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Real noncommutative convexity I

by

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In memory of Zhong-Jin Ruan (1955–2025)

Abstract. We initiate the theory of real noncommutative (nc) convex sets, the real case of the recent and profound complex theory developed by Davidson and Kennedy (2025). The present paper focuses on the real case of the topics from the first several sections of their memoir. Later results will be discussed in future papers. We develop here some of the infrastructure of real nc convexity, giving many foundational structural results for real operator systems and their associated nc convex sets, and elucidate how the complexification interacts with the basic convexity theory constructions. Several new features appear in the real case, including the novel notion of the complexification of a nc convex set.

1. Introduction. We open with the categorical duality of compact convex sets and operator systems, beginning with the classical case. Let X be a compact Hausdorff space. A *concrete function system* is a selfadjoint unital subspace V of $C(X)$, where $C(X)$ is the abelian C^*-algebra of scalar valued continuous functions on X. Our scalar field $\mathbb{F}$ will be either $\mathbb{R}$ or $\mathbb{C}$. An element $f \in V$ is called *positive* if for all $x \in X$ we have $f(x) \geqslant 0$. A *state* on V is a scalar-valued linear functional on V which is unital, selfadjoint, and positive in the sense that it maps positive functions to positive numbers. Equivalently, these are the unital contractive functionals on V. The collection of states on V, denoted by $S(V)$, is a convex set. By the Banach–Alaoglu theorem it is also compact with the weak* topology. Conversely, given a compact convex set K the set $A(K)$ of affine scalar-valued functions on K is a *function system* inside $C(K)$. There is also an abstract characterization of function systems which will be discussed in a later section. Kadison's rep-

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resentation theorem shows that there is a duality between function systems and compact convex sets. Indeed, we see that for all function systems V and compact convex sets K,

$$A(S(V)) \cong V, \quad S(A(K)) \cong K,$$

where the isomorphism in both cases is given by evaluation. The maps taking $V \mapsto S(V)$ and $K \mapsto A(K)$ are contravariant functors, so that the category of convex sets is dually equivalent to the category of function systems. All the above works for convex sets in both real or complex vector spaces, and for real or complex function systems (see e.g. [1, 33]). However, the relationship between the real and complex theory is complicated in places. For example, many convexity-theoretic results about complex function spaces V are proved via $\text{Re}(V)$, i.e. in the real setting, since the direct complex variant can be messy.

In [15] Davidson and Kennedy establish a profound noncommutative (nc) convexity theory in the complex case. Early in this work they exhibit a categorical equivalence similar to the one above, but for noncommutative convex sets and complex operator systems, giving a noncommutative analogue of Kadison's duality. This is built on previous work of Webster and Winkler [40], who use complex matrix convex sets instead. Much of the matrix convexity in this sense was hitherto done in the complex case. See also [41, 13] for e.g. the original definition of matrix convexity in the real case. However, there has been very substantial and remarkable work from a different perspective on this matrix convexity in the real case, some of it quite recent, motivated in part by connections to system engineering and the matrix inequalities and convexity found there. See e.g. [21] and references therein, where the theory of matrix convex sets has been developed from the point of view of positivity domains of (affine linear) polynomial maps (note that all polynomially convex matrix convex sets are defined by an operator system). For example, the papers [19, 20, 24, 28, 17] develop many beautiful aspects of real matrix convexity, especially for finite-dimensional classes of particular interest like spectrahedra and their variant of matrix convexity. (We thank Scott McCullough and James Pascoe for discussions on this history and recent developments.) Aspects of this work will definitely interact with our program in the future, and it furnishes very many interesting and important examples. Indeed, for this reason we will not spend much time on new examples in our paper: there is already a great abundance of interesting examples in the literature. Many more may be derived from the huge existing classical convexity literature, as we hope to present elsewhere. However, even basic examples are very interesting as we shall see.

Earlier this year the first author and Russell developed the theory of real operator systems in [10]. The present paper is a sequel to this, which in turn was a sequel of [3]. Given any real operator system, there is a very natural way to complexify to get a complex operator system. The process of complexification is functorial in the sense that many of the constructions done with operator systems (for instance, max/min operator systems, duals, and tensor products) usually commute with complexification. This allows much of the complex theory of operator systems and nc convex sets to be applied in the real case. Therefore it is natural to ask if there is a categorical equivalence between real compact noncommutative (nc) convex sets and real operator systems. More generally, does the theory of Davidson and Kennedy carry over to the real case? There are several motivations for investigating this. For example, classical convexity is in many ways essentially a real theory, as inspection of foundational texts such as [1, 35] shows, or e.g. as one sees in nearly all graduate courses on convexity theory, or in some aspects of the list in the previous paragraph. Thus the real theory is likely to play a future role in operator algebras and mathematical physics. As mentioned in more detail in the second paragraph of [10], real structure occurs naturally and crucially in very many areas of mathematics and mathematical physics, and in several deep mathematical theories at some point a crucial advance has been made by switching to the real case (e.g. in K-theory and the Baum–Connes conjecture; see also for example [36]). This is sometimes because the real category is bigger and hence allows more freedom. In our case, every complex operator system (resp. nc convex set) is clearly a real operator system (resp. real nc convex set), but there are many interesting real operator systems (resp. nc convex sets) that are not complex operator systems (resp. complex nc convex sets). An obvious example of this is the selfadjoint matrices: the real nc convex set associated with these is not a nc convex set in the sense of [15]. This is a somewhat trivial example (much better examples may be retrieved from the list in the last paragraph), but it illustrates the point.

In this paper we initiate the theory of real nc convex sets in the sense above, investigating the real case of Davidson and Kennedy's theory, at least up to Chapter 5 of [15] (since our paper is already lengthy, later results will be discussed in future papers). We also have many complementary results, and several new features appear in the real case. Many of these are connected to the fact that, as opposed to the classical case of convex sets and function systems, it turns out that there is a very natural way to complexify a nc convex set. Because this complexification is functorial this will give us an efficient way to generalize the theory to the real case. We give many foundational structural results for real operator systems and their associated nc convex sets. In particular we elucidate how the complexification interacts with the basic convexity theory constructions. In addition, we include some results about nc convex sets in the real and complex case that do not seem to be in the literature. Most of our results and proofs about nc convex sets apply verbatim to matrix convex sets in an obvious way, but we focus on the nc convex category.

The differences between the real and complex case discovered in [10] show up for us too, here and in the sequel papers [8]. For example, the casual reader may miss some of the following items, some of which also mention some of our novel contributions:

(1) Some of the major techniques in the complex case are not available in the real case. For example, a common trick in [15] and its sequels, is that the complex nc affine function space $A(K)$ is unitally order isomorphic to the classical affine function space on K_1 (the 'first level' of K). In particular the classical state space faithfully reproduces the affine structure and order at level 1. This is never true in the real case unless K is what we call 'symmetric', that is, the corresponding real operator system has trivial (i.e. identity) involution (see Theorem 6.12 and Remark 2 after it). This has significant ramifications for us; we will often have to find other routes, here and in the sequels. This is related to the next item:

(2) The absence of a Min and Max functor for nonsymmetric nc convex sets, e.g. those corresponding to general real operator systems such as the quaternions. Indeed, we shall see examples of minimal and maximal complex nc convex sets with a real counterpart which is not minimal nor maximal.

(3) The use of complexification is usually pervasive for us as alluded to above. We spend much time in our series of papers on developing various technical features of the complexification, and developing other effective tools in the real case.

(4) There are places where the existing complex proofs do not work, and we will usually point such out. Sometimes one is then saved by complexification (point (3)), as is the case for example in the proof of one of our categorical dualities. Also 'proof by complexification' is often much quicker, and sometimes safer.

(5) Even a cursory survey of our proofs will reveal that in the real case many calculations have their own technicalities which are quite different to the complex case. As just one example of this, the reader will notice in our papers a repeated use of a certain isometry u and map c in many proofs.

With all this in mind we found it remarkable that things work out so well; so that we were able to establish the real case of so much of the complex theory. The reader might keep in mind as illustrations when encountering the main results of our paper, a few basic examples (see Examples 2.1, 2.2, 3.5, 3.8, 6.9, 6.10). We often weave in such examples, however they may be exposited further in a different section of the paper, and in later papers. Our advances in the present paper are of course a primary foundation for the sequel papers. In [8] we focus on the theory of nc extreme (and pure and maximal) points and the nc Choquet boundary in the real case, and on the theory of real nc convex and semicontinuous functions and real nc convex envelopes. Again our main emphasis there is on how these interact with complexification, for example complexifying nc convex functions and their convex envelopes. A third paper under construction is focused on nc real Choquet theory and complementary topics.

Turning to the structure of our paper, Section 2 gives some background on real and complex operator systems and their complexifications. In keeping with the task and nature of our paper we do however expect the reader to be reading alongside with parts of [10, 15]. The very recent textbook [14] contains an account of many results from [15]. See also e.g. [16, 22, 26, 27] for more on complex nc convexity. Because of this we also do not need to be very pedantic or overly careful with definitions, preliminaries, or the history of the subject, which may usually be found in detail in those sources. In Section 2.2 we define real and complex noncommutative (nc) convex sets and give basic examples such as the real noncommutative state space.

Section 3 describes the complexification of a real nc convex set. This can be done intrinsically by specifying what elements will be in the complexification, or extrinsically by taking a suitable complex nc convex hull of a real nc convex set. These two constructions will be equivalent. We show that there is a unique reasonable complexification of a real nc convex set. We also prove functorial properties of the complexification. For instance, if K is a nc convex set and $A(K)$ are the nc affine functions on K, then

$$A(K_c) = A(K)_c.$$

In Section 4, we show the real version of Davidson and Kennedy's categorical duality. Some useful techniques here are using a real version of the nc separation theorem (see Theorem 3.6), or the functoriality of complexification from Section 3. This has many applications.

Section 6 begins with some facts about function systems. A classical compact convex set K may be turned into a compact nc complex convex set using the fact that the function system $A(K)$ can be given a minimum and maximum operator system structure OMIN and OMAX [34, 23, 42, 10], and then employing the categorical duality between compact nc convex sets and operator systems. Thus we define $\mathrm{Min}(K)$ and $\mathrm{Max}(K)$ via the relations

$$A(\mathrm{Min}(K)) = \mathrm{OMIN}(A(K)) \quad \text{and} \quad A(\mathrm{Max}(K)) = \mathrm{OMAX}(A(K)).$$

As with operator systems, the min and max structure given to a real convex set commutes with complexification. This process uses the bipolar of a nc convex set, and so we develop that in Section 5. Section 7 develops the important notion of *noncommutative functions* in the real case. We may avoid many of the complications in the proofs by e.g. proving key theorems such as [15, Theorem 4.3.3] (or [14, Theorem 16.8.10]) in the real case by complexification.

2. Preliminaries

2.1. Operator systems and operator spaces. For general background on operator systems and spaces, and in particular on the definitions etc. in the rest of this section, we refer the reader to e.g. [32, 7, 15, 14] and in the real case to e.g. [3, 10]. It might also be helpful to also browse some of the other existing real operator space theory, e.g. [37, 38, 39, 11, 5]. Some basic real C^*- and von Neumann algebra theory may be found in [29].

We write $M_n(\mathbb{R})$ for the real $n \times n$ matrices, or sometimes simply M_n when the context is clear. Similarly in the complex case. We sometimes use the quaternions $\mathbb{H}$ as an example: this is simultaneously a real operator system, a real Hilbert space, and a real C^*-algebra, usually thought of as a real $*$-subalgebra of $M_4(\mathbb{R})$ or $M_2(\mathbb{C})$. Its complexification is $M_2(\mathbb{C})$. The letters H, L are usually reserved for real or complex Hilbert spaces, and K for (nc) compact convex sets. Every complex Hilbert space H is a real Hilbert space, i.e. we forget the complex structure. More generally we write X_r for a complex Banach space regarded as a real Banach space. We write $X_{\rm sa}$ for the selfadjoint elements in a $*$-vector space X. In the complex case $M_n(X)_{\rm sa} \cong (M_n)_{\rm sa} \otimes X_{\rm sa}$, but this fails for real spaces. A subspace of $B(H)$ is *unital* if it contains the identity, and a map T is *unital* if $T(1) = 1$. Our identities 1 always have norm 1. We write $\mathfrak{R}\, a$ for $\frac{1}{2}(a + a^*)$, while for $z \in M_n(\mathbb{C})$ we write ${\rm Re}\, z$ for $x \in M_n(\mathbb{R})$ where $z = x + iy$ for $y \in M_n(\mathbb{R})$. Finally, for a cardinal n we define the isometry $u_n = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1_n \\ -i \cdot 1_n \end{smallmatrix} \right]$ where 1_n is the n-dimensional identity operator. We sometimes also write this as u.

A *concrete complex* (resp. *real*) *operator system* V is a unital selfadjoint subspace of $B(H)$ for H a complex (resp. real) Hilbert space. For $n \in \mathbb{N}$ we have the identification $M_n(B(H)) \cong B(H^{(n)})$ where $H^{(n)}$ is the n-fold direct sum of H. From this identification, $M_n(V)$ inherits a norm and positive cone. The latter is the set

$$M_n(V)^+ := \{x \in M_n(V) : x = x^* \geqslant 0 \text{ in } B(H^{(n)})\}.$$

For $n \in \mathbb{N}$ we define the *amplification* of a linear map $\varphi : V \to W$ by

$$\varphi^{(n)} : M_n(V) \to M_n(W), \qquad [x_{ij}] \mapsto [\varphi(x_{ij})].$$

The natural morphisms between operator systems are *unital completely positive* (ucp) functions [2], which are linear maps $\varphi : V \to W$ that are unital and every amplification is positive (or equivalently selfadjoint and contractive). The isomorphisms (resp. embeddings) of operator systems which are used in this paper are bijective (resp. injective) ucp maps whose inverse (resp. inverse in its range) is ucp. These are called *unital complete order isomorphisms* (resp. *unital complete order embeddings*), or ucoi (resp. ucoe) for short.

Similarly a *concrete operator space* E is a subspace of $B(H)$ with norms on $M_n(E)$ inherited from $B(H^{(n)})$. The natural morphisms between operator spaces are the *completely bounded maps*, that is, the linear maps φ between operator spaces such that the amplifications of φ are uniformly bounded. If the uniform bound is $\leqslant 1$ then φ is called a *complete contraction.* If the amplifications of φ are isometries then φ is a *complete isometry.* For a possibly infinite cardinal n, $M_n(E)$ is the space of matrices whose 'finitely supported' submatrices have uniformly bounded norm. In the case E is the scalar field we simply write M_n; thus $M_n \cong B(l^2_n)$. Indeed, for every Hilbert space H, $B(H) \cong M_n$ $*$-isomorphically for some n, after one chooses an orthonormal basis. The above definitions hold for both real and complex operator systems.

There are abstract characterizations of real/complex operator spaces and operator systems. An abstract real/complex operator space is a vector space E with a sequence $\{\|\cdot\|_n\}_{n=1}^{\infty}$ of matrix norms satisfying Ruan's axioms. Operator systems on the other hand are usually abstractly characterized using the notion of an *order unit* e: these are such that for any selfadjoint elements x there is a $t > 0$ satisfying $x + te \geqslant 0$. We say that e is *archimedean* if $x + \epsilon e \geqslant 0$ for all $\epsilon > 0$ implies $x \geqslant 0$. For abstract complex operator systems, we begin with a complex $*$-*vector space* V (a vector space with a period 2 conjugate linear map $* : V \to V$), with a *matrix ordering* $M_n(V)^+$ and an *archimedean matrix order unit* (or *AOU*) e. The definition is the same in the real case, with conjugate linear replaced by linear. The matrix ordering consists of cones $M_n(V)^+$ in the $n \times n$ matrices of V, which are selfadjoint, proper, and closed under compressions by matrices $\beta \in M_{n,m}(\mathbb{C})$. An archimedean matrix order unit is an element $e \in V$ such that $e \otimes 1_n$ (where 1_n is the identity of the $n \times n$ matrices) is an archimedean order unit for each n. These conditions define an abstract operator system. One may then prove that there is a unital complete order embedding of this space into $B(H)$ for some H. See e.g. [14, 32], or [10] in the real case (although sadly we omitted to say in Definition 2.1 there that the $C_n = M_n(V)^+$ are cones).

A real operator system can naturally be made into a complex operator system by complexification. To do this, we start with a real abstract operator system, call it V, with involution $*$, matrix ordering $M_n(V)^+$, and Archimedean matrix order unit e. The complexification of V is the complex vector space V_c consisting of elements $x+iy$ for $x, y \in V$. We give this a conjugate linear involution $(x + iy)^* = x^* - iy^*$. The matrix ordering $M_n(V_c)^+$ will be defined by

$$M_n(V_c)^+ = \{x + iy \in M_n(V_c) : c(x, y) \geqslant 0\}$$

where

$$c(x,y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

We also sometimes write $c(x+iy)$ for $c(x,y)$. The element $e + i0$ will be an archimedean order unit. With this, the complexification becomes an abstract complex operator system. Moreover, one can show that this operator system structure on the complexification is the unique one satisfying Ruan's completely *reasonable* condition, namely, that the map $\theta_V(x+iy) = x - iy$ is a ucoi (or equivalently is completely contractive).

In part of [10] it was checked that many of the basic theorems and constructions for complex operator systems also hold for real operator systems. Very many foundational structural results for real operator systems were developed, and it was shown how the complexification interacts with the basic constructions in the subject. In certain parts of our paper we will need real operator systems V with trivial involution $x^* = x$. It is easy to see that these coincide with the operator systems which are the selfadjoint part of complex operator systems (or of real operator systems). Thus they form an important class of real operator systems. Note however that in this case $M_n(V)$ has a nontrivial involution for $n \geqslant 2$, the transpose. At the end of Section 4 we characterize the nc convex sets associated with such operator systems: they are the *symmetric* nc convex sets. Just as 'level 1' (that is, $M_1(V)$) of the complex operator systems is exactly the complex function systems (see e.g. [25, Section 4.3] and [23, 33, 34]), the real function systems (or real (unital) function spaces) are exactly 'level 1' of the real operator systems with trivial (i.e. identity) involution [10, Section 9]. It is shown in [10] that a real operator system V can be given a minimum or a maximum operator system structure if and only if V has trivial involution. More precisely, by Remarks after Examples 9.6 and 9.14 in that paper, and by [10, Proposition 9.19], $\mathrm{OMAX}(V)$ and $\mathrm{OMIN}(V)$ are operator systems if and only if the involution on the real operator system V is trivial, i.e. the identity.

2.2. Noncommutative real convex sets and affine functions. As stated in e.g. [3, 10], every positive functional on a real operator system is a multiple of a state, and every contractive unital functional is a state. The norm of a positive functional (resp. cb norm of a completely positive map) is its (resp. the norm of its) value at 1. The real states φ on a real operator system V are precisely the real parts of complex states on V_c (such as φ_c), or of a complex C^*-algebra generated by V_c. However, the real parts of two different such complex states may coincide on V. Similarly, the real matrix states φ on V are precisely the 'real parts' $\mathrm{Re} \circ \psi$ of complex matrix states ψ on V_c (such as φ_c).

See [7, Sections 1.3 and 1.6.1–1.6.4] for basics about dual operator spaces and their theory. The real case is almost identical (see e.g. [3]). We say a little more about the weak* topology: For E a real dual operator space with operator space predual E_*, $M_n(E)$ is also a dual operator space, and we have

$$M_n(E) \cong M_n(CB(E_*,\mathbb{R})) \cong CB(E_*, M_n).$$

So, for $[f^\alpha_{st}] \in M_n(CB(E_*,\mathbb{R}))$ and $[f_{st}] \in M_n(CB(E_*,\mathbb{R}))$ we find that $[f^\alpha_{st}] \to [f_{st}]$ weak* if and only if for all $[x_{kl}] \in M_m(E_*)$ we have $[f^\alpha_{st}(x_{kl})] \to [f_{st}(x_{kl})]$ in M_{nm}. Also, $M_n(E_*) \cong w^*CB(E, M_n)$, the weak* continuous completely bounded maps.

For a real operator space E, as in [15] we define $\mathcal{M}(E) = \bigsqcup_n M_n(E)$ (for n cardinals bounded by some cardinal κ) with $M_n(E)$ the matrix space of E. Here $\bigsqcup$ is disjoint union. For $X \subseteq \mathcal{M}(E)$ define $X_n = X \cap M_n(E)$. We call this the *nth level* of the nc set X. In the case $E = \mathbb{R}$ write $\mathcal{M} = \mathcal{M}(\mathbb{R})$. A *real noncommutative convex set* over E is a subset $K = \bigsqcup K_n \subseteq \mathcal{M}(E)$ such that:

(1) K is graded: $K_n \subseteq M_n(E)$ for all n.
(2) K is closed under direct sums: $\sum \alpha_i x_i \alpha_i^{\mathsf{T}} \in K_n$ for all bounded families $\{x_i \in K_{n_i}\}$ and every family $\{\alpha_i \in M_{n,n_i}\}$ of isometries where $\sum \alpha_i \alpha_i^{\mathsf{T}} = 1_n$.
(3) K is closed under compressions: $\beta^{\mathsf{T}} x \beta \in K_m$ for every $x \in K_n$ and every isometry $\beta \in M_{n,m}$.

As in [15], we say that K is *closed/compact* if E is a dual operator space and K_n is closed/compact in the weak* topology in $M_n(E)$.

For $\{x_i \in M_{n_i}(E)\}$ bounded and $\{\alpha_i \in M_{n_i,n}(\mathbb{R})\}$ such that $\sum \alpha_i^{\mathsf{T}} \alpha_i = 1_n$, a *nc convex combination* of x_i is defined as $\sum \alpha_i^{\mathsf{T}} x_i \alpha_i \in M_n(E)$. As in the complex case (see [15, Proposition 2.2.8] or [14, Remark 16.4.3(3)]), a subset $K \subseteq \mathcal{M}(E)$ is nc convex if and only if it is closed under nc convex combinations.

Example 2.1. Let $a, b \in \mathbb{R}$ with $a < b$. Then for $n \in \mathbb{N}$ let $K_n = [a1_n, b1_n]$ where

$$[a1_n, b1_n] = \{\alpha \in M_n(\mathbb{R})_{\mathrm{sa}} : a1_n \leqslant \alpha \leqslant b1_n\}.$$

Then $K = \bigsqcup_{n\in\mathbb{N}} K_n$ is a real compact convex set over $\mathbb{R}$ called the *real compact operator interval*, and written as $\mathbb{I} = [[a, b]]$. If we replace $\mathbb{R}$ in the above definition with $\mathbb{C}$ then we get the *complex compact operator interval.* We have $M_n(\mathbb{R}) \subseteq M_n(\mathbb{C})$ and $M_n(\mathbb{R})^+ \subseteq M_n(\mathbb{C})^+$ and therefore the real operator interval is a subset of the complex operator interval. Here we see that K_1 agree in the real and complex case because K_1 is the interval $[a, b] \subseteq \mathbb{R}$.

Example 2.2. Let V be a real operator system, then the *real nc state space* $K = \mathrm{ncS}(V) = \bigsqcup_n \mathrm{UCP}(V, M_n(\mathbb{R}))$ is a point-weak* compact nc convex set over V^*. This norms V. Indeed, as in the complex case

$$\|[v_{ij}]\| = \sup\{\|[\varphi(v_{ij})]\| : n \in \mathbb{N},\ \varphi \in \mathrm{UCP}(V, M_n(\mathbb{R}))\}.$$

To see this quickly, note that taking a ucoe $\varphi : V \to B(H) \cong M_\kappa$ does this in one shot. For finite-dimensional subspaces K of H the compressions $P_K\varphi(\cdot)_{|K}$ achieve the above norm in the limit, identifying $B(K) \cong M_n$.

Example 2.3. For a real dual operator space E, the 'sequence' of matrix unit balls clearly constitutes a nc convex matrix set $\mathbb{B}(E)$.

The following proposition is useful for extending arguments about matrix convex sets to noncommutative convex sets.

Proposition 2.4. *Suppose we have a closed nc convex set K over a dual operator space E, a net $\{x_i \in K_{n_i}\}$, and a net $\{\alpha_i \in \mathcal{M}_{n,n_i}\}$ of isometries such that* $\lim \alpha_i\alpha_i^\mathsf{T} = 1_n$ *and* $\lim \alpha_i x_i \alpha_i^\mathsf{T} = x \in M_n(E)$. *Then $x \in K_n$.*

Proof. Same as in the complex case. See [15, Proposition 2.2.9]. ∎

As in [14, Proposition 16.4.5] (or [15, Proposition 2.2.10]), this result implies the following.

Proposition 2.5. *Suppose that K and L are closed nc convex sets over a dual operator space E. If $K_n = L_n$ for all $n < \infty$ then $K = L$.*

The natural morphisms between real nc convex sets are *real nc affine functions*. These will be maps $\theta : K \to L$ between real nc convex sets which are graded, respect direct sums, and are equivariant with respect to isometries. That is, for all n,

(1) $\theta(K_n) \subseteq L_n$,
(2) $\theta(\sum \alpha_i x_i \alpha_i^\mathsf{T}) = \sum \alpha_i \theta(x_i)\alpha_i^\mathsf{T}$ for all bounded families $\{x_i \in K_{n_i}\}$ and every family $\{\alpha_i \in M_{n,n_i}\}$ of isometries where $\sum \alpha_i\alpha_i^\mathsf{T} = 1_n$,
(3) $\theta(\beta^\mathsf{T} x\beta) = \beta^\mathsf{T}\theta(x)\beta$ for every $x \in K_n$ and isometry $\beta \in M_{n,m}$.

We say that θ is *continuous* if $\theta|_{K_n} : K_n \to M_n(\mathbb{R})$ is continuous for every n. $A(K)$ is the space of all continuous affine nc functions from K into $\mathcal{M}(\mathbb{R})$.

For K, L classical convex sets and for $\varphi : K \to L$ a bijective function which is affine, its inverse is easily seen to be affine. The same will be true in the noncommutative case. Indeed, φ^{-1} is graded because if $l \in M_n(L)$ then $\varphi^{-1}(l) \in M_n(K)$, and for $y \in L_n$ and an isometry $\beta \in M_{n,m}$ we have $\beta^\mathsf{T} y\beta = \beta^\mathsf{T}\varphi(\varphi^{-1}(y))\beta = \varphi(\beta\varphi^{-1}(y)\beta)$. Taking φ^{-1} of the left and right hand sides gives the result. A similar proof holds to show (2).

2.3. Some relations between the real and complex case

Lemma 2.6. *The function* $\mathrm{Re} : \mathcal{M}(\mathbb{C}) \to \mathcal{M}(\mathbb{R})$ *taking a complex matrix $A + iB$ (where $A, B \in M_n(\mathbb{R})$) to the real matrix A is real affine and completely contractive. The same is true for the map* $\mathrm{Im} : \mathcal{M}(\mathbb{C}) \to \mathcal{M}(\mathbb{R})$ *taking $a + ib$ to b.*

Proof. The map Re is well defined and graded because it sends $N \times N$ matrices over the complex numbers to $N \times N$ matrices over the reals. We leave it to the reader to prove that it is completely contractive. To show (3) in the definition of 'affine', let $A + iB \in M_N(\mathbb{C})$ and $\beta \in M_{N,M}(\mathbb{R})$ be an isometry. Then we have

$$\begin{aligned}\mathrm{Re}(\beta^{\mathsf{T}}(A+iB)\beta) &= \mathrm{Re}(\beta^{\mathsf{T}}A\beta + i\beta^{\mathsf{T}}B\beta) \\ &= \beta^{\mathsf{T}}A\beta = \beta^{\mathsf{T}}\,\mathrm{Re}(A+iB)\beta.\end{aligned}$$

The same proof holds to show condition (2) and for the imaginary part. ∎

If $V = M_n(\mathbb{C})_{\mathrm{sa}}$ then V_c may be identified (via a unital complex complete order isomorphism, and of course complete isometry) with a canonical subspace of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. Namely, if $x, y \in V$ then $z = x + iy$ in V_c is identified with $(z, z^T) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. For a general complex operator system W, if $V = W_{\mathrm{sa}}$ then the canonical complex linear map $u : V_c \to W$ is an isometric and unital identification, since $x + iy$ for $x, y \in V$ in both cases may be identified with $c(x, y) \in M_2(W)$. For a complex selfadjoint operator space E, we claim that $(E_{\mathrm{sa}})_c = E^{\mathrm{sym}}$ completely isometrically. Here E^{sym} is E but with matrix norm $\|[x_{ij}]\|_{\mathrm{sym}} = \max\{\|[x_{ij}]\|, \|[x_{ji}]\|\}$. Indeed, $E_{\mathrm{sa}} \subset E^{\mathrm{sym}}$ and $E_{\mathrm{sa}} + iE_{\mathrm{sa}} = E^{\mathrm{sym}}$. The map $x \mapsto x^*$ on E^{sym} is a period 2, conjugate linear, complete isometry with fixed points E_{sa}. So $(E_{\mathrm{sa}})_c = E^{\mathrm{sym}}$. Moreover, this identification is as operator systems if E is an operator system.

LEMMA 2.7. *For a complex operator system V the 'identity map' taking $x + iy \in (V_{\mathrm{sa}})_c$ to $x + iy \in V$, for $x, y \in V_{\mathrm{sa}}$, is ucp, and is a complex linear bijective isometric order isomorphism.*

Proof. The complexification of the inclusion $V_{\mathrm{sa}} \to V$ is a canonical unital completely isometric complex map $(V_{\mathrm{sa}})_c \to V_c$. If we compose this with the canonical complex quotient map $V_c \to V$ (see e.g. the third paragraph of [10, Section 11]), we obtain a ucp map $(V_{\mathrm{sa}})_c \to V$. This agrees with the 'identity map'. It clearly is complex linear and bijective. To see that it is an isometric order isomorphism, note that $x + iy$ for $x, y \in V_{\mathrm{sa}}$ in both cases may be identified with $c(x, y) \in M_2(V)$. ∎

Note that the above is an order isomorphism, but not necessarily a complete order isomorphism.

LEMMA 2.8. *For a complex operator system V the complex nc state space of V is real nc affinely homeomorphic to the closed nc subset $\{\varphi \in \mathrm{ncS}_{\mathbb{R}}(V) : \varphi(i1) = 0\}$ in the real nc state space, via the 'real part' operation.*

Proof. Since Re is completely contractive and nc affine by Lemma 2.6, $\varphi \mapsto \mathrm{Re} \circ \varphi$ is a continuous nc affine map $\mathrm{ncS}_{\mathbb{C}}(V) \to \mathrm{ncS}_{\mathbb{R}}(V)$. Conversely, we define a map $\epsilon : \mathrm{ncS}_{\mathbb{R}}(V) \to \mathrm{ncS}_{\mathbb{C}}(V)$ by

$$\epsilon(\varphi)(x) = \varphi(x) - i\varphi(ix) = 2(\varphi_c \circ j)(x), \qquad \varphi \in \mathrm{ncS}_{\mathbb{R}}(V) x \in V,$$

where $j : V \to V_c$ is the canonical complex linear inclusion (discussed e.g. in the third paragraph of [10, Section 11]). It is easy to check that $\epsilon(\varphi)$ is selfadjoint and completely positive (since j and φ_c are), and that $\mathrm{Re} \circ \epsilon(\varphi) = \varphi$. So if $\varphi(i1) = 0$ then $\epsilon(\varphi) \in \mathrm{ncS}_{\mathbb{C}}(V)$. Clearly, ϵ is a continuous nc affine map $\mathrm{ncS}_{\mathbb{C}}(V) \to \mathrm{ncS}_{\mathbb{R}}(V)$. ∎

2.4. Affine maps as an operator system. Let K be a real compact nc convex set and let $A(K)$ be the collection of continuous real nc affine maps from K to $\mathcal{M}(\mathbb{R})$. As in the complex case, this is a $*$-vector space with adjoint given by $f^*(k) = f(k)^{\mathsf{T}}$ for $f \in A(K)$ and $k \in K$. We identify $M_n(A(K))$ and $A(K, M_n(\mathbb{R}))$. We define the positive cone $M_n(A(K))^+$ by saying $[f_{ij}] \in M_n(A(K))$ is positive if $[f_{ij}(k)]$ is positive for all $k \in K$.

The matrix order unit will be the constant function 1 which sends everything in K to the corresponding identity in $\mathcal{M}(\mathbb{R})$. This is a matrix order unit because an element a of $M_n(A(K))_{\mathrm{sa}}$ will be bounded by a number $c < \infty$ by the proof of [15, Proposition 2.5.3] (or [14, Remark 16.6.2(2)]), which is the same in the real case. Hence $-cI \leqslant a(k) \leqslant cI$ for each $k \in K$, so that $c1 - a \geqslant 0$. Suppose that for some $n \in \mathbb{N}$ and $f \in M_n(A(K))_{\mathrm{sa}}$ we have $f + \epsilon 1_n \geqslant 0$ for all ϵ. Evaluating at $k \in K$ we get $f(k) + \epsilon I \geqslant 0$. Letting $\epsilon \to 0$ shows that $f(k) \geqslant 0$ for all k, and so f is positive. Therefore, 1 is an archimedean matrix order unit and $A(K)$ is a real operator system.

Remark. Because $M_n(\mathbb{R}) \subseteq M_n(\mathbb{C})$, every complex nc convex set $K \subseteq \bigsqcup M_n(E)$ can be regarded as a real nc convex set $K \subseteq \bigsqcup M_n(E_r)$, where E_r is E regarded as a real vector space. Note that in this case complex affine functions with domain K are real affine. We saw that the real nc affine functions on K are a real operator system, and it contains the complex affine functions as a real subsystem.

3. Complexification. Given an operator space E, we define its complexification E_c to have matrix norms $M_n(E_c)$ inherited from the embedding $c : M_n(E_c) \to M_{2n}(E)$,

$$c : [x_{nm} + iy_{nm}] \mapsto \begin{bmatrix} [x_{nm}] & -[y_{nm}] \\ [y_{nm}] & [x_{nm}] \end{bmatrix}.$$

If E is a dual space then so is E_c, because $E_c = ((E_*)^*)_c = ((E_*)_c)^*$, and then it is easy to see that c is a bicontinuous embedding for the weak* topologies.

Let K be a real nc convex subset of E. Define the *complexification of* K as the set $K_c \subseteq \bigsqcup M_n(E_c)$ by $[z_{ij}] \in (K_c)_n$ if and only if $c([z_{ij}]) \in K_{2n}$.

Theorem 3.1. *Given a real nc convex set $K \subseteq \bigsqcup E_n$, the complexification $K_c \subseteq \bigsqcup (E_c)_n$ is a complex nc convex set with K canonically embedded* *in* K_c *via a real continuous nc affine map* ι. *We have* $K_c = \mathrm{co}_{\mathbb{C}}(\iota(K))$, *the noncommutative convex hull. Also,* $x+iy \in K_c$ *if and only if* $x-iy \in K_c$, *for* $x,y \in M_n(E)$. *Moreover, if* E *is a dual operator space* K *is closed* (*resp. compact*) *if and only if* K_c *is closed* (*resp. compact*).

Proof. Clearly, K_c is graded. To show (2) and (3) in the definition of 'affine' we need the map c to behave well. Specifically, if $[x_{nm}+y_{nm}] \in (K_c)_N$ and $[a_{nm}+ib_{nm}] \in M_{K,N}(\mathbb{C})$, where $a_{nm}, b_{nm} \in \mathbb{R}$, then

$$\begin{aligned} c([a_{nm}+ib_{nm}][x_{nm}+iy_{nm}]) &\\ &= c\Big(\Big[\sum_k a_{nk}x_{km} - b_{nk}y_{km} + ib_{nk}x_{km} + ia_{nk}y_{km}\Big]\Big) \\ &= \sum_k \begin{bmatrix} a_{nk}x_{km} - b_{nk}y_{km} & -(b_{nk}x_{km}+a_{nk}y_{km}) \\ b_{nk}x_{km}+a_{nk}y_{km} & a_{nk}x_{km} - b_{nk}y_{km} \end{bmatrix} \\ &= \begin{bmatrix} a_{nm} & -b_{nm} \\ b_{nm} & a_{nm} \end{bmatrix} \begin{bmatrix} x_{nm} & -y_{nm} \\ y_{nm} & x_{nm} \end{bmatrix} \\ &= c([a_{nm}+ib_{nm}])c([x_{nm}+iy_{nm}]). \end{aligned}$$

We also have

$$\begin{aligned} c([a_{nm}+ib_{nm}]^*) &= c([a_{mn}-ib_{mn}]) \\ &= \begin{bmatrix} [a_{mn}] & [b_{mn}] \\ -[b_{mn}] & [a_{mn}] \end{bmatrix} = \begin{bmatrix} [a_{nm}]^{\mathsf{T}} & [b_{nm}]^{\mathsf{T}} \\ -[b_{nm}]^{\mathsf{T}} & [a_{nm}]^{\mathsf{T}} \end{bmatrix} \\ &= c([a_{nm}+ib_{nm}])^{\mathsf{T}}. \end{aligned}$$

Let $x_i \in (K_c)_{n_i}$ and $\alpha_i \in M_{n,n_i}(\mathbb{C})$ be a family of isometries such that $\sum \alpha_i\alpha_i^* = 1_n$. Then we have $c(\sum \alpha_i x_i \alpha_i^*) = \sum c(\alpha_i)c(x_i)c(\alpha_i)^{\mathsf{T}}$ where $c(\alpha_i)$ will be a family of real isometries such that $c(\alpha_i)c(\alpha_i)^{\mathsf{T}}$ sum to 1. So, $c(\sum \alpha_i x_i \alpha_i^*) \in K_{2n}$, which means $\sum \alpha_i x_i \alpha_i^* \in (K_c)_n$. This verifies condition (2). A similar proof works for condition (3), showing K_c is a complex noncommutative convex set. Therefore, the complexification of a real nc convex set is a complex nc convex set.

The map $\iota : K \hookrightarrow K_c$ taking $[x_{nm}]$ to $[x_{nm} + i\,0]$ is a real continuous nc affine map. Indeed, it is graded and satisfies properties (2) and (3) in the affine map definition because $\iota(\beta^{\mathsf{T}} x \beta) = \beta^{\mathsf{T}} \iota(x)\beta$, where the latter are viewed as elements of K_c. This map is well defined because if $x \in K$ then $c(x+i0) = x \oplus x \in K$ and therefore $x + i0 \in K_c$. It is also continuous at each level.

Clearly $\mathrm{co}_{\mathbb{C}}(\iota(K)) \subseteq K_c$ since K_c is convex and contains $\iota(K)$. For the reverse inequality, if $x+iy \in K_c$ then $x+iy = u^* i_{2n}(c(x,y))u$ where u is the usual isometry $(1/\sqrt{2})[I_n \ \ -iI_n]^{\mathsf{T}}$. This is a nc convex combination of an element from $\iota(K)$.

The first 'if and only if' is clear from the definitions, the nc convexity, and the fact that $c(x,-y) = wc(x,y)w$ where w is the selfadjoint unitary $I \oplus (-I)$.

Finally, suppose that E is a dual real operator space and each K_n is closed in the weak* topology. Suppose that $(x_t + iy_t)$ is a net in $(K_c)_n$ with weak* limit $x+iy$ in $M_n(E_c) \cong M_n(E)_c$. Then $x_t \to x$ and $y_t \to y$ weak* (see [3, Lemma 5.2] and its proof). So $(c(x_t, y_t))$ is a net in K_{2n} with weak* limit $c(x,y)$. Thus $c(x,y) \in K_{2n}$ and $x + iy \in (K_c)_n$. Thus K_c is closed. A similar argument works for compactness. The converse is easier. (E.g. suppose that (x_t) is a net in K_n with weak* limit x. Then $(\iota(x_t))$ is a net in $(K_c)_n$, and $\iota(x_t) \to x$ weak* in $M_n(E_c)$.) ∎

Remarks. (1) Similar considerations show that if K_c is nc convex then so is K. Define $r : K_c \to K$ by $r(x+iy) = x$. By simple calculations (similar to the last proof and the proof of Lemma 2.6) this is continuous and real nc affine, and $r \circ \iota = I_K$.

(2) If E is a dual operator space then $c : K_c \to K$ is a bicontinuous embedding satisfying (2) and (3) in the definition of an affine function.

We may thus define $\theta_K : K_c \to K_c$ as the restriction of the canonical period 2 automorphism θ_E of E_c taking $x + iy \to x - iy$. Then θ_K is easily seen to be a period 2 real nc affine homeomorphism of K_c whose fixed points are K. Conversely, if C is a complex nc convex set in E_c possessing a period 2 real nc affine homeomorphism, then the set K of its fixed points is easily seen to be a real nc convex set.

Lemma 3.2. *For real nc convex sets K, L, every real nc affine map $f : K \to L$ has a unique complex nc affine extension $f_c : K_c \to L_c$. If L is complex nc convex, there is a unique complex nc affine extension $K_c \to L$. These extensions are continuous if f is continuous.*

In particular, every real nc affine isomorphism (resp. homeomorphism) $f : K \to L$ has a unique complex nc affine bijective (resp. homeomorphism) extension $f_c : K_c \to L_c$.

Proof. Define $f_c(x+iy) = u^* f(c(x,y))u$ if $x+iy \in (K_c)_n$, where $u = u_n$ is the isometry above ($u_n = (1/\sqrt{2})[I_n \ \ -iI_n]^\intercal$). Note that $c(\beta)u_n = u_m\beta$ for $\beta \in M_{m,n}(\mathbb{C})$. Then

$$f_c(\beta^*(x+iy)\beta) = u^* f(c(\beta^*(x+iy)\beta)u = u^* f(c(\beta^*)c(x,y)c(\beta))u.$$

Since $c(\beta^*) = c(\beta)^\intercal$ and $c(\beta)u = u\beta$, we obtain $f_c(\beta^*(x+iy)\beta) = \beta^* f_c(x+iy)\beta$. So f_c is affine. A similar argument works if L is complex nc affine. If f is continuous and $x_n + iy_n \to x + iy$ in $(K_c)_n$ then $x_n \to x$, $y_n \to y$, $c(x_n, y_n) \to c(x,y)$. So it is clear from the formula at the start of the proof that $f_c(x_n + iy_n) \to f_c(x+iy)$, hence f_c is continuous.

The uniqueness is clear from the above and the relation $f_c(x+iy) = f_c(u^*c(x,y)u)$. The isomorphism case evidently follows. ∎

LEMMA 3.3. *If $f : K \to L$ is a one-to-one continuous nc affine map between closed nc convex sets, and if K is compact, then f_n is a homeomorphism onto its* (*compact*) *range for all n, and $f(K)$ is a compact nc convex set.*

Proof. A continuous one-to-one map on a compact space is a homeomorphism onto its compact range. ■

We call f in the last result a *nc topological affine embedding.* Note that if f is a real nc topological affine embedding, then f_c is one-to-one and is a complex nc topological affine embedding.

We say that a complex compact nc convex set L is an (abstract) *reasonable complexification* of a real compact nc convex set K if it (or more properly, (L, ϵ_L)) satisfies any one of the equivalent conditions in the next result.

THEOREM 3.4. *Let $\epsilon = \epsilon_L : K \to L$ be a real nc affine topological embedding from a real compact nc convex set to a complex compact nc convex set, with*

$$L_n = \{u^* \epsilon_L(c(x, y))u : c(x, y) \in K_{2n}\}$$

for each n. (*That is, L_n consists of the $x+iy$ for $c(x, y) \in K_{2n}$.*) *The following statements are equivalent:*

(1) *L is a complex compact nc convex set in a complex space F, and F has a real subspace Y with $Y \cap iY = 0$ such that $\epsilon_L(K) \subset Y$.*
(2) *The map*

$$u^* \epsilon_L(c(x, y))u \mapsto x$$

is well defined on L. That is, the 'real part function' on L is well defined.
(3) *The map $\theta_L : L \to L$ with*

$$u^* \epsilon_L(c(x, y))u \mapsto u^* \epsilon_L(c(x, -y))u, \qquad c(x, y) \in K_{2n},$$

is well defined.
(4) *The map θ_L is a well defined period 2 real nc affine homeomorphism with fixed point set $\epsilon_L(K)$.*

Up to real nc affine homeomorphism there is a unique L satisfying these conditions. That is, K has a unique reasonable complexification.

Proof. For $\epsilon_L : K \to L$ as in the statement, let $\widetilde{\epsilon} = \widetilde{\epsilon}_L : K_c \to L$ be its continuous nc affine extension from Lemma 3.2. It satisfies

$$\widetilde{\epsilon}(x + iy) = u^* \epsilon_L(c(x, y))u, \qquad c(x, y) \in K_{2n},$$

and in particular $\widetilde{\epsilon}_L \circ \iota = \epsilon_L$. Also $\widetilde{\epsilon}$ is surjective, since

$$u^* \epsilon_L(c(x, y))u = \widetilde{\epsilon}(u^* i_{2n}(c(x, y))u).$$

We will show that $\widetilde{\epsilon}_L$ is one-to-one if and only if any one of conditions (1)–(4) holds. Indeed, if $\widetilde{\epsilon}_L$ is one-to-one then it is a nc affine homeomorphism by

Lemma 3.3. Thus $L \cong K_c$, via a map taking ϵ_L to the canonical embedding $K \to K_c$. Hence (1)–(4) all hold since they hold for K_c (e.g. in (1) one may take $Y = E$ and $F = E_c$, using notation from the definition of K_c above).

Clearly (4) implies (3). If (3) holds and $u^*\epsilon_L(c(x,y))u = u^*\epsilon_L(c(x',y'))u$ then applying θ_L to this condition gives $u^*\epsilon_L(c(x,-y))u = u^*\epsilon_L(c(x',-y'))u$. Averaging these we obtain

$$\epsilon_L(x) = u^*\epsilon_L(c(x,0))u = u^*\epsilon_L(c(x',0))u = \epsilon_L(x').$$

So $x = x'$. Thus (2) holds.

Since ϵ_L is nc affine we have $\epsilon_L(c(x,y)) = W^*\epsilon_L(c(x,y))W$ where W is the matrix with rows $[0,-I]$ and $[I,0]$. It follows that $\epsilon_L(c(x,y)) = c(a,b)$ for some a, b. We have

$$a = \vec{e}_1^{\intercal}\epsilon_L(c(x,y))\vec{e}_1 = \epsilon_L(\vec{e}_1^{\intercal}c(x,y)\vec{e}_1) = \epsilon_L(x).$$

Thus $u^*\epsilon_L(c(x,y))u=\epsilon_L(x)+i\,b$. Supposing $u^*\epsilon_L(c(x,y))u=u^*\epsilon_L(c(x',y'))u$, we get

$$u^*\epsilon_L(c(x',y'))u = \epsilon_L(x') + i\,z \quad \text{ if } \epsilon_L(c(x',y')) = c(\epsilon_L(x'),z).$$

If (2) holds then $x = x'$ and so $b = z$, so that $\epsilon_L(c(x,y)) = c(a,b) = \epsilon_L(c(x,y'))$, hence $y = y'$. Thus $\widetilde{\epsilon}_L$ is one-to-one. Similarly, assuming (1) note that $\epsilon_L(c(x,y)) \in M_{2n}(Y)$, so that b in the last lines is in $M_n(Y)$, as is $\epsilon(x)$. Thus $\epsilon(x) + ib = \epsilon(x') + iz$ implies $b = z$ and $x = x'$, and $y = y'$. ∎

Example 3.8 below shows that complexification can be complicated, and for instance can change the first level of a nc convex set quite a bit. For now we give a simpler example.

Example 3.5. Consider the real compact operator interval from Example 2.1. The complexification of the real operator interval will be the complex compact operator interval. To see this, first let a, b be real numbers. Let $\bigsqcup [a1_n, b1_n]_{\mathbb{R}}$ be the real operator interval and $\bigsqcup [a1_n, b1_n]_{\mathbb{C}}$ be the complex operator interval. Take $x+iy \in [a1_n, b1_n]_c$; we want to show it is in $[a1_n, b1_n]_{\mathbb{C}}$. By the definition of the complexification, $c(x+iy) \in [a1_{2n}, b1_{2n}]$ and so we have

$$a1_{2n} \leqslant \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \leqslant b1_{2n}.$$

However, the above will hold if and only if

$$a1_n \leqslant x + iy \leqslant b1_n$$

and therefore $x + iy \in [a1_{2n}, b1_{2n}]_{\mathbb{C}}$. Conversely, if we take $z \in [a1_n, b1_n]_{\mathbb{C}}$ then z can be written as the sum of a real and imaginary part, say $z = x+iy$, satisfying the centered equation above and therefore $z \in [a1_n, b1_n]_c$.

As in [15, Theorem 2.4.1] or [14, Corollary 16.5.3] we have a real noncommutative separation theorem.

THEOREM 3.6. *Let K be a real closed nc convex set over a real dual operator space E. Suppose there are n and $y \in M_n(E)$ such that $y \notin K_n$. Then there exists $\gamma \in M_n(\mathbb{R})_{\rm sa}$ and a normal completely bounded map $\varphi : E \to M_n(\mathbb{R})$ such that $\mathfrak{R}\,\varphi_n(y) \nleqslant 1_n \otimes \gamma$ but for all p and $x \in K_p$ we have $\mathfrak{R}\,\varphi_p(x) \leqslant 1_p \otimes \gamma$. If $0_E \in K$ we can take $\gamma = 1_n$. If E is a real operator system and $K \cup \{y\}$ consists of selfadjoint elements, then φ can be chosen to be selfadjoint.*

Proof. This follows as in the complex case in [15] from the Effros–Winkler separation theorem [18, Theorem 5.4] (see [14, Theorem 16.5.2] for a different account of the latter). The real version of the latter is proved exactly as in the complex case (needing the real version of the GNS representation theorem [29, Theorem 3.3.4] applied to a faithful real state of M_n). ∎

Let V be a real operator system and consider $\mathrm{ncS}(V)_c$. This will be a complex nc convex set, as is $\mathrm{ncS}(V_c)$. There is a canonical map

$$\psi : \mathrm{ncS}(V)_c \to \mathrm{ncS}(V_c)$$

induced by the canonical isomorphism $CB(V,W)_c \cong CB(V_c,W_c)$ (see for example [5, Theorem 2.3]). Indeed, for $x, y \in V$ and $f, g \in M_n(\mathrm{CB}(V,\mathbb{C}))$ we have

$$\psi(f+ig)(x+iy) = f(x) - g(y) + if(y) + ig(x).$$

The inverse map takes $u \in CB(V_c,W_c)$ to $\mathrm{Re}\, u_{|V} + i\, \mathrm{Im}\, u_{|V}$, where Re, Im here denote the two canonical projections $W_c \to W$. This map will be called γ.

LEMMA 3.7. *The map $\psi : \mathrm{ncS}(V)_c \to \mathrm{ncS}(V_c)$ is a bijective continuous complex affine nc function with continuous inverse γ.*

Proof. For the reader's convenience and because we will need some of the details later, such as certain specific maps, we give two proofs, and mention a third. Since these are closed nc convex sets in a dual operator space, we may use the idea in [15, Proposition 2.2.10] (or [14, Proposition 16.4.5]) to see that it suffices to check this at the nth level, for all $n \in \mathbb{N}$. Since ψ is a restriction of the canonical isomorphism $CB(V,\mathbb{R})_c \cong CB(V_c,\mathbb{C})$, it is a continuous complex nc affine function with continuous inverse. To see that this maps $\mathrm{ncS}(V)_c$ onto $\mathrm{ncS}(V_c)$, note that by [10, Lemma 3.1] a map $u : V_c \to M_n(\mathbb{C})$ is a complex matrix state if and only if its restriction to V is real ucp. However, the real ucp maps $h : V \to M_n(\mathbb{C})$ are identifiable with the elements $f + ig \in \mathrm{ncS}(V)_c$. To see this, notice that the latter are precisely the $f + ig$ such that $c(f,g)(x) = c(f(x), g(x))$ defines a real matrix state on V. Indeed, these matrix states are precisely the ones which can be identified (via composition with the canonical identification $c_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$) with a real ucp map $h : V \to M_n(\mathbb{C})$ with $h(x) = f(x) + ig(x)$ for $x \in V$. These identifications are another way of describing the map ψ above, and its inverse.

More detailed proof: The reason why ψ is well defined and continuous is as above. Similarly, ψ is graded because it sends certain elements of $M_n(CB(V,\mathbb{R})_c)$ to elements of $M_n(CB(V_c,\mathbb{C}))$, and is complex nc affine being the restriction of a $\mathbb{C}$-linear map. Let $f+ig \in (\mathrm{ncS}(V)_c)_N$ so that $c(f+ig)$ will be a ucp map from V to $M_{2N}(\mathbb{R})$. The complexification of $c(f+ig)$ is a ucp map from V_c to $M_{2N}(\mathbb{C})$ given by

$$\big(c(f+ig)_c\big)(x+iy) = \begin{bmatrix} f(x) & -g(x) \\ g(x) & f(x) \end{bmatrix} + i \begin{bmatrix} f(y) & -g(y) \\ g(y) & f(y) \end{bmatrix}.$$

Taking the rth amplification shows that for $0 \leqslant [x_{nm}+iy_{nm}] \in M_r(V_c)$ we have

$$\begin{aligned} 0 &\leqslant \big(c(f+ig)_c\big)^{(r)}([x_{nm}+iy_{nm}]) \\ &= \left[\begin{bmatrix} f(x_{nm}) & -g(x_{nm}) \\ g(x_{nm}) & f(x_{nm}) \end{bmatrix} + i \begin{bmatrix} f(y_{nm}) & -g(y_{nm}) \\ g(y_{nm}) & f(y_{nm}) \end{bmatrix}\right]. \end{aligned}$$

Compressing the last matrix by $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1_n \\ -i\cdot 1_n \end{smallmatrix}\right]$ implies that

$$\begin{aligned} 0 \leqslant & \left(\begin{bmatrix} f(x_{nm}) & -g(x_{nm}) \\ g(x_{nm}) & f(x_{nm}) \end{bmatrix} + i \begin{bmatrix} f(y_{nm}) & -g(y_{nm}) \\ g(x_{nm}) & f(y_{nm}) \end{bmatrix}\right) \frac{1}{\sqrt{2}}[1_n \;\; i\cdot 1_n] \frac{1}{\sqrt{2}} \begin{bmatrix} 1_n \\ -i\cdot 1_n \end{bmatrix} \\ &= [f(x_{nm}) - g(y_{nm}) + if(y_{nm}) + ig(x_{nm})] \\ &= \psi(f+ig)^{(M)}([x_{nm}+iy_{nm}]) \end{aligned}$$

or $\psi(f+ig)$ is completely positive. It is unital because $c(f+ig)$ is unital and therefore ψ sends elements of $\mathrm{ncS}(V)_c$ to matrix states.

The inverse of ψ is γ, since for $\omega \in \mathrm{ncS}(K_c)$ and $x, y \in V$ we have

$$\begin{aligned} \psi(\operatorname{Re}\omega + i\operatorname{Im}\omega)(x+iy) &= \operatorname{Re}\omega(x) - \operatorname{Im}\omega(y) + i\operatorname{Re}\omega(y) + i\operatorname{Im}\omega(x) \\ &= (\operatorname{Re}\omega + i\operatorname{Im}\omega)(x+iy) = \omega(x+iy) \end{aligned}$$

and conversely, for $f+ig \in \mathrm{ncS}(V)_c$ we get

$$\gamma(\psi(f+ig)) = \operatorname{Re}\psi(f+ig) + i\operatorname{Im}\psi(f+ig) = f+ig.$$

The inverse is a well defined map because $\operatorname{Re}\omega + i\operatorname{Im}\omega$ is in $\mathrm{ncS}(V)_c$. Indeed, $c(\operatorname{Re}\omega + i\operatorname{Im}\omega)$ is ucp because for $N \in \mathbb{N}$ and $0 \leqslant [x_{nm}] \in M_N(V)$ we have

$$c(\operatorname{Re}\omega + i\operatorname{Im}\omega)^{(N)}([x_{nm}]) = \begin{bmatrix} \operatorname{Re}\omega(x_{nm}) & -\operatorname{Im}\omega(x_{nm}) \\ \operatorname{Im}\omega(x_{nm}) & \operatorname{Re}\omega(x_{nm}) \end{bmatrix} \in M_{2N}(\mathbb{R})$$

and the latter is positive if and only if $[\operatorname{Re}\omega(x_{nm}) + i\operatorname{Im}\omega(x_{nm})] = [\omega(x_{nm})]$ is positive. However, this is the Kth amplification applied to $[x_{nm}]$ which is positive. Finally, the inverse is continuous because ψ restricted to any $(\mathrm{ncS}(V)_c)_N$ is a bijective continuous map with compact domain. ∎

Remarks. (1) The last result also follows from Theorem 3.4, and checking that $\mathrm{ncS}(V_c)$ is a reasonable complexification of $\mathrm{ncS}(V)$. Indeed, take $Y = V^*$ and $F = (V_c)^*$ there, with $Y \subset F$ via $\psi \mapsto \psi_c$.

(2)A similar proof shows that for a real operator system V, and cardinal n, on $M_n((V_c)^*) = CB(V_c, M_n(\mathbb{C}))$ the canonical cone $(\mathfrak{C}_c)_n$ coincides with the expected cone $\mathrm{CP}(V_c, M_n(\mathbb{C}))$. Here $(\mathfrak{C}_n)$ are the usual cones $(\mathrm{CP}(V, M_n(\mathbb{R})))$. Thus if $\varphi \in M_n((V_c)^*) \cong CB(V_c, M_n(\mathbb{C})) = CB(V, M_n(\mathbb{R}))_c$, so that (uniquely) $\varphi = \psi_c + i\rho_c$ for $\psi, \rho \in CB(V, M_n(\mathbb{R}))$, then $\varphi \in M_n((V_c)^*)^+ = \mathrm{CP}(V_c, M_n(\mathbb{C}))$ if and only if $c(\varphi) = c(\psi, \rho) \in M_{2n}(V^*)^+ = \mathrm{CP}(V, M_{2n}(\mathbb{R}))$. This may also be deduced from the statement of Lemma 3.7.

Example 3.8. Consider the quaternions $\mathbb{H}$ as a real C^*-algebra. This example will recur repeatedly through the series of papers, so we will be brief here. The state space of $\mathbb{H}$ is trivial, a singleton containing only the map $(a + ib + jc + kd \mapsto a)$. However, the noncommutative state space at the higher levels make up for this deficit, and is much more interesting. The complexification of $\mathbb{H}$ is $M_2(\mathbb{C})$. So the first level of $\mathrm{ncS}(\mathbb{H}_c) \cong \mathrm{ncS}(M_2(\mathbb{C}))$ is all states on $M_2(\mathbb{C})$, which correspond to positive 2×2 trace 1 matrices – i.e. the first level is affine isomorphic to the Bloch sphere. By Lemma 3.7 we have $\mathrm{ncS}(\mathbb{H}_c) \cong \mathrm{ncS}(\mathbb{H})_c$ and so through complexification the first level of $\mathrm{ncS}(\mathbb{H})$ went from having a single element to containing a three-dimensional ball's worth of elements.

Next, for K real compact nc convex we consider $A(K)_c$ as a complexification of an operator system versus $A(K_c)$ as a complex operator system. Let $f, g : K \to \mathcal{M}(\mathbb{R})$ be real affine maps and $x + iy \in M_N(K_c)$. Define the map $\Psi : A(K)_c \to A(K_c)$ by

$$\Psi(f + ig)(x + iy) = u_n^*\big(f(c(x + iy)) + ig(c(x + iy))\big)u_n.$$

We will see that this has inverse $\Gamma : A(K_c) \to A(K)_c$ taking $\omega \in A(K_c)$ to $\mathrm{Re}\,\omega_{|K} + i\,\mathrm{Im}\,\omega_{|K}$.

For a real compact nc convex set K there is a canonical map $\epsilon : A(K) \to A_{\mathbb{C}}(K_c)$ defined by $\epsilon(f) = f_c$, where f_c is as above.

Theorem 3.9. *The map $\Psi : A(K)_c \to A(K_c)$ is a ucoi with inverse Γ. Indeed, $A(K)$ may be identified with the fixed points of the period 2 conjugate linear complete order automorphism a_θ of $A(K_c)$ defined by $a_\theta(f) = \theta_{\mathbb{C}} \circ f \circ \theta_K$, where θ_K is as defined after Theorem 3.1.*

Proof. For the same reason as before, and also to exhibit a complementary result, we give two proofs. Since θ_K is affine and continuous it follows that $a_\theta : A(K_c) \to A(K_c)$. Since θ_K is period 2, so clearly is a_θ too.

Clearly, a_θ is unital, and it is not hard to see that it is conjugate linear since $\theta_{\mathbb{C}}$ is conjugate linear: for example if $f \in A(K_c)$ then

$$a_\theta(if)(k_1 + ik_2) = \theta_{\mathbb{C}}((if)(k_1 - ik_2)) = ia_\theta(f)(k_1 + ik_2).$$

For $x, y \in M_n(\mathbb{R})$ we have

$$\theta_{\mathbb{C}}((x + iy)^*) = \theta_{\mathbb{C}}(x^* - iy^*) = x^* + iy^*,$$

which equals $(\theta_{\mathbb{C}}(x + iy))^* = (x - iy)^*$. Thus $a_\theta(f^*) = a_\theta(f)^*$, since for example

$$a_\theta(f^*)(k_1 + ik_2) = \theta_{\mathbb{C}}(f^*((k_1 - ik_2)) = \theta_{\mathbb{C}}(f(k_1 - ik_2)^*).$$

If $f \in M_n(A(K))^+ = A(K, M_n)^+$ then

$$[a_\theta(f_{ij})(k)] = [\theta_{\mathbb{C}}(f_{ij}(\theta_K(k)))], \qquad k \in K_c.$$

Since $\theta_{\mathbb{C}}$ and its amplifications are completely positive, and $f(K) \geqslant 0$, we have

$$[a_\theta(f_{ij})(k)] \geqslant 0,$$

so that a_θ is completely positive.

The fixed points of a_θ clearly include $\epsilon(A(K))$. Indeed, $a_\theta(f_c) = \theta_{\mathbb{C}} \circ f_c \circ \theta_K$ is an affine extension of f and so $a_\theta(f_c) = f_c$ by the uniqueness in Lemma 3.2. Conversely, suppose that $a_\theta(g) = g$ for $g \in A(K_c)$. Then $\theta_{\mathbb{C}}(g(\iota(k))) = g(\iota(k))$ for $k \in K$. Thus $g(\iota(k)) \in \iota(K)$. Let $f = \mathrm{Re}\ g_{|K}$, a real nc affine map on K. Then $g = f_c = \epsilon(f)$ by the uniqueness in Lemma 3.2, since these are both nc affine extensions of f.

More detailed proof: The function Ψ is complex linear. We show that this map is well defined. First, $\Psi(f + ig)$ will be continuous because f, g are continuous, and $\Psi(f + ig)$ is graded because f, g are graded. Now, notice that for matrices $a, b \in M_N(\mathbb{R})$ we have

$$u_n^* c(a + ib)^\mathsf{T} = (a + ib)^* u_n^*.$$

Similarly $c(a+ib)u_n = u_n(a+ib)$. Let $x+iy \in M_N(K_c)$ and $a, b \in M_{N,k}(\mathbb{R})$ such that $a + ib$ is an isometry, then using the properties of the function c in Theorem 3.1 and the fact that f, g are real affine gives

$$\begin{aligned}
\Psi(f + ig)&((a + ib)^*(x + iy)(a + ib)) \\
&= u_n^*(f(c(a + ib)^\mathsf{T} c(x + iy)c(a + ib)) \\
&\quad + ig(c(a + ib)^\mathsf{T} c(x + iy)c(a + ib)))u_n \\
&= (a + ib)^* \big(\Psi(f + ig)(x + iy)\big)(a + ib).
\end{aligned}$$

Therefore, $\Psi(f + ig)$ preserves compressions. A similar proof shows that it preserves direct sums and therefore $\Psi(f + ig)$ is affine. If $f + ig$ is positive in $A(K)_c$ then $c(f + ig)$ is positive in $M_2(A(K))$, or for any $k \in K$ we have $c(f + ig)(k) \geqslant 0$. Compressing this matrix by u_n shows that $f(k) + ig(k) \geqslant 0$ for all $k \in K$. From this we see that for any $x + iy \in K_c$ we have $\Psi(f+ig)(x+iy) \geqslant 0$ as we are just taking the adjoint of a positive matrix. Therefore, Ψ is positive, and a similar proof shows that our map is completely positive. The unit of $A(K)_c$ is $1+i0$ where 1 is the constant function on K. Note $\Psi(1+i0)(x+iy) = 1_N$ and so Ψ is unital.

To show that Ψ has inverse Γ we see that for bounded $\omega : K_c \to \mathcal{M}(\mathbb{C})$ and $x+iy \in (K_c)_n$ we have

$$\begin{aligned}\omega(x+iy) &= \omega(u_n^* c(x+iy)u_n) = u_n^* \omega(c(x+iy))u_n \\ &= \Psi((\mathrm{Re}\circ \omega_{|K}) + i(\mathrm{Im}\circ \omega_{|K}))(x+iy) = \Psi(\Gamma(\omega))(x+iy).\end{aligned}$$

Conversely for $f+ig \in A(K)_c$ we have

$$\Gamma(\Psi(f+ig)) = \mathrm{Re}\,\Psi(f+ig)_{|K} + i\,\mathrm{Im}\,\Psi(f+ig)_{|K} = f+ig$$

where the last equality comes for instance from the fact that

$$\begin{aligned}\mathrm{Re}\,\Psi(f+ig)(x) &= \mathrm{Re}\,\frac{1}{\sqrt{2}}[1_n \quad i\cdot 1_n]\big(f(x\oplus x)+ig(x\oplus x)\big)\frac{1}{\sqrt{2}}\begin{bmatrix}1_n \\ -i\cdot 1_n\end{bmatrix} \\ &= \mathrm{Re}(f(x)+ig(x)) = f(x).\end{aligned}$$

Finally, we need to show that Γ is ucp. It is unital because Ψ is unital. If $[\omega_{nm}] \in M_N(A(K_c))$ is positive then it maps all $x+iy \in K_c$ to positive matrices. So, for $x \in K$ we have

$$\begin{aligned}c([\mathrm{Re}\circ \omega_{nm} + i\,\mathrm{Im}\circ \omega_{nm}])(x) &= c([\mathrm{Re}\,\omega_{nm}(x) + i\,\mathrm{Im}\,\omega_{nm}(x)]) \\ &= c([\omega_{nm}(x)]).\end{aligned}$$

Now $[\omega_{nm}(x)]$ is positive in $M_N(\mathbb{C})$ and so $c([\omega_{nm}(x)]$ is positive in $M_{2N}(\mathbb{R})$. Therefore, $c([\Gamma(\omega_{nm})])$ is a positive element of $M_2(A(K))$ which by definition means $[\Gamma(\omega_{nm})]$ is positive in $A(K)_c$. ■

4. Real categorical duality. For any real compact nc convex set K, $A(K)$ will be a real operator system by Section 2.4. On the other hand, given a real operator system V, $\mathrm{ncS}(V)$ will be a real compact nc convex set. We have the following duality extending the complex case in [15, Theorem 3.2.2]:

THEOREM 4.1. *Let K be a (real or complex) compact nc convex set, then $K \cong \mathrm{ncS}(A(K))$ via the complex affine homeomorphism $\Lambda : K \to \mathrm{ncS}(A(K))$ where*

$$\Lambda(x)(\varphi) = \varphi(x)$$

for $x \in K$ and $\varphi \in A(K)$.

Conversely, we have (extending [15, Theorem 3.2.3]):

THEOREM 4.2. *Let V be a closed (real or complex) operator system. For $v \in V$ define the function $\hat{v} : \mathrm{ncS}(V) \to \mathcal{M}(\mathbb{F})$ by*

$$\hat{v}(\varphi) = \varphi(v) \quad \textit{for } \varphi \in \mathrm{ncS}(V).$$

This map is a continuous nc affine function. The map $\hat{}: V \to A(\mathrm{ncS}(V))$ *taking* v *to* $\hat{v}$ *is a ucoi.*

We first prove Theorem 4.1 in the real case. For a real compact nc convex set, we know that K_c is a complex nc convex set and therefore isomorphic to $\mathrm{ncS}(A(K_c))$, which in turn is isomorphic to $\mathrm{ncS}(A(K))_c$. We have the embedding ι of our real nc convex sets into their complexifications, so we just need to make sure $\iota(K)$ is mapped onto $\iota(\mathrm{ncS}(A(K))$ through the above isomorphisms. Or equivalently, we need to show that the following diagram commutes:

$$\begin{array}{ccccccc} K_c & \xrightarrow{\Lambda} & \mathrm{ncS}(A(K_c)) & \xrightarrow{\Psi^*} & \mathrm{ncS}(A(K)_c) & \xleftarrow{\psi} & \mathrm{ncS}(A(K))_c \\ {\scriptstyle\iota}\uparrow & & & & & & \uparrow{\scriptstyle\iota} \\ K & & & \xrightarrow{\qquad\qquad\Lambda\qquad\qquad} & & & \mathrm{ncS}(A(K)) \end{array}$$

Here, $\Psi^* : \mathrm{ncS}(A(K_c)) \to \mathrm{ncS}(A(K)_c)$ is defined by setting $\Psi^*(f)(x+iy) = f(\Psi(x+iy))$. So now we just diagram-chase. Let $k \in K_N$; then going to the right, for $f + ig \in A(K)_c$ we have

$$\psi(\Lambda(k) + i0)(f + ig) = \Lambda(k)(f) + i\Lambda(k)(g) = f(k) + ig(k).$$

Going up we have

$$\begin{aligned} \Psi^*(\Lambda(k+i0))(f+ig) &= \Lambda(k+i0)(\Psi(f+ig)) = \Psi(f+ig)(k+i0) \\ &= \frac{1}{2}[1_N \quad i\cdot 1_N]\big(f(c(k+i0)) + ig(c(k+i0))\big)\begin{bmatrix} 1_N \\ -i\cdot 1_N \end{bmatrix} \\ &= \frac{1}{2}[1_N \quad i\cdot 1_N]\big(f(k\oplus k) + ig(k\oplus k)\big)\begin{bmatrix} 1_N \\ -i\cdot 1_N \end{bmatrix} \\ &= \frac{1}{2}[1_N \quad i\cdot 1_N]\big(f(k)\oplus f(k) + ig(k)\oplus g(k)\big)\begin{bmatrix} 1_N \\ -i\cdot 1_N \end{bmatrix} \\ &= f(k) + ig(k). \end{aligned}$$

Therefore, starting at the bottom left and going clockwise is the same as going right and then anticlockwise in the diagram. That is, the diagram commutes, resulting in a nc affine homeomorphism between K and $\mathrm{ncS}(A(K))$. For the reader's convenience we check surjectivity. If $\varphi \in \mathrm{ncS}(A(K))$ then $\iota(\varphi) \in \mathrm{ncS}(A(K))_c = \mathrm{ncS}_{\mathbb{C}}(A_{\mathbb{C}}(K_c))$. (Indeed, φ_c is a matrix state of $A(K)_c$, and thus gives a matrix state of $A_{\mathbb{C}}(K_c)$ by composition with the canonical map $A_{\mathbb{C}}(K_c) \to A(K)_c$.) Thus, by [15, Theorem 3.2.2] (or [14, Theorem 16.6.4]) there exists $x + iy \in K_c$ mapping to $\iota(\varphi)$. That is, for all $f \in A_{\mathbb{C}}(K_c)$ we have

$$f(x+iy) = \iota(\varphi)(f) = \varphi(\mathrm{Re}\, f_{|K}) + i\varphi(\mathrm{Im}\, f_{|K}).$$

Here Re, Im are the real affine functions coming from Lemma 2.6. In particular, replacing f by $\epsilon(f) = f_c$ for $f \in A(K)$, taking real parts, and remembering that $x \in K_n$, we have $f(x) = \mathrm{Re}(f(x) + if(y)) = \varphi(f)$ for $f \in A(K)$. That is, $\Lambda(x) = \varphi$, so that Λ is surjective.

REMARK. Alternatively, the proof of [15, Theorem 3.2.2] works in the real case.

The proof of Theorem 4.2 in the real case uses a similar idea (although the proof in [15, 14] does not work in the real case). We want to show the following diagram commutes:

$$\begin{array}{ccccccc}
V_c & \xrightarrow{\wedge} & A(\mathrm{ncS}(V_c)) & \xrightarrow{\psi^*} & A(\mathrm{ncS}(V)_c) & \xleftarrow{\Psi} & A(\mathrm{ncS}(V))_c \\
\uparrow{\scriptstyle\iota} & & & & & & \uparrow{\scriptstyle\iota} \\
V & & & \xrightarrow{\qquad\quad\wedge\quad\qquad} & & & A(\mathrm{ncS}(V))
\end{array}$$

To show the diagram commutes, let $v \in V$. Going to the right we have

$$\begin{aligned}
\Psi(\hat{v} + i0)(x + iy) &= \frac{1}{2}[1_n \quad i \cdot 1_n]\hat{v}(c(x+iy))\begin{bmatrix} 1_n \\ -i \cdot 1_n \end{bmatrix} \\
&= \frac{1}{2}[1_n \quad i \cdot 1_n]c(x(v) + iy(v))\begin{bmatrix} 1_n \\ -i \cdot 1_n \end{bmatrix} \\
&= x(v) + iy(v),
\end{aligned}$$

and going up we get

$$\begin{aligned}
\psi^*(\widehat{v + i0})(x + iy) &= (\widehat{v + i0})(\psi(x + iy)) \\
&= \psi(x + iy)(v + i0) = x(v) + iy(v).
\end{aligned}$$

To show the surjectivity of $\wedge$ from the diagram, first take $f + i0 \in A(\mathrm{ncS}(V))_c$. Via the diagram, $f + i0$ corresponds to evaluation at some $x + iy \in V_c$. Via the canonical evaluation map and ψ^* we find that

$$\psi^*(\widehat{x + iy})(\varphi + i\theta) = \varphi(x) - \theta(y) + i\varphi(y) + i\theta(x).$$

Applying Γ to this gives $(\varphi \mapsto \varphi(x)) + i(\varphi \mapsto \varphi(y)) \in A(\mathrm{ncS}(V))_c$. This must equal $f + i0 \in A(\mathrm{ncS}(V))_c$. This forces $f = \hat{x}$ and $y = 0$. Indeed, for any element $\varphi \in \mathrm{ncS}(V)$ we have $\varphi(y) = 0$. This implies $y = 0$ by the fact in Example 2.2 (or e.g. [10, Corollary 3.2]). This shows surjectivity.

It follows as in [15, 14] that the categories $\mathrm{NCConv}_{\mathbb{R}}$ (real compact nc convex sets and continuous affine nc maps) and $\mathrm{OpSy}_{\mathbb{R}}$ (real operator systems and ucp maps) are dually equivalent via a contravariant functor. Thus, for example $f \mapsto f \circ \tau : A(K_2) \to A(K_1)$, composition of $f \in A(K_2)$ with a continuous affine nc map $\tau : K_1 \to K_2$, is ucp. For compact nc convex sets K and L we have $A(K)$ and $A(L)$ unitally complete order isomorphic if and only if K and L are affinely homeomorphic. Hence two operator systems are isomorphic if and only if their nc state spaces are affinely homeomorphic.

Remark. If we view a complex nc compact convex K as a real nc compact convex K_r, then K_r corresponds by this duality to a real operator system W. Can one describe W simply in terms of the complex operator system V associated with K?

We now characterize the real compact nc convex sets which correspond to real operator systems which are the selfadjoint part of another operator system.

Corollary 4.3. *A real compact nc set K corresponds under the duality above to a real operator system with trivial involution if and only if every $k \in K$ is symmetric (that is, $k = k^{\mathsf{T}}$).*

Proof. For ucp $\varphi : V \to M_n$ we have

$$\varphi(x)^{\mathsf{T}} = \varphi(x^*), \qquad x \in V.$$

Thus if V has trivial involution then $K = \mathrm{ncS}(V)$ is symmetric. Similarly, if the latter is symmetric then $\varphi(x^*) = \varphi(x)$ for all such φ and $x \in V$, so that $x = x^*$. ∎

We shall call real compact nc sets that satisfy the condition in the last result *symmetric*.

If K is a real nc convex set then the complexification K_c of K consists of elements $z = x + iy$ where $c(x, y) \in K$. We call the set of such x (resp. y) the *real part* (resp. *imaginary part*) of K_c. These are also nc convex sets. The imaginary part of K_c is a bit mysterious, and the following discussion is intended to clarify part of the mystery. If K is also compact then we may assume by the duality above that K is the nc state space $\mathrm{ncS}(V)$ of a real operator system V. That is,

$$K_n = \mathrm{UCP}(V, M_n(\mathbb{R})).$$

Let $W = B(H) = M_n(\mathbb{R})$ for a cardinal n. Let C be the real nc convex set with $C_n = \mathrm{CP}(V, M_n(\mathbb{R}))$. Write

$$D = \{u \in \mathrm{CB}(V, M_n(\mathbb{R})) : u^* = -u,\ u(1) = 0\},$$

with its canonical operator space structure, and let $\mathbb{B}_0$ be the set of matrix unit balls for D (see Example 2.3), a real nc convex compact set. We mention a result from [9]:

Theorem 4.4 ([9, Theorem 4.2]). *The set $\mathbb{B}_0$ is the 'imaginary part' of the complexification K_c of the real nc compact convex set K above. Also, D is the imaginary part of the complexification C_c of the real nc convex set C above.*

Of course the 'real parts' of these nc sets are respectively K and C. It seems remarkable that every map in $\mathbb{B}(D)$ is the imaginary part of a ucp map.

5. Bipolar Theorem

5.1. Real Bipolar Theorem. Let E be a real dual operator space and K a real nc set over E. The *polar* of K is a real nc convex set over E^* defined by

$$K_n^\circ = \{\varphi \in M_n(E^*) : \mathfrak{R}\,\varphi^{(m)}(v) \leqslant 1_{nm} \text{ for all } v \in K_m, m \leqslant \kappa\}.$$

The set

$$K^\circ = \bigsqcup_n K_n^\circ$$

is a closed real nc convex set. This definition is the same as in the complex case, except that Effros and Winkler only consider finite n, m in their definition. Note though that if $\mathfrak{R}\,\varphi^{(m)}(v) \leqslant 1_{nm}$ for all finite m and $v \in K_m$, then it is easy to argue that we have the same relation for infinite m. Thus we may take $m < \infty$ in the definition above. Hence the definition makes sense and produces a closed real nc convex set even if K is only a matrix convex set.

PROPOSITION 5.1. *Let K be a real nc set* (*or real matrix set*) *in the real dual operator space E. Then*

$$(K^\circ)_c \cong (K_c)^\circ$$

via the same maps as in Lemma 3.7.

Proof. Of course K_c is a complex nc convex set containing $0_{E_c} \in E_c$. Let $\gamma : (K_c)^\circ \to (K^\circ)_c$ taking ω to $\operatorname{Re}\omega_{|E} + i \operatorname{Im}\omega_{|E}$, which will have inverse ψ. To see that γ maps into $(K^\circ)_c$, let $\omega \in (K_c)_n^\circ$ and $v \in K_m$. We have

$$\begin{aligned} \mathfrak{R}\,c(\gamma(\omega))^{(m)}(v) &= \mathfrak{R}\,c(\operatorname{Re}\omega^{(m)}(v+i0) + i \operatorname{Im}\omega^{(m)}(v+i0)) \\ &= \mathfrak{R}\,c(\omega^{(m)}(v+i0)) \\ &= c(\mathfrak{R}\,\omega^{(m)}(v+i0)). \end{aligned}$$

We also have

$$c(\mathfrak{R}\,\omega^{(m)}(v+i0)) \leqslant 1_{m\cdot 2n} \iff \mathfrak{R}\,\omega^{(m)}(v+i0) \leqslant 1_{mn}.$$

However, if $v \in K$ then $v + i0 \in K_c$, and so this is true. So γ maps into $(K^\circ)_c$. We saw in Lemma 3.7 that γ is a bijective continuous affine map with continuous inverse ψ. Hence we will be done if $\psi((K^\circ)_c) \subseteq (K^\circ)_c$.

The inverse map $\psi : (K^\circ)_c \to (K_c)^\circ$ takes $f + ig \in ((K^\circ)_c)_N$ and $x + iy \in E_c$ to

$$\psi(f+ig)(x+iy) = f(x) - g(y) + if(y) + ig(x).$$

To show this indeed maps into $(K_c)^\circ$, suppose that $x+iy=[x_{nm}+iy_{nm}]\in (K_c)_M$, so that $c(x+iy)\in K_{2M}$. We want to show that

$$\Re\,[f(x_{nm})-g(y_{nm})+if(y_{nm})+ig(x_{nm})]\leqslant 1_{NM}.$$

However, as shown in Lemma 3.7, we have

$$\begin{aligned}\Re\,[f(x_{nm})&-g(y_{nm})+if(y_{nm})+ig(x_{nm})]\\ &=\Re\,u_N^*\left(\begin{bmatrix}f(x_{nm}) & -g(x_{nm})\\ g(x_{nm}) & f(x_{nm})\end{bmatrix}+i\begin{bmatrix}f(y_{nm}) & -g(y_{nm})\\ g(y_{nm}) & f(y_{nm})\end{bmatrix}\right)u_N\\ &=\Re\,u_N^*\big(c(f+ig)^{(M)}(x)+ic(f+ig)^{(M)}(y)\big)u_N\\ &=u_N^*\Re\,\big(c(f+ig)^{(M)}(x)+ic(f+ig)^{(M)}(y)\big)u_N.\end{aligned}$$

Since u_n is an isometry we see that the latter is $\leqslant 1_{NM}$ because for $f+ig\in (K^\circ)_c$ and $x+iy\in K_c$ we have

$$\begin{aligned}\Re\,c(f+ig)^{(2N)}(c(x+iy))=\Re\begin{bmatrix}c(f+ig)^{(N)}(x) & -c(f+ig)^{(N)}(y)\\ c(f+ig)^{(N)}(y) & c(f+ig)^{(N)}(x)\end{bmatrix}\\ \leqslant 1_{4NM}.\end{aligned}$$

This completes the proof. ■

THEOREM 5.2 (Bipolar Theorem). *Let $K\subseteq E$ be a closed real or complex nc convex set containing $0_E\in E$. Then $K^{\circ\circ}\cong K$.*

Proof. Clearly $K\subseteq K^{\circ\circ}$. If $x\in (K^{\circ\circ})_n\setminus K_n$ then by Theorem 3.6 there exists a normal completely bounded map $\varphi: E\to M_n(\mathbb{F})$ such that $\Re\,\varphi_n(x)\not\leqslant 1_n\otimes 1_n$ but for all p and $k\in K_p$ we have $\Re\,\varphi_p(k)\leqslant 1_p\otimes 1_n$. Then $\varphi\in (K^\circ)_n$, and we obtain the contradiction $\varphi_n(x)\leqslant 1_n\otimes 1_n$. So $K=K^{\circ\circ}$. ■

Effros and Winkler's application of the Bipolar Theorem in [18, Section 5] essentially works for us too. That is, a weakly compact nc convex set L with $L_1=K$ is sandwiched between minimal and maximal nc convex sets which are K at level 1. Effros and Winkler write these as $\hat{L}$ and $\check{L}$. We recall their construction. We will assume that L is a weak* closed (or weak* compact) real or complex nc convex set in a dual operator space E. By translating if necessary by a fixed vector in K one may assume that $0\in L$. We then define the desired minimal nc convex set $\hat{L}$ to be the closed convex hull C in E of $L_1=K$. The 'maximal one' is defined as the prepolar of "the 'minimal one' of the polar of L". That is, $\check{L}=\widehat{L^\circ}$. If we had previously translated to ensure that $0\in L$, we must then apply the inverse of the above translation to these minimal and maximal nc sets.

By the Bipolar Theorem 5.2 as in [18] we see that $\hat{L}$ and $\check{L}$ are respectively the smallest and largest closed nc convex set D in E with $D_1=K$. In the real case, though, this is sometimes more helpful under the restriction that the compact nc convex set corresponds to an operator system with trivial involution, as we will explain at the end of the next section. Under this last restriction $\hat{L}$ agrees with a familiar nc set. On the other hand, suppose that L is the noncommutative state space of the quaternions or of the real C^*-algebra $\mathbb{C}_r$ (these systems have nontrivial involution). The reader can check in these cases that $\hat{L}$ is a trivial compact nc convex set (a translate of (0)), which conveys little useful information.

6. Max and Min nc convex sets. By a real function system (in Kadison's original sense) we mean (concretely) a unital subspace $\mathcal{S}$ of $C_{\mathbb{R}}(K)$ for a compact Hausdorff K, or abstractly (via Kadison's theorem), a (real) ordered vector space V with an archimedean order unit. Note that no involution is mentioned here. Indeed, the canonical involution on $\mathcal{S}$ is trivial (the identity map). We may view V as having this trivial involution. Similarly a complex function system is (concretely) a unital selfadjoint subspace of $C_{\mathbb{C}}(K)$ for compact K, or abstractly [33, 40], a complex $*$-vector space which is ordered (i.e. with a proper selfadjoint cone $E^+ \subset E_{\mathrm{sa}}$), and possesses an archimedean order unit for E_{sa}. The complex $*$-vector space version of Kadison's characterization of archimedean order unit spaces as function systems also follows immediately from the real case. (See e.g. [4, Lemma 1.2 and the lines after it] and [6, Theorem 1.1 and Lemma 1.2].)

These form categories, with the morphisms being unital selfadjoint positive maps, or equivalently (by basic results in e.g. [32, Section 2] or [1, Section II.1]) unital selfadjoint contractions.

Proposition 6.1. *The category of complex function systems is equivalent to the category of real function systems. Moreover, every real function system has a unique reasonable function system complexification, and every complex function system has unique real structure, that is, it is the complexification of an essentially unique real function system.*

Proof. For real-valued functions f, g on a set K we have

$$\begin{aligned}\|f + ig\| &= \sup\{\sqrt{f(x)^2 + g(x)^2} : x \in K\}\\ &= \sup\{\|sf + tg\|_{C(K,\mathbb{R})} : s^2 + t^2 \leqslant 1\}.\end{aligned}$$

Here s, t are real. It follows that the 'function system complexification' of a real function system S is unique and reasonable (we assume that the embedding of S into the function system complexification is as real-valued functions). This complexification is the so-called Taylor complexification (see e.g. [30]).

Conversely, every complex function system V is a reasonable complexification of a real function system. Indeed, suppose that V is a unital selfadjoint subspace of $C(K, \mathbb{C})$. Then V_{sa} is a real function system with the inherited cone (for example it is clearly an ordered real space with archimedean order unit 1_V). Moreover, V is a reasonable complexification of V_{sa}.

Clearly, the above defines two functors between the categories. Notice that a unital positive map $T : V \to W$ in the complex category is completely positive, and selfadjoint, so $T(V_{\rm sa}) \subseteq W_{\rm sa}$. Thus it is clear that the category of complex function systems is isomorphic to the category of real function systems.

Any operator system S whose complexification is V is the set of fixed points for some period 2 conjugate linear unital order isomorphism $u : V \to V$. Note that u is selfadjoint since it is positive. Let $w = u_{|V_{\rm sa}}$, which is a period 2 order automorphism of $V_{\rm sa}$. Then $u \circ w_c$ is a period 2 conjugate linear unital order isomorphism $V \to V$ whose fixed points are exactly $V_{\rm sa}$. That is, $u(w_c(v)) = \bar{v}$ for $v \in V$. Thus up to the unital order isomorphism w_c, the complexification of S can be identified with the complexification of $V_{\rm sa}$. ∎

Remark. Of course, the analogues for operator systems of most of the assertions in the last result are (badly) false in general. In particular, a complex operator system V need not be a reasonable complexification of $V_{\rm sa}$. For example, $M_2(\mathbb{C})$ is not a reasonable complexification of $M_2(\mathbb{C})_{\rm sa}$ (see the discussion after Lemma 2.6). Lemma 2.7 however is a partial result along these lines. We also always have a canonical one-to-one real continuous nc affine map $\mathrm{ncS}_{\mathbb{C}}(V) \to \mathrm{ncS}_{\mathbb{R}}(V_{\rm sa})$, taking $\varphi \to \mathrm{Re}\,\varphi$. Since Re is completely contractive, this map is well defined. It is one-to-one since $\mathrm{Re}\ \varphi = 0$ implies that $\mathrm{Re}(i\varphi) = 0$. It is surjective if $V = \mathrm{OMAX}(V)$ for example.

We now consider the nc convex sets canonically associated with the function system and its complexification.

Let K be a classical compact convex set in a real dual Banach space E. Since the beginnings of the subject of matrix and nc convexity, authors have shown that in the complex case there is a smallest and largest matrix/nc convex set which agrees at the first level with K (see e.g. [18, Section 5], or [31, Section 1.2.3] and references therein). It seems to us that in general these sets depend on the particular embedding of K into a LCTVS operator space. In this paper we will define, in both the real and complex case, $\mathrm{Min}(K)$ to be the closed nc convex hull in $\mathrm{Max}(E)$ of $(K, \varnothing, \varnothing, \ldots)$ (for the definition of Min and Max of operator spaces and their properties see e.g. [7, 39]). This is the smallest compact nc convex set containing $(K, \varnothing, \varnothing, \ldots)$. We remark that our notation conflicts with that of Kennedy and Shamovich [27, Section 5], who call this $\max(K)$ perhaps because they want to regard it as the largest compact nc convex in their ordering.

Lemma 6.2. *Let K be a classical compact convex set as above. At the first level $\mathrm{Min}(K)$ is simply K, at the nth level it (that is, $(\mathrm{Min}(K)_n)$ is the weak* closure in $M_n(E)$ of the ordinary convex hull $\mathrm{co}(C)$ of the set C of terms $a \otimes k$ for a (trace 1 positive selfadjoint) density matrix $a \in M_n(\mathbb{F})^+$ and $x \in K$.*

Proof. To see this, first note that

$$\operatorname{co}(C) = \Big\{ \sum_{j=1}^{m} v_j^* k_j v_j : m \in \mathbb{N},\, k_j \in K,\, v_j \in M_{1,n},\, \sum_j v_j^* v_j = I_n \Big\}.$$

Indeed, if x is in the weak* closure W_n of $\operatorname{co}(C)$ in $M_n(E)$, a limit of x_t, where $x_t \in \operatorname{co}(C)$, and $\beta \in M_{m,n}$ is an isometry, then $\beta^* x_t \beta \to \beta^* x \beta$ weak*. Thus W_n satisfies (3) in the definition of a nc convex set, and similarly it satisfies (2) there, if we also use the fact that $\sum_i \alpha_i x_i \alpha_i^*$ converges weak*. Hence $W = (W_n)$ is a nc convex set. It is closed, and hence compact. Indeed, we may assume that $K \subseteq \operatorname{Ball}(E)$, and hence in $M_n(E)$ the nc convex combinations are in $\operatorname{Ball}(M_n(\operatorname{Max}(E)))$. Since this last ball is weak* compact, so is W. Clearly then this is the smallest closed nc convex set which agrees at the first level with K. ∎

We remark that the weak* closure is unnecessary if $A(K)$ is finite-dimensional and $n < \infty$. For the convex hull of a compact set in a finite-dimensional LCTVS is compact, and the set C above is easily seen to be compact in $M_n(E)$.

If K is the classical state space of an operator system and n is a cardinal, then the weak* closed convex hull W_n of the set C defined in Lemma 6.2 may be called the *separable* (i.e. *nonentangled*) matrix states of V. The lemma asserts that these nonentangled states (for all levels n, i.e. $W = (W_n)$) are the elements of the nc convex set $\operatorname{Min}(K)$.

We shall see next that this is also exactly the nc/matrix state space of $\operatorname{OMIN}(A(K))$.

LEMMA 6.3. *Let $\varphi : V \to B(H)$ be a completely positive selfadjoint map on a real operator system. Let $a = \varphi(1)^{1/2}$. Then there exists a ucp $\Psi : V \to B(H)$ such that $\varphi = a\Psi(\cdot)a$.*

Proof. Note that $\varphi : V_c \to B(H)_c$ is completely positive, so by e.g. [13, Lemma 2.2] there exists a ucp $\Psi : V_c \to B(H)_c$ with $\varphi_c = a\Psi(\cdot)a$. Inspecting the proof of the last cited result we see that $\Psi(V) \subseteq B(H)$ and $\varphi = a\Psi(\cdot)a$. ∎

PROPOSITION 6.4. *Let K be a classical compact convex set. Then $A(\operatorname{Min}(K)) = \operatorname{OMIN}(A(K))$. That is, $\operatorname{Min}(K)$ is the nc convex set corresponding to $\operatorname{OMIN}(A(K))$ via the functorial correspondence between compact nc convex sets and operator systems. In particular, $\operatorname{Min}(K)$ consists of the ucp maps $\operatorname{OMIN}(A(K)) \to M_n$, for all $n \leqslant \kappa$.*

Proof. We just prove the real case. Note that if $f \in A(\operatorname{Min}(K))$ then clearly $f \in A(K)$. Conversely, suppose that $f \in A(K)$. We may assume that K is a subset of a real dual Banach space such that f extends to a linear continuous $\varphi \in E^*$. Indeed, we can take $E = \operatorname{Max}(A(K)^*)$. The restriction of φ_n to K_n defines the desired function from $\operatorname{Min}(K)_n$ to M_n extending f. Call this function f_n; then $f_n(R \otimes k) = Rf(k)$ for a density matrix $R \in M_n^+$

and $k \in K$. This defines the (unique) nc affine function $\hat{f}$ in $A(\mathrm{Min}(K))$, which at first level is $f : K \to \mathbb{F}$.

Note that $\|\hat{f}_n\| = \|f\|$. To see this, note that for $x = \sum_j v_j^* k_j v_j$ with $k_j \in K$, $v_j \in M_{1,n}$ and $\sum_j v_j^* v_j = I_n$ we have

$$\|\hat{f}_n(x)\| = \Big\| \sum_j v_j^* f(k_j) v_j \Big\| \leqslant \max_j |f(k_j)| \leqslant \|f\|.$$

By density and continuity we see that $\|\hat{f}_n(x)\| \leqslant \|f\|$ for all $x \in (\mathrm{Min}(K))_n$.

This proves that the canonical map $A(\mathrm{Min}(K)) \to \mathrm{OMIN}(A(K))$ is a unital isometry, and hence by a basic property of OMIN it is a completely positive complete contraction. Conversely, suppose that $f = [f_{ij}] \in M_n(\mathrm{OMIN}(A(K)))^+$, so that $f(k) \geqslant 0$ for all $k \in K$. We claim that $[\widehat{f_{ij}}] \in M_n(A(\mathrm{Min}(K)))^+$. Indeed, for x as in the last paragraph we have, for an appropriate scalar matrix V,

$$[\widehat{f_{ij}}(x)] = \Big[\sum_j v_j^* f_{ij}(k_j) v_j \Big] = V^* \mathrm{diag}(f(k_1), \dots, f(k_n)) V \geqslant 0.$$

Thus $[\widehat{f_{ij}}] \geqslant 0$, by density and continuity. This proves the claim, so that the canonical map $A(\mathrm{Min}(K)) \to \mathrm{OMIN}(A(K))$ is a unital complete order isomorphism, hence also a complete isometry. ∎

Remark. If K is a classical compact convex set *regularly embedded* in a dual operator space E (see e.g. [1, II.2] or [4]), then one may define a variant of $\mathrm{Min}(K)$ as the closed nc convex hull of K in E. Since it is regularly embedded, any $f \in A(K)$ extends to a weak* continuous $\varphi \in E^*$. The last proof then works to show that this nc convex set is topologically affine nc isomorphic to the nc convex set corresponding to $\mathrm{OMIN}(A(K))$, and hence also to $\mathrm{Min}(K)$ as defined earlier (above Lemma 6.2).

Most of the last result in the complex case and for $n < \infty$ can also be deduced from an assertion in [34, Theorem 4.8 and Remark 4.5], and is equivalent to that assertion. Indeed, we use the above to give a generalization of this result:

Corollary 6.5. *Let V be a real or complex function system, H a real or complex Hilbert space, and $n < \infty$. Any element of $M_n(\mathrm{OMIN}(V)^d)^+$, or any completely positive selfadjoint map $\mathrm{OMIN}(V) \to B(H)$, is a point-weak* limit of a uniformly bounded net of maps of the form $\sum_j v_j^* \varphi_j v_j = \sum_j (v_j^* v_j) \otimes \varphi_j$, for (scalar-valued) states φ_j on V and row vectors v_i with real or complex entries and $\sum_j v_j^* v_j$ strongly convergent.*

Proof. Let $\varphi : V \to B(H)$ be completely positive and selfadjoint. Let $a = \varphi(1)^{1/2}$. By Lemma 6.3 there is a ucp $\Psi : V \to B(H)$ such that $\varphi = a\Psi(\cdot)a$. By Lemma 6.2 and its proof we find that Ψ is a point-weak* limit of maps of the form $\sum_i v_i^* \varphi_j v_j$ for states φ_j and with $\sum_i v_i^* v_j = 1$. Hence φ is a point-weak* limit of maps of the form $\sum_j a v_j^* \, \varphi_j(\cdot) \, v_j a$. Note that $\| \sum_j a v_j^* v_j a\| = \|\varphi(1)\|$, so the net is uniformly bounded by $\|\varphi(1)\|$. ∎

LEMMA 6.6. *Let K be a classical compact convex set (in a real dual Banach space). Then* $\mathrm{Min}_{\mathbb{R}}(K)_c = \mathrm{Min}_{\mathbb{C}}(K)$ *as nc convex sets, with both equaling K at the first level.*

Proof. Complexifying the relation $A_{\mathbb{R}}(\mathrm{Min}_{\mathbb{R}}(K)) = \mathrm{OMIN}_{\mathbb{R}}(A(K))$, we have

$$A_{\mathbb{C}}((\mathrm{Min}_{\mathbb{R}}(K))_c) = \mathrm{OMIN}_{\mathbb{R}}(A(K))_c = \mathrm{OMIN}_{\mathbb{C}}(A_{\mathbb{C}}(K)).$$

In the last equality we used [10, Proposition 9.19] taking $V = \mathrm{OMIN}(A_{\mathbb{R}}(K))$, so that $V_c = A_{\mathbb{C}}(K)$ (note that $A_{\mathbb{R}}(K)$ has the trivial involution at the first level). However, $\mathrm{OMIN}_{\mathbb{C}}(A_{\mathbb{C}}(K)) = A_{\mathbb{C}}(\mathrm{Min}_{\mathbb{C}}(K))$ by the complex case of Proposition 6.4. It follows by the functorial correspondence between compact nc convex sets and operator systems that $\mathrm{Min}_{\mathbb{R}}(K)_c = \mathrm{Min}_{\mathbb{C}}(K)$. ∎

An *OMIN* (resp. *OMAX*) *operator system* is just a (real or complex) function system with OMIN (resp. OMAX) operator system structure, or equivalently equals $\mathrm{OMIN}(A(K))$ (resp. $\mathrm{OMAX}(A(K))$) for a classical closed convex set K. As discovered in [10, Section 9] (see the lines preceding our Section 2.2 above), a real AOU $*$-space (resp. real operator system V) with nontrivial involution cannot be an (resp. cannot have an associated) OMIN (resp. OMAX) operator system. We remind the reader that an OMIN (resp. OMAX) operator system has the property that unital positive maps into (resp. out of) it are ucp.

We define the maximal quantization $\mathrm{Max}(K)$ by the functorial correspondence between compact nc convex sets and operator systems, via $A(\mathrm{Max}(K)) = \mathrm{OMAX}(A(K))$. We know that $\mathrm{Max}(K)$ is the nc set (K^n) with K^n in $M_n(E^*)$ the set of unital positive (selfadjoint) linear maps $\varphi : A_{\mathbb{F}}(K) \to M_n$. (Cf. e.g. [18, end of Section 5], where the maximal one is defined by duality or by the Bipolar Theorem as the prepolar of $(\mathrm{Min}(K))^\circ$. For an appropriate choice of E this will coincide with ours because both are the largest compact nc convex set agreeing with K at 'level 1'.) This is nc convex and nc compact. Indeed, by a basic property of OMAX, (K^n) is the nc matrix state space of $\mathrm{OMAX}(A(K))$. That is:

LEMMA 6.7. *Let K be a classical compact convex set (in a dual Banach space). Then* $\mathrm{Max}(K) = \mathrm{ncS}(\mathrm{OMAX}(A(K)))$.

PROPOSITION 6.8. *Let K be a classical compact convex set (in a real dual Banach space). Then* $\mathrm{Max}_{\mathbb{R}}(K)_c = \mathrm{Max}_{\mathbb{C}}(K)$ *as nc convex sets, with both equaling K at the first level.*

Proof. This is almost identical to the OMIN case. Complexifying the relation $A_{\mathbb{R}}(\mathrm{Max}_{\mathbb{R}}(K)) = \mathrm{OMAX}_{\mathbb{R}}(A(K))$, we have

$$A_{\mathbb{C}}((\mathrm{Max}_{\mathbb{R}}(K))_c) = \mathrm{OMAX}_{\mathbb{R}}(A(K))_c = \mathrm{OMAX}_{\mathbb{C}}(A_{\mathbb{C}}(K)).$$

In the last equality we used [10, Proposition 9.19] with $V = \mathrm{OMAX}(A_{\mathbb{R}}(K))$, so that $V_c = A_{\mathbb{C}}(K)$. However, $\mathrm{OMAX}_{\mathbb{C}}(A_{\mathbb{C}}(K)) = A_{\mathbb{C}}(\mathrm{Max}_{\mathbb{C}}(K))$ by the discussion above the lemma. It follows by the functorial correspondence between compact nc convex sets and operator systems that $(\mathrm{Max}_{\mathbb{R}}(K))_c = \mathrm{Max}_{\mathbb{C}}(K)$. ∎

Remark. We saw above that $(\mathrm{OMAX}_{\mathbb{R}}(A(K)))_c = \mathrm{OMAX}_{\mathbb{C}}(A_{\mathbb{C}}(K))$. This is related to the fact that the 'function system complexification' of a real function system is unique.

For a complex operator system V we get 'minimal and maximal nc convex sets' $\mathrm{Min}(K)$ and $\mathrm{Max}(K)$ as exactly the nc convex sets corresponding to $\mathrm{OMIN}(V)$ and $\mathrm{OMAX}(V)$. In the real case this never works, as we have said repeatedly, unless V has trivial (i.e. identity) involution.

Example 6.9. As an illustration of Lemma 6.6 and Proposition 6.7, consider the real compact operator interval $\mathbb{I}$ from Examples 2.1 and 3.5. As we said there, the complexification of the real operator interval is the complex compact operator interval $\mathbb{J}$. If K is the classical interval then $K = \mathbb{I}_1 = \mathbb{J}_1$. It is probably well known that the complex operator system corresponding to $\mathbb{J}$ is the two-dimensional function system and C^*-algebra $\ell^\infty_2(\mathbb{C})$, and similarly, the real operator system corresponding to $\mathbb{I}$ is $\ell^\infty_2(\mathbb{R})$. Indeed, the reader can find a proof that works in both cases in [8, Example 3.5]. The latter operator system has trivial involution. It is also well known that the usual (C^*-algebraic) operator system structure on $\ell^\infty_2(\mathbb{C})$ is both an OMAX and an OMIN. Such equality is very rare. Thus $\mathrm{Max}_{\mathbb{C}}(K) = \mathrm{Min}_{\mathbb{C}}(K)$ (see [14, Example 16.4.4(1)]). Similarly $\mathbb{I} = \mathrm{Max}_{\mathbb{R}}(K) = \mathrm{Min}_{\mathbb{R}}(K)$. Thus indeed,

$$\mathbb{J} = \mathrm{Min}_{\mathbb{R}}(K)_c = \mathrm{Min}_{\mathbb{C}}(K) = \mathrm{Max}_{\mathbb{R}}(K)_c = \mathrm{Max}_{\mathbb{C}}(K)$$

as nc convex sets, with all equaling K at the first level.

Example 6.10. Consider the real unitary matrix $W = c(i) = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, and the noncommutative convex set of real matrices

$$K = \mathrm{ncconv}\,\{W, -W\} = \mathrm{ncconv}\,\{W\} \subset \mathcal{M}(\mathbb{R}).$$

Somewhat surprisingly, and in stark contrast to the operator interval which, as we just saw, is a minimal nc convex set, K is not a minimal nc convex set, because the first level does not generate the entire nc convex set K. However, its complexification is a minimal nc convex set. From Theorem 3.1 we deduce that $K_c = \mathrm{ncconv}_{\mathbb{C}}\,\{\mathrm{ncconv}_{\mathbb{R}}\,\{W\}\} = \mathrm{ncconv}_{\mathbb{C}}\,\{W\}$. A compression by u

shows $\pm i \in K_c$ so that $\mathrm{ncconv}_{\mathbb{C}}\{\pm i\} \subseteq K_c$. Also,

$$W = \frac{1}{2}\begin{bmatrix}1\\ i\end{bmatrix} i[1 \quad -i] + \frac{1}{2}\begin{bmatrix}1\\ -i\end{bmatrix}(-i)[1 \quad i],$$

so that $K_c = \mathrm{ncconv}_{\mathbb{C}}\{\pm i\}$. It is easy to see (multiplying by i) that this is nc affine isomorphic to the complex operator interval $\mathbb{J}$, so it is a minimal nc convex set. Since $\mathbb{J}$ is compact, so is K_c, and thus K is a compact nc convex set by Theorem 3.1. We work more with these nc convex sets in the sequel [8], where for instance we determine their corresponding operator systems and extreme points. The real operator system corresponding to K is the two-dimensional real C^*-algebra $\mathbb{C}_r$. From this it also follows that K is not a maximal nc convex set, since the operator system corresponding to a maximal nc convex set has trivial involution.

One might think that it is obvious that $\mathrm{Min}(K) \subseteq \mathrm{Max}(K)$ since $\mathrm{Max}(K)$ is a nc convex set containing $(K, \varnothing, \varnothing, \ldots)$. However, recall that $\mathrm{Min}(K)$ naturally 'lives in' $\mathrm{Max}(A(K)^*) = (\mathrm{Min}(A(K))^*$, being the noncommutative states on $\mathrm{OMIN}(A(K))$, while $\mathrm{Max}(K)$ corresponds to the set of noncommutative states on $\mathrm{OMAX}(A(K))$, and naturally 'lives in' $\mathrm{Min}(A(K)^*) = (\mathrm{Max}(A(K))^*$. Of course $\mathrm{Max}(K)$ may be viewed as consisting of the unital selfadjoint positive maps from $\mathrm{OMIN}(A(K))$ into M_n, while $\mathrm{Min}(K)$ is the subset of ucp maps from $\mathrm{OMIN}(A(K))$ into M_n. This is not a problem for finite n, since one may just use the 'product topology' (i.e. work entrywise, see e.g. [7, 1.6.4]); the matrix spaces are isomorphic. For infinite n this suggests that perhaps the 'ambient LCTVS operator space' for both $\mathrm{Min}(K)$ and $\mathrm{Max}(K)$ is $\mathrm{Min}(A(K)^*)$. In any case, the identity map, which is the adjoint of the canonical ucp map $\Phi : \mathrm{OMAX}(A(K)) \to \mathrm{OMIN}(A(K))$, yields a canonical nc affine continuous map $\epsilon : \mathrm{Min}(K) \to \mathrm{Max}(K)$. It is surjective at the first level of course.

PROPOSITION 6.11. *Let K be a classical compact convex set* (*in a dual Banach space*). *The canonical nc affine embedding $\epsilon : \mathrm{Min}(K) \to \mathrm{Max}(K)$ is a nc topological affine embedding. That is, ϵ is a homeomorphism onto its* (*compact*) *range for all n. Similarly for a closed nc convex set L in a dual operator space E with $L_1 = K$, with L symmetric if $\mathbb{F} = \mathbb{R}$, there are canonical nc topological affine embeddings* $\mathrm{Min}(K) \subseteq L \subseteq \mathrm{Max}(K)$. *These maps are the identity map on K at level* 1.

Proof. To see that ϵ is one-to-one note that if $\varphi : \mathrm{OMIN}(A(K)) \to M_n$ satisfies $\varphi \circ \Phi = 0$ for Φ as above, then $\varphi = 0$. The statement then follows from Lemma 3.3.

For the last assertion we suppose that L corresponds to an operator system V. If L is symmetric, V has trivial involution. 'Level 1' of V is a function system with state space K. The canonical ucp maps

$$\mathrm{OMAX}(A(K)) \to V = A(L) \to \mathrm{OMIN}(A(K))$$

dualize to give continuous nc affine maps

$$\mathrm{Min}(K) \to L \to \mathrm{Max}(K) \subseteq \mathrm{Min}(A(K)^*).$$

These maps are the identity map on K at level 1. As in the previous paragraph, these maps are one-to-one, and nc topological affine embeddings by Lemma 3.3. ∎

The following is the real version of [15, Theorem 2.5.8]. We define $A(K_2, M_2(\mathbb{R}))$ to be the (classical) affine continuous maps $K_2 \to M_2(\mathbb{R})$. (In the result below we can also assume if desired that these functions satisfy the compatibility conditions (2), (3) in the definition of $A(K)$ but with all integers $\leqslant 2$.)

THEOREM 6.12. *If K is a symmetric real compact nc convex set then the canonical restriction map $\rho : A(K) \to A(K_1)$ is an isometric unital order isomorphism. More generally, if K is any real compact nc convex set then the canonical restriction map $\rho_2 : A(K) \to A(K_2, M_2(\mathbb{R}))$ is a contractive selfadjoint unital order embedding, and $\|f\| \leqslant 2\|f_{|K_2}\|$ for $f \in A(K)$.*

Proof. Clearly ρ is contractive, unital and positive. Indeed, ρ is simply Kadison's function representation (see e.g. [25, Section 4.3]). For the second assertion, certainly $\rho_2 : A(K) \to A(K_2, M_2(\mathbb{R}))$ is contractive selfadjoint unital and positive. Let $V = A(K)$, $v \in V$, and recall that we may take $K_2 = \mathrm{UCP}(V, M_2(\mathbb{R}))$. Suppose that $\Phi(v) \geqslant 0$ (resp. $\|\Phi(v)\| \leqslant 1$) for all ucp $\Phi : V \to M_2(\mathbb{R})$. For a complex state φ on V_c the map $c \circ \varphi_{|V}$ is a ucp $V \to M_2(\mathbb{R})$. Thus, $c(\varphi(v)) \geqslant 0$ (resp. $\|c(\varphi(v))\| \leqslant 1$), and so $\varphi(v) \geqslant 0$ (resp. $|\varphi(v)| \leqslant 1$). By [15, Theorem 2.5.8] we have $v \geqslant 0$ (resp. $\|v\| \leqslant 2$). ∎

REMARKS. (1) The first assertions of [15, Theorem 2.5.8] are true for all real symmetric compact nc convex sets by a similar proof (e.g. using the real form of the polarization identity).

(2) The necessity of the 'symmetric' condition here, and the use of K_2 versus K_1, is clear by examining the case of the quaternions. Matt Kennedy asked us if one could characterize exactly when ρ is an order isomorphism. In fact, this holds exactly when K is symmetric. Indeed, suppose that K is not symmetric, so that $V = A(K)$ has nontrivial involution by Corollary 4.3, or equivalently possesses nonzero antisymmetric elements. If $0 \neq x \in V_{\mathrm{as}}$ then $\rho(x)(\varphi) = x(\varphi)$ for $\varphi \in K$, while $\rho(x^*)(\varphi) = x(\varphi)$. Since $x^* = -x$ we have $x(\varphi) = 0$. That is, $\rho(x) = 0$. Hence ρ is not even bijective.

The following consequence, extracted from the last proof, may be viewed as an 'improvement' on part of [10, Corollary 3.2]. The main idea in the proof we gave for that though in [10, Corollary 3.2] is essentially the same as our proof above.

Corollary 6.13. *Let V be a real operator system, and $v \in V$. If $\Phi(v) \geqslant 0$ (resp. $\|\Phi(v)\| \leqslant 1$) for all ucp $\Phi : V \to M_2(\mathbb{R})$ then $v \geqslant 0$ (resp. $\|v\| \leqslant 2$).*

Similarly, one may describe the noncommutative state spaces $\mathrm{Min}_k(K)$ and $\mathrm{Max}_k(K)$ of $\mathrm{OMIN}_k(V)$ and $\mathrm{OMAX}_k(V)$ for a real or complex operator system $V = A(K)$ and $k \in \mathbb{N}$. E.g. we define $\mathrm{Max}_k(K)$ by the functorial correspondence between compact nc convex sets and operator systems, via $A(\mathrm{Max}_k(K)) = \mathrm{OMAX}_k(A(K))$. So $\mathrm{Max}_k(K) = \mathrm{ncS}(\mathrm{OMAX}_k(A(K)))$. This agrees with K up to level k, since n-positive states into M_n are ucp, hence are in K_n for $n \leqslant k$. We will not however take the time to add the details here.

Remarks. (1) One cannot however expect analogues of Propositions 6.6 and 6.8 to hold in general for Min_k and Max_k. Indeed, if K is a compact nc convex set (in a real dual operator space) then often $\mathrm{Max}_{\mathbb{R},k}(K)_c \neq \mathrm{Max}_{\mathbb{C},k}(K_c)$. Indeed, this fails in general, as do the matching operator system equalities (matching via the functorial correspondence between compact nc convex sets and operator systems). For $k > 1$ it fails because of the problems with complexifying entanglement breaking maps as seen in [12] and [10, Section 9] (this is spelled out in more detail in later revisions of [10]). For $k = 1$ it can fail because of the existence of real entangled states that are complex separable (i.e. nonentangled) (see [12] and [10, Section 9]); and of course $\mathrm{OMAX}(A(K))$ may not exist as we have said. Indeed, for the quaternions, $\mathrm{Max}_{\mathbb{R}}(K)$ has one point, while $(K_c)_1$ consists of 2×2 complex density matrices.

(2) The minimal and maximal nc convex envelopes $\hat{L}$ and $\check{L}$ of a weak* closed nc convex set in a dual operator space E, constructed at the end of Section 5 in connection with Effros and Winkler's application of the Bipolar Theorem, are not in full generality necessarily well related to Min and Max as defined above. We now explain what we mean by this, together with an example. Indeed, if L is a closed nc convex set in E with $L_1 = K$, then one cannot in general expect $\mathrm{Min}(K) \subseteq L \subseteq \mathrm{Max}(K)$. Here '$\subseteq$' indicates a canonical nc topological affine embedding. For instance, the second inclusion is certainly not valid if L is the nc state space of the quaternions, where K is a singleton. In this case $\mathrm{Max}(K)$ is affine isomorphic to the noncommutative state space of $\mathbb{R}$, a nc convex set with singleton matrix levels, while L_2 is not a singleton. It is interesting to compute $\check{L}$ in this example: we leave it to the reader to check that its nth level is the set of unital maps $\varphi : \mathbb{H} \to M_n$ with $\varphi(b)$ skew (antisymmetric) for $b \in \{i, j, k\}$. That $\mathrm{Min}(K) \subseteq L$ is usually clear in examples, but may not be very helpful (as we saw in the example at the end of Section 5).

On the other hand, if L is a closed nc convex set in a dual operator space E, with L symmetric in the real case, then we showed in Proposition 6.11 that there are topological nc affine embeddings

$$\mathrm{Min}(K) \to L \to \mathrm{Max}(K).$$

The range of the first will be $\hat{L}$ since the nc convex hull of K is dense in both $\mathrm{Min}(K)$ and $\hat{L}$. So we may identify $\mathrm{Min}(K)$ and $\hat{L}$. Alternatively, this also follows from the Remark after Proposition 6.4. We imagine that there is also a way to identify $\mathrm{Max}(K)$ and $\check{L}$ in this case, but will not pursue this further here.

7. Noncommutative functions

7.1. Real nc functions. Let K be a real compact nc convex set. A *nc function* is a map $f : K \to \mathcal{M}(\mathbb{R})$ that is graded, preserves direct sums, and is unitarily equivariant. More specifically, it satisfies the following properties:

(1) $f(K_n) \subseteq M_n(\mathbb{R})$.
(2) $f(\sum \alpha_i x_i \alpha_i^\mathsf{T}) = \sum \alpha_i f(x_i) \alpha_i^\mathsf{T}$ for every family $\{x_i \in K_{n_i}\}$ and collection $\{\alpha_i \in M_{n,n_i}(\mathbb{R})\}$ of isometries such that $\sum \alpha_i \alpha_i^\mathsf{T} = 1_n$.
(3) $f(\beta x \beta^\mathsf{T}) = \beta f(x) \beta^\mathsf{T}$ for every $x \in K_n$ and unitary $\beta \in M_n(\mathbb{R})$.

(Note that (3) is in fact a special case of (2).) We say that f is *bounded* if it is uniformly bounded for all $k \in K$. The space of all real bounded nc functions on K is $B(K)$. This has the uniform norm

$$\|f\| = \sup_{k \in K} \|f(k)\|.$$

We can similarly define $B(K, L)$ for K, L real nc convex sets to be the nc functions from K to L which are bounded. Here, a nc function has the same definition as above, but with $\mathcal{M}(\mathbb{R})$ replaced by L.

As in [15], we have

LEMMA 7.1. *If K is a real compact nc convex set then $B(K)$ is a real C^*-algebra with the uniform norm and pointwise adjoint/product.*

Proof. The main difficulty in showing $B(K)$ is a C^*-algebra is showing it is complete. For the reader's convenience we give the argument to show that it works in the real case. Let $f^r \in B(K)$ be a sequence such that

$$\sum_{r=1}^{\infty} \|f^r\| < \infty.$$

Define $f : K \to \mathcal{M}(\mathbb{F})$ by $f(x) = \sum_{r=1}^{\infty} f^r(x)$ for $x \in K_m$. This converges (absolutely) since $M_n(\mathbb{F})$ is complete. Because the f^r are all graded and unitarily equivariant, so is f. Condition (3) in the definition above is easy to check. For (2), let $\{x_i \in K_{n_i}\}$ be a set of elements in K, and $\alpha_i \in M_{n,n_i}(\mathbb{F})$

a family of isometries such that $\sum_i \alpha_i\alpha_i^* = 1_n$. Take $\xi, \eta \in \ell^2_n$. Then we have

$$\begin{aligned}\sum_{r=1}^{\infty}\sum_i |\langle \alpha_i f^r(x_i)\alpha_i^*\xi,\eta\rangle| &\leqslant \sum_{n=1}^{\infty}\sum_i |\langle \alpha_i f^r(x_i)\alpha_i^*\xi,\eta\rangle| \\ &\leqslant \sum_{n=1}^{\infty}\Big(\sum_i \|f^r(x_i)\alpha_i^*\xi)\|^2\Big)^{1/2}\Big(\sum_i \|\alpha_i^*\eta\|^2\Big)^{1/2} \\ &\leqslant \sum_{n=1}^{\infty}\|f^r\|\Big(\sum_i \|\alpha_i^*\xi)\|^2\Big)^{1/2}\Big(\sum_i \|\alpha_i^*\eta\|^2\Big)^{1/2} \\ &= \sum_{n=1}^{\infty}\|f^r\|\,\|\xi\|\,\|\eta\|,\end{aligned}$$

since e.g. $(\sum_i \|\alpha_i^*\xi\|^2)^{1/2} = (\sum_i\langle\alpha_i\alpha_i^*\xi,\xi\rangle)^{1/2} = \|\xi\|$. Thus we may interchange the order of summation in $\sum_{r=1}^{\infty}\sum_i\langle\alpha_i f^r(x_i)\alpha_i^*\xi,\eta\rangle$. We see that

$$\Big\langle f\Big(\sum_i \alpha_i x_i\alpha_i^*\Big)\xi,\eta\Big\rangle = \Big\langle \sum \alpha_i f(x_i)\alpha_i^*\xi,\eta\Big\rangle$$

as desired, so that (2) holds.

We check that $1 + f^*f$ is invertible for all $f \in B(K)$. Indeed, $g(x) = (1 + f(x)^*f(x))^{-1}$ clearly defines a bounded graded function, and checking item (3) in the definition of nc function is easy. As for item (2) in that definition, suppose that we have $\alpha_i \in M_{n,n_i}(\mathbb{R})$ such that $\sum \alpha_i\alpha_i^{\mathsf{T}} = 1_n$ and $x_i \in K_{n_i}$. Because $p_i = \alpha_i\alpha_i^{\mathsf{T}}$ are mutually orthogonal projections which sum to 1, we have $\alpha_i^{\mathsf{T}}\alpha_j = \delta_{ij}1_{n_i}$. So,

$$\begin{aligned}g\Big(\sum \alpha_i x_i\alpha_i^{\mathsf{T}}\Big) &= \Big(1 + f\Big(\sum \alpha_i x_i\alpha_i^{\mathsf{T}}\Big)^* f\Big(\sum \alpha_j x_j\alpha_j^{\mathsf{T}}\Big)\Big)^{-1} \\ &= \Big(1 + \sum \alpha_i f(x_i)^* f(x_i)\alpha_i^{\mathsf{T}}\Big)^{-1} \\ &= \Big(\sum \alpha_i(1 + f(x_i)^* f(x_i))\alpha_i^{\mathsf{T}}\Big)^{-1},\end{aligned}$$

where the second equality is because of orthogonality. Here the inverse of $\sum \alpha_i(1+f(x_i)^*f(x_i))\alpha_i^{\mathsf{T}}$ is $\sum \alpha_i(1+f(x_i)^*f(x_i))^{-1}\alpha_i^{\mathsf{T}}$. Indeed, $(\sum p_iz_ip_i)^{-1} = \sum p_iw_ip_i$ if $z_iw_i = w_iz_i = p_i$ and $z_i, w_i \in p_iM_np_i$. This completes the proof. ∎

As in the complex case, $A(K) \hookrightarrow B(K)$ and we define $C(K)$ to be the C^*-algebra generated by $A(K)$ in $B(K)$.

7.2. Maximal C^*-algebra. Let S be a real (or complex) operator system. The maximal C^*-algebra generated by S, denoted $C^*_{\max}(S)$, is the

C^*-algebra satisfying the following universal property:

$$\begin{array}{ccc} S & \overset{\iota}{\hookrightarrow} & C^*_{\max}(S) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \pi} \\ & & A \end{array}$$

where ι is a ucoi into a C^*-algebra A such that $C^*(\iota(S)) = C^*_{\max}(S)$, φ is a ucoe such that $C^*(\varphi(S)) = A$, and π is an induced $*$-homomorphism.

Lemma 7.2 ([10, Lemma 5.1]). *For a real operator system S we have $C^*_{\max}(S_c) \cong C^*_{\max}(S)_c$ where the prior is the complex maximal C^*-algebra of the complex operator system S_c and the latter is the complexification of the real maximal C^*-algebra of S.*

For any compact nc convex set K we have $K \cong \mathrm{ncS}(A(K))$, and therefore any $k \in K_n$ corresponds to a nc state from $A(K)$ to M_n. The universal property of $C^*_{\max}(A(K))$ gives a $*$-homomorphism

$$\delta_x : C^*_{\max}(A(K)) \to M_n$$

such that $\delta_x \circ \iota = \hat{x}$. Taking the double adjoint gives a normal $*$-homomorphism $\delta_x^{**} : C^*_{\max}(A(K))^{**} \to M_n$ and with this we define the map $\sigma : C^*_{\max}(A(K))^{**} \to B(K)$ by

$$\sigma(b)(x) = \delta_x^{**}(b) \quad \text{for } b \in C^*_{\max}(A(K))^{**} \text{ and } x \in K.$$

Theorem 4.3.3 of [15] (or [14, Theorem 16.8.10]) shows that for a complex compact nc convex set, $B(K)$ is von Neumann algebraically isomorphic to $C^*_{\max,\mathbb{C}}(A(K))^{**}$ via the map σ. The proof also shows that σ restricts to an isomorphism between $C^*_{\max,\mathbb{C}}(A(K))$ and $C(K)$ and that elements of $C(K)$ are the point-strong continuous nc functions on K. We will prove the real analogue of Theorem 4.3.3 using complexification. Therefore, we need the following lemmas:

Lemma 7.3. *For real nc convex sets K, L, every real bounded nc map $f : K \to L$ has a unique complex bounded nc extension $f_c : K_c \to L_c$. If L is complex bounded nc affine, there is a complex bounded nc extension $K_c \to L$. These extensions are strongly continuous if f is strongly continuous.*

Proof. For $x + iy \in (K_c)_n$ define

$$f_c(x+iy) = \frac{1}{\sqrt{2}}[1_n \quad i \cdot 1_n]\, f(c(x+iy))\, \frac{1}{\sqrt{2}} \begin{bmatrix} 1_n \\ -i \cdot 1_n \end{bmatrix}.$$

This function is a complex nc function by a proof similar to Theorem 3.9. First, it is clearly graded. Then, for $\beta \in M_n(\mathbb{C})$ unitary and $x + iy \in (K_c)_n$, we have

$$\begin{aligned} f_c(\beta(x+iy)\beta^*) &= \frac{1}{2}[1_n \quad i\cdot 1_n]\, f(c(\beta)c(x+iy)c(\beta)^{\mathsf T}) \begin{bmatrix} 1_n \\ -i\cdot 1_n \end{bmatrix} \\ &= \frac{1}{2}[1_n \quad i\cdot 1_n]\, c(\beta) f(c(x+iy))c(\beta)^{\mathsf T} \begin{bmatrix} 1_n \\ -i\cdot 1_n \end{bmatrix} \\ &= \beta\frac{1}{2}[1_n \quad i\cdot 1_n]\, f(c(x+iy)) \begin{bmatrix} 1_n \\ -i\cdot 1_n \end{bmatrix}\beta^* \\ &= \beta f_c(x+iy)\beta^*, \end{aligned}$$

which implies (3). A similar proof shows (2). If f is bounded, then f_c will be bounded by Ruan's first axiom. If f is SOT continuous then so is f_c because the function c is SOT continuous and so is adjoining the matrix $[1_n \quad -i\cdot 1_n]$.

This will be the unique extension because for any complex nc function on K_c extending f, say g, we have

$$(x+iy)\oplus(x-iy) = \frac{1}{2}\begin{bmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{bmatrix} c(x+iy) \begin{bmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{bmatrix},$$

so

$$\begin{aligned} g(x+iy)\oplus g(x-iy) &= g\left(\frac{1}{2}\begin{bmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{bmatrix} c(x+iy) \begin{bmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{bmatrix} f(c(x+iy)) \begin{bmatrix} 1_n & -i\cdot 1_n \\ -i\cdot 1_n & 1_n \end{bmatrix}. \end{aligned}$$

Here we first used the fact that $g(x\oplus y) = g(x)\oplus g(y)$. The last part of the above equation is a $2n\times 2n$ matrix with top-right and bottom-left corners being 0. Comparing the top-left corners of the matrices in the above equation gives

$$g(x+iy) = \frac{1}{\sqrt{2}}[1_n \quad i\cdot 1_n]\, f(c(x+iy))\,\frac{1}{\sqrt{2}} \begin{bmatrix} 1_n \\ -i\cdot 1_n \end{bmatrix}$$

as desired. ∎

LEMMA 7.4. *The algebra $B(K)$ is a real W^*-algebra, and $B(K_c) \cong B(K)_c$ as complex von Neumann algebras.*

Proof. We use the same maps as in Theorem 3.9, where we have a map

$$\Gamma : B(K_c) \to B(K)_c, \qquad \omega \mapsto \mathrm{Re}\circ\omega_{|K} + i\,\mathrm{Im}\circ\omega_{|K},$$

with inverse Ψ. To see that Γ is well defined note

$$\|\mathrm{Re}(\omega(x+i0))\| \leqslant \|\omega(x)\| \leqslant \|\omega\|, \qquad x\in K,$$

and similarly for the imaginary part. Therefore,

$$\|c(\operatorname{Re}\omega + i\operatorname{Im}\omega)\| \leqslant \|\operatorname{Re}\omega\| + \|\operatorname{Im}\omega\| \leqslant 2\|\omega\|.$$

A similar proof to that of Lemma 2.6 shows that Re and Im are unitarily equivariant. Also Γ is a unital complete order isomorphism, as may be shown similarly to the proof of Theorem 3.9, and therefore is a $*$-isomorphism. It follows that $B(K)_c$ is a W^*-algebra, and Γ is automatically normal. The canonical period 2 real $*$-automorphism θ on $B(K)_c$ is weak* continuous, and so its fixed point algebra, hence $B(K)$, is a real W^*-algebra. Therefore $B(K_c) \cong B(K)_c$ via Γ. Note that the proof in Theorem 3.9 that the inverse of Γ is Ψ does not quite work here because in showing $\Psi \circ \Gamma = \mathrm{Id}$ we have used compressions. However, we can adjust that proof to use unitaries as we did in the proof of Lemma 7.3 to make it work. Namely, for $\omega \in B(K_c)$ and $x + iy \in (K_c)_n$ we have $\omega(x+iy) = u_n^*\omega(c(x+iy))u_n$ exactly as at the end of the proof of Lemma 7.3. It follows that

$$\begin{aligned}\omega(x+iy) &= u_n^*\omega(c(x+iy))u_n \\ &= u_n^*[(\operatorname{Re}\circ\omega)(c(x+iy)) + i(\operatorname{Im}\circ\omega)(c(x+iy))]u_n \\ &= \Psi((\operatorname{Re}\circ\omega) + i(\operatorname{Im}\circ\omega))(x+iy) = \Psi(\Gamma(\omega))(x+iy).\end{aligned}$$

The equality $\Gamma \circ \Psi = \mathrm{Id}$ is proved exactly as at the end of the proof of Theorem 3.9. ∎

Proposition 7.5. *$C(K_c) \cong C(K)_c$ as complex C^*-algebras.*

Proof. We have

$$C(K_c) = C^*(A(K_c)) \cong C^*(A(K)_c) \cong C^*(A(K))_c = C(K)_c.$$

We have used the fact that if A is a subsystem of a real C^*-algebra B then $C^*(A_c) = C^*(A) + iC^*(A)$ in B_c. ∎

Lemma 7.6. *Let $S(K)$ be the point-strong continuous functions in $B(K)$. Then $S(K_c) \cong S(K)_c$ as C^*-algebras coming from the isomorphism $B(K_c) \cong B(K)_c$.*

Proof. We just need to show that the map Γ in the proof of Lemma 7.4 satisfies $\Gamma(S(K_c)) = S(K)_c$. Let $\omega \in S(K_c)$ and $x_\lambda \to x \in K_n$. Because Re is a contraction, $\operatorname{Re}(\omega(x_\lambda + i0)) \to \operatorname{Re}(\omega(x + i0))$ in the strong operator topology, and similarly for Im. Therefore,

$$c(\operatorname{Re}\omega + i\operatorname{Im}\omega)(x_\lambda) \xrightarrow{\mathrm{SOT}} c(\operatorname{Re}\omega + i\operatorname{Im}\omega)(x).$$

Thus $\Gamma(\omega) \in S(K)_c$. The proof of the converse inclusion is similar, using the map Ψ defined above Theorem 3.9. ∎

Theorem 7.7 (Real case of [15, Theorem 4.3.3]). *Let K be a real compact nc convex set. Then the map $\sigma : C^*_{\max}(A(K))^{**} \to B(K)$ is a real linear normal $*$-isomorphism, which restricts to a $*$-isomorphism from $C^*_{\max}(A(K))$*

onto $C(K)$. The elements of $C(K)$ are the point-strong continuous nc functions on K. Also, $\sigma \circ \iota$ is the identity map on $A(K)$.

Proof. For $\mathcal{A}$ a real C^*-algebra, we have $(\mathcal{A}_c)^* \cong (\mathcal{A}^*)_c$ by [38]. The ensuing map $(\mathcal{A}_c)^{**} \to (\mathcal{A}^{**})_c$ is a unital complete order isomorphism and normal $*$-isomorphism of complex von Neumann algebras [29]. Moreover,

$$(C^*_{\max,\mathbb{R}}(A(K))^{**})_c \cong (C^*_{\max,\mathbb{R}}(A(K))_c)^{**} \cong C^*_{\max,\mathbb{C}}(A(K)_c)^{**} \\ \cong C^*_{\max,\mathbb{C}}(A(K_c))^{**}.$$

Consequently, Lemma 7.2 and Theorem 3.9 give the complex case of this theorem, and Lemma 7.4 gives

$$C^*_{\max,\mathbb{C}}(A(K_c))^{**} \cong B(K_c) \cong B(K)_c,$$

with the composition $C^*_{\max,\mathbb{C}}(A(K))^{**} \cong B(K)_c \cong B(K_c)$ being via a (complex) normal $*$-isomorphism, say π. There is a real embedding of $C^*_{\max,\mathbb{R}}(A(K))^{**}$ into $(C^*_{\max,\mathbb{R}}(A(K))^{**})_c$, and similarly for $B(K)$ into $B(K)_c$. A diagram chase shows that the restriction of the complex normal $*$-isomorphism above is a real normal $*$-isomorphism $C^*_{\max,\mathbb{R}}(A(K))^{**} \cong B(K)$, which is the 'identity map' on the copies of $A(K)$. From this we see again that $B(K)$ is a von Neumann algebra.

To do the diagram chase, let $f \in A(K)$. This embeds into $C^*_{\max,\mathbb{R}}(A(K))^{**}$ and is denoted by $\theta(f)$. Going 'up' in the diagram gives an element in $(C^*_{\max,\mathbb{R}}(A(K))^{**})_c$, namely $\theta(f)+i0$. The isomorphism $(C^*_{\max,\mathbb{R}}(A(K))^{**})_c \cong C^*_{\max,\mathbb{C}}(A(K)_c)^{**}$ will take $\theta(f)+i0$ to $\theta(f+i0)$. The third isomorphism in the above centered equations takes our element to $\theta(\psi(f+i0))$. Taking σ of this element and evaluating at $x+i0 \in K_c$ for $x \in K_n$ will give an element of $M_n(\mathbb{C})^{**}$. We evaluate this functional at $A \in M_n(\mathbb{C})^*$ and get

$$\sigma(\theta(\psi(f+i0)))(x+i0)(A) = \delta^{**}_{x+i0}(\theta(\psi(f+i0)))(A) = A(\delta_{x+i0}(\psi(f+i0))) \\ = A(\psi(f+i0)(x+i0)) = A(f(x)).$$

On the other hand,

$$\sigma(\theta(f))(x)(A) = \delta^{**}_x(\theta(f))(A) = A(f(x)).$$

Therefore, the diagram chase shows that it is the 'identity map' on the copies of $A(K)$ that extends to π. Since $\pi(A(K))$ is the copy of $A(K)$ in $B(K)$ inside $B(K_c)$, it follows that $\pi(C^*_{\max}(A(K))^{**})$ is a C^*-subalgebra D of $B(K)$ with $D+iD = B(K)_c$. Hence $D = B(K) = \pi(C^*_{\max}(A(K))^{**})$. Also, $I_{A(K)}$ extends to a $*$-isomorphism between the C^*-algebra generated by $A(K)$ in both sets, so $C^*_{\max}(A(K)) \cong C(K)$. It also shows that $\sigma \circ \iota$ is the identity map on $A(K)$. The fact that $C(K)$ consists of the point-strong continuous nc functions follows from the complex case and Lemma 7.6. Indeed, $C(K) = B(K) \cap C(K)_c = B(K) \cap C(K_c)$. ∎

As we saw after Lemma 7.2, any element k of K defines a nc/matrix state on $C(K)$, and a weak* continuous matrix state on $B(K)$. In particular, $f \mapsto f(k)$ is weak* continuous on $B(K)$.

Corollary 7.8. *Let K be a real compact nc convex set. The real enveloping von Neumann algebra $C(K)^{**}$ of $C(K)$ is $*$-isomorphic to the real von Neumann algebra $B(K)$ of bounded nc functions on K. The real dual operator system $A(K)^{**}$ is completely order isomorphic to the real operator system of bounded nc affine functions on K. The latter space has as complexification the bounded complex nc affine functions on A_c.*

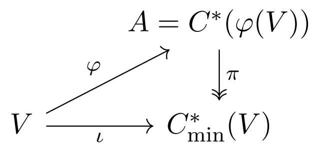
Proof. To see that $A(K)^{**}$ is completely order isomorphic to the real operator system of bounded nc affine functions on K, note that $A(K)^{\perp\perp} \subseteq B(K)$. We need to show $f \in B(K)$ is in $A(K)^{\perp\perp}$ if and only if $f(\beta^* x \beta) = \beta^* f(x) \beta$ for every $x \in K_m$ and isometry $\beta \in M_{m,n}(\mathbb{R})$. Since this is true for $f \in A(K)$, it will also be true by a weak* approximation argument for $f \in A(K)^{\perp\perp}$, using the fact above the corollary. This gives a weak* continuous ucoe $\nu : A(K)^{\perp\perp} \to BA(K)$, where $BA(K)$ is the operator subsystem of bounded nc affine functions in $B(K)$. Then $BA(K)_c = BA(K) + iBA(K) = BA_{\mathbb{C}}(K_c)$ in $B(K)_c = B(K_c)$. The rest follows by complexification from [15, Theorem 4.3.3] (see also [14, Theorem 16.8.10]). Indeed, if ν were not surjective then its complexification would not be either, contradicting $BA_{\mathbb{C}}(K_c) \cong A_{\mathbb{C}}(K_c)^{\perp\perp}$. ∎

Proposition 7.9. *Let K be a real compact nc convex set and $f \in C(K)$ a continuous nc function. Then f is bounded with*

$$\|f\| = \sup_{n<\infty} \|f|_{K_n}\|.$$

In [15, Proposition 2.5.3] (or [14, Remark 16.6.2(2)]) there is a more general version of the last result in the complex case; this holds with the same proof in the real case.

7.3. Minimal C^*- algebra. As in the complex case, every real operator system V has a C^*-envelope or minimal C^*-algebra denoted by $C^*_{\min}(V)$. There is a ucoe $\iota : V \to C^*_{\min}(V)$ which satisfies the following universal property:



where φ is a ucoe of V into another real C^*-algebra A such that $\varphi(V)$ generates A as a C^*-algebra, and π is an induced surjective $*$-homomorphism making the diagram commute.

Lemma 7.10 ([5, Corollary 4.3]). *For a real operator system S we have $C^*_{\min}(S_c) \cong C^*_{\min}(S)_c$ where the prior is the complex minimal C^*-algebra of the complex operator system S_c and the latter is the complexification of the real minimal C^*-algebra of S.*

Example 7.11. If $\mathcal{A}$ is a real/complex C^*-algebra viewed as an operator system, the universal property shows $C^*_{\min}(\mathcal{A}) = \mathcal{A}$. There exist C^*-algebras that are not complexifications of real C^*-algebras by [36, Problem 1.5] and we can use this to construct complex operator systems and compact nc convex sets which are not complexifications. For instance, let A be such a complex C^*-algebra viewed as an operator system; then if it were the complexification of a real operator system V we would get

$$\mathcal{A} = C^*_{\min}(\mathcal{A}) = C^*_{\min}(V_c) \cong C^*_{\min}(V)_c,$$

and the latter is the complexification of a real C^*-algebra, which is a contradiction. Similarly, if every complex compact nc convex set were a complexification, then $\mathrm{ncS}(\mathcal{A})$ would be L_c for some real compact nc convex set L. By Theorem 4.1 we deduce that $\mathcal{A} \cong A(L_c)$ as complex operator systems and so

$$\mathcal{A} \cong C^*_{\min}(A(L_c)) \cong C^*_{\min}(A(L))_c,$$

which is a contradiction.

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Similarly we thank Scott McCullough for very kind input (solicited by us) on our paper, and on real matrix convexity, and for drawing our attention to several recent papers in that area. He mentioned to us for example that there is some complexification procedure in the finite-dimensional setting of [19, Section 4.1]. We also thank James Pascoe for some comments at a similar time, and Ken Davidson for drawing our attention to [14].

We also thank the referee for identifying some things that were not clearly expressed in our preprint, and for several comments and useful suggestions.

Finally, we mention two related recent papers by the first author released in January 2026: [6] (with D. M. Hay), and [4]. For example, we saw in the present paper that a real nc compact convex set corresponds to a real nc state space. However, this is essentially also the real *nc base* for a real *nc base norm space*, and similarly in the complex case.

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