overrightarrow{\Delta}_{\rm d})(I-e^{-r_B^2\Delta_d})^M(a\cdot 1_{S_j(B)})]\|^2_{\ell^2(S_\ell(2^jB))}\\
&\qquad\lesssim\sum_{k\in\mathbb{Z}}2^{kd}\min\{1,(2^kr_B)^{4M}\}(1+2^{k+j+\ell}r_B)^{-2s}2^{-2j\epsilon}|2^jB|^{2-2/p}\\
&\qquad\lesssim 2^{-2j\epsilon}2^{-2s(j+\ell)}|2^jB|^{2-2/p}|B|^{-1}.
\end{aligned}
$$

Hence, for $\ell \geq 4$,

$$J_{j\ell} \lesssim 2^{-jp\epsilon} 2^{-sp(j+\ell)} |2^j B|^{p-1} |B|^{-p/2} |2^{j+\ell} B|^{(2-p)/2}$$
$$\lesssim 2^{-jp\epsilon} 2^{-jp(s-d/2)} 2^{-\ell p[s-d(1/p-1/2)]}.$$

Since $s > d(1/p - 1/2) \geq d/2$, we have

$$J \leq \sum_{j,\ell \geq 0} J_{j\ell} \lesssim 1,$$

which implies

$$\|S_{\Delta_{\mathrm{d}},\psi}[F(\overrightarrow{\Delta}_{\mathrm{d}})(I - e^{-r_B^2 \Delta_d})^M a]\|_{\ell^p(\mathbb{Z}^d)}^p \lesssim 1.$$

This completes our proof. ∎

We are now ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. We will split the proof into two steps. In the first step, we will show the boundedness on the Hardy spaces $H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$ for $0 < p \leq 1$. Then we will use the complex interpolation method to extend the boundedness to $1 < p < \infty$ in the second step.

Step 1. Set

$$F(\lambda_1, \ldots, \lambda_d) = (1 + \lambda_1 + \cdots + \lambda_d)^{-s} e^{it(\lambda_1 + \cdots + \lambda_d)}, \qquad t > 0.$$

If $s > d(1/p - 1/2)$, then we write

$$F(\lambda_1, \ldots, \lambda_d) = G(\lambda_1, \ldots, \lambda_d)\,(1 + \lambda_1 + \cdots + \lambda_d)^{-d(1/p-1/2)} e^{it(\lambda_1 + \cdots + \lambda_d)},$$

where $G(\lambda_1, \ldots, \lambda_d) = (1 + \lambda_1 + \cdots + \lambda_d)^{-[s-d(1/p-1/2)]}$.

It follows that

$$F(\overrightarrow{\Delta}_{\mathrm{d}}) = G(\overrightarrow{\Delta}_{\mathrm{d}}) \circ (I + \Delta_{\mathrm{d}})^{-d(1/p-1/2)} e^{it\Delta_{\mathrm{d}}}.$$

Moreover, for each $\alpha \in \mathbb{N}^d$,

$$|\partial^\alpha G(\lambda_1, \ldots, \lambda_d)| \lesssim (1 + \lambda_1 + \cdots + \lambda_d)^{-|\alpha|}$$

for all $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ with $\lambda_1 + \cdots + \lambda_d > 0$.

Consequently, for a fixed non-zero function $\eta \in C_c^\infty(0, \infty)$ and any $k \in \mathbb{N}$, we have

$$\sup_{j \in \mathbb{Z}} \|m_\eta\, \delta_{2^{2j}} G\|_{W^k_\infty(\mathbb{R}^d)} < \infty,$$

where $m_\eta(\lambda_1, \ldots, \lambda_d) = \eta(\lambda_1 + \cdots + \lambda_d)$ and $\delta_t G(\lambda_1, \ldots, \lambda_d) = G(t\lambda_1, \ldots, t\lambda_d)$ for $t > 0$. Therefore, by Theorem 4.1, $G(\overrightarrow{\Delta}_{\mathrm{d}})$ is bounded on the Hardy space $H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$.

It then remains to prove that $(I + \Delta_{\mathrm{d}})^{-d(1/p-1/2)} e^{it\Delta_{\mathrm{d}}}$ acts boundedly on $H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$ with norm controlled by a multiple of $(1 + |t|)^{d(1/p-1/2)}$, which is strictly less than $(1 + |t|)^s$. This explains why it suffices to deal with the case $s = d(1/p - 1/2)$.

Assuming $s = d(1/p - 1/2)$, let $\varphi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [1/4, 4]$ and $\varphi \neq 0$. Due to Theorem 3.4, it suffices to show that there exists $C > 0$ such that

$$\|S_{\Delta_{\rm d},\varphi}[F(\overrightarrow{\Delta}_{\rm d})a]\|_{\ell^p(\mathbb{Z}^d)} \le C(1+|t|)^{d(1/p-1/2)}$$

for every $(p, M, \epsilon)_{\Delta_{\rm d}}$-molecule a with $\epsilon > 0$ and $M > d(1/p - 1/2) + 1$.

To do this, assume that $a = \Delta_{\rm d}^M b$ is a $(p, M, \epsilon)_{\Delta_{\rm d}}$-molecule associated with a ball B. Using the following identity:

$$\mathrm{Id} = (I - e^{-r_B^2\Delta_{\rm d}})^M + \sum_{k=1}^{M}(-1)^{k+1}C_k^M e^{-kr_B^2\Delta_{\rm d}} =: (I - e^{-r_B^2\Delta_{\rm d}})^M + P(r_B^2\Delta_{\rm d}),$$

we can write

$$\begin{aligned} S_{\Delta_{\rm d},\varphi}(F(\overrightarrow{\Delta}_{\rm d})a) &= S_{\Delta_{\rm d},\varphi}[(I - e^{-r_B^2\Delta_{\rm d}})^M F(\overrightarrow{\Delta}_{\rm d})a] \\ &\quad + S_{\Delta_{\rm d},\varphi}[(r_B^2\Delta_{\rm d})^M P(r_B^2\Delta_{\rm d})F(\overrightarrow{\Delta}_{\rm d})r_B^{-2M}b] \\ &\lesssim \sum_{k\ge 0} S_{\Delta_{\rm d},\varphi}[(I - e^{-r_B^2\Delta_{\rm d}})^M F(\overrightarrow{\Delta}_{\rm d})a_k] \\ &\quad + \sum_{k\ge 0} S_{\Delta_{\rm d},\varphi}[(r_B^2\Delta_{\rm d})^M P(r_B^2\Delta_{\rm d})F(\overrightarrow{\Delta}_{\rm d})r_B^{-2M}b_k] \\ &=: \sum_{k\ge 0} E_1^k + \sum_{k\ge 0} E_2^k, \end{aligned}$$

where $a_k = a \cdot 1_{S_k(B)}$ and $b_k = b \cdot 1_{S_k(B)}$.

Therefore, it suffices to prove that

$$\|E_1^k\|_p + \|E_2^k\|_p \lesssim 2^{-k\epsilon}(1+t)^{d(1/p-1/2)} \tag{4.2}$$

for each $k \in \mathbb{N} \cup \{0\}$.

We take care of E_1^k first by showing that

$$\|E_1^k\|_p \lesssim 2^{-k\epsilon}(1+t)^{d(1/p-1/2)}, \quad k \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

For each $k \ge 0$ set $B_{t,k} = (1+t)2^k B$. Then we have

$$\begin{aligned} \|E_1^k\|_p^p &= \|S_{\Delta_{\rm d},\varphi}[(I - e^{-r_B^2\Delta_{\rm d}})^M F(\overrightarrow{\Delta}_{\rm d})a_k]\|_{\ell^p(4B_{t,k})}^p \\ &\quad + \|S_{\Delta_{\rm d},\varphi}[(I - e^{-r_B^2\Delta_{\rm d}})^M F(\overrightarrow{\Delta}_{\rm d})a_k]\|_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}^p \\ &=: E_{11}^k + E_{12}^k. \end{aligned}$$

Using Hölder's inequality and the ℓ^2-boundedness of $S_{\Delta_{\rm d},\varphi}$, $(I - e^{-r_B^2\Delta_{\rm d}})^M$ and $F(\overrightarrow{\Delta}_{\rm d})$, we obtain

$$
\begin{aligned}
\|E_{11}^k\|_p^p &\lesssim |2^k(1+t)B|^{(2-p)/2}\|S_{\Delta_{\rm d},\varphi}[(I-e^{-r_B^2\Delta_{\rm d}})^M F(\overrightarrow{\Delta}_{\rm d})a_k]\|_{\ell^2(4B_{t,k})}^p\\
&\lesssim |(1+t)2^kB|^{1-p/2}\|a_k\|_{\ell^2}^p \lesssim 2^{-\epsilon kp}|(1+t)2^kB|^{1-p/2}|2^kB|^{p/2-1}\\
&\lesssim 2^{-\epsilon kp}(1+t)^{d(1-p/2)}.
\end{aligned}
$$

For the second term E_{12}^k, set

$$
F_{\ell,r_B}(\lambda_1,\dots,\lambda_d)=\varphi_\ell(\lambda_1+\cdots+\lambda_d)(I-e^{-r_B^2(\lambda_1+\cdots+\lambda_d)})^M F(\lambda_1,\dots,\lambda_d),
$$

where $\varphi_\ell(\cdot)=\varphi(2^{-2\ell}\cdot)$.

Let ℓ_0 be the largest integer such that $2^{\ell_0}\le 2^k r_B$. We now write

$$
\begin{aligned}
\|E_{12}^k\|_p^p &= \Big\|\Big(\sum_{\ell\in\mathbb{Z}}|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}^p\\
&\lesssim \Big\|\sum_{\ell\in\mathbb{Z}}|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k|\Big\|_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}^p\\
&\lesssim \sum_{\ell\in\mathbb{Z}}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k\|_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}^p\\
&= \sum_{\ell\ge\ell_0}\cdots+\sum_{\ell<\ell_0}\cdots =: F_1^k+F_2^k,
\end{aligned}
\tag{4.4}
$$

We split the term F_1^k into two terms as follows:

$$
\begin{aligned}
F_1^k &= \sum_{\ell\ge\ell_0}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k\|_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}^p\\
&\le \sum_{\ell\ge\ell_0}\sum_{j\ge\ell-\ell_0}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k\|_{\ell^p(S_j(B_{t,k}))}^p\\
&\quad+\sum_{\ell\ge\ell_0}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k\|_{\ell^p(B(x_B,2^\ell(1+t)))}^p\\
&=: F_{11}^k+F_{12}^k.
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
F_{12}^k &\lesssim \sum_{\ell\ge\ell_0}|B(x_B,2^\ell(1+t))|^{1-p/2}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\rm d})a_k\|_{\ell^2(B(x_B,2^\ell(1+t)))}^p\\
&\lesssim \sum_{\ell\ge\ell_0}|B(x_B,2^\ell(1+t))|^{1-p/2}\|F_{\ell,r_B}\|_\infty^p\|a_k\|_{\ell^2(\mathbb{Z}^d)}^p.
\end{aligned}
$$

Note that $\|F_{\ell,r_B}\|_\infty\lesssim \min\{1,(2^\ell r_B)^{2M}\}\,2^{-2\ell s}$ and $s=d(1/p-1/2)$. Hence,

$$
\begin{aligned}
F_{12}^k &\lesssim \sum_{\ell\ge\ell_0}2^{-kp\epsilon}|B(x_B,2^\ell(1+t))|^{1-p/2}\min\{1,(2^\ell r_B)^{2pM}\}\,2^{-2\ell sp}|2^kB|^{p/2-1}\\
&\lesssim \sum_{\ell\ge\ell_0}2^{-kp\epsilon}(1+t)^{sp}2^{\ell ps}(2^kr_B)^{-ps}\min\{1,(2^\ell r_B)^{2pM}\}\,2^{-2\ell sp}
\end{aligned}
$$

$$\lesssim \sum_{\ell\ge\ell_0} 2^{-kp\epsilon}(1+t)^{sp}(2^\ell r_B)^{-ps}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim 2^{-kp\epsilon}(1+t)^{sp}.$$

We now turn to the term F^k_{11}. For $\ell\ge\ell_0$ and $j\ge\ell-\ell_0$, we have

$$F^k_{11}\le \sum_{\ell\ge\ell_0}\sum_{j\ge\ell-\ell_0}\|F_{\ell,r_B}a_k\|^p_{\ell^2(S_j(B_{t,k}))}|2^jB_{t,k}|^{1-p/2} \tag{4.5}$$
$$\lesssim \sum_{\ell\in\mathbb{N}}\sum_{j\ge\ell-\ell_0}\|F_{r_B,\ell}(\overrightarrow{\Delta}_{\rm d})\|^p_{\ell^2(S_k(B))\to\ell^2(S_j(B_{t,k}))}\|a_k\|_2^p|2^jB_{t,k}|^{1-p/2}$$
$$\lesssim \sum_{\ell\in\mathbb{N}}\sum_{j\ge\ell-\ell_0}2^{-kp\epsilon}\|F_{r_B,\ell}(\overrightarrow{\Delta}_{\rm d})\|^p_{\ell^2(S_k(B))\to\ell^2(S_j(B_{t,k}))}|2^kB|^{p/2-1}|2^jB_{t,k}|^{1-p/2}$$
$$\lesssim \sum_{\ell\ge\ell_0}\sum_{j\ge\ell-\ell_0}2^{-kp\epsilon}[2^j(1+t)]^{ps}\|F_{r_B,\ell}(\overrightarrow{\Delta}_{\rm d})\|^p_{\ell^2(S_k(B))\to\ell^2(S_j(B_{t,k}))}.$$

By Proposition 2.1, for $\alpha=s+\theta\in\mathbb{N}$ with some $\theta>0$, we have

$$\|F_{r_B,\ell}(\overrightarrow{\Delta}_{\rm d})\|_{\ell^2(S_k(B))\to\ell^2(S_j(B_{t,k}))} \lesssim (2^{\ell+j}(1+t)2^kr_B)^{-\alpha}\|\delta_{2^{2\ell}}F_{r_B,\ell}\|_{W^\alpha_\infty(\mathbb{R}^d)}. \tag{4.6}$$

On the other hand, it is easy to check that

$$\|\delta_{2^{2\ell}}F_{r_B,\ell}\|_{W^\alpha_\infty(\mathbb{R}^d)}\lesssim \max\{1,2^{2\ell(\alpha-s)}\}(1+t)^\alpha\min\{1,(2^\ell r_B)^{2M}\}. \tag{4.7}$$

From (4.7) and (4.6), we conclude that

$$\|F_{r_B,\ell}(\overrightarrow{\Delta}_{\rm d})\|_{\ell^2(S_k(B))\to\ell^2(S_j(B_{t,k}))}$$
$$\lesssim (2^{\ell+j+k}r_B)^{-\alpha}\max\{1,2^{2\ell(\alpha-s)}\}\min\{1,(2^\ell r_B)^{2M}\}$$
$$\lesssim (2^{\ell+j+k}r_B)^{-\alpha}\max\{1,2^{2\ell\theta}\}\min\{1,(2^\ell r_B)^{2M}\}.$$

Inserting this into (4.5), we arrive at

$$F^k_{11}\lesssim \sum_{\ell\ge\ell_0}\sum_{j\ge\ell-\ell_0}2^{-kp\epsilon}(1+t)^{ps}2^{-jp\theta}(2^{\ell+k}r_B)^{-\alpha p}$$
$$\times\max\{1,2^{2\ell\theta p}\}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$=:\sum_{\ell\ge\ell_0:\,\ell<0}\cdots+\sum_{\ell\ge\ell_0:\,\ell\ge0}\cdots.$$

For the first sum, we have

$$\sum_{\ell\ge\ell_0:\,\ell<0}\cdots\lesssim \sum_{\ell\ge\ell_0:\,\ell<0}2^{-kp\epsilon}(1+t)^{ps}(2^{\ell+k}r_B)^{-\alpha p}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim \sum_{\ell\ge\ell_0:\,\ell<0}2^{-kp\epsilon}(1+t)^{d(1-p/2)}(2^\ell r_B)^{-\alpha p}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim 2^{-kp\epsilon}(1+t)^{ps}$$

as long as $M>\alpha/2$.

For the second sum we have

$$\begin{aligned}
&\sum_{\ell\geq\ell_0:\,\ell\geq 0}\cdots\\
&\quad\lesssim \sum_{\ell\geq\ell_0:\,\ell\geq 0} 2^{-kp\epsilon}(1+t)^{ps}2^{-p\theta(\ell-\ell_0)}(2^{\ell+k}r_B)^{-\alpha p}2^{2\ell p\theta}\min\{1,(2^\ell r_B)^{2pM}\}\\
&\quad\lesssim \sum_{\ell\geq\ell_0} 2^{-kp\epsilon}(1+t)^{ps}[2^\ell(2^k r_B)^{-1}]^{-\theta p}(2^{\ell+k}r_B)^{-\alpha p}2^{2\ell p\theta}\min\{1,(2^\ell r_B)^{2pM}\}\\
&\quad\lesssim \sum_{\ell\geq\ell_0\vee 0} 2^{-kp(\epsilon+\alpha-\theta)}(1+t)^{ps}(2^\ell r_B)^{-p(\alpha-\theta)}\min\{1,(2^\ell r_B)^{pM}\}\\
&\quad\lesssim 2^{-kp\epsilon}(1+t)^{ps},
\end{aligned}$$

where have we used the fact that $2^{\ell_0}\sim 2^k r_B$ in the second inequality. Therefore,

$$F_{11}^k \lesssim 2^{-k\epsilon}(1+t)^{d(1-p/2)}$$

for some $\epsilon>0$.

Taking the estimates of F_{11}^k and F_{12}^k into account, we obtain

$$F_1^k \lesssim 2^{-k\epsilon}(1+t)^{d(1-p/2)}$$

for some $\epsilon>0$.

It remains to handle the term F_2^k. Indeed, we have

$$F_2^k=\sum_{\ell<\ell_0}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\mathrm{d}})a_k\|^p_{\ell^p(\mathbb{Z}^d\setminus 4B_{t,k})}=\sum_{\ell<\ell_0}\sum_{j\geq 3}\|F_{\ell,r_B}(\overrightarrow{\Delta}_{\mathrm{d}})a_k\|^p_{\ell^p(S_j(B_{t,k}))}.$$

Arguing similarly to the estimate of F_{11}, we have

$$\begin{aligned}
F_2^k &\lesssim \sum_{\ell<\ell_0}\sum_{j\geq 3} 2^{-kp\epsilon}(1+t)^{ps}2^{-jp\theta}(2^{\ell+k}r_B)^{-\alpha p}\\
&\qquad\qquad\times\max\{1,2^{2\ell p\theta}\}\min\{1,(2^\ell r_B)^{2pM}\}\\
&\lesssim \sum_{\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{ps}(2^{\ell+k}r_B)^{-\alpha p}\max\{1,2^{2\ell p\theta}\}\min\{1,(2^\ell r_B)^{2pM}\}\\
&\lesssim \sum_{\ell<0} 2^{-kp\epsilon}(1+t)^{ps}(2^\ell r_B)^{-\alpha p}\min\{1,(2^\ell r_B)^{2pM}\}\\
&\quad+\sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{ps}(2^{\ell+k}r_B)^{-\alpha p}2^{2\ell p\theta}\min\{1,(2^\ell r_B)^{2pM}\},
\end{aligned}$$

where the last term will be zero if $\ell_0\leq 0$.

It is clear that

$$\sum_{\ell<0} 2^{-kp\epsilon}(1+t)^{ps}(2^\ell r_B)^{-\alpha p}\min\{1,(2^\ell r_B)^{2pM}\}\lesssim 2^{-kp\epsilon}(1+t)^{ps}$$

as long as $M>\alpha$.

For the second sum, we have

$$\sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{ps}(2^{\ell+k}r_B)^{-\alpha p}2^{2\ell p\theta}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim \sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{ps}[2^\ell(2^k r_B)^{-1}]^{-\theta p}(2^\ell 2^k r_B)^{-\alpha p}2^{2\ell p\theta}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim \sum_{\ell>0} 2^{-kp(\epsilon-\theta+\alpha)}(1+t)^{ps}(2^\ell r_B)^{-p(\alpha-\theta)}\min\{1,(2^\ell r_B)^{2pM}\}$$
$$\lesssim 2^{-kp\epsilon}(1+t)^{ps},$$

where in the second inequality we have used the fact that

$$2^\ell(2^k r_B)^{-1} \le 2^{\ell-\ell_0}\times[2^{\ell_0}(2^k r_B)^{-1}] \le 1,$$

as long as $\ell<\ell_0$ and $2^{\ell_0}\le 2^k r_B$.

Therefore we may conclude

$$F_2^k \lesssim 2^{-kp\epsilon}(1+t)^{ps}.$$

This, along with the estimate of F_1^k and (4.4), implies that

$$E_{12}^k \lesssim 2^{-kp\epsilon}(1+t)^{ps},$$

completing the proof of (4.3).

It remains to estimate the term E_2^k. We now show that

$$\|E_2^k\|_p \lesssim 2^{-k\epsilon}(1+t)^{d(1/p-1/2)}, \quad k=0,1,\dots, \tag{4.8}$$

for some $\epsilon>0$.

Set

$$G_{\ell,r_B}(\lambda_1,\dots,\lambda_d)$$
$$=\varphi_\ell(\lambda_1+\dots+\lambda_d)[r_B^2(\lambda_1+\dots+\lambda_d)]^M P(r_B^2(\lambda_1+\dots+\lambda_d))F(\lambda_1,\dots,\lambda_d).$$

Then we have

$$\|G_{\ell,r_B}\|_\infty \lesssim \min\{(2^\ell r_B)^{-2M},(2^\ell r_B)^{2M}\}\,2^{-2\ell s}.$$

Similarly to (4.7), we see that

$$\|\delta_{2^\ell}G_{r_B,\ell}\|_{W^\infty_\alpha(\mathbb{R}^d)} \lesssim \max\{1,2^{2\ell(\alpha-s)}\}\,(1+t)^\alpha\min\{(2^\ell r_B)^{-2M},(2^\ell r_B)^{2M}\}.$$

At this stage, proceed along the same lines as in the proof of (4.3) to obtain (4.8). This completes the proof of (4.2).

Thus, we have just proved that for $p\in(0,1]$ and $s=d(1/p-1/2)$,

$$\|(I+\Delta_{\rm d})^{-s}e^{it\Delta_{\rm d}}\|_{H^p_{\Delta_{\rm d}}(\mathbb{Z}^d)} \le C(1+t)^{d(1/p-1/2)}\|f\|_{H^p_{\Delta_{\rm d}}(\mathbb{Z}^d)}, \quad t\in\mathbb{R}.$$

This completes the proof of Step 1.

Step 2. Similarly to Step 1, it suffices to consider $s=d|1/p-1/2|$ and $1<p<\infty$. We now use the standard method making use of the complex interpolation as in [26]. However, for the sake of completeness, we provide the details of the proof.

Setting $X_0 = H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$ and $X_1 = \ell^2(\mathbb{Z}^d)$, we now define

$$X_0 + X_1 = \{x : \exists x_0 \in X_0,\, \exists x_1 \in X_1,\, x = x_0 + x_1\}$$

with the norm

$$\|a\|_{X_0+X_1} = \inf_{x=x_0+x_1,\, x_i\in X_i} (\|x_0\|_{X_0} + \|x_1\|_{X_1}),$$

and

$$X_0 \cap X_1 = \{x : x \in X_0,\, x \in X_1\}$$

with the norm

$$\|x\|_{X_0\cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}.$$

These two are Banach spaces. It is clear that

$$X_0 \cap X_1 \hookrightarrow X_i \hookrightarrow X_0 + X_1.$$

Definition 4.2. Let $S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$. We define $G(X_0, X_1)$ to be the set of all functions $g : S \to X_0 + X_1$ such that

(i) g is holomorphic in $\mathring{S}$ and bounded on S with values in $X_0 + X_1$;
(ii) $u \mapsto g(iu)$ and $u \mapsto g(1 + iu)$ are continuous, and

$$\|f\|_{G(X_0,X_1)} = \max\Big\{\sup_{u\in\mathbb{R}} \|g(iu)\|_{X_0}, \sup_{u\in\mathbb{R}} \|g(1 + iu)\|_{X_1}\Big\} < \infty.$$

Let X, Y be Banach spaces. We denote by $\mathscr{B}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we write $\mathscr{B}(X)$ for short.

We recall the following result of [23, Theorem 2.1.7].

Theorem 4.3. *For every $z \in S$, let $T_z \in \mathscr{B}(X_0 \cap X_1, X_0 + X_1)$ be such that $z \mapsto T_z x$ is in $G(X_0, X_1)$ for every $x \in X_0 \cap X_1$. Moreover, assume that $T_{iu} \in \mathscr{B}(X_0)$ and $T_{1+iu} \in \mathscr{B}(X_1)$ for every $u \in \mathbb{R}$. Assume further that the following suprema are finite:*

$$M_0 = \sup_{u\in\mathbb{R}} \|T_{iu}\|_{\mathscr{B}(X_0)}, \qquad M_1 := \sup_{u\in\mathbb{R}} \|f(1 + iu)\|_{\mathscr{B}(X_1)}.$$

Then for every $\theta \in (0, 1)$ and $x \in X_0 \cap X_1$, we have

$$\|T_\theta x\|_{[X_0,X_1]_\theta} \le M_0^{1-\theta} M_1^\theta \|x\|_{[X_0,X_1]_\theta},$$

so that T_θ extends to an operator in $\mathscr{B}([X_0, X_1]_\theta)$, where $[\cdot, \cdot]_\theta$ is the complex interpolation bracket.

We now turn to the proof. We will apply the above theorem to our setting. To do this, for every $z \in S$, define

$$T_z = e^{z^2}(\mathrm{Id} + \Delta_{\mathrm{d}})^{-sz} e^{it\Delta_{\mathrm{d}}}, \qquad s = d/2.$$

It is clear that for every $f \in X_0 \cap X_1$, the function $z \mapsto T_z f$ is $(X_0 + X_1)$-valued, and holomorphic in $\mathring{S}$. This function is also bounded on S since, for all $z \in S$,

$$\begin{aligned}(4.9)\qquad \|T_z f\|_{X_0+X_1} &\le \|T_z f\|_{X_1} = \|T_z f\|_{\ell^2(\mathbb{Z}^d)} \\ &\le \|f\|_{\ell^2(\mathbb{Z}^d)} \le \|f\|_{X_1\cap X_2}.\end{aligned}$$

It is also clear that the functions $u \mapsto T_{iu} f$ and $u \mapsto T_{1+iu} f$ are continuous. Moreover, using the fact that $\|m(\Delta_d)\|_{\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)} = \|m\|_\infty$, we have

$$\begin{aligned}M_0 := \sup_{u\in\mathbb{R}} \|T_{iu}\|_{X_1\to X_1} &= \sup_{u\in\mathbb{R}} \|T_{iu}\|_{\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)} \\ &= \sup_{u\in\mathbb{R}} \sup_{\lambda\in\mathbb{R}} |e^{(iu)^2}(1+\lambda)^{-isu} e^{it\lambda}| \le 1.\end{aligned}$$

In addition,

$$\begin{aligned}M_1 := \sup_{u\in\mathbb{R}} \|T_{1+iu}\|_{X_0\to X_0} &= \sup_{u\in\mathbb{R}} \|T_{1+iu}\|_{H^1_L(X)\to H^1_L(X)} \\ &\le \sup_{u\in\mathbb{R}} [e^{1-u^2} \|(\mathrm{Id}+\Delta_{\mathrm{d}})^{-s} e^{it\Delta_{\mathrm{d}}}\|_{H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)\to H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)} \\ &\qquad\qquad \times \|(\mathrm{Id}+\Delta_{\mathrm{d}})^{-ius}\|_{H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)\to H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)}].\end{aligned}$$

On the other hand, using the result from Step 1 we have

$$\|(\mathrm{Id}+\Delta_{\mathrm{d}})^{-s} e^{it\Delta_{\mathrm{d}}}\|_{H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)\to H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)} \lesssim (1+|t|)^{d/2}.$$

In addition, for $F(\lambda_1,\dots,\lambda_d) = (1+\lambda_1+\dots+\lambda_d)^{-ius}$, we can check that for each $\alpha \in \mathbb{N}^d$,

$$|\partial^\alpha F(\lambda_1,\dots,\lambda_d)| \lesssim (1+|us|)^{|\alpha|}(1+\lambda_1+\dots+\lambda_d)^{-|\alpha|}$$

for all $(\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d$ with $\lambda_1+\ldots+\lambda_d > 0$.

Hence, for a fixed nonzero function $\eta \in C_c^\infty(0,\infty)$ and any $k \in \mathbb{N}$, we have

$$\sup_{j\in\mathbb{Z}} \|m_\eta\, \delta_{2^{2j}} F\|_{W^k_\infty(\mathbb{R}^d)} \lesssim (1+|us|)^k,$$

where $m_\eta(\lambda_1,\dots,\lambda_d) = \eta(\lambda_1+\dots+\lambda_d)$ and $\delta_t F(\lambda_1,\dots,\lambda_d) = F(t\lambda_1,\dots,t\lambda_d)$ for $t>0$. By Theorem 4.1, $F(\overrightarrow{\Delta}_{\mathrm{d}})$ is bounded on the Hardy space $H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)$ and

$$\begin{aligned}\|(\mathrm{Id}+\Delta_{\mathrm{d}})^{-ius}\|_{H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)\to H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)} &= \|F(\overrightarrow{\Delta}_{\mathrm{d}})\|_{H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)\to H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d)} \\ &\lesssim \sup_{j\in\mathbb{Z}} \|m_\eta\, \delta_{2^{2j}} F\|_{W^d_\infty(\mathbb{R}^d)} \lesssim (1+|us|)^d,\end{aligned}$$

since $d > d(1-1/2) = d/2$.

Consequently,

$$\begin{aligned} M_1 &\le C \sup_{u\in\mathbb{R}} e^{1-u^2}(1+|t|)^{d/2}(1+|us|)^d \\ &\le C_s(1+|t|)^{d/2}. \end{aligned}$$

Thus, we deduce that for every $f \in H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d) \cap \ell^2(\mathbb{Z}^d)$, the function $z \mapsto T_z f$ is in $G(X_0, X_1)$.

The estimate (4.9) also implies that for every $z \in S$, the mapping $f \mapsto T_z f$ is in $\mathscr{B}(X_0 \cap X_1, X_0 + X_1)$. Therefore, applying Theorem 4.3 we have, for every $\theta \in (0,1)$,

$$\|T_\theta\|_{[X_0,X_1]_\theta \to [X_0,X_1]_\theta} \le M_0^{1-\theta} M_1^\theta \lesssim (1+|t|)^{\theta d/2}. \tag{4.10}$$

For any $p \in (1,2)$, letting $\theta = (2-p)/p$ in (4.10) and using the interpolation theorem (Theorem 4.3) and Theorem 3.5, we have

$$[X_0, X_1]_\theta = [H^1_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d), \ell^2(\mathbb{Z}^d)]_\theta = H^p_{\Delta_{\mathrm{d}}}(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d).$$

This, along with (4.10), implies that

$$\|(I+\Delta_{\mathrm{d}})^{-\tilde{s}} e^{it\Delta_{\mathrm{d}}}\|_{\ell^p(\mathbb{Z}^d)\to\ell^p(\mathbb{Z}^d)} \lesssim (1+|t|)^{d(1/p-1/2)}, \qquad \tilde{s} = d(1/p-1/2),$$

and this holds for all $p \in (1,2)$. By duality, the inequality holds true for all $p \in (2,\infty)$.

This completes the proof. ∎

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