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Quantum automorphism groups of direct sums of Cuntz algebras

by

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Abstract. We explore the quantum symmetry of the direct sum of a finite family $\{\mathcal{O}_{n_i}\}_{i=1}^m$ of Cuntz algebras, viewing them as graph C^*-algebras associated to the graphs $\{L_{n_i}\}_{i=1}^m$ (where L_n denotes the graph containing n loops based at a single vertex), in the category introduced by Joardar and Mandal (2018). We show that the quantum automorphism group of the direct sum of non-isomorphic Cuntz algebras is $U_{n_1}^+ * \cdots * U_{n_m}^+$ for distinct n_i's, i.e.

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m L_{n_i}\Big) \cong \mathop{*}_{i=1}^m Q_\tau^{\mathrm{Lin}}(L_{n_i}) \cong U_{n_1}^+ * \cdots * U_{n_m}^+,$$

where $Q_\tau^{\mathrm{Lin}}(\Gamma)$ denotes the quantum automorphism group of the graph C^*-algebra associated to Γ. Also, the quantum automorphism group of the direct sum of m copies of isomorphic Cuntz algebra $\mathcal{O}_n$ is $U_n^+ \wr_* S_m^+$, i.e.

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m L_n\Big) \cong Q_\tau^{\mathrm{Lin}}(L_n) \wr_* S_m^+ \cong U_n^+ \wr_* S_m^+.$$

Furthermore, we provide counterexamples to demonstrate that the isomorphisms above cannot be generalized to arbitrary graph C^*-algebras, whereas analogous relations do extend to quantum automorphism groups of graphs in the sense of Banica and Bichon.

1. Introduction. The Cuntz algebra, introduced by Joachim Cuntz in 1977 [Cu77], is an interesting example of a C^*-algebra that can be described as a universal C^*-algebra in terms of generators and algebraic relations. Formally, the Cuntz algebra with n generators (denoted by $\mathcal{O}_n$) is the universal C^*-algebra with generators $s_1, \ldots, s_n$ and relations $s_i^* s_i = 1$ for all $i \in \{1, \ldots, n\}$ (i.e. all s_i's are isometries) and $\sum_{i=1}^n s_i s_i^* = 1$. By construction, $\mathcal{O}_n$ is commutative if and only if $n = 1$, and $\mathcal{O}_1$ is isomorphic to $C(S^1)$. Moreover, $\mathcal{O}_n$ and $\mathcal{O}_m$ are isomorphic iff $m = n$. Interestingly, the Cuntz

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algebras can be viewed as graph C^*-algebras. A graph C^*-algebra (denoted by $C^*(\Gamma)$) is a universal C^*-algebra generated by some orthogonal projections and partial isometries coming from a given directed graph Γ. Though not every C^*-algebra can be thought of as a graph C^*-algebra (for instance $C(S^1 \times S^1)$ cannot), many important examples of C^*-algebras, including matrix algebras, Cuntz algebras, Toeplitz algebra etc. can be recognized as graph C^*-algebras. In particular, $\mathcal{O}_n$ can be realized as a graph C^*-algebra with respect to the graph containing n distinct loops based at a single vertex. Moreover, a graph C^*-algebra is a Cuntz–Krieger algebra (introduced in [CK80]) if the underlying finite graph contains no sink [KP$^+$97]. The importance of graph C^*-algebras lies in the fact that some graph-theoretic properties of the underlying graph Γ can be recovered from the operator-algebraic properties of $C^*(\Gamma)$ and vice versa (see [KPR98, To06, Ra] for more details).

On the other hand, compact quantum groups (for short, CQG) appeared in mathematics in the 20th century, almost a hundred years after the appearance of the group. Using an analytic construction, S. L. Woronowicz constructed a few initial examples of CQGs in [Wo87]. In non-commutative geometry, mathematicians were always highly determined to capture an appropriate notion of symmetry for non-commutative spaces. The aim was to generalize the concept of classical group symmetry to develop a 'non-commutative version of symmetry' [Co94]. In 1998, Shuzhou Wang defined a quantum automorphism group for a finite space X_n (containing n points) as the 'universal object' in a category of compact quantum groups acting on the unital C^*-algebra $C(X_n)$. He showed that the quantum automorphism group of X_n is larger than the classical automorphism group of X_n for $n > 3$ and its associated C^*-algebra is infinite-dimensional and non-commutative. Moreover, he classified the quantum automorphism groups for any finite-dimensional C^*-algebras (see [Wa98]). In 2003, J. Bichon [Bi03] introduced the quantum symmetry structure for a finite graph Γ; we denote the corresponding quantum automorphism group of Γ by $\mathrm{QAut}_{\mathrm{Bic}}(\Gamma)$. Later, quantum symmetry for a finite graph Γ was also extended by T. Banica [Ba05] and the quantum automorphism group of Γ in the sense of Banica is denoted by $\mathrm{QAut}_{\mathrm{Ban}}(\Gamma)$. The notion of quantum isometry group (in an infinite-dimensional set-up) was defined by Goswami [Go09]. A few years later, adopting the key ideas from their works, T. Banica and A. Skalski proposed the notion of orthogonal filtration on a C^*-algebra, equipped with a faithful state, and its quantum symmetry [BS13]. Since the graph C^*-algebra associated to a finite, directed graph may indeed be infinite-dimensional, in 2017, an interesting study about the quantum symmetry of a graph C^*-algebra $C^*(\Gamma)$ was presented by S. Schmidt and M. Weber [SW17]. Inspired by their work, in 2018–2021, S. Joardar and A. Mandal studied the quantum symmetry of a graph C^*-algebra $C^*(\Gamma)$ for a finite graph Γ without isolated vertices, in a categorical framework considering two different categories, $\mathfrak{C}_\tau^{\mathrm{Lin}}(\Gamma)$ and $\mathfrak{C}_{\mathrm{KMS}}^{\mathrm{Lin}}(\Gamma)$, introduced in [JM18] and [JM21] respectively. The quantum symmetry group $\mathrm{QAut}_{\mathrm{SW}}(\Gamma)$ of the graph C^*-algebra coincides with the quantum automorphism group (of the underlying graph) $\mathrm{QAut}_{\mathrm{Ban}}(\Gamma)$; $Q_\tau^{\mathrm{Lin}}(\Gamma)$ is strictly larger than $\mathrm{QAut}_{\mathrm{Ban}}(\Gamma)$. Moreover, for a finite graph Γ without a sink, one can also consider the category $\mathfrak{C}_{\mathrm{KMS}}^{\mathrm{Lin}}(\Gamma)$ and under certain conditions, the category $\mathfrak{C}_\tau^{\mathrm{Lin}}(\Gamma)$ coincides with $\mathfrak{C}_{\mathrm{KMS}}^{\mathrm{Lin}}(\Gamma)$ (see [JM21]).

Now, if we take n graphs $\{\Gamma_i\}_{i=1}^n$ and consider their disjoint union $\bigsqcup_{i=1}^n \Gamma_i$, then it is natural to ask about the relation between the quantum symmetry of the graph $\bigsqcup_{i=1}^n \Gamma_i$ and the individual quantum symmetries of the graphs Γ_i in the sense of Banica or Bichon: is there any relation between the groups $\mathrm{QAut}_{\mathrm{Ban}}(\bigsqcup_{i=1}^n \Gamma_i)$ $[\mathrm{QAut}_{\mathrm{Bic}}(\bigsqcup_{i=1}^n \Gamma_i)]$ and $\{\mathrm{QAut}_{\mathrm{Ban}}(\Gamma_i)\}_{i=1}^n$ $[\{\mathrm{QAut}_{\mathrm{Bic}}(\Gamma_i)\}_{i=1}^n]$? It is well known that if Γ_i's are connected and mutually 'quantum non-isomorphic', then

$$\mathrm{QAut}_{\mathrm{Ban}}\Big(\bigsqcup_{i=1}^n \Gamma_i\Big) \cong \mathop{\ast}_{i=1}^{n} \mathrm{QAut}_{\mathrm{Ban}}(\Gamma_i)$$

(we refer to [LMR20, Sc20, DK$^+$26, Me] for details). Moreover, if all Γ_i's are connected and isomorphic to each other, then

$$\mathrm{QAut}_{\mathrm{Ban}}\Big(\bigsqcup_{i=1}^n \Gamma_i\Big) \cong \mathrm{QAut}_{\mathrm{Ban}}(\Gamma_1) \wr_* S_n^+ \quad \text{([BB07])},$$

$$\mathrm{QAut}_{\mathrm{Bic}}\Big(\bigsqcup_{i=1}^n \Gamma_i\Big) \cong \mathrm{QAut}_{\mathrm{Bic}}(\Gamma_1) \wr_* S_n^+ \quad \text{([Bi04])}.$$

Similar questions also arise in a graph C^*-algebraic context, i.e. what is the relation between $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^n \Gamma_i)$ and $\{Q_\tau^{\mathrm{Lin}}(\Gamma_i)\}_{i=1}^n$? Do analogous results hold for the quantum symmetries of graph C^*-algebras? More precisely, are the following true: (i) $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^n \Gamma_i) \cong Q_\tau^{\mathrm{Lin}}(\Gamma_1) * \cdots * Q_\tau^{\mathrm{Lin}}(\Gamma_n)$ if Γ_i's are mutually non-isomorphic graphs? (ii) $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^n \Gamma_i) \cong Q_\tau^{\mathrm{Lin}}(\Gamma_1) \wr_* S_n^+$ if all Γ_i's are isomorphic? However, the answers are negative in both scenarios. We have found non-isomorphic (even 'quantum non-isomorphic') graphs, specifically P_1 and So_2 (see Figure 4 in Section 3.4 for details), for which relation (i) does not hold in general. Interestingly, (ii) does not hold either when considering two disjoint copies of P_1 (see Figure 6). But we have also found a class of graph C^*-algebras, namely Cuntz algebras $\mathcal{O}_n$ (whose underlying graph is L_n), for which both (i) and (ii) hold in the sense that

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m L_{n_i}\Big) \cong \mathop{\ast}_{i=1}^{m} Q_\tau^{\mathrm{Lin}}(L_{n_i}) \cong U_{n_1}^+ * \cdots * U_{n_m}^+,$$

if L_{n_i}'s are mutually non-isomorphic graphs (i.e. n_i's are distinct), and

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^{m} L_n\Big) \cong Q_\tau^{\mathrm{Lin}}(L_n) \wr_* S_m^+ \cong U_n^+ \wr_* S_m^+.$$

Now, we briefly discuss the organization of this article: In Section 2, some prerequisites are recalled about directed graphs, graph C^*-algebras, compact quantum groups and their action on a C^*-algebra, orthogonal filtrations, quantum automorphism groups etc. Moreover, we recall the quantum symmetry of a graph C^*-algebra in the category introduced in [JM18, JM21]. We consider three categories on the direct sum of Cuntz algebras: (i) the category $\mathfrak{C}_\tau^{\mathrm{Lin}}$ from [JM18], (ii) a modified category of KMS states on the direct sum of Cuntz algebras, and (iii) the orthogonal filtration preserving category. In Section 3, we describe the quantum automorphism group of the direct sum of non-isomorphic Cuntz algebras in the above categories. Additionally, we provide a counterexample to demonstrate that an analogous relation does not hold in general. In Section 4, we explore the quantum automorphism group of isomorphic Cuntz algebras and provide a counterexample to illustrate that the same formula cannot be extended to all graph C^*-algebras.

2. Preliminaries

2.1. Notations and conventions. For a set X, $|X|$ will denote the cardinality of X and id_X will denote the identity function on X. The n-set $\{1,\dots,n\}$ will be denoted by $[n]$, and $I_{n\times n}$ is the identity matrix on $M_n(\mathbb{C})$. For a C^*-algebra $\mathcal{B}$, $\mathcal{B}^*$ is the set of all bounded linear functionals on $\mathcal{B}$. For a set X, $\operatorname{span} X$ will denote the linear space spanned by the elements of X. The tensor product '$\otimes$' is the spatial or minimal tensor product between two C^*-algebras.

For us, all the C^*-algebras are unital.

2.2. The Cuntz algebra. The Cuntz algebra $\mathcal{O}_n$ was introduced by Cuntz [Cu77] in 1977. An element s in a C^*-algebra is an *isometry* if $s^*s = 1$.

Definition 2.1. The *Cuntz algebra* $\mathcal{O}_n$ (with n generators) is the universal C^*-algebra generated by isometries $s_1,\dots,s_n$ such that $\sum_{i=1}^n s_i s_i^* = 1$, i.e.

$$\mathcal{O}_n := C^*\Big\{s_1,\dots,s_n \;\Big|\; s_i^*s_i = 1\ \forall i\in[n],\ \sum_{i=1}^{n} s_i s_i^* = 1\Big\}.$$

Clearly, $\mathcal{O}_1$ is C^*-isomorphic to $C(S^1)$. Moreover, since $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ for all $n\in\mathbb{N}$, $\mathcal{O}_n$ and $\mathcal{O}_m$ are isomorphic iff $m = n$ [Cu81].

An important fact is that one can also view the Cuntz algebra $\mathcal{O}_n$ as a graph C^*-algebra. Next, we introduce the graph C^*-algebra for a finite, directed graph.

A directed graph $\Gamma = \{V(\Gamma), E(\Gamma), s, r\}$ consists of countable sets $V(\Gamma)$ of vertices and $E(\Gamma)$ of edges together with the maps $s, r : E(\Gamma) \to V(\Gamma)$ describing the source and range of the edges. We say that a vertex $v \in V(\Gamma)$ is *adjacent to* $w \in V(\Gamma)$ (denoted by $v \to w$) if there exists an edge $e \in E(\Gamma)$ such that $v = s(e)$ and $w = r(e)$. A graph is said to be *finite* if both $|V(\Gamma)|$ and $|E(\Gamma)|$ are finite. A directed *graph without isolated vertices* means that for every vertex $v \in V(\Gamma)$, $s^{-1}(v)$ or $r^{-1}(v)$ is non-empty. A *directed path* α of length n in a directed graph Γ is a sequence $\alpha = e_1 \dots e_n$ of edges in Γ such that $r(e_i) = s(e_{i+1})$ for $1 \le i \le n-1$. We define $s(\alpha) := s(e_1)$ and $r(\alpha) := r(e_n)$. Let $E^{<\infty}(\Gamma)$ denote the set of all finite length paths on Γ. A *loop* is a graph with a vertex v and an edge e such that $s(e) = r(e) = v$. Let $\Gamma = \{V(\Gamma), E(\Gamma), s, r\}$ be a finite, directed graph with $|V(\Gamma)| = n$. The *adjacency matrix* of Γ with respect to the ordering $(v_1, \dots, v_n)$ of its vertices is a matrix $A(\Gamma) = (a_{ij})_{i,j=1}^n$ with

$$a_{ij} = \begin{cases} n(v_i, v_j) & \text{if } v_i \to v_j, \\ 0 & \text{otherwise,} \end{cases}$$

where $n(v_i, v_j)$ denotes the number of edges joining v_i to v_j.

In this article, we will consider graph C^*-algebras only for finite, directed graphs. For more details about the theory of graph C^*-algebras, consult [BH$^+$02, BP$^+$00, KP$^+$97, KPR98, PR06, Ra, MS].

Definition 2.2. Given a finite, directed graph Γ, the *graph C^*-algebra* $C^*(\Gamma)$ is a universal C^*-algebra generated by orthogonal projections $\{p_v : v \in V(\Gamma)\}$ and partial isometries $\{S_e : e \in E(\Gamma)\}$ such that

(i) $S_e^* S_e = p_{r(e)}$ for all $e \in E(\Gamma)$,
(ii) $p_v = \sum_{\{f:\, s(f)=v\}} S_f S_f^*$ for all $v \in V(\Gamma)$ if $s^{-1}(v) \neq \emptyset$.

Examples. (1) The Cuntz algebra $\mathcal{O}_n$ can be thought of as the graph C^*-algebra with respect to L_n (Figure 1), a graph containing n loops based at a single vertex, i.e. $C^*(L_n)$ is C^*-isomorphic to $\mathcal{O}_n$.

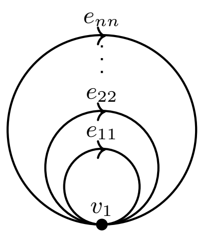


Fig. 1. L_n

(2) For a directed path P_n (Figure 2) of length n containing $n+1$ vertices, the graph C^*-algebra $C^*(P_n)$ is isomorphic to the C^*-algebra $M_{n+1}(\mathbb{C})$.

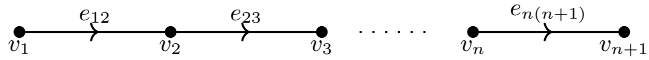


Fig. 2. P_n

For any graph C^*-algebra $C^*(\Gamma)$, we have the following results.

PROPOSITION 2.3. *Let $\Gamma = \{V(\Gamma), E(\Gamma), s, r\}$ be a finite, directed graph. For a path $\gamma = e_1 \dots e_n \in E^{<\infty}(\Gamma)$, define $S_\gamma := S_{e_1} \dots S_{e_n}$, where $e_1, \dots, e_n \in E(\Gamma)$.*

(i) $S_\gamma^* S_\mu = 0$ *for all* $\gamma, \mu \in E^{<\infty}(\Gamma)$ *with* $\gamma \neq \mu$. *In particular,* $S_e^* S_f = 0$ *for all distinct* $e, f \in E(\Gamma)$.
(ii) $\sum_{v \in V(\Gamma)} p_v = 1$.
(iii) $S_e S_f \neq 0 \Leftrightarrow r(e) = s(f)$, *i.e.* ef *is a path of length* 2. *Moreover,* $S_{e_1} \dots S_{e_k} \neq 0 \Leftrightarrow r(e_i) = s(e_{i+1})$ *for* $i = 1, \dots, k-1$, *i.e.* $S_\gamma \neq 0$ *for all* $\gamma \in E^{<\infty}(\Gamma)$.
(iv) $S_\gamma S_\mu^* \neq 0 \Leftrightarrow r(\gamma) = r(\mu)$ *for all* $\gamma, \mu \in E^{<\infty}(\Gamma)$. *In particular, for all* $e, f \in E$, $S_e S_f^* \neq 0 \Leftrightarrow r(e) = r(f)$.
(v) $p_{s(e)} S_e = S_e p_{r(e)} = S_e$ *for* $e \in E(\Gamma)$.
(vi) $\operatorname{span}\{S_\gamma S_\mu^* : \gamma, \mu \in E^{<\infty}(\Gamma)$ *with* $r(\gamma) = r(\mu)\}$ *is dense in* $C^*(\Gamma)$.

2.3. Direct sum of C^*-algebras. The direct sum of C^*-algebras $A_1, \dots, A_n$ is the C^*-algebra whose underlying set is $A_1 \oplus \dots \oplus A_n := \{(a_1, \dots, a_n) : a_i \in A_i \text{ for all } i \in [n]\}$ together with coordinatewise operations and the unique C^*-norm is given by $\|(a_1, \dots, a_n)\| = \max\{\|a_i\| : i \in [n]\}$.

If all A_i's are unital with units 1_{A_i} respectively, then $(1_{A_1}, \dots, 1_{A_n})$ is the unit of $A_1 \oplus A_2 \oplus \dots \oplus A_n$.

2.3.1. *A state on direct sum of C^*-algebras.* Since every C^*-algebra always admits a state, we can naturally define a state on the direct sum. Let ϕ_i be a state on a C^*-algebra A_i for all $i \in [n]$. We can naturally define a state $\bigoplus_{i=1}^n \phi_i$ on $A_1 \oplus \dots \oplus A_n$ by $\bigoplus_{i=1}^n \phi_i(a_1, \dots, a_n) = \frac{1}{n}[\phi_1(a_1) + \dots + \phi_n(a_n)]$.

2.3.2. *Direct sum of graph C^*-algebras.* Let $\{\Gamma_i\}_{i=1}^m$ be disjoint graphs. Then

$$C^*\Big(\bigsqcup_{i=1}^m \Gamma_i\Big) \cong \bigoplus_{i=1}^m C^*(\Gamma_i).$$

In particular, $C^*(\bigsqcup_{i=1}^m L_{n_i}) \cong \bigoplus_{i=1}^m \mathcal{O}_{n_i}$ and $C^*(\bigsqcup_{i=1}^m P_{n_i}) \cong \bigoplus_{i=1}^m M_{n_i+1}$.

2.3.3. *KMS state on a graph C^*-algebra at a critical inverse temperature.* For a finite directed graph Γ without sinks and isolated vertices, the spectral radius of the adjacency matrix of Γ is denoted by $\rho(A(\Gamma))$. Let $\mathcal{P}$ be a probability measure on $V(\Gamma)$. Let $\mathcal{F}_0 = \mathbb{C}1$ and $\mathcal{F}_i = \operatorname{span}\{S_\gamma S_\mu^* : |\gamma| = |\mu| = i\}$ for $i \geq 1$; these are finite-dimensional algebras such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$. For the proof of the following proposition, see [HLR13, Proposition 4.1] and [JM21, Proposition 2.21].

PROPOSITION 2.4. *For a directed graph $\Gamma = \{V(\Gamma), E(\Gamma), s, r\}$, the following hold:*

(1) *There exists a* $\mathrm{KMS}_{\ln\rho(A(\Gamma))}$ *state on* $C^*(\Gamma)$ *such that*

$$\mathrm{KMS}_{\ln\rho(A(\Gamma))}(S_\gamma S_\mu^*) = \begin{cases} \rho(A(\Gamma))^{-|\gamma|}\mathcal{P}_{r(\gamma)} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

iff $A(\Gamma)\mathcal{P} = \rho(A(\Gamma))\mathcal{P}$.

(2) *The graph* C^**-algebra* $C^*(\Gamma)$ *has a state* $\mathrm{KMS}_{\ln\rho(A(\Gamma))}$ *which is faithful on each* $\mathcal{F}_k$ *iff* $\rho(A(\Gamma))$ *is an eigenvalue of* $A(\Gamma)$ *corresponding to an eigenvector* $(\mathcal{P}_1, \dots, \mathcal{P}_{|V(\Gamma)|})$ *with* $\mathcal{P}_i > 0$ *for all* $i \in [|V(\Gamma)|]$. *If the eigenvector* $(\mathcal{P}_1, \dots, \mathcal{P}_{|V(\Gamma)|})$ *is normalized with* $\sum_{i\in[|V(\Gamma)|]}\mathcal{P}_i = 1$, *then*

$$\mathrm{KMS}_{\ln\rho(A(\Gamma))}(S_\gamma S_\mu^*) = \begin{cases} \rho(A(\Gamma))^{-|\gamma|}\mathcal{P}_{r(\gamma)} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY 2.5. *For the Cuntz algebras* $\{\mathcal{O}_{n_i}\}_{i=1}^m$, *the direct sum* $\bigoplus_{i=1}^m \mathcal{O}_{n_i}$ $(\cong C^*(\bigsqcup_{i=1}^m L_{n_i}))$ *has a* $\mathrm{KMS}_{\ln(\max\{n_i\}_{i=1}^m)}$ *state. Moreover, if all* n_i*'s are equal, then* $C^*(\bigsqcup_{i=1}^m L_{n_i})$ *has a* $\mathrm{KMS}_{\ln(n_1)}$ *state which is faithful on each* $\mathcal{F}_k$ *such that*

$$\mathrm{KMS}_{\ln(n_1)}(S_\gamma S_\mu^*) = \begin{cases} \frac{n_1{}^{-|\gamma|}}{m} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{KMS}_{\ln\rho(A(\bigsqcup_{i=1}^m L_{n_i}))} = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln\rho(A(L_{n_i}))}.$$

Proof. The existence of a $\mathrm{KMS}_{\ln(\max\{n_i\}_{i=1}^m)}$ state follows directly from Proposition 2.4(1).

If all n_i's are equal, then $\frac{1}{m}(1, \dots, 1)$ is an eigenvector corresponding to the eigenvalue n_1. Hence, the faithfulness of the state $\mathrm{KMS}_{\ln(n_1)}$ clearly follows from Proposition 2.4(2).

Lastly, we need to show that $\mathrm{KMS}_{\ln\rho(A(\bigsqcup_{i=1}^m L_{n_i}))} = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln\rho(A(L_{n_i}))}$. Observe that $\{\gamma : \gamma \text{ is a path on } \bigsqcup_{i=1}^m L_{n_i}\} = \bigsqcup_{i=1}^m\{\gamma : \gamma \text{ is a path on } L_{n_i}\}$. Using the fact that $S_\gamma S_\mu = S_\gamma^* S_\mu = S_\gamma S_\mu^* = 0$ for all $\gamma \in L_{n_i}, \mu \in L_{n_j}$ with $i \neq j$ (by Proposition 2.3), one can show that $\overline{\mathrm{span}}\,\{S_\gamma S_\mu^* : \gamma, \mu \in E^{<\infty}(\bigsqcup_{i=1}^m L_{n_i})\} = \bigoplus_{i=1}^m \overline{\mathrm{span}}\,\{S_\gamma S_\mu^* : \gamma, \mu \in E^{<\infty}(L_{n_i})\}$. Hence, by Proposition 2.3(vi), the states $\mathrm{KMS}_{\ln\rho(A(\bigsqcup_{i=1}^m L_{n_i}))}$ and $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln\rho(A(L_{n_i}))}$ have the same domain set. Moreover, for $\gamma, \mu \in L_{n_j}$,

$$\begin{aligned}\mathrm{KMS}_{\ln \rho(A(\bigsqcup_{i=1}^m L_{n_i}))}(S_\gamma S_\mu^*) &= \begin{cases} \frac{\rho(A(\bigsqcup_{i=1}^m L_{n_i}))^{-|\gamma|}}{m} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{n_1^{-|\gamma|}}{m} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise,} \end{cases} \\ &= \bigoplus_{i=1}^m \mathrm{KMS}_{\ln \rho(A(L_{n_i}))}(S_\gamma S_\mu^*). \ \blacksquare\end{aligned}$$

2.4. Compact quantum groups and quantum automorphism groups. In this subsection, we recall some facts related to compact quantum groups and their actions on a given C^*-algebra. We refer the readers to [MV98, Wa98, Wo87, Ti08, NT13, Fr23] for more details.

DEFINITION 2.6. A *compact quantum group* (CQG) is a pair $(\mathcal{Q}, \Delta)$, where $\mathcal{Q}$ is a unital C^*-algebra and $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ is a unital C^*-homomorphism such that

(i) $(\mathrm{id}_{\mathcal{Q}} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_{\mathcal{Q}})\Delta$,
(ii) span $\{\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})\}$ and span $\{\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)\}$ are dense in $(\mathcal{Q} \otimes \mathcal{Q})$.

Given two compact quantum groups, $(\mathcal{Q}_1, \Delta_1)$ and $(\mathcal{Q}_2, \Delta_2)$, a *compact quantum group morphism* (CQG morphism) between $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is a C^*-homomorphism $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $(\phi \otimes \phi)\Delta_1 = \Delta_2 \phi$.

For any CQG $\mathcal{Q}$, there exists a canonical dense Hopf $*$-algebra $\mathcal{Q}_0 \subseteq \mathcal{Q}$ in which one can define an antipode κ and a counit ϵ.

In this article, we are interested in a special class of CQG called compact matrix quantum groups.

DEFINITION 2.7. Let A be a unital C^*-algebra and $q = (q_{ij})_{n \times n}$ be a matrix with entries from A. Then (A, q) is called a *compact matrix quantum group* (CMQG) if there exists a $*$-subalgebra A' generated by the entries of q such that

(i) A' dense in A,
(ii) there exists a C^*-homomorphism $\Delta : A \to A \otimes A$ such that $\Delta(q_{ij}) = \sum_{k=1}^n q_{ik} \otimes q_{kj}$ for all $i, j \in [n]$,
(iii) There exists a linear anti-multiplicative map $\kappa : A' \to A'$ such that $\kappa(\kappa(a^*))^* = a$ for all $a \in A'$, and $q_\kappa := (\kappa(q_{ij}))_{n \times n}$ is the inverse of q.

The following definition is equivalent to Definition 2.7 (see [Wo91] for more details).

DEFINITION 2.8. Let A be a unital C^*-algebra and $q = (q_{ij})_{n \times n}$ be a matrix with entries from A. Then (A, q) is called a *compact matrix quantum group* (CMQG) if

(i) A is generated by the entries of q,
(ii) both q and $q^t := (q_{ji})_{n\times n}$ are invertible in $M_n(A)$,
(iii) there exists a C^*-homomorphism $\Delta : A \to A \otimes A$ such that $\Delta(q_{ij}) = \sum_{k=1}^{n} q_{ik} \otimes q_{kj}$ for all $i, j \in [n]$.

The matrix q is called the *fundamental representation* of the CMQG (A, q).

DEFINITION 2.9. Let (A, q) and (A', q') be two CMQGs, where A and A' are unital C^*-algebras with fundamental representations $q = (q_{ij})_{n\times n}$ and $q' = (q'_{ij})_{n\times n}$ respectively.

(1) (A, q) and (A', q') are said to be *identical* (denoted $(A, q) \approx (A', q')$) if there exists a C^*-isomorphism $\phi : A \to A'$ such that $\phi(q_{ij}) = q'_{ij}$.
(2) (A', q') is said to be a *quantum subgroup* of (A, q) (denoted $A' \subset A$) if there exists a surjective C^*-homomorphism $\phi : A \to A'$ such that $\phi(q_{ij}) = q'_{ij}$.

Next, we present some examples of CMQGs which will appear in this article.

EXAMPLES. (1) For $n \in \mathbb{N}$, $C(S_n^+)$ is the universal C^*-algebra generated by $\{u_{ij}\}_{i,j\in[n]}$ such that

(i) $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j \in [n]$,
(ii) $\sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{kj} = 1$ for all $i, j \in [n]$.

Define a coproduct $\Delta : C(S_n^+) \to C(S_n^+) \otimes C(S_n^+)$ on generators by $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. Then (S_n^+, Δ) [respectively, (S_n^+, u)] is called a CQG [respectively, CMQG] whose underlying C^*-algebra is $C(S_n^+)$ (see [Wa98, BBC07] for more details).

(2) Let $F \in \mathbb{GL}_n(\mathbb{C})$ and let $A_{U^t}(F)$ be the universal C^*-algebra generated by $\{q_{ij} : i, j \in [n]\}$ such that the matrix $U = (q_{ij})_{n\times n}$ satisfies the following conditions:

- $(U^t)(U^t)^* = (U^t)^*(U^t) = I_{n\times n}$, i.e. U^t is unitary.
- $UF^{-1}U^*F = F^{-1}U^*FU = I_{n\times n}$.

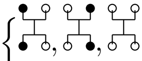
Again, the coproduct on the generators $\{q_{ij} : i, j \in [n]\}$ is given by $\Delta(q_{ij}) = \sum_{k=1}^{n} q_{ik} \otimes q_{kj}$. It can be shown that $(A_{U^t}(F), \Delta)$ is a CQG (as well as a CMQG with respect to the fundamental representation U).

If $F = I_{n\times n}$, then we write U_n^+ for $A_{U^t}(F)$, i.e. $(U_n^+, \Delta) := (A_{U^t}(I_{n\times n}), \Delta)$ (consult [MV98] for details).

(3) $C(H_n^{\infty+})$ is defined to be the universal C^*-algebra generated by $\{u_{ij} : i, j \in [n]\}$ such that

- $u = (u_{ij})_{n\times n}$ and $(u_{ij}^*)_{n\times n}$ are unitary matrices,
- u_{ij}'s are normal partial isometries for all i, j.

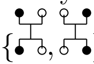
The coproduct $\varDelta$ on generators is again given by $\varDelta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. Then $(C(H_n^{\infty+}), \varDelta)$ forms a CQG. Moreover, the CMQG $(C(H_n^{\infty+}), u)$ forms a unitary easy quantum group with respect to the category of partition

$\mathcal{C}_0 = \Big\{$  $\Big\}$

(see [TW17] for the details on the unitary easy quantum group). We denote this CQG (or CMQG) by $H_n^{\infty+}$.

(4) Consider the universal C^*-algebra generated by $\{u_{ij} : i, j \in [n]\}$ such that

(i) $(u_{ij})_{n\times n}$ and $(u_{ij}^*)_{n\times n}$ are unitary matrices,
(ii) u_{ij}'s are partial isometries for all i, j,

and denote it by $C(SH_n^{\infty+})$. Observe that condition (ii) can be replaced by "$u_{ik}u_{jk}^* = u_{ik}^*u_{jk} = 0$ for all $i, j, k \in [n]$ with $i \neq j$". Similarly, the coproduct $\varDelta$ on generators is given by $\varDelta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. Thus, the CMQG $(C(SH_n^{\infty+}), u)$ constitutes a unitary easy quantum group with respect to the category of partition $\mathcal{C}_0 = \{$  $\}$ (see [TW17, Section 4] for details). We denote this CQG (or CMQG) by $SH_n^{\infty+}$.

(5) Let $n \in \mathbb{N}$ and let $(Q, \varDelta_Q)$ be a CQG. The *free wreath product* of Q by the quantum permutation group S_n^+, denoted by $Q \wr_* S_n^+$, is the quotient of the algebra $\underbrace{Q * \cdots * Q}_{n\,\text{times}} * S_n^+$ by the ideals of the form $(i_k(a)t_{kl} - t_{kl}i_k(a))$ for $k, l \in [n]$, $a \in Q$, where $i_k : Q \to Q * \cdots * Q * S_n^+$ is the natural inclusion. Define the coproduct $\varDelta : Q \wr_* S_n^+ \to (Q \wr_* S_n^+) \otimes (Q \wr_* S_n^+)$ by

$$\varDelta(i_k(a)) = \sum_{l=1}^{n} i_k \otimes i_l(\varDelta_Q(a))(t_{kl} \otimes 1) \quad \text{and} \quad \varDelta(t_{kl}) = \sum_{r=1}^{n} t_{kr} \otimes t_{rl}.$$

Moreover, the counit satisfies $\epsilon(t_{kl}) = \delta_{kl}$ and $\epsilon(i_k(a)) = \epsilon_Q(a)$, where ϵ_Q is the counit of $(Q, \varDelta_Q)$ (see [Wa95, Bi04]).

Next, we will discuss the CQG action and the quantum symmetry of a C^*-algebra from the categorical viewpoint. The readers are referred to [Wa98, Bi03] for the following definitions and discussions.

Definition 2.10. A CQG $(\mathcal{Q}, \varDelta)$ is said to be *acting faithfully* on a unital C^*-algebra $\mathcal{C}$ if there exists a unital C^*-homomorphism $\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$ such that:

(i) (action equation) $(\alpha \otimes \mathrm{id}_{\mathcal{Q}})\alpha = (\mathrm{id}_{\mathcal{C}} \otimes \varDelta)\alpha$,
(ii) (Podleś condition) span $\{\alpha(\mathcal{C})(1 \otimes \mathcal{Q})\}$ is dense in $\mathcal{C} \otimes \mathcal{Q}$,
(iii) (faithfulness) the $*$-algebra generated by the set $\{(\theta \otimes \mathrm{id})\alpha(\mathcal{C}) : \theta \in \mathcal{C}^*\}$ is norm-dense in $\mathcal{Q}$.

$((\mathcal{Q}, \varDelta), \alpha)$ is also called a *quantum transformation group of* $\mathcal{C}$.

Given a unital C^*-algebra $\mathcal{C}$, the *category* $\mathfrak{C}$ *of quantum transformation groups of* $\mathcal{C}$ is a category having quantum transformation groups of $\mathcal{C}$ as objects, and a morphism from $((\mathcal{Q}_1, \Delta_1), \alpha_1)$ to $((\mathcal{Q}_2, \Delta_2), \alpha_2)$ is a CQG morphism $\phi : (\mathcal{Q}_1, \Delta_1) \to (\mathcal{Q}_2, \Delta_2)$ such that $(\mathrm{id}_{\mathcal{C}} \otimes \phi)\alpha_1 = \alpha_2$.

The *universal object* of $\mathfrak{C}$ is a quantum transformation group of $\mathcal{C}$, denoted by $((\widehat{\mathcal{Q}}, \widehat{\Delta}), \widehat{\alpha})$, satisfying the following universal property: For any object $((\mathcal{D}, \Delta_{\mathcal{D}}), \delta)$ of $\mathcal{C}$, there is a surjective CQG morphism $\widehat{\phi} : (\widehat{\mathcal{Q}}, \widehat{\Delta}) \to (\mathcal{D}, \Delta_{\mathcal{D}})$ such that $(\mathrm{id}_{\mathcal{C}} \otimes \widehat{\phi})\widehat{\alpha} = \delta$.

DEFINITION 2.11. Given a unital C^*-algebra $\mathcal{C}$, the *quantum automorphism group* of $\mathcal{C}$ is the underlying CQG of the universal object of the category $\mathcal{C}$ of quantum transformation groups of $\mathcal{C}$ if the universal object exists.

REMARK 2.12. In $\mathcal{C}$, the universal object might fail to exist in general. One can remedy this by restricting the category to a subcategory so that a universal object would exist. Take a linear functional $\tau : \mathcal{C} \to \mathbb{C}$. Define a subcategory $\mathfrak{C}_\tau$ whose objects are those quantum transformation groups of $\mathcal{C}$, $((\mathcal{Q}, \Delta), \alpha)$, for which $(\tau \otimes \mathrm{id})\alpha(\cdot) = \tau(\cdot)1$ on a suitable subspace of $\mathcal{C}$ and morphisms are as above.

EXAMPLES. (1) For the n-point space X_n, the universal object in the category of quantum transformation groups of $C(X_n)$ exists and is isomorphic to the quantum permutation group S_n^+ (see [Wa98, BBC07] for more details).

(2) For the C^*-algebra $M_n(\mathbb{C})$, the universal object in the category of quantum transformation groups of $M_n(\mathbb{C})$ (for $n \geq 2$) does not exist. But if we fix a linear functional τ' on $M_n(\mathbb{C})$ which is defined by $\tau'(A) = \mathrm{Tr}(A)$ and assume that any object of the category also preserves τ', i.e. $(\tau' \otimes \mathrm{id})\alpha(\cdot) = \tau'(\cdot)1$ on $M_n(\mathbb{C})$, then the universal object exists in $\mathfrak{C}_{\tau'}$ (see [Wa98] for more details).

2.5. Quantum symmetry in an orthogonal filtration preserving way. In this subsection, we recall the quantum symmetry of a C^*-algebra equipped with an orthogonal filtration with respect to a given state ϕ on that C^*-algebra, introduced by Banica and Skalski [BS13].

DEFINITION 2.13. Let $\mathcal{C}$ be a unital C^*-algebra together with a faithful state ϕ and a family $\{F_i\}_{i\in\mathcal{I}}$ of finite-dimensional subspaces of $\mathcal{C}$ (where $\mathcal{I}$ is the index set containing a distinguished element 0). The collection $(\{F_i\}_{i\in\mathcal{I}}, \phi)$ defines an *orthogonal filtration* on $\mathcal{C}$ if

(1) $F_0 = \mathbb{C}1_{\mathcal{C}}$,
(2) if $a \in F_i$ and $b \in F_j$ for $i, j \in \mathcal{I}$ with $i \neq j$, then $\phi(a^*b) = 0$,
(3) $\mathrm{span}(\bigcup_{i\in\mathcal{I}} F_i)$ is dense in $\mathcal{C}$.

EXAMPLES. (1) Any unital separable AF algebra admits an orthogonal filtration.

(2) In [JM21, Section 5], the authors provide an orthogonal filtration for the Cuntz algebra $\mathcal{O}_n$ (viewing it as a graph C^*-algebra with respect to the graph L_n). We will similarly define an orthogonal filtration on the direct sum $\bigoplus_{i=1}^m \mathcal{O}_{n_i}$ (whose underlying graph is $\bigsqcup_{i=1}^m L_{n_i}$). Set $\mathcal{F}_0 = \mathbb{C}1$ and $\mathcal{F}_i = \text{span}\,\{S_\gamma S_\mu^* : |\gamma| = |\mu| = i\}$ for $i \geq 1$. Note that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$. Consider the KMS state $\text{KMS}_{\ln(n_i)}$ on $C^*(L_{n_i})$ and $\bigoplus_{i=1}^m \text{KMS}_{\ln(n_i)}$ on $C^*(\bigsqcup_{i=1}^m L_{n_i})$. Now, define finite-dimensional vector subspaces $\mathcal{W}_0 = \mathcal{F}_0 = \mathbb{C}1$ and $\mathcal{W}_i = \mathcal{F}_i \ominus \mathcal{F}_{i-1}$, the orthogonal complement of $\mathcal{F}_{i-1}$ in $\mathcal{F}_i$ for $i \geq 1$. Moreover, we define the finite-dimensional subspaces

$$\mathcal{M}_{k,l}^{(1)} := \text{span}\,\{S_\mu x : |\mu| = l,\, x \in \mathcal{W}_k\},$$
$$\mathcal{M}_{k,l}^{(2)} := \text{span}\,\{y S_\nu^* : |\nu| = l,\, y \in \mathcal{W}_k\}$$

for $(k,l) \in \mathbb{N}_0 \times \mathbb{N}$.

Now, the collection $(\{\mathcal{W}_i, \mathcal{M}_{p,q}^{(1)}, \mathcal{M}_{r,s}^{(2)} : i \in \mathbb{N}_0,\, (p,q) \in \mathbb{N}_0 \times \mathbb{N},\, (r,s) \in \mathbb{N}_0 \times \mathbb{N}\}, \bigoplus_{i=1}^m \text{KMS}_{\ln(n_i)})$ is an orthogonal filtration on $\bigoplus_{i=1}^m \mathcal{O}_{n_i}$.

For the proof, we mention a key fact: if we take any two paths γ and μ from L_k and L_l respectively (for $k \neq l$), then $S_\gamma S_\mu = S_\gamma^* S_\mu = S_\gamma S_\mu^* = 0$ (by Proposition 2.3). Now, the arguments can be adopted from [JM21, Theorem 5.3]. Though Lemma 5.1 of [JM21] does not hold for $\bigoplus_{i=1}^m \mathcal{O}_n$ with respect to the state $\bigoplus_{i=1}^m \text{KMS}_{\ln(n_i)}$, it is true for each individual $\mathcal{O}_{n_i}$ with the state $\text{KMS}_{\ln(n_i)}$. Hence, the fact mentioned above ensures that a similar proof works.

Definition 2.14. For an orthogonal filtration $\mathfrak{F} = (\{F_i\}_{i\in\mathcal{I}}, \phi)$ on a unital C^*-algebra $\mathcal{C}$, we say that a CQG $(\mathcal{Q}, \Delta)$ acts on $\mathcal{C}$ by α *in a filtration preserving way* if

$$\alpha(F_i) \in F_i \otimes \mathcal{Q} \quad \text{for all } i \in \mathcal{I}.$$

Definition 2.15. We define a category $\mathfrak{C}_{\mathfrak{F}}(\mathcal{C})$ whose objects are $((\mathcal{Q}, \Delta), \alpha)$ such that $(\mathcal{Q}, \Delta)$ acts on $\mathcal{C}$ in an orthogonal filtration preserving way in the sense of Definition 2.14, and a morphism from $((\mathcal{Q}_1, \Delta_1), \alpha_1)$ to $((\mathcal{Q}_2, \Delta_2), \alpha_2)$ is a CQG morphism $\Phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $(\text{id}_{\mathcal{C}} \otimes \phi)\alpha_1 = \alpha_2$.

For the proof of the theorem below, consult [BS13, Theorem 2.7].

Theorem 2.16. *For an orthogonal filtration $\mathfrak{F} = (\{F_i\}_{i\in\mathcal{I}}, \phi)$ on a unital C^*-algebra $\mathcal{C}$, there exists a universal object in the category $\mathfrak{C}_{\mathfrak{F}}(\mathcal{C})$.*

We denote by $Q_{\mathfrak{F}}(\mathcal{C})$ the underlying CQG of the universal object in $\mathfrak{C}_{\mathfrak{F}}(\mathcal{C})$.

2.6. Quantum symmetry of a graph C^*-algebra. Let $\Gamma = \{V(\Gamma), E(\Gamma), s, r\}$ be a finite directed graph without isolated vertices. To define a CQG action on $C^*(\Gamma)$, it is enough to define an action on the partial isometries corresponding to the edges of Γ. To describe the quantum symmetries of a graph C^*-algebra, *we restrict ourselves to finite directed graphs without isolated vertices* and we simply call them *graphs*.

2.6.1. *Quantum symmetry of a graph C^*-algebra in the category $\mathfrak{C}_\tau^{\mathrm{Lin}}$ introduced in* [JM18]

Definition 2.17 ([JM18, Definition 3.4]). Given a graph Γ, a faithful action α of a CQG $\mathcal{Q}$ on a C^*-algebra $C^*(\Gamma)$ is said to be *linear* if $\alpha(S_e) = \sum_{f\in E(\Gamma)} S_f \otimes q_{fe}$, where $q_{ef} \in \mathcal{Q}$ for all $e, f \in E(\Gamma)$.

Set

$$\begin{aligned}\mathcal{V} &= \{u \in V(\Gamma) : u \text{ is not a source of any edge of } \Gamma\},\\ \mathcal{E} &= \{(e,f) \in E(\Gamma) \times E(\Gamma) : S_e S_f^* \neq 0\}\\ &= \{(e,f) \in E(\Gamma) \times E(\Gamma) : r(e) = r(f)\}.\end{aligned}$$

In [JM18, Lemma 3.2], it was shown that $\{p_u, S_e S_f^* : u \in \mathcal{V},\ (e,f) \in \mathcal{E}\}$ is a linearly independent set.

Now, define $\mathcal{V}_{2,+} = \operatorname{span}\{p_u, S_e S_f^* : u \in \mathcal{V},\ (e,f) \in \mathcal{E}\}$ and a linear functional $\tau : \mathcal{V}_{2,+} \to \mathbb{C}$ by $\tau(S_e S_f^*) = \delta_{ef}$, $\tau(p_u) = 1$ for all $(e,f) \in \mathcal{E}$ and $u \in \mathcal{V}$ (see [JM18, Section 3.1]).

Since $\alpha(\mathcal{V}_{2,+}) \subseteq \mathcal{V}_{2,+} \otimes \mathcal{Q}$ by [JM18, Lemma 3.6], the equation $(\tau \otimes \mathrm{id})\alpha(\cdot) = \tau(\cdot)1$ on $\mathcal{V}_{2,+}$ makes sense.

Definition 2.18 ([JM18, Definition 3.7]). For a graph Γ, define a category $\mathfrak{C}_\tau^{\mathrm{Lin}}$ whose objects are $((\mathcal{Q}, \Delta), \alpha)$, quantum transformation groups of $C^*(\Gamma)$ such that $(\tau\otimes\mathrm{id})\alpha(\cdot) = \tau(\cdot)1$ on $\mathcal{V}_{2,+}$. A morphism from $((\mathcal{Q}_1, \Delta_1), \alpha_1)$ to $((\mathcal{Q}_2, \Delta_2), \alpha_2)$ is a CQG morphism $\Phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $(\mathrm{id}_{C^*(\Gamma)} \otimes \phi)\alpha_1 = \alpha_2$.

F^Γ is an $|E(\Gamma)| \times |E(\Gamma)|$ matrix such that $(F^\Gamma)_{ef} = \tau(S_e^* S_f)$. It can be shown that F^Γ is an invertible diagonal matrix. Therefore, $A_{U^t}(F^\Gamma)$ is a CQG. We refer to [JM18, Proposition 3.8 and Theorem 3.9] for the proof of the following theorem.

Theorem 2.19. *For a graph Γ,*

(1) *there is a surjective C^*-homomorphism from $A_{U^t}(F^\Gamma)$ to any object in the category $\mathfrak{C}_\tau^{\mathrm{Lin}}$,*
(2) *$\mathfrak{C}_\tau^{\mathrm{Lin}}$ admits a universal object.*

We denote the underlying CQG and the respective action of the universal object by $Q_\tau^{\mathrm{Lin}}(\Gamma)$ and α respectively, with respect to the category $\mathfrak{C}_\tau^{\mathrm{Lin}}$. Note from [JM18, proof of Theorem 3.9] that $Q_\tau^{\mathrm{Lin}}(\Gamma)$ is essentially a CMQG with the fundamental representation $q = (q_{ef})_{|E(\Gamma)|\times|E(\Gamma)|}$ such that $\alpha(S_e) = \sum_{f\in E(\Gamma)} S_f \otimes q_{fe}$.

2.6.2. *Quantum symmetry of a graph C^*-algebra in the* KMS *state category introduced in* [JM21]

Definition 2.20 ([JM21, Definition 3.1]). For a graph Γ without a sink, $\mathfrak{C}_{\mathrm{KMS}}^{\mathrm{Lin}}$ is a category whose objects are $((Q, \Delta), \alpha)$, where (Q, Δ) is a CQG and α is a linear action on $C^*(\varGamma)$ which preserves the state $\mathrm{KMS}_{\ln\rho(A(\varGamma))}$, i.e.

$$\alpha(S_e) = \sum_{f\in E(\varGamma)} S_f \otimes q_{fe}$$

and on $C^*(\varGamma)$,

$$(\mathrm{KMS}_{\ln\rho(A(\varGamma))} \otimes \mathrm{id}_{C^*(\varGamma)}) \circ \alpha(\cdot) = \mathrm{KMS}_{\ln\rho(A(\varGamma))}(\cdot)1.$$

A morphism from $((\mathcal{Q}_1, \varDelta_1), \alpha_1)$ to $((\mathcal{Q}_2, \varDelta_2), \alpha_2)$ is a CQG morphism $\varPhi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $(\mathrm{id}_{C^*(\varGamma)} \otimes \varPhi)\alpha_1 = \alpha_2$.

Theorem 2.21 ([JM21, Proposition 3.2]). *The category* $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$ *has a universal object.*

We denote by $Q^{\mathrm{Lin}}_{\mathrm{KMS}}(\varGamma)$ the underlying CMQG of the universal object in $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$.

The next result provides a sufficient condition for the isomorphism between the categories $\mathfrak{C}^{\mathrm{Lin}}_{\tau}$ and $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$.

Theorem 2.22 ([JM21, Theorem 3.6]). *For a graph* $\varGamma$ *without a sink, if all the row sums of* $A(\varGamma)$ *are* $\rho(A(\varGamma))$*, then the* $\mathrm{KMS}_{\ln\rho(A(\varGamma))}$ *state exists on* $C^*(\varGamma)$ *such that*

$$\mathrm{KMS}_{\ln\rho(A(\varGamma))}(S_\gamma S^*_\mu) = \begin{cases} \frac{\rho(A(\varGamma))^{-|\gamma|}}{|V(\varGamma)|} & \text{if } \gamma = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and in this case the categories $\mathfrak{C}^{\mathrm{Lin}}_{\tau}$ *and* $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$ *coincide.*

Corollary 2.23. *For the Cuntz algebras* $\{\mathcal{O}_{n_i}\}_{i=1}^m$ *where all* n_i*'s are equal, the categories* $\mathfrak{C}^{\mathrm{Lin}}_{\tau}$ *and* $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$ *coincide for* $\bigoplus_{i=1}^m \mathcal{O}_{n_i} (\cong C^*(\bigsqcup_{i=1}^m L_{n_i}))$.

2.6.3. *Quantum symmetry of direct sum of Cuntz algebras in a natural state preserving category.* We will adopt the same idea to define a new state preserving category only for the direct sum of Cuntz algebras. Since each $C^*(L_{n_i})$ has a natural $\mathrm{KMS}_{\ln(n_i)}$ state, we can define a state $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}$ on $C^*(\bigsqcup_{i=1}^m L_{n_i})$.

Definition 2.24. For $\bigoplus_{i=1}^m \mathcal{O}_{n_i}$ $(\cong C^*(\bigsqcup_{i=1}^m L_{n_i}))$, $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ is a category whose objects are $((Q, \varDelta), \alpha)$, where $(Q, \varDelta)$ is a CQG and α is a linear action on $C^*(\bigsqcup_{i=1}^m L_{n_i})$ which preserves $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}$, i.e.

$$\alpha(S_e) = \sum_{f\in E(\bigsqcup_{i=1}^m L_{n_i})} S_f \otimes q_{fe},$$

and on $C^*(\bigsqcup_{i=1}^m L_{n_i})$,

$$\Big(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\varGamma)}\Big) \circ \alpha(\cdot) = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(\cdot)1.$$

A morphism from $((\mathcal{Q}_1,\Delta_1),\alpha_1)$ to $((\mathcal{Q}_2,\Delta_2),\alpha_2)$ is a CQG morphism $\Phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $(\mathrm{id}_{C^*(\bigsqcup_{i=1}^m L_{n_i})} \otimes \Phi)\alpha_1 = \alpha_2$.

LEMMA 2.25. *There exists a surjective C^*-homomorphism from $A_U(F^\Gamma)$ to any object of $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ corresponding to the graph $\Gamma = \bigsqcup_{i=1}^m L_{n_i}$.*

Proof. For convenience, we denote the state $\mathrm{KMS}_{\ln(n_i)}$ simply by ϕ_i and $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} = \bigoplus_{i=1}^m \phi_i$ by ϕ. Also, we define $E_i := E(L_{n_i})$ and $E := \bigsqcup_{i=1}^m E_i = E(\bigsqcup_{i=1}^m L_{n_i})$. Hence, $|E_i| = n_i$ and $|E| = \sum_{i=1}^m n_i =: n$.

Let $((Q,\Delta),\alpha)$ be an object of $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ with $\alpha(S_e) = \sum_{f\in E} S_f \otimes q_{fe}$, where $q_{fe} \in Q$ for $e, f \in E$. Since α preserves ϕ, for each $p \in [m]$ and for all $e, f \in E_p$, we have (here and below, " $\Longrightarrow$ " means "therefore")

$$
\begin{aligned}
&(\phi \otimes \mathrm{id}_{C^*(\Gamma)}) \circ \alpha(S_e S_f^*) = \phi(S_e S_f^*)1\\
\Longrightarrow\ &\sum_{g,h\in E} \phi(S_g S_h^*) q_{ge} q_{hf}^* = \frac{1}{m}\phi_i(S_e S_f^*) = \frac{1}{mn_p}\delta_{ef}\\
\Longrightarrow\ &\sum_{i\in[m]} \sum_{g,h\in E_i} \frac{1}{m}\phi_i(S_g S_h^*) q_{ge} q_{hf}^* = \frac{1}{m}\phi_i(S_e S_f^*) = \frac{1}{mn_p}\delta_{ef}\\
\Longrightarrow\ &\sum_{i\in[m]} \sum_{g,h\in E_i} \frac{1}{mn_i}\delta_{gh} q_{ge} q_{hf}^* = \frac{1}{mn_p}\delta_{ef}\\
\Longrightarrow\ &\sum_{i\in[m]} \sum_{g,h\in E_i} \frac{1}{n_i}\delta_{gh} q_{ge} q_{hf}^* = \frac{1}{n_p}\delta_{ef}.
\end{aligned}
$$

The last equation implies $U^t (F^\Gamma)^{-1} U^{t*} = F^{\Gamma^{-1}}$, where $U := (q_{ef})_{n\times n}$. Since we are considering a CQG action, the invertibility of U^t guarantees $F^{\Gamma^{-1}} U^{t*} F^\Gamma U^t = I$. Hence, $U^{t*} F^\Gamma U^t = F^\Gamma$.

Now, observe that since $S_e^* S_e = \sum_{f\in E_p} S_f S_f^*$ for all $e \in E_p$ and $p \in [m]$, we have

$$
\phi(S_e^* S_e) = \sum_{f\in E_p} \frac{1}{m}\phi_p(S_f S_f^*) = \sum_{f\in E_p} \frac{1}{m}\frac{1}{n_p} = \frac{1}{m}.
$$

Again using the fact that α preserves ϕ, for all $e \in E_p$ we get

$$
\begin{aligned}
&(\phi \otimes \mathrm{id}_{C^*(\Gamma)}) \circ \alpha(S_e^* S_e) = \phi(S_e^* S_e)1\\
\Longrightarrow\ &\sum_{k\in E} \phi(S_k^* S_k) q_{ke}^* q_{ke} = \frac{1}{m}\\
\Longrightarrow\ &\sum_{i\in[m]} \sum_{k\in E_i} \frac{1}{m} q_{ke}^* q_{ke} = \frac{1}{m}\\
\Longrightarrow\ &\sum_{i\in[m]} \sum_{k\in E_i} q_{ke}^* q_{ke} = 1.
\end{aligned}
$$

Thus, $U^*U = I_n$. Since we are considering a CQG action, invertibility of U again yields $UU^* = I_n$. Then the universal property of $A_U(F^\Gamma)$ ensures the statement of the lemma. ■

Now, using similar arguments to those in [JM18, Theorem 3.9], one can easily prove the following theorem.

THEOREM 2.26. *The category* $\mathfrak{C}^{\mathrm{Lin}}_{\oplus \mathrm{KMS}}$ *always admits a universal object.*

For the graph $\bigsqcup_{i=1}^m L_{n_i}$, we denote the underlying CQG of the universal object of $\mathfrak{C}^{\mathrm{Lin}}_{\oplus \mathrm{KMS}}$ by $Q^{\mathrm{Lin}}_{\oplus \mathrm{KMS}}(\bigsqcup_{i=1}^m L_{n_i})$. From the proof of the theorem above, one can notice that the universal object $Q^{\mathrm{Lin}}_{\oplus \mathrm{KMS}}(\bigsqcup_{i=1}^m L_{n_i})$ appears as a CMQG with the fundamental representation $U := (q_{ef})_{n\times n}$ (where $n = \sum_{i=1}^m n_i$) such that $\alpha(S_e) = \sum_{f\in E(\bigsqcup_{i=1}^m L_{n_i})} S_f \otimes q_{fe}$. Hence, as a particular case of Lemma 2.25, we have $U^{t^*} F^\Gamma U^t = F^\Gamma$ or equivalently

$$\sum_{i=1}^m \sum_{g\in E(L_{n_i})} n_i q^*_{eg} q_{fg} = n_p \delta_{ef} \quad \text{for all } e, f \in E(L_{n_p}) \text{ and } p \in [m]. \tag{2.1}$$

We will compute the quantum symmetry of the direct sum of Cuntz algebras (viewing it as a graph C^*-algebra) with respect to the categories introduced in [JM18] and [JM21].

3. Direct sum of non-isomorphic Cuntz algebras

THEOREM 3.1. *Let* $\{\mathcal{O}_{n_i}\}_{i=1}^m$ *be a finite family of Cuntz algebras where all* n_i*'s are distinct. Then*

$$Q^{\mathrm{Lin}}_\tau\Big(\bigsqcup_{i=1}^m L_{n_i}\Big) \cong \mathop{*}_{i=1}^m Q^{\mathrm{Lin}}_\tau(L_{n_i}) \cong \mathop{*}_{i=1}^m U^+_{n_i}.$$

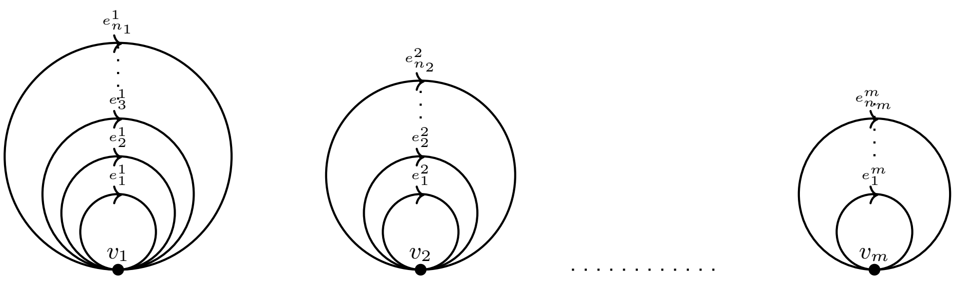


Fig. 3. $\bigsqcup_{i=1}^m L_{n_i}$

Proof. Let $V(L_{n_i}) = \{i\}$ for all $i \in [m]$ and $E_i := E(L_{n_i}) = \{e^i_1, \dots, e^i_{n_i}\}$ (see Figure 3). Then $E := E(\bigsqcup_{i=1}^m L_{n_i}) = \bigsqcup_{i=1}^m E_i$. Without loss of generality, we may assume that $n_1 > \cdots > n_m$.

The linear action $\alpha : C^*(\bigsqcup_{i=1}^m L_{n_i}) \to C^*(\bigsqcup_{i=1}^m L_{n_i}) \otimes Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^m L_{n_i})$ is given by $\alpha(S_{e_l^j}) = \sum_{f\in E} S_f \otimes q_{fe_l^j}$. For each $i \neq j$, it is enough to show that $q_{e_k^i e_l^j} = 0$ for all $k \in [n_i]$ and all $l \in [n_j]$ because in this scenario the above action reduces to

$$\alpha(S_{e_l^j}) = \sum_{f\in E_j} S_f \otimes q_{fe_l^j} = \sum_{k\in[n_j]} S_{e_k^j} \otimes q_{e_k^j e_l^j}$$

and the result follows using [JM18, Proposition 4.12]. Since $q_{e_k^i e_l^j} = 0 \Rightarrow \kappa(q_{e_k^i e_l^j}) = 0 \Rightarrow q_{e_l^j e_k^i} = 0$, it suffices to show $q_{e_k^i e_l^j} = 0$ for all $k \in [n_i]$ and all $l \in [n_j]$ with $i < j$.

We will show this inductively. Let

$\mathcal{S}(i) : q_{e_k^i f} = 0$ for all $k \in [n_i]$ and all $f \in \bigsqcup_{j=i+1}^m E_j$.

Also, for convenience, we define $Q_{ef} = q_{ef}^* q_{ef}$.

Now, we will show that $\mathcal{S}(1)$ is true. Since $\sum_{i=1}^m p_i = 1$, we have

$$S^*_{e^1_{j_1}} S_{e^1_{j_1}} + S^*_{e^2_{j_2}} S_{e^2_{j_2}} + \cdots + S^*_{e^m_{j_m}} S_{e^m_{j_m}} = 1 \quad \forall j_i \in [n_i] \text{ and } \forall i \in [m].$$

Applying the action α on both sides of the above equation, we obtain

$$\alpha(S^*_{e^1_{j_1}} S_{e^1_{j_1}} + S^*_{e^2_{j_2}} S_{e^2_{j_2}} + \cdots + S^*_{e^m_{j_m}} S_{e^m_{j_m}}) = 1 \otimes 1 \qquad \forall j_i \in [n_i] \text{ and } \forall i \in [m]$$

so that

$$\sum_{e\in E} S_e^* S_e \otimes (Q_{ee^1_{j_1}} + Q_{ee^2_{j_2}} + \cdots + Q_{ee^m_{j_m}}) = 1 \otimes 1 \qquad \forall j_i \in [n_i] \text{ and } \forall i \in [m].$$

Now, for each $i \in [m]$, multiplying both sides by $(p_i \otimes 1)$, we get m equations of the form

$$\sum_{k\in[n_i]} (Q_{e_k^i e^1_{j_1}} + Q_{e_k^i e^2_{j_2}} + \cdots + Q_{e_k^i e^m_{j_m}}) = 1. \tag{3.1}$$

For $i = 1$, we have

$$\sum_{k\in[n_1]} (Q_{e_k^1 e^1_{j_1}} + Q_{e_k^1 e^2_{j_2}} + \cdots + Q_{e_k^1 e^m_{j_m}}) = 1$$

for each $j_t \in [n_t]$.

Now taking the sum over $j_t \in [n_t]$ for each $t \in [m]$, we get

$$\sum_{j_m\in[n_m]} \cdots \sum_{j_2\in[n_2]} \sum_{j_1\in[n_1]} \Big[\sum_{k\in[n_1]} (Q_{e_k^1 e^1_{j_1}} + Q_{e_k^1 e^2_{j_2}} + \cdots + Q_{e_k^1 e^m_{j_m}}) \Big] = \prod_{i=1}^m n_i$$

and hence

$$\sum_{i=1}^{m}\Big(\prod_{k\in[m],\,k\neq i} n_k\Big)\sum_{k\in[n_1]}\sum_{j_i\in[n_i]} Q_{e^1_k e^i_{j_i}} = \prod_{i=1}^{m} n_i.$$

Dividing both sides by the scalar $\prod_{i=1}^{m} n_i$ yields

$$\frac{1}{n_1}\sum_{k\in[n_1]}\sum_{j_1\in[n_1]} Q_{e^1_k e^1_{j_1}} + \frac{1}{n_2}\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \frac{1}{n_m}\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}} = 1. \tag{3.2}$$

Consequently,

$$\frac{1}{n_1}\sum_{k\in[n_1]}\Big[\sum_{j_1\in[n_1]} Q_{e^1_k e^1_{j_1}} + \sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}}\Big] + \Big(\frac{1}{n_2}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \Big(\frac{1}{n_m}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}} = 1. \tag{3.3}$$

Since for each $k\in[n_1]$,

$$\sum_{j_1\in[n_1]} Q_{e^1_k e^1_{j_1}} + \sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}} = 1 \text{ (as } U^t \text{ is unitary)},$$

the above equation can be written as

$$\frac{1}{n_1}\sum_{k\in[n_1]} 1 + \Big(\frac{1}{n_2}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \Big(\frac{1}{n_m}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}} = 1,$$

which further implies

$$\Big(\frac{1}{n_2}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} Q_{e^1_k e^2_{j_2}} + \cdots + \Big(\frac{1}{n_m}-\frac{1}{n_1}\Big)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} Q_{e^1_k e^m_{j_m}} = 0$$

(because $\sum_{k\in[n_1]} 1 = n_1$). Since $\frac{1}{n_i}-\frac{1}{n_1} > 0$ for all $i\in\{2,\dots,m\}$, we conclude that $Q_{e^1_k e^2_{j_2}} = Q_{e^1_k e^3_{j_3}} = \cdots = Q_{e^1_k e^m_{j_m}} = 0$ for all $k\in[n_1]$ and $j_i\in[n_i]$ whenever $i\in\{2,\dots,m\}$.

In other words, $q_{e^1_k f} = 0$ for all $k\in[n_1]$ and all $f\in\bigsqcup_{i=2}^{m} E_i$. Therefore, $\mathcal{S}(1)$ is true.

Assume that $\mathcal{S}(i)$ is true for $i = 1,\dots,l-1$, i.e. for each such i, we already have $q_{e^i_k f} = 0$ for all $k\in[n_i]$ and all $f\in\bigsqcup_{s=i+1}^{m} E_s$. We now need to show that $\mathcal{S}(l)$ is also true.

Putting $i = l$ in (3.1), we get

$$\sum_{k\in[n_l]} (Q_{e^l_k e^1_{j_1}} + Q_{e^l_k e^2_{j_2}} + \cdots + Q_{e^l_k e^m_{j_m}}) = 1.$$

By induction hypothesis, for each $k \in [n_l]$, $Q_{e^l_k e^1_{j_1}} = Q_{e^l_k e^2_{j_2}} = \cdots = Q_{e^l_k e^{l-1}_{j_{l-1}}} = 0$ for any $j_1 \in [n_1], \ldots, j_{l-1} \in [n_{l-1}]$. Therefore, the above equation reduces to

$$\sum_{k\in[n_l]} (Q_{e^l_k e^l_{j_l}} + Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + Q_{e^l_k e^m_{j_m}}) = 1.$$

Taking the sum over $j_i \in [n_i]$ for each $i \in \{l, (l+1), \ldots, m\}$, we obtain

$$\sum_{j_m\in[n_m]} \cdots \sum_{j_{l+1}\in[n_{l+1}]} \sum_{j_l\in[n_l]} \Big[\sum_{k\in[n_l]} (Q_{e^l_k e^l_{j_l}} + Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + Q_{e^l_k e^m_{j_m}}) \Big] = \prod_{i=l}^{m} n_i.$$

Consequently,

$$\text{(3.4)}\quad \frac{1}{n_l} \sum_{k\in[n_l]} \sum_{j_l\in[n_l]} Q_{e^l_k e^1_{j_l}} + \frac{1}{n_{l+1}} \sum_{k\in[n_l]} \sum_{j_{l+1}\in[n_{l+1}]} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \frac{1}{n_m} \sum_{k\in[n_l]} \sum_{j_m\in[n_m]} Q_{e^l_k e^m_{j_m}} = 1.$$

Hence

$$\text{(3.5)}\quad \frac{1}{n_l} \sum_{k\in[n_l]} \Big[\sum_{j_l\in[n_l]} Q_{e^l_k e^l_{j_l}} + \sum_{j_{l+1}\in[n_{l+1}]} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \sum_{j_m\in[n_m]} Q_{e^l_k e^m_{j_m}} \Big] + \left(\frac{1}{n_{l+1}} - \frac{1}{n_l}\right) \sum_{k\in[n_l]} \sum_{j_{l+1}\in[n_{l+1}]} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \left(\frac{1}{n_m} - \frac{1}{n_l}\right) \sum_{k\in[n_l]} \sum_{j_m\in[n_m]} Q_{e^l_k e^m_{j_m}} = 1.$$

Using the fact that $U^t = (q_{ef})^t$ is unitary and the induction hypothesis again, one can easily find that

$$\sum_{j_l\in[n_l]} Q_{e^l_k e^l_{j_l}} + \sum_{j_{l+1}\in[n_{l+1}]} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \sum_{j_m\in[n_m]} Q_{e^l_k e^m_{j_m}} = 1.$$

Therefore, one can rewrite (3.5) as before:

$$\left(\frac{1}{n_{l+1}} - \frac{1}{n_l}\right) \sum_{k\in[n_l]} \sum_{j_{l+1}\in[n_{l+1}]} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \left(\frac{1}{n_m} - \frac{1}{n_l}\right) \sum_{k\in[n_l]} \sum_{j_m\in[n_m]} Q_{e^l_k e^m_{j_m}} = 0.$$

Hence, for all $k \in [n_l]$, we get $Q_{e^l_k e^{l+1}_{j_{l+1}}} = \cdots = Q_{e^l_k e^m_{j_m}} = 0$ for all $j_i \in [n_i]$ with $i \in \{l+1, \ldots, m\}$. Therefore, $\mathcal{S}(l)$ is also true. ■

REMARK 3.2. Though the statement of Theorem 3.1 is written in the sense of 'CGQ isomorphic', the proof suggests that the result is also true in the 'identical' sense, i.e. under the hypothesis of Theorem 3.1, we have $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^m L_{n_i}) \approx *_{i=1}^m Q_\tau^{\mathrm{Lin}}(L_{n_i}) \approx *_{i=1}^m U_{n_i}^+$.

Remark 3.3. It is known that if we take m connected graphs $\{\Gamma_i\}_{i=1}^m$ which are 'quantum non-isomorphic' to each other, then

$$\mathrm{QAut}_{\mathrm{Ban}}\Big(\bigsqcup_{i=1}^m \Gamma_i\Big) \approx \mathop{*}_{i=1}^m \mathrm{QAut}_{\mathrm{Ban}}(\Gamma_i)$$

with respect to their standard fundamental matrix representation (consult [LMR20, Sc20, DK+26, Me] for details). From Theorem 3.1, for non-isomorphic Cuntz algebras $\{\mathcal{O}_{n_i}\}$ we get that

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m L_{n_i}\Big) \approx \mathop{*}_{i=1}^m Q_\tau^{\mathrm{Lin}}(L_{n_i}).$$

It is natural to ask: does a similar result hold if we replace $\{L_{n_i}\}$ by any non-isomorphic class of graphs? But the answer is negative in graph C^*-algebraic context, even if we take 'quantum non-isomorphic' graphs (see Counterexample 3.4). But for non-isomorphic matrix algebras $\{M_{n_i}\}$ (for distinct n_i's), we also have an analogous result, i.e.

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m P_{n_i}\Big) \approx \mathop{*}_{i=1}^m Q_\tau^{\mathrm{Lin}}(P_{n_i}).$$

The proof is a simple application of [KM24, Lemmas 4.4 and 4.5]; we omit the details of the proof.

Counterexample 3.4. In this subsection, we provide two non-isomorphic graphs Γ_1 and Γ_2 to show that $Q_\tau^{\mathrm{Lin}}(\Gamma_1 \sqcup \Gamma_2)$ is not always identical to $Q_\tau^{\mathrm{Lin}}(\Gamma_1) * Q_\tau^{\mathrm{Lin}}(\Gamma_2)$ in general.

Consider the graph $P_1 \sqcup So_2$ (as shown in Figure 4), which is the disjoint union of two non-isomorphic (even 'quantum non-isomorphic') graphs P_1 and So_2, where P_1 and So_2 denote the first and second connected components of the graph shown in Figure 4.

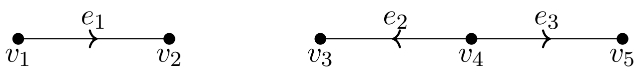


Fig. 4. $P_1 \sqcup So_2$

By [KM25, Proposition 3.4], $(Q_\tau^{\mathrm{Lin}}(P_1 \sqcup So_2), q) \approx SH_3^{\infty+}$ and $(Q_\tau^{\mathrm{Lin}}(So_2), q) \approx SH_2^{\infty+}$. Thus, it is evident that $Q_\tau^{\mathrm{Lin}}(P_1 \sqcup So_2) \approx SH_3^{\infty+}$ is not identical to $C(S^1) * SH_2^{\infty+} \approx Q_\tau^{\mathrm{Lin}}(P_1) * Q_\tau^{\mathrm{Lin}}(So_2)$, where all CMQGs are taken with respect to the standard fundamental representations as introduced in Sections 2.4 and 2.6.

We now remark that in the context of the quantum automorphism group of a graph (in the sense of Banica)

$$\mathrm{QAut}_{\mathrm{Ban}}(P_1 \sqcup So_2) \approx S_2^+ \approx \mathrm{QAut}_{\mathrm{Ban}}(P_1) * \mathrm{QAut}_{\mathrm{Ban}}(So_2),$$

whereas in the graph C^*-algebraic scenario $Q_\tau^{\mathrm{Lin}}(P_1 \sqcup So_2)$ is not identical to $Q_\tau^{\mathrm{Lin}}(P_1) * Q_\tau^{\mathrm{Lin}}(So_2)$.

Now, we demonstrate that an analogue of Theorem 3.1 also holds with respect to the 'KMS state' and the 'orthogonal filtration' preserving categories for the direct sum of Cuntz algebras.

THEOREM 3.5. *Let $\{\mathcal{O}_{n_i}\}_{i=1}^m$ be a finite family of Cuntz algebras, where all n_i's are distinct. Then*

$$Q_{\oplus\mathrm{KMS}}^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^m L_{n_i}\Big) \cong \mathop{*}_{i=1}^m U_{n_i}^+ \cong \mathop{*}_{i=1}^m Q_{\mathrm{KMS}}^{\mathrm{Lin}}(L_{n_i}).$$

Proof. We will use the same notations and conventions as in the proof of Theorem 3.1. First, we will show that if $(Q_{\oplus\mathrm{KMS}}^{\mathrm{Lin}}(\bigsqcup_{i=1}^m L_{n_i}), \alpha)$ is the universal object with the fundamental representation $U = (q_{e_k^i e_l^j})_{n\times n}$ of underlying CMQG, then $q_{e_k^i e_l^j} = 0$ for all $k \in [n_i]$ and $l \in [n_j]$ whenever $i \neq j$. The strategy is similar to that for Theorem 3.1 (see the induction statement $\mathcal{S}(i)$ there). We just need to modify a few computational tricks. To show $\mathcal{S}(1)$, just as for the previous theorem, again starting with $\sum_{i=1}^m p_i = 1$, we arrive at (3.2), i.e.

$$\frac{1}{n_1}\sum_{k\in[n_1]}\sum_{j_1\in[n_1]} Q_{e_k^1 e_{j_1}^1} + \frac{1}{n_2}\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} Q_{e_k^1 e_{j_2}^2} + \cdots + \frac{1}{n_m}\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} Q_{e_k^1 e_{j_m}^m} = 1.$$

This can be rewritten as

$$\frac{1}{n_1^2}\sum_{k\in[n_1]}\sum_{j_1\in[n_1]} n_1 Q_{e_k^1 e_{j_1}^1} + \frac{1}{n_2^2}\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} n_2 Q_{e_k^1 e_{j_2}^2} + \cdots + \frac{1}{n_m^2}\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} n_m Q_{e_k^1 e_{j_m}^m} = 1,$$

which implies

$$\frac{1}{n_1^2}\sum_{k\in[n_1]}\Big[\sum_{j_1\in[n_1]} n_1 Q_{e_k^1 e_{j_1}^1} + \sum_{j_2\in[n_2]} n_2 Q_{e_k^1 e_{j_2}^2} + \cdots + \sum_{j_m\in[n_m]} n_m Q_{e_k^1 e_{j_m}^m}\Big] + \Big(\frac{1}{n_2^2} - \frac{1}{n_1^2}\Big)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} n_2 Q_{e_k^1 e_{j_2}^2} + \cdots + \Big(\frac{1}{n_m^2} - \frac{1}{n_1^2}\Big)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} n_m Q_{e_k^1 e_{j_m}^m} = 1. \tag{3.6}$$

Since

$$\sum_{j_1\in[n_1]} n_1 Q_{e^1_k e^1_{j_1}} + \sum_{j_2\in[n_2]} n_2 Q_{e^1_k e^2_{j_2}} + \cdots + \sum_{j_m\in[n_m]} n_m Q_{e^1_k e^m_{j_m}} = n_1$$

for each $k \in [n_1]$ (using (2.1)), equation (3.6) reduces to

$$\frac{1}{n_1^2}\sum_{k\in[n_1]} n_1 + \left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} n_2 Q_{e^1_k e^2_{j_2}} + \cdots + \left(\frac{1}{n_m^2}-\frac{1}{n_1^2}\right)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} n_m Q_{e^1_k e^m_{j_m}} = 1,$$

and consequently

$$\left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right)\sum_{k\in[n_1]}\sum_{j_2\in[n_2]} n_2 Q_{e^1_k e^2_{j_2}} + \cdots + \left(\frac{1}{n_m^2}-\frac{1}{n_1^2}\right)\sum_{k\in[n_1]}\sum_{j_m\in[n_m]} n_m Q_{e^1_k e^m_{j_m}} = 0.$$

Now, $\frac{1}{n_i^2} - \frac{1}{n_1^2} > 0$ for all $i \in \{2, \ldots, m\}$ ensures that $Q_{e^1_k e^2_{j_2}} = Q_{e^1_k e^3_{j_3}} = \cdots = Q_{e^1_k e^m_{j_m}} = 0$ for all $k \in [n_1]$ and $j_i \in [n_i]$ whenever $i \in \{2, \ldots, m\}$, which proves the base case $\mathcal{S}(1)$.

Assuming $\mathcal{S}(i)$ holds for each $i \in [l-1]$, we now show that $\mathcal{S}(l)$ is true. Just as in the proof of Theorem 3.1, starting with (3.1) (by putting $i = l$) and using the induction hypothesis, we get (3.4), which implies

(3.7)

$$\frac{1}{n_l^2}\sum_{k\in[n_l]}\Big[\sum_{j_l\in[n_l]} n_l Q_{e^l_k e^l_{j_l}} + \sum_{j_{l+1}\in[n_{l+1}]} n_{l+1} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \sum_{j_m\in[n_m]} n_m Q_{e^l_k e^m_{j_m}}\Big] + \left(\frac{1}{n_{l+1}^2}-\frac{1}{n_l^2}\right)\sum_{k\in[n_l]}\sum_{j_{l+1}\in[n_{l+1}]} n_{l+1} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \left(\frac{1}{n_m^2}-\frac{1}{n_l^2}\right)\sum_{k\in[n_l]}\sum_{j_m\in[n_m]} n_m Q_{e^l_k e^m_{j_m}} = 1$$

(which is just a modification of (3.5)).

Again using the induction hypothesis, (2.1) reduces to

$$\sum_{j_l\in[n_l]} n_l Q_{e^l_k e^l_{j_l}} + \sum_{j_{l+1}\in[n_{l+1}]} n_{l+1} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \sum_{j_m\in[n_m]} n_m Q_{e^l_k e^m_{j_m}} = n_l.$$

Therefore, (3.7) implies

$$\left(\frac{1}{n_{l+1}^2}-\frac{1}{n_l^2}\right)\sum_{k\in[n_l]}\sum_{j_{l+1}\in[n_{l+1}]} n_{l+1} Q_{e^l_k e^{l+1}_{j_{l+1}}} + \cdots + \left(\frac{1}{n_m^2}-\frac{1}{n_l^2}\right)\sum_{k\in[n_l]}\sum_{j_m\in[n_m]} n_m Q_{e^l_k e^m_{j_m}} = 0.$$

Hence, for each $k \in [n_l]$, we have $Q_{e^l_k e^{l+1}_{j_{l+1}}} = \cdots = Q_{e^l_k e^m_{j_m}} = 0$ for all $j_i \in [n_i]$ with $i \in \{l+1, \ldots, m\}$. Therefore, $\mathcal{S}(l)$ is also true.

Finally, the universal property of $*_{i=1}^m U_{n_i}^+$ ensures that $Q^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}(\bigsqcup_{i=1}^m L_{n_i})$ is a quantum subgroup of $*_{i=1}^m U_{n_i}^+$ with respect to the standard fundamental representation.

Conversely, we will show that the CMQG $*_{i=1}^m U_{n_i}^+$ (whose fundamental representation is $q = (q_{ef})_{n\times n}$, where q is a unitary matrix and $q_{ef} = 0$ iff e and f based on two distinct vertices) acts linearly, faithfully and in $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}$ preserving way on $C^*(\bigsqcup_{i=1}^m L_{n_i})$, where we denote $\bigsqcup_{i=1}^m L_{n_i}$ $= \{V, E, s, r\}$ with $|V| = m$ and $n := |E| = \sum_{i=1}^m n_i$. Also, we write $E_k :=$ $\{e \in E : s(e) = r(e) = k\}$. Let us define a linear faithful action α by

$$\alpha(S_e) = \sum_{f\in E_i} S_f \otimes q_{ef} \quad \text{for all } e \in E_i \text{ and } i \in [m]. \tag{3.8}$$

We have to show that α preserves the state $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}$.

Let γ, μ be finite paths in $\bigsqcup_{i=1}^m L_{n_i}$. Hence, γ, μ are paths in L_{n_k} and L_{n_l} respectively, for some $k, l \in [m]$.

If $k \neq l$, then $S_\gamma S_\mu^* = 0$. Therefore, $(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\Gamma)}) \circ \alpha(S_\gamma S_\mu^*)$ $= 0 = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(S_\gamma S_\mu^*)1$ follows trivially.

If $k = l$, then $S_\gamma S_\mu^* \neq 0$ (since $r(\gamma) = r(\mu)$). Assume $\gamma = \gamma_1 \ldots \gamma_p$ and $\mu = \mu_1 \ldots \mu_r$. Then

$$\alpha(S_\gamma S_\mu^*) = \sum_{i_1,\ldots,i_p,j_1,\ldots,j_r\in E_k} S_{i_1}\ldots S_{i_p} S_{j_r}^* \ldots S_{j_1}^* \otimes q_{i_1\gamma_1}\ldots q_{i_p\gamma_p} q_{j_r\mu_r}^* \ldots q_{j_1\mu_1}^*.$$

Case 1: $p \neq r$. Then $\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(S_\gamma S_\mu^*) = \frac{1}{m}\mathrm{KMS}_{\ln(n_k)}(S_\gamma S_\mu^*) = 0$. On the other hand, $S_{i_1}\ldots S_{i_p} S_{j_r}^* \ldots S_{j_1}^* \neq 0$, since all the loops $i_1, \ldots, i_p$, $j_1, \ldots, j_r$ are based at a single vertex k. By definition,

$$\mathrm{KMS}_{\ln(n_k)}(S_{i_1}\ldots S_{i_p} S_{j_r}^* \ldots S_{j_1}^*) = 0.$$

Hence,

$$\Big(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\Gamma)}\Big) \circ \alpha(S_\gamma S_\mu^*) = 0 = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(S_\gamma S_\mu^*)1.$$

Case 2: $p = r$. In this case,

$$\begin{aligned} &\Big(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\Gamma)}\Big) \circ \alpha(S_\gamma S_\mu^*) \\ &\qquad = \sum_{i_1,\ldots,i_p\in E_k} \frac{1}{mn_k^p} q_{i_1\gamma_1}\ldots q_{i_p\gamma_p} q_{i_p\mu_p}^* \ldots q_{i_1\mu_1}^* \\ &\qquad = \frac{1}{m{n_k}^p} \sum_{i_1,\ldots,i_p\in E_k} q_{i_1\gamma_1}\ldots q_{i_p\gamma_p} q_{i_p\mu_p}^* \ldots q_{i_1\mu_1}^*. \end{aligned} \tag{3.9}$$

Depending on the paths γ, μ, we consider the following two subcases:

CASE (A): $\gamma = \mu$. Then $\gamma_t = \mu_t$ for all $t \in [p]$. Since $\sum_{i_t \in E_k} q_{i_t\gamma_t} q^*_{i_t\gamma_t} = 1$ for each $t \in [p]$, we have

$$\sum_{i_1,\ldots,i_p \in E_k} q_{i_1\gamma_1} \ldots q_{i_p\gamma_p} q^*_{i_p\gamma_p} \ldots q^*_{i_1\gamma_1} = 1.$$

Therefore, from (3.9),

$$\Big(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\varGamma)}\Big) \circ \alpha(S_\gamma S^*_\gamma) = \frac{1}{m.{n_k}^p} = \bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(S_\gamma S^*_\gamma)1.$$

CASE (B): $\gamma \neq \mu$. Assume that $\gamma_t = \mu_t$ for $t = p, p-1, \ldots, p-j$ but $\gamma_{p-j-1} = \mu_{p-j-1}$. Now, since $\sum_{i_t \in E_k} q_{i_t\gamma_t} q^*_{i_t\mu_t} = 1$ for $t = p, p-1, \ldots, p-j$ and $\sum_{i_{p-j-1} \in E_k} q_{i_{p-j-1}\gamma_{p-j-1}} q^*_{i_{p-j-1}\mu_{p-j-1}} = 0$, the sum on the RHS of (3.9) is 0. Therefore,

$$\begin{aligned}\Big(\bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)} \otimes \mathrm{id}_{C^*(\varGamma)}\Big) \circ \alpha(S_\gamma S^*_\mu) = 0 &= \frac{1}{m}\mathrm{KMS}_{\ln(n_k)}(S_\gamma S^*_\mu)1 \\ &= \bigoplus_{i=1}^m \mathrm{KMS}_{\ln(n_i)}(S_\gamma S^*_\mu)1. \quad \blacksquare\end{aligned}$$

Recall that

$$\mathfrak{F} = \Big(\{\mathcal{W}_i, \mathcal{M}^{(1)}_{p,q}, \mathcal{M}^{(2)}_{r,s} : i \in \mathbb{N}_0,\ (p,q) \in \mathbb{N}_0 \times \mathbb{N},\ (r,s) \in \mathbb{N}_0 \times \mathbb{N}\}, \bigoplus_{k=1}^m \mathrm{KMS}_{\ln(n_k)}\Big)$$

is an orthogonal filtration on $\bigoplus_{i=1}^m \mathcal{O}_{n_i}$ and $\mathfrak{F}_i$ denotes the orthogonal filtration on $\mathcal{O}_i \cong C^*(L_{n_i})$ with respect to the KMS state $\mathrm{KMS}_{\ln(n_i)}$ (for $i \in [m]$) as introduced in [JM21, Section 5].

The same idea as in the proof of [JM21, Theorem 5.4] will be used to prove Theorem 3.6 below.

Let $\bigsqcup_{i=1}^m L_{n_i} = \{V, E, s, r\}$, where $|V| = m$. Before going to the proof, observe that since $\mathcal{M}^{(1)}_{0,1}$ generates $C^*(\bigsqcup_{i=1}^m L_{n_i})$ (as a C^*-algebra) and $\{\sqrt{m}S_e : e \in E\}$ is an orthonormal basis for $\mathcal{M}^{(1)}_{0,1}$, Theorem 2.10(ii)–(iii) of [BS13] ensures that $Q_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$ is essentially a CMQG with the fundamental representation $q = (q_{ef})_{|E|\times|E|}$ such that $\alpha(S_e) = \sum_{f \in E} S_f \otimes q_{fe}$.

THEOREM 3.6. *Let $\{\mathcal{O}_{n_i}\}_{i=1}^m$ be a finite family of Cuntz algebras, where all n_i's are distinct. Then*

$$Q_{\mathfrak{F}}\Big(C^*\Big(\bigsqcup_{i=1}^m L_{n_i}\Big)\Big) \cong \mathop{*}_{i=1}^{m} U^+_{n_i} \cong \mathop{*}_{i=1}^{m} Q_{\mathfrak{F}_i}(C^*(L_{n_i})).$$

Proof. First, we claim that there exists a surjective C^*-homomorphism from $*_{i=1}^m U_{n_i}^+$ to $Q_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$ that takes generators of $*_{i=1}^m U_{n_i}^+$ to generators of $Q_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$. To verify this, suppose $((Q,\Delta),\alpha)$ is an object in the category $\mathfrak{C}_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$. Then, by [BS13, (2.1)], the action α preserves the state $\bigoplus_{k=1}^m \mathrm{KMS}_{\ln(n_k)}$. Moreover, $\alpha(\mathcal{M}_{0,1}^{(1)}) \subset \mathcal{M}_{0,1}^{(1)} \otimes Q$ implies that α is linear. Hence, $((Q,\Delta),\alpha)$ is an object of $\mathfrak{C}_{\oplus \mathrm{KMS}}^{\mathrm{Lin}}$, and the claim follows from Theorem 3.5.

For the converse, we have to show that the CMQG $(*_{i=1}^m U_{n_i}^+, q)$ (as described in Theorem 3.5) acts on $C^*(\bigsqcup_{i=1}^m L_{n_i})$ by α as in (3.8), and preserves the above orthogonal filtration $\mathfrak{F}$. Using (3.8), we get

$$\alpha(S_\gamma S_\mu^*) = \sum_{i_1,\dots,i_k;j_1,\dots,j_k \in E_l} S_{i_1}\dots S_{i_k} S_{j_k}^* \dots S_{j_1}^* \otimes q_{i_1\gamma_1}\dots q_{i_k\gamma_k} q_{j_k\mu_k}^* \dots q_{j_1\mu_1}^*,$$

where $\gamma = \gamma_1 \dots \gamma_k$ and $\mu = \mu_1 \dots \mu_k$ are paths in L_{n_l}. Therefore,

$$\alpha(\mathcal{F}_k) \subset \mathcal{F}_k \otimes Q.$$

Since the fundamental representation $q = (q_{ef})_{n\times n}$ is unitary, the action restricted to each $\mathcal{F}_k$, $\alpha|_{\mathcal{F}_k}$, is actually a unitary (co)representation on $\mathcal{F}_k$. Hence, α preserves the orthogonal complement

$$\mathcal{W}_k = \mathcal{F}_k \ominus \mathcal{F}_{k-1}.$$

Lastly, to show $\alpha(\mathcal{M}_{r,s}^{(1)}) \subset \mathcal{M}_{r,s}^{(1)} \otimes Q$, take a non-zero element

$$S_\mu x = S_{\mu_1}\dots S_{\mu_s} x \in \mathcal{M}_{r,s}^{(1)},$$

where $x \in \mathcal{W}_r$ and μ is a path on $E(L_l)$. Since

$$\alpha(S_\mu x) = \Big(\sum_{i_1,\dots,i_s \in E_l} S_{i_1}\dots S_{i_s} \otimes q_{i_1\mu_1}\dots q_{i_s\mu_s}\Big)(\alpha(x)),$$

where $\alpha(x) \in \mathcal{W}_r \otimes Q$, the claim follows. Also, the $*$-preserving property of α ensures that similar arguments hold for $\alpha(\mathcal{M}_{r,s}^{(2)}) \subset \mathcal{M}_{r,s}^{(2)} \otimes Q$. Hence, there exists a surjective C^*-homomorphism from $Q_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$ to $*_{i=1}^m U_{n_i}^+$ that takes generators to generators, which proves the first isomorphism in the statement of the theorem.

The second isomorphism follows from [JM21, Corollary 5.5]. ∎

Observation 3.7. We mention two key observations from the above proof that will help us to understand Theorem 4.4 below.

(i) Any object of the category $\mathfrak{C}_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m L_{n_i}))$ is also an object of $\mathfrak{C}_{\oplus \mathrm{KMS}}^{\mathrm{Lin}}$ (whether the n_i's are distinct or not).
(ii) In the second half of the above proof, to show that the action α is filtration preserving, we have just used the fact that the fundamental matrix representation q of the acting CMQG is unitary.

4. Direct sum of isomorphic Cuntz algebras

THEOREM 4.1. *Let $\{O_N\}_{i=1}^K$ be a finite family of isomorphic Cuntz algebras (with N generators) whose underlying graphs are L_N. Then*

$$Q_\tau^{\mathrm{Lin}}\Big(\bigsqcup_{i=1}^K L_N\Big) \cong Q_\tau^{\mathrm{Lin}}(L_N) \wr_* S_K^+ \cong U_N^+ \wr_* S_K^+.$$

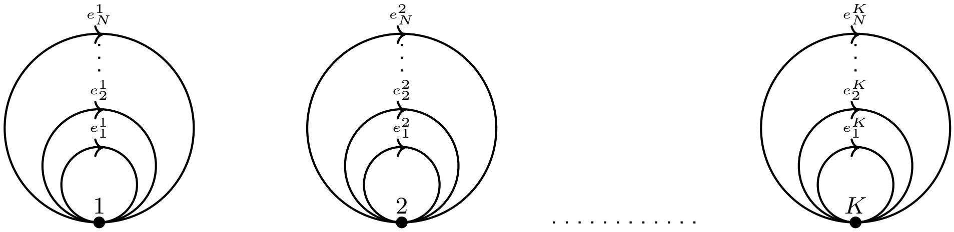


Fig. 5. $\bigsqcup_{i=1}^K L_N$

Proof. Let $V' := V(\bigsqcup_{i=1}^K L_N)$ and $E' := E(\bigsqcup_{i=1}^K L_N)$ (see Figure 5). Denote by e_i^a (for $i \in [N]$) the ith loop based at $a \in V(\bigsqcup_{i=1}^K O_N) =: V'$. The action α can be given by

$$\alpha(S_{e_j^b}) = \sum_{a\in[K],\, i\in[N]} S_{e_i^a} \otimes q_{e_i^a e_j^b}$$

for $b \in [K]$ and $j \in [N]$.

For convenience, we denote $q_{e_i^a e_j^b}$ by q_{ij}^{ab}. Then the above equation can be written as

$$\alpha(S_{e_j^b}) = \sum_{a\in[K],\, i\in[N]} S_{e_i^a} \otimes q_{ij}^{ab}.$$

With respect to the ordering of the edges given by

$$\{e_1^1, \dots, e_N^1; e_1^2, \dots, e_N^2; \dots; e_1^K, \dots, e_N^K\},$$

the fundamental matrix formed by the generators of $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$ is $U := [(U_{ab})]_{NK\times NK}$, where each $U_{ab} := (q_{ij}^{ab})_{i,j\in[N]}$ is an $N \times N$ block matrix.

We divide the proof into two parts. In the first part, we derive the relations among the generators of $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$. In the second part, we use these relations to identify this group with $U_N^+ \wr_* S_K^+$.

Finding the relations among the generators of $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$. To determine the group $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$, we derive some relations among its generators $\{q_{ij}^{ab} : i, j \in [N],\, a, b \in [K]\}$ arising from the action α described above.

• First, since $F^{\bigsqcup_{i=1}^{K} L_N} = NI_{NK\times NK}$, one can easily observe that U and U^t are both unitary matrices. Therefore,

$$\sum_{y\in[K],\, s\in[N]} q_{rs}^{xy}(q_{ts}^{zy})^* = \delta_{(x,r)(z,t)}, \tag{4.1}$$

$$\sum_{y\in[K],\, s\in[N]} (q_{sr}^{yx})^* q_{st}^{yz} = \delta_{(x,r)(z,t)}, \tag{4.2}$$

$$\sum_{y\in[K],\, s\in[N]} q_{sr}^{yx}(q_{st}^{yz})^* = \delta_{(x,r)(z,t)}, \tag{4.3}$$

$$\sum_{y\in[K],\, s\in[N]} (q_{rs}^{xy})^* q_{ts}^{zy} = \delta_{(x,r)(z,t)}. \tag{4.4}$$

• Next, we will show that for all $a, b \in V'$,

$$\sum_{i\in[N]} (q_{ij}^{ab})^* q_{ij}^{ab} = \sum_{l\in[N]} q_{kl}^{ab}(q_{kl}^{ab})^* \quad \text{for any } j,k\in[N], \tag{4.5}$$

$$\sum_{i\in[N]} (q_{ji}^{ba})^* q_{ji}^{ba} = \sum_{l\in[N]} q_{lk}^{ba}(q_{lk}^{ba})^* \quad \text{for any } j,k\in[N]. \tag{4.6}$$

We denote the above term in (4.5) by c_{ab}. Therefore, for all $a, b \in V'$,

$$\begin{aligned} c_{ab} := \sum_{i\in[N]} (q_{i1}^{ab})^* q_{i1}^{ab} &= \sum_{i\in[N]} (q_{i2}^{ab})^* q_{i2}^{ab} = \cdots = \sum_{i\in[N]} (q_{iN}^{ab})^* q_{iN}^{ab} \\ &= \sum_{l\in[N]} q_{1l}^{ab}(q_{1l}^{ab})^* = \sum_{l\in[N]} q_{2l}^{ab}(q_{2l}^{ab})^* = \cdots = \sum_{l\in[N]} q_{Nl}^{ab}(q_{Nl}^{ab})^*. \end{aligned} \tag{4.7}$$

For fixed $a, b \in V'$,

$$S_e^* S_e = \sum_{f\colon s(f)=b} S_f S_f^* \quad \forall e \in E' \text{ with } r(e) = b.$$

Applying the action α to both sides, we get

$$\sum_{g\in E'} S_g^* S_g \otimes q_{ge}^* q_{ge} = \sum_{g,h\in E'} S_g S_h^* \otimes \Big(\sum_{f\colon s(f)=b} q_{gf} q_{hf}^* \Big) \quad \forall e \in E' \text{ with } r(e) = b,$$

and so

$$\sum_{g\in E'} p_{r(g)} \otimes q_{ge}^* q_{ge} = \sum_{g,h\in E'} S_g S_h^* \otimes \Big(\sum_{f\colon s(f)=b} q_{gf} q_{hf}^* \Big) \quad \forall e \in E' \text{ with } r(e) = b.$$

Take any loop $g' \in E'$ with $r(g') = s(g') = a$. Multiplying the above equation by $S_{g'}^* \otimes 1$ on the left and by $S_{g'} \otimes 1$ on the right, we get

$$\sum_{g\in E'} S_{g'}^* p_{r(g)} S_{g'} \otimes q_{ge}^* q_{ge} = \sum_{g,h\in E'} S_{g'}^* S_g S_h^* S_{g'} \otimes \Big(\sum_{f\colon s(f)=b} q_{gf} q_{hf}^* \Big)$$

for all $e \in E'$ such that $r(e) = b$. This yields

$$\sum_{g\colon r(g)=s(g)=a} p_a \otimes q_{ge}^* q_{ge} = S_{g'}^* S_{g'} S_{g'}^* S_{g'} \otimes \Big(\sum_{f\colon s(f)=b} q_{g'f} q_{g'f}^* \Big)$$

and consequently

$$p_a \otimes \Big(\sum_{g\colon r(g)=s(g)=a} q_{ge}^* q_{ge} \Big) = p_a \otimes \Big(\sum_{f\colon s(f)=b} q_{g'f} q_{g'f}^* \Big)$$

for all such e. Therefore, for all $a, b \in V'$,

$$\Big(\sum_{g\colon r(g)=s(g)=a} q_{ge}^* q_{ge} \Big) = \Big(\sum_{f\colon s(f)=b} q_{g'f} q_{g'f}^* \Big)$$

for all $e, g' \in E'$ such that $r(e) = b$ and $r(g') = s(g') = a$. Hence, in our notations, we have

$$\sum_{i\in[N]} (q_{e_i^a e_j^b})^* q_{e_i^a e_j^b} = \sum_{l\in[N]} q_{e_k^a e_l^b} (q_{e_k^a e_l^b})^*$$

and (4.5) follows from the convention $q_{e_i^a e_j^b} := q_{ij}^{ab}$. Applying antipode κ to both sides of (4.5), we get (4.6).

- Moreover, we claim that for all $a, b, b' \in V'$ with $b \neq b'$,

$$q_{ij}^{ab} q_{kl}^{ab'} = 0 \quad \forall i, j, k, l \in [N], \tag{4.8}$$

$$q_{ji}^{ba} q_{lk}^{b'a} = 0 \quad \forall i, j, k, l \in [N]. \tag{4.9}$$

Clearly, (4.9) follows from (4.8) simply by applying the antipode κ to (4.8). To show (4.8), observe that $S_{e_j^b} S_{e_l^{b'}} = 0$ for all $j, l \in [N]$ and $b \neq b'$. Applying the action α, we get

$$\sum_{g,h\in E'} S_g S_h \otimes q_{ge_j^b} q_{he_l^{b'}} = 0.$$

Since $\{S_g S_h : g, h \in E' \text{ with } r(g) = s(h)\}$ is a linearly independent set (by [KM25, Lemma 2.3]), we get

$$q_{ge_j^b} q_{he_l^{b'}} = 0 \quad \forall g, h \in E' \text{ with } r(g) = s(h) \in [K]$$

and so

$$q_{e_i^a e_j^b} q_{e_k^a e_l^{b'}} = 0 \quad \forall i, j, k, l, \in [N] \text{ and } a, b, b' \in [K] \text{ with } b \neq b',$$

i.e. $q_{ij}^{ab} q_{kl}^{ab'} = 0$ for all $i, j, k, l \in [N]$ and $a, b, b' \in [K]$ for all $b \neq b'$.

Using the relations above, namely (4.1)–(4.6), (4.8) and (4.9), we will derive some more relations among $\{q_{ij}^{ab}\}$.

- For $a, b, b' \in V'$ with $b \neq b'$ and $i, j, k, l \in [N]$,

$$(q_{ij}^{ab})^* q_{kl}^{ab'} = 0, \tag{4.10}$$

$$(q_{ji}^{ba})^* q_{lk}^{b'a} = 0, \tag{4.11}$$

$$q_{ij}^{ab} (q_{kl}^{ab'})^* = 0, \tag{4.12}$$

$$q_{ji}^{ba} (q_{lk}^{b'a})^* = 0. \tag{4.13}$$

We will prove only (4.10) and (4.12). A similar proof works for (4.11) and (4.13). To show (4.10), observe that

$$\begin{aligned} (q_{ij}^{ab})^* q_{kl}^{ab'} &= (q_{ij}^{ab})^* \Big(\sum_{y \in [K],\, s \in [N]} (q_{is}^{ay})^* q_{is}^{ay} \Big) q_{kl}^{ab'} \qquad \text{[using (4.4)]} \\ &= \sum_{y \in [K],\, s \in [N]} (q_{is}^{ay} q_{ij}^{ab})^* (q_{is}^{ay} q_{kl}^{ab'}) \\ &= \sum_{s \in [N]} (q_{is}^{ab} q_{ij}^{ab})^* (q_{is}^{ab} q_{kl}^{ab'}) = 0 \qquad \text{[using (4.8)]}. \end{aligned}$$

Similarly, for (4.12), we have

$$\begin{aligned} q_{ij}^{ab} (q_{kl}^{ab'})^* &= q_{ij}^{ab} \Big(\sum_{y \in [K],\, s \in [N]} q_{is}^{ay} (q_{is}^{ay})^* \Big) (q_{kl}^{ab'})^* \qquad \text{[using (4.1)]} \\ &= \sum_{y \in [K],\, s \in [N]} (q_{ij}^{ab} q_{is}^{ay})(q_{kl}^{ab'} q_{is}^{ay})^* \\ &= \sum_{s \in [N]} (q_{ij}^{ab} q_{is}^{ab})(q_{kl}^{ab'} q_{is}^{ab})^* = 0 \qquad \text{[using (4.8)]}. \end{aligned}$$

- For all $a, b \in V'$, $(c_{ab})_{K \times K}$ is a magic unitary matrix such that

$$q_{ij}^{ab} c_{ab} = c_{ab} q_{ij}^{ab} = q_{ij}^{ab} \quad \forall a, b \in [K] \text{ and } i, j \in [N]. \tag{4.14}$$

Observe that for $a, a' \in [K]$ with $a \neq a'$,

$$\begin{aligned} c_{ab} c_{a'b} &= \Big(\sum_{i \in [N]} (q_{i1}^{ab})^* q_{i1}^{ab} \Big) \Big(\sum_{i' \in [N]} (q_{i'1}^{a'b})^* q_{i'1}^{a'b} \Big) \\ &= \sum_{i, i' \in [N]} (q_{i1}^{ab})^* q_{i1}^{ab} (q_{i'1}^{a'b})^* q_{i'1}^{a'b} = 0 \qquad \text{[by (4.13)]}. \end{aligned} \tag{4.15}$$

Since $\sum_{x \in [K]} c_{xb} = 1$ (using (4.2)), multiplying both sides by c_{ab}, we have $\sum_{x \in [K]} c_{ab} c_{xb} = c_{ab}$ and so $c_{ab}^2 = c_{ab}$ [by (4.15)]. Moreover, clearly $c_{ab}^* = c_{ab}$. Hence, each c_{ab} is a projection.

Since $\sum_{x\in[K]} c_{xb} = 1$ again, multiplying by q_{ij}^{ab} on the left, we have

$$\sum_{x\in[K],\, i'\in[N]} q_{ij}^{ab}(q_{i'1}^{xb})^* q_{i'1}^{xb} = q_{ij}^{ab}$$

$$\implies \sum_{i'\in[N]} q_{ij}^{ab}(q_{i'1}^{ab})^* q_{i'1}^{ab} = q_{ij}^{ab} \quad [\text{by } (4.13)]$$

$$\implies q_{ij}^{ab}\Big(\sum_{i'\in[N]} (q_{i'1}^{ab})^* q_{i'1}^{ab}\Big) = q_{ij}^{ab} \implies q_{ij}^{ab} c_{ab} = q_{ij}^{ab}.$$

Again, multiplying by q_{ij}^{ab} on the right, one can get

$$\sum_{x\in[K],\, i'\in[N]} (q_{i'1}^{xb})^* q_{i'1}^{xb} q_{ij}^{ab} = q_{ij}^{ab}$$

$$\implies \sum_{i'\in[N]} (q_{i'1}^{ab})^* q_{i'1}^{ab} q_{ij}^{ab} = q_{ij}^{ab} \quad [\text{by } (4.9)]$$

$$\implies \Big(\sum_{i'\in[N]} (q_{i'1}^{ab})^* q_{i'1}^{ab}\Big) q_{ij}^{ab} = q_{ij}^{ab} \implies c_{ab} q_{ij}^{ab} = q_{ij}^{ab}.$$

Hence, (4.14) holds.

Therefore, the generators $\{q_{ij}^{ab} : a, b \in [K] \text{ and } i, j \in [N]\}$ of the group $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$ satisfy the relations specified in (4.1)–(4.6) and (4.8)–(4.14). Moreover, the coproduct on these generators is given by

$$\Delta_1(q_{ij}^{ab}) = \sum_{c\in[K],\, k\in[N]} q_{ik}^{ac} \otimes q_{kj}^{cb}.$$

Finding the CQG isomorphism between $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$ and $U_N^+ \wr_* S_K^+$. We will now identify $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$ with the compact quantum group $(U_N^+ \wr_* S_K^+, \Delta_2)$, which is described as follows. $U_N^+ \wr_* S_K^+$ is defined as the universal C^*-algebra generated by the entries of the matrices $U^{(1)} = (u_{ij}^1)_{N\times N}, U^{(2)} = (u_{ij}^2)_{N\times N}, \ldots, U^{(K)} = (u_{ij}^K)_{N\times N}, T = (t_{ab})_{K\times K}$ such that

(a) for each $p \in [K]$, both $U^{(p)}$ and $U^{(p)^t}$ are unitary matrices,
(b) T is a magic unitary matrix,
(c) $u_{ij}^a t_{ab} = t_{ab} u_{ij}^a$ for all $a, b \in [K]$, $i, j \in [N]$.

Moreover, the coproduct Δ_2 is given on generators by

$$\Delta_2(u_{ij}^a) = \sum_{c'\in[K],\, k\in[N]} (u_{ik}^a \otimes u_{kj}^{c'})(t_{ac'} \otimes 1) \quad \text{and} \quad \Delta_2(t_{ab}) = \sum_{c\in[K]} t_{ac} \otimes t_{cb}.$$

Define $\phi : Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N) \to U_N^+ \wr_* S_K^+$ on generators by $\phi(q_{ij}^{ab}) = u_{ij}^a t_{ab} =: w_{ij}^{ab}$. We will prove that ϕ is a CQG isomorphism in three steps.

STEP 1. We first show that ϕ is a surjective *-homomorphism. To do this, we construct a linear τ-preserving action α such that $(U_N^+ \wr_* S_K^+, \Delta_2)$ acts faithfully on $C^*(\bigsqcup_{i=1}^K L_N)$; in other words, $((U_N^+ \wr_* S_K^+, \Delta_2), \alpha)$ is an object of the category $\mathfrak{C}_\tau^{\mathrm{Lin}}$.

For convenience, we derive several relations among $\{w_{ij}^{ab}\}$ and $\{t_{ab}\}$, which will later be used to show that α is a well-defined τ-preserving action. For all $x, z \in [K]$ and $i, j \in [N]$, we have

$$
\begin{aligned}
(4.16)\quad & \sum_{y\in[K],\, s\in[N]} w_{is}^{xy}(w_{js}^{zy})^* \\
&= \sum_{y\in[K],\, s\in[N]} u_{is}^x t_{xy}(u_{js}^z t_{zy})^* = \sum_{y\in[K],\, s\in[N]} u_{is}^x t_{xy} t_{zy}(u_{js}^z)^* \\
&= \sum_{s\in[N]} u_{is}^x \Big(\sum_{y\in[K]} t_{xy}t_{zy}\Big)(u_{js}^z)^* = \sum_{s\in[N]} u_{is}^x(\delta_{xz})(u_{js}^z)^* \qquad \text{[by (b)]} \\
&= \delta_{xz}\sum_{s\in[N]} u_{is}^x(u_{js}^z)^* = \begin{cases} \sum_{s\in[N]} u_{is}^x(u_{js}^x)^* = \delta_{ij} & \text{if } x = z \\ 0 & \text{otherwise} \end{cases} \qquad \text{[by (a)]} \\
&= \begin{cases} 1 & \text{if } x = z,\ i = j \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{(x,i)(z,j)},
\end{aligned}
$$

and

$$
\begin{aligned}
(4.17)\quad & \sum_{y\in[K],\, s\in[N]} w_{si}^{yx}(w_{sj}^{yz})^* \\
&= \sum_{y\in[K],\, s\in[N]} u_{si}^y t_{yx} t_{yz}(u_{sj}^y)^* = \sum_{y\in[K],\, s\in[N]} t_{yx}t_{yz}u_{si}^y(u_{sj}^y)^* \qquad \text{[by (c)]} \\
&= \sum_{y\in[K]} t_{yx}t_{yz}\Big(\sum_{s\in[N]} u_{si}^y(u_{sj}^y)^*\Big) = \sum_{y\in[K]} t_{yx}t_{yz}(\delta_{ij}) \qquad \text{[by (a)]} \\
&= \delta_{ij}\delta_{xz} \qquad \text{[by (b)]} \\
&= \delta_{(x,i)(z,j)}.
\end{aligned}
$$

Similarly, one can show that

$$
(4.18)\quad \sum_{y\in[K],\, s\in[N]} (w_{si}^{yx})^* w_{sj}^{yz} = \delta_{(x,i)(z,j)}, \qquad \sum_{y\in[K],\, s\in[N]} (w_{is}^{xy})^* w_{js}^{zy} = \delta_{(x,i)(z,j)}.
$$

Furthermore, for all $a, b \in [K]$ and $i, j \in [N]$, we have

$$
\begin{aligned}
(4.19)\quad \sum_{l\in[N]} (w_{li}^{ab})^* w_{lj}^{ab} &= \sum_{l\in[N]} (u_{li}^a t_{ab})^*\, u_{lj}^a t_{ab} = \sum_{l\in[N]} t_{ab}(u_{li}^a)^* u_{lj}^a t_{ab} \\
&= t_{ab}\Big(\sum_{l\in[N]} (u_{li}^a)^* u_{lj}^a\Big) t_{ab} = \delta_{ij} t_{ab} \qquad \text{[by (a) and (b)]},
\end{aligned}
$$

and

$$\begin{aligned}
(4.20) \qquad \sum_{l\in[N]} w_{il}^{ab}(w_{jl}^{ab})^* &= \sum_{l\in[N]} u_{il}^a t_{ab}(u_{jl}^a)^* && \text{[by (b)]}\\
&= \sum_{l\in[N]} t_{ab}u_{il}^a(u_{jl}^a)^* && \text{[by (c)]}\\
&= t_{ab}\sum_{l\in[N]} u_{il}^a(u_{jl}^a)^* = \delta_{ij}t_{ab} && \text{[by (a)].}
\end{aligned}$$

We now define an action

$$\alpha : C^*\Big(\bigsqcup_{i=1}^{K} L_N\Big) \to C^*\Big(\bigsqcup_{i=1}^{K} L_N\Big) \otimes (U_N^+ \wr_* S_K^+)$$

on generators by

$$\alpha(S_{e_j^b}) := \sum_{a\in[K],\, i\in[N]} S_{e_i^a} \otimes w_{ij}^{ab}, \qquad \alpha(p_b) := \sum_{a\in[K]} p_a \otimes t_{ab}$$

for all $b \in [K]$ and $j \in [N]$. To show the existence of α, we verify that the elements of $\{\alpha(S_{e_j^b}), \alpha(p_c) : j \in [N] \text{ and } b, c \in [K]\}$ satisfy the defining relations (i) and (ii) of Definition 2.2 for $C^*(\bigsqcup_{i=1}^K L_N)$:

$$\begin{aligned}
\alpha(S_{e_j^b})^*\alpha(S_{e_j^b}) &= \sum_{\substack{a,c\in[K]\\ i,k\in[N]}} S_{e_i^a}^* S_{e_k^c} \otimes (w_{ij}^{ab})^* w_{kj}^{cb}\\
&= \sum_{a\in[K],\, i\in[N]} S_{e_i^a}^* S_{e_i^a} \otimes (w_{ij}^{ab})^* w_{ij}^{ab} \qquad \text{[by Proposition 2.3(i)]}\\
&= \sum_{a\in[K]} p_a \otimes \Big(\sum_{i\in[N]} (w_{ij}^{ab})^* w_{ij}^{ab}\Big) = \sum_{a\in[K]} p_a \otimes t_{ab} \qquad \text{[by (4.19)]}\\
&= \alpha(p_b)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j\in[N]} \alpha(S_{e_j^b})\alpha(S_{e_j^b})^*\\
&\quad= \sum_{j\in[N]} \sum_{\substack{a,c\in[K]\\ i,k\in[N]}} S_{e_i^a} S_{e_k^c}^* \otimes w_{ij}^{ab}(w_{kj}^{cb})^*\\
&\quad= \sum_{j\in[N]} \sum_{\substack{a\in[K]\\ i,k\in[N]}} S_{e_i^a} S_{e_k^a}^* \otimes w_{ij}^{ab}(w_{kj}^{ab})^* \qquad \text{[by Proposition 2.3(iv)]}\\
&\quad= \sum_{\substack{a\in[K]\\ i,k\in[N]}} S_{e_i^a} S_{e_k^a}^* \otimes \delta_{ik}t_{ab} \qquad \text{[by (4.20)]}
\end{aligned}$$

$$= \sum_{a\in[K],\, i\in[N]} S_{e_i^a} S_{e_i^a}^* \otimes t_{ab} = \sum_{a\in[K]} \Big(\sum_{i\in[N]} S_{e_i^a} S_{e_i^a}^* \Big) \otimes t_{ab}$$
$$= \sum_{a\in[K]} p_a \otimes t_{ab} = \alpha(p_b).$$

Since α is a linear action, it is straightforward to verify that α satisfies both the action equation and the faithfulness condition.

To show the Podleś condition, we have to ensure that both $S_{e_i^a} \otimes 1$ and $S_{e_i^a}^* \otimes 1$ belong to $\operatorname{span}[\alpha(C^*(\bigsqcup_{i=1}^K L_N))(1 \otimes U_N^+ \wr_* S_K^+)]$ for all $a \in [K]$ and $i \in [N]$. Indeed, observe that

$$\begin{aligned}
\sum_{c\in[K],\, s\in[N]} \alpha(S_{e_s^c})(1 \otimes (w_{is}^{ac})^*) &= \sum_{\substack{c\in[K]\\ s\in[N]}} \Big(\sum_{\substack{p\in[K]\\ k\in[N]}} S_{e_k^p} \otimes w_{ks}^{pc} \Big) (1 \otimes (w_{is}^{ac})^*) \\
&= \sum_{\substack{c,p\in[K]\\ s,k\in[N]}} S_{e_k^p} \otimes w_{ks}^{pc} (w_{is}^{ac})^* \\
&= \sum_{p\in[K],\, k\in[N]} S_{e_k^p} \otimes \Big(\sum_{c\in[K],\, s\in[N]} w_{ks}^{pc} (w_{is}^{ac})^* \Big) \\
&= \sum_{p\in[K],\, k\in[N]} S_{e_k^p} \otimes \delta_{(p,k)(a,i)} \quad \text{[by (4.16)]} \\
&= S_{e_i^a} \otimes 1,
\end{aligned}$$

and (4.18) similarly ensures that

$$\sum_{c\in[K],\, s\in[N]} \alpha(S_{e_s^c}^*)(1 \otimes w_{is}^{ac}) = S_{e_i^a}^* \otimes 1.$$

Moreover, since α is a unital action, it follows that $1 \otimes q = \alpha(1)(1 \otimes q) \in \alpha(C^*(\bigsqcup_{i=1}^K L_N))(1 \otimes U_N^+ \wr_* S_K^+)$ for all $q \in U_N^+ \wr_* S_K^+$.

The rest of the argument now follows by invoking standard facts about CQG actions (for further details, one may consult [SW17, 4.2.3 and 4.2.4]).

Finally, since the subspace $\mathcal{V}_{2,+}$ for the graph $\bigsqcup_{i=1}^K L_N$ is spanned by $\{S_{e_i^a} S_{e_j^a}^* : a \in [K] \text{ and } i, j \in [N]\}$, one can get the τ-preserving condition from

$$\begin{aligned}
(\tau \otimes \mathrm{id})\alpha(S_{e_i^a} S_{e_j^a}^*) &= \sum_{\substack{c,d\in[K]\\ k,l\in[N]}} \tau(S_{e_k^c} S_{e_l^d}^*) w_{ki}^{ca} {w_{lj}^{da}}^* = \sum_{\substack{c,d\in[K]\\ k,l\in[N]}} \delta_{(c,k)(d,l)} w_{ki}^{ca} {w_{lj}^{da}}^* \\
&= \sum_{c\in[K],\, k\in[N]} w_{ki}^{ca} {w_{kj}^{ca}}^* = \delta_{ij} 1 \quad \text{[by (4.17)]} \\
&= \tau(S_{e_i^a} S_{e_j^a}^*) 1.
\end{aligned}$$

Therefore, by the universal property of $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$, there exists a surjective C^*-homomorphism $\phi : Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N) \to U_N^+ \wr_* S_K^+$ such that $\phi(q_{ij}^{ab}) = w_{ij}^{ab} = u_{ij}^a t_{ab}$.

STEP 2. We define a *-homomorphism $\psi : U_N^+ \wr_* S_K^+ \to Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K L_N)$ such that ψ is the inverse of ϕ. On the generators of $U_N^+ \wr_* S_K^+$ we set

$$\psi(u_{ij}^a) = \sum_{b\in[K]} q_{ij}^{ab} =: v_{ij}^a \quad \text{and} \quad \psi(t_{ab}) = \sum_{i\in[N]} (q_{ij}^{ab})^* q_{ij}^{ab} = c_{ab}.$$

We need to verify that $\{v_{ij}^a, c_{ab} : a, b \in [K] \text{ and } i, j \in [N]\}$ satisfies all three defining relations (a), (b) and (c).

We start by observing that

$$\begin{aligned}\sum_{k\in[N]} v_{ik}^a (v_{jk}^a)^* &= \sum_{k\in[N]} \sum_{b,b'\in[K]} q_{ik}^{ab} (q_{jk}^{ab'})^* = \sum_{k\in[N],b\in[K]} q_{ik}^{ab} (q_{jk}^{ab})^* \qquad \text{[by (4.12)]}\\ &= \delta_{ij} \qquad \text{[by (4.1)]},\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in[N]} (v_{ki}^a)^* v_{kj}^a &= \sum_{k\in[N]} \sum_{b,b'\in[K]} (q_{ki}^{ab})^* q_{kj}^{ab'} = \sum_{k\in[N],\, b\in[K]} (q_{ki}^{ab})^* q_{kj}^{ab} \qquad \text{[by (4.10)]}\\ &= \sum_{k\in[N],\, b\in[K]} q_{ik}^{ab} (q_{jk}^{ab})^* \qquad \text{[by (4.7)]}\\ &= \delta_{ij}.\end{aligned}$$

A similar computation also shows that

$$\sum_{k\in[N]} v_{ki}^a (v_{kj}^a)^* = \delta_{ij}, \quad \text{and} \quad \sum_{k\in[N]} v_{ik}^{a\,*} v_{jk}^a = \delta_{ij}.$$

Therefore, for each $p \in [K]$, both $V^{(p)} := (v_{ij}^p)_{N\times N}$ and ${V^{(p)}}^t$ are unitary matrices.

Since each c_{ab} is a projection with $\sum_{x\in[K]} c_{xb} = 1$ and

$$\begin{aligned}\sum_{y\in[K]} c_{ay} &= \sum_{y\in[K]} \sum_{i\in[N]} (q_{ij}^{ay})^* q_{ij}^{ay} = \sum_{y\in[K]} \sum_{j\in[N]} q_{ij}^{ay} (q_{ij}^{ay})^* \quad \text{[using (4.7)]}\\ &= 1,\end{aligned}$$

$(c_{ab})_{K\times K}$ is a magic unitary matrix.

Lastly, we show that $v_{ij}^a c_{ab} = c_{ab} v_{ij}^a$ for all $a, b \in [K]$ and $i, j \in [N]$. Indeed,

$$\begin{aligned}v_{ij}^a c_{ab} &= \sum_{b'\in[K],\, i'\in[N]} q_{ij}^{ab'} (q_{i'1}^{ab})^* q_{i'1}^{ab} = \sum_{i'\in[N]} q_{ij}^{ab} (q_{i'1}^{ab})^* q_{i'1}^{ab} \qquad \text{[by (4.12)]}\\ &= q_{ij}^{ab} c_{ab} = q_{ij}^{ab} \qquad \text{[by (4.14)]}.\end{aligned}$$

Similarly, using (4.8) and (4.14), one can obtain $c_{ab}v^a_{ij} = q^{ab}_{ij}$. Hence, $v^a_{ij}c_{ab} = c_{ab}v^a_{ij}$ for all $a, b \in [K]$ and $i, j \in [N]$.

The above calculations show that the elements of $\{v^a_{ij}, c_{ab} : a, b \in [K]$ and $i, j \in [N]\} \subset Q^{\mathrm{Lin}}_\tau(\bigsqcup_{i=1}^K L_N)$ satisfy all the defining relations (a)–(c) of $U^+_N \wr_* S^+_K$.

Therefore, by the universal property of $U^+_N \wr_* S^+_K$, there exists a surjective C^*-homomorphism $\psi : U^+_N \wr_* S^+_K \to Q^{\mathrm{Lin}}_\tau(\bigsqcup_{i=1}^K L_N)$ such that $\psi(u^a_{ij}) = v^a_{ij} = \sum_{b\in[K]} q^{ab}_{ij}$ and $\psi(t_{ab}) = c_{ab} = \sum_{i\in[N]}(q^{ab}_{ij})^* q^{ab}_{ij}$.

Moreover, it is evident that $\phi \circ \psi = \mathrm{id}_{U^+_N \wr_* S^+_K}$ and $\psi \circ \phi = \mathrm{id}_{Q^{\mathrm{Lin}}_\tau(\bigsqcup_{i=1}^K L_N)}$. Hence, $\phi : Q^{\mathrm{Lin}}_\tau(\bigsqcup_{i=1}^K L_N) \to U^+_N \wr_* S^+_K$ is a C^*-isomorphism.

Step 3. Finally, to show ϕ is a CQG morphism, it is sufficient to verify that $(\phi\otimes\phi)\Delta_1 = \Delta_2\phi$ on the generators. Starting with the left-hand side, we have

$$\begin{aligned}(\phi\otimes\phi)\Delta_1(q^{ab}_{ij}) &= (\phi\otimes\phi)\Big(\sum_{k\in[N],\, c\in[K]} q^{ac}_{ik}\otimes q^{cb}_{kj}\Big)\\ &= \sum_{k\in[N],\, c\in[K]} \phi(q^{ac}_{ik})\otimes\phi(q^{cb}_{kj}) = \sum_{k\in[N],\, c\in[K]} u^a_{ik}t_{ac}\otimes u^c_{kj}t_{cb}.\end{aligned}$$

On the other hand, the right-hand side gives

$$\begin{aligned}\Delta_2\phi(q^{ab}_{ij}) &= \Delta_2(u^a_{ij}t_{ab}) = \Delta_2(u^a_{ij})\Delta_2(t_{ab})\\ &= \Big[\sum_{c'\in[K],\, k\in[N]} (u^a_{ik}\otimes u^{c'}_{kj})(t_{ac'}\otimes 1)\Big]\Big[\sum_{c\in[K]}(t_{ac}\otimes t_{cb})\Big]\\ &= \sum_{c,c'\in[K],\, k\in[N]} (u^a_{ik}\otimes u^{c'}_{kj})(t_{ac'}\otimes 1)(t_{ac}\otimes t_{cb})\\ &= \sum_{c'\in[K],\, k\in[N]} (u^a_{ik}\otimes u^{c'}_{kj})\sum_{c\in[K]}(t_{ac'}t_{ac}\otimes t_{cb})\\ &= \sum_{c'\in[K],\, k\in[N]} (u^a_{ik}\otimes u^{c'}_{kj})(t_{ac'}\otimes t_{c'b}) = \sum_{c'\in[K],\, k\in[N]} (u^a_{ik}t_{ac'}\otimes u^{c'}_{kj}t_{c'b}).\end{aligned}$$

Thus, $(\phi\otimes\phi)\Delta_1 = \Delta_2\phi$, and the proof is complete. ∎

Remark 4.2. For a connected graph Γ, if we consider K disjoint copies of Γ, namely $\bigsqcup_{i=1}^K \Gamma$, then the relevant quantum automorphism group in the sense of Banica (respectively, Bichon) is isomorphic to the free wreath product of the quantum automorphism group of Γ in the sense of Banica (respectively, Bichon) with S^+_K (see [BB07, Bi04] for more details), i.e.

$$\mathrm{QAut}_{\mathrm{Ban}}\Big(\bigsqcup_{i=1}^K \Gamma\Big) \cong \mathrm{QAut}_{\mathrm{Ban}}(\Gamma) \wr_* S^+_K$$

and

$$\mathrm{QAut}_{\mathrm{Bic}}\Big(\bigsqcup_{i=1}^{K}\Gamma\Big)\cong \mathrm{QAut}_{\mathrm{Bic}}(\Gamma)\wr_* S_K^+.$$

Though Theorem 4.1 ensures that a similar result is also true for the graph L_n (i.e. for the underlying Cuntz algebra $\mathcal{O}_n$) in the context of graph C^*-algebras, the same is not always true for an arbitrary graph in the category discussed above. In the next subsection, we will provide a counterexample to show the above claim.

Now, we will justify the analogous results in terms of the categories $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ and $\mathfrak{C}_{\mathfrak{F}}(\bigsqcup_{i=1}^{K} L_N)$.

Theorem 4.3. *Let $\{\mathcal{O}_N\}_{i=1}^K$ be a finite family of isomorphic Cuntz algebras (with N generators) whose underlying graphs are L_N. Then*

$$Q^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}\Big(\bigsqcup_{i=1}^{K}L_N\Big)\cong Q^{\mathrm{Lin}}_{\tau}\Big(\bigsqcup_{i=1}^{K}L_N\Big)\cong U_N^+\wr_* S_K^+\cong Q^{\mathrm{Lin}}_{\mathrm{KMS}}(L_N)\wr_* S_K^+.$$

Proof. In this case, since $\mathrm{KMS}_{\ln\rho(A(\bigsqcup_{i=1}^{K}L_N))}=\bigoplus_{i=1}^{K}\mathrm{KMS}_{\ln\rho(A(L_N))}$ (by Corollary 2.5), $\mathfrak{C}^{\mathrm{Lin}}_{\mathrm{KMS}}$ and $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ coincide for $\bigoplus_{i=1}^{K}\mathcal{O}_N$. Hence, by Theorem 2.21, $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ has a universal object. Now, using Corollary 2.23, we can say that the universal object of $\mathfrak{C}^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}$ is isomorphic to the universal object of $\mathfrak{C}^{\mathrm{Lin}}_{\tau}$. Hence, the result follows. ∎

Theorem 4.4. *Let $\{\mathcal{O}_N\}_{i=1}^K$ be a finite family of isomorphic Cuntz algebras (with N generators) whose underlying graphs are L_N. Then*

$$Q_{\mathfrak{F}}\Big(C^*\Big(\bigsqcup_{i=1}^{K}L_N\Big)\Big)\cong Q^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}\Big(\bigsqcup_{i=1}^{K}L_N\Big)\cong U_N^+\wr_* S_K^+\cong Q_{\mathfrak{F}_i}(C^*(L_N))\wr_* S_K^+.$$

Recall that the fundamental representation $q=(q_{ij}^{ab})_{NK\times NK}$ of CMQG

$$\Big(Q^{\mathrm{Lin}}_{\oplus\mathrm{KMS}}\Big(\bigsqcup_{i=1}^{K}L_N\Big),q\Big)\cong\Big(Q^{\mathrm{Lin}}_{\tau}\Big(\bigsqcup_{i=1}^{K}L_N\Big),q\Big)$$

satisfies (4.1)–(4.4), which imply q is unitary. Now, Observation 3.7 ensures that exactly the same arguments as in Theorem 3.6 work to show the first isomorphism mentioned in the above theorem.

Counterexample 4.5. We provide a graph Γ such that $Q^{\mathrm{Lin}}_{\tau}(\bigsqcup_{i=1}^{K}\Gamma)$ is not the same as $Q^{\mathrm{Lin}}_{\tau}(\Gamma)\wr_* S_K^+$.

Consider the graph $P_1\sqcup P_1$, a disjoint union of two distinct edges e_1 and e_2 (see Figure 6). Here, P_1 represents a simple directed path of length 1 (which is effectively a directed edge). Note that $C^*(P_1\sqcup P_1)$ is C^*-isomorphic to $M_2(\mathbb{C})\oplus M_2(\mathbb{C})$.

Fig. 6. $P_1 \sqcup P_1$

Again, Proposition 3.4 of [KM25] tells us that $Q_\tau^{\mathrm{Lin}}(P_1 \sqcup P_1) \approx SH_2^{\infty+}$ with respect to their standard fundamental representations, as mentioned in Sections 2.4 and 2.6. On the other hand, $Q_\tau^{\mathrm{Lin}}(P_1) \wr_* S_2^+ \cong C(S^1) \wr_* S_2^+ \cong H_2^{\infty+}$. But clearly, $SH_2^{\infty+}$ is not identical to $H_2^{\infty+}$.

Remark 4.6. Though from Remark 4.2, the quantum automorphism group (in the sense of Banica or Bichon) of K disjoint copies of a connected graph Γ is isomorphic to the free wreath product of the quantum automorphism group of Γ with S_K^+, Counterexample 4.5 tells us that a similar result may not be true if we move from the context of graphs to graph C^*-algebras. We can ask how one can classify the graph C^*-algebras $C^*(\Gamma)$ for which $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^K \Gamma) \cong Q_\tau^{\mathrm{Lin}}(\Gamma) \wr_* S_K^+$. (Note that the class is non-empty due to Theorem 4.1.)

Remark 4.7. Using similar ideas to what we have used in the main theorems in Sections 3 and 4, one can extend the results to the quantum symmetry of the direct sum of any family of Cuntz algebras.

If there are k_i copies of L_{n_i} for all $i \in [m]$, where all n_i's are distinct, then the quantum symmetries of the direct sum Cuntz algebras, namely $Q_\tau^{\mathrm{Lin}}(\bigsqcup_{i=1}^m \bigsqcup_{k=1}^{k_i} L_{n_i})$, $Q_{\oplus\mathrm{KMS}}^{\mathrm{Lin}}(\bigsqcup_{i=1}^m \bigsqcup_{k=1}^{k_i} L_{n_i})$ and $Q_{\mathfrak{F}}(C^*(\bigsqcup_{i=1}^m \bigsqcup_{k=1}^{k_i} L_{n_i}))$ (with respect to the categories mentioned before), are isomorphic to $*_{i=1}^m (U_{n_i}^+ \wr_* S_{k_i}^+)$.

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