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Cosine families, invariant subspaces, and boundary conditions for a class of diffusions on star graphs

by

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Abstract. This paper explores the interplay between boundary conditions and invariant subspaces for one-dimensional Laplacians, extending these concepts to Walsh's spider process on a star-like graph. We establish a precise correspondence between the transmission condition characterizing this process and a specific subspace within a larger function space. This correspondence is facilitated by relating the cosine family associated with the spider process to the basic cosine family of unrestricted Brownian motion. Furthermore, we introduce a complementary subspace, leading to a novel decomposition of the function space that generalizes known results for simpler boundary conditions. This decomposition reveals a fundamental relationship between two distinct transmission conditions, highlighting their complementary nature. Our findings provide new insights into the structure of Walsh's spider process and offer a framework for further analysis, including the study of its limiting behavior as the stickiness parameter varies.

1. Motivation. It has been known for some time that in the cosine family (and, consequently, the semigroup) describing one-dimensional *unrestricted* Brownian motion, harbors hidden treasures: a collection of invariant subspaces related to Brownian motions on a half-line with various types of boundary behavior. Specifically, there is a one-to-one correspondence between Feller–Wentzell boundary conditions at $x = 0$ and certain subspaces that are invariant under the *basic cosine family* $\{C(t) : t \in \mathbb{R}\}$ defined by

$$C(t)f(x) = \tfrac{1}{2}[f(x+t) + f(x-t)], \quad f \in C[-\infty,\infty],\ x,t \in \mathbb{R},$$

where $C[-\infty,\infty]$ denotes the space of continuous functions on $\mathbb{R}$ with finite limits at $\pm\infty$, equipped with the usual supremum norm.

The origins of this observation can be traced back to Feller [13, pp. 340–343], who used the invariant subspace $C_{\mathrm{even}}[-\infty,\infty]$ of even functions in

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$C[-\infty,\infty]$ to construct the *reflecting* Brownian motion semigroup in $C[0,\infty]$, the space of continuous functions on $\mathbb{R}_+ := [0,\infty)$ having finite limits at infinity. Furthermore, there exists a corresponding cosine family $\{C_{\mathrm{ref}}(t) : t \in \mathbb{R}\}$ in $C[0,\infty]$ which is similar (isomorphic) to the basic cosine family restricted to the subspace $C_{\mathrm{even}}[-\infty,\infty]$,

$$C_{\mathrm{ref}}(t) \cong C(t)\big|_{C_{\mathrm{even}}[-\infty,\infty]}, \qquad t \in \mathbb{R}.$$

The underlying isometric isomorphism is the operator E_{ref} mapping a function to its even extension. Thus

$$C_{\mathrm{ref}}(t) = RC(t)E_{\mathrm{ref}}, \qquad t \in \mathbb{R}, \tag{1.1}$$

where R maps any function on $\mathbb{R}$ to its restriction to $\mathbb{R}_+$. We note that this cosine family is generated by the Laplace operator $f \mapsto f''$ with domain consisting of functions $f \in C[0,\infty]$ that are twice continuously differentiable with $f'' \in C[0,\infty]$ and satisfy the Neumann boundary condition $f'(0) = 0$.

Another invariant subspace, the space $C_{\mathrm{odd}}[-\infty,\infty]$ of odd functions in $C[-\infty,\infty]$, is associated with the Dirichlet boundary condition $f(0) = 0$ and the cosine family $\{C_{\mathrm{min}}(t) : t \in \mathbb{R}\}$ of the *minimal* Brownian motion on $\mathbb{R}_+$. Clearly, this family acts in the space $C_0[0,\infty] \subset C[0,\infty]$ of functions that vanish at 0. The *abstract Kelvin formula* analogous to (1.1) also holds, but the extension operator E_{ref} must be replaced by E_{min}, which maps a function to its odd extension.

In [5], a similar construction was performed to demonstrate that one-dimensional Laplacians with more general boundary conditions at the origin generate cosine families in $C[0,\infty]$. In particular, for $\alpha > 0$, the so-called *elastic* and *sticky* boundary conditions correspond precisely to the following invariant subspaces (see Section 3 for more details):

$$\begin{aligned} f'(0) = \alpha f(0) &\leftrightarrow C_{\mathrm{el}}[-\infty,\infty] := \{f \in C[-\infty,\infty] : f^o = \alpha e_\alpha * f\}, \\ f''(0) = \alpha f'(0) &\leftrightarrow C_{\mathrm{st}}[-\infty,\infty] \\ &\qquad := \{f \in C[-\infty,\infty] : f^e = f(0)e_\alpha + \alpha e_\alpha * f\}, \end{aligned} \tag{1.2}$$

where $\leftrightarrow$ denotes correspondence. Here and throughout, for $f \in C[-\infty,\infty]$, f^e and f^o represent the even and odd parts of f, respectively, that is,

$$f^e(x) := \tfrac{1}{2}(f(x) + f(-x)) \quad \text{and} \quad f^o(x) := \tfrac{1}{2}(f(x) - f(-x)), \qquad x \in \mathbb{R}.$$

Furthermore, we define

$$e_\alpha(x) := \mathrm{e}^{-\alpha x}, \qquad x \in \mathbb{R},$$

and, for any real-valued continuous functions g, h, we set

$$g * h(x) := \int_0^x g(x-y)h(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

A particularly intriguing result arising from the correspondence between boundary conditions and invariant subspaces is that the standard decomposition

$$C[-\infty,\infty] = C_{\text{even}}[-\infty,\infty] \oplus C_{\text{odd}}[-\infty,\infty]$$

of subspaces related to Neumann and Dirichlet boundary conditions can be seen as the limiting case, when $\alpha \to \infty$, of the decompositions

$$C[-\infty,\infty] = C_{\text{st}}[-\infty,\infty] \oplus C_{\text{el}}[-\infty,\infty], \tag{1.3}$$

established in [7], demonstrating that also sticky and elastic boundary conditions are complementary. In a sense, these conditions are perpendicular, as the related projections, which map each $f \in C[-\infty,\infty]$ to functions $Pf \in C_{\text{st}}[-\infty,\infty]$ and $Qf \in C_{\text{el}}[-\infty,\infty]$ given by

$$Pf(x) = f^o(x) + \alpha \int_x^\infty e^{\alpha(x-y)} f(-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}, \tag{1.4}$$

and $Qf = f - Pf$ have their origins in orthogonal projections in $L^2(\mathbb{R})$. Specifically, Pf minimizes the functional $L(g) = \int_{-y}^{y} [f(x) - g(x)]^2\,\mathrm{d}x$ among all $g \in C_{\text{st}}[-\infty,\infty]$ for any positive y.

2. Transmission conditions on the infinite star-like graph: our main results. Let $k \in \mathbb{N}$ and let $K = \{1,\dots,k\}$. Let $\alpha > 0$ and let $\beta = (\beta_i)_{i\in K}$ be a sequence of positive numbers such that

$$\sum_{i\in K} \beta_i = 1. \tag{2.1}$$

The primary objective of this paper is to extend the ideas presented in the previous section to the case of Walsh's spider process with a sticky vertex. This entails investigating the transmission condition

$$\alpha^{-1} f''(0) - \sum_{i\in K} \beta_i f_i'(0) = 0 \tag{2.2}$$

for the Laplace operator on the star-like graph S_k with k infinitely long edges emanating from the origin (see Figure 1).

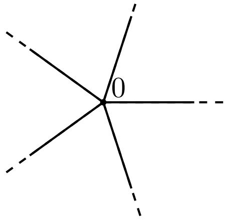

Fig. 1. The infinite star-like graph S_k with $k = 5$ edges

To elaborate, let $\mathscr{X}$ be the Cartesian product of k copies of $C[0,\infty]$,

$$\mathscr{X} := (C[0,\infty])^k.$$

Since all edges of S_k can be identified with $\mathbb{R}_+$, we will work in the space

$$\mathscr{X}_0 := \{(f_i)_{i\in K} \in \mathscr{X} : f_i(0) = f_j(0) \text{ for all } i,j \in K\}$$

and treat each element of $\mathscr{X}_0$ as a single continuous function f on S_k. Consequently, we will adopt the convention that $f(0)$ denotes the common value of $f_i(0)$, $i \in K$.

We then define an operator $\mathsf{A}_{\alpha,\beta}$ in $\mathscr{X}_0$ by

$$\mathsf{A}_{\alpha,\beta} f = \left(f_i''\right)_{i\in K}$$

with domain consisting of all $f = (f_i)_{i\in K} \in \mathscr{X}_0$ satisfying the following three properties:

(a) each f_i is twice continuously differentiable with $f_i'' \in C[0,\infty]$,
(b) $f_i''(0) = f_j''(0)$ for all $i,j \in K$,
(c) condition (2.2) holds.

Note that $\mathsf{A}_{\alpha,\beta} f \in \mathscr{X}_0$ and in particular it is meaningful to speak of $f''(0)$, even though $f'(0)$ may not be well-defined.

It is known that $\mathsf{A}_{\alpha,\beta}$ is a generator of Feller semigroup in $\mathscr{X}$ [9, Proposition 2.1] and [3, Remark on p. 281]. The associated process is a diffusion in S_k that, away from the origin, behaves like one-dimensional Brownian motion along each ray, but exhibits a peculiar behavior at the origin, often referred to as a “roundhouse singularity” [26]. In essence, this means that whenever the process reaches the center of the graph, it immediately departs in all directions simultaneously, embarking on a standard Brownian excursion into the ith edge with probability β_i, $i \in K$. Furthermore, the graph's center exhibits a certain degree of stickiness, causing the time spent by a diffusing particle at the origin to form a generalized Cantor set, which is nowhere dense yet has positive Lebesgue measure depending on α [22, 23], [17, p. 307]. This process was initially introduced by Walsh in the non-sticky case as a generalization of skew Brownian motion on the real line, where the sign of each excursion from the origin is determined by an independent Bernoulli random variable [18]. Since then, it has been extensively studied in various contexts [4, 11, 14, 21, 24], including its application to solving applied problems [2, 12, 16], providing a counterexample in studies on Brownian filtrations [25], and serving as a building block in the construction of Brownian motions on general metric graphs [15, 19].

In the first main result of the paper we establish a precise correspondence between the transmission condition (2.2) of Walsh's process and the subspace

$$\mathscr{Y}_{\alpha,\beta} := \{f \in \mathscr{Y} : \overline{f}^e = \overline{f}(0)e_\alpha + \alpha e_\alpha * \overline{f},\ f_i^e = f_j^e \text{ for all } i,j \in K\},$$

where

$$\mathscr{Y} := (C[-\infty,\infty])^k \quad \text{and} \quad \overline{f} := \sum_{i\in K} \beta_i f_i. \tag{2.3}$$

Specifically, in Section 4 we prove that the operator $\mathsf{A}_{\alpha,\beta}$ generates a strongly continuous cosine family $\{\mathsf{C}_{\alpha,\beta}(t) : t \in \mathbb{R}\}$ in $\mathscr{X}_0$. To this end, we consider k copies of the unresticted Brownian motion on $\mathbb{R}$ and the *Cartesian product basic cosine family* $\{\mathsf{C}_{\mathrm{D}}(t) : t \in \mathbb{R}\}$ ('D' for 'Descartes') defined in the space $\mathscr{Y}$ by the formula

$$\mathsf{C}_{\mathrm{D}}(t)f = (C(t)f_i)_{i\in K}, \qquad f = (f_i)_{i\in K} \in \mathscr{Y},\ t \in \mathbb{R}. \tag{2.4}$$

We show that the subspace $\mathscr{Y}_{\alpha,\beta}$ is isomorphic to $\mathscr{X}_0$ and invariant under $\{\mathsf{C}_{\mathrm{D}}(t) : t \in \mathbb{R}\}$. Furthermore, we demonstrate that the Cartesian product basic cosine family restricted to $\mathscr{Y}_{\alpha,\beta}$ is similar to the cosine family generated by $\mathsf{A}_{\alpha,\beta}$,

$$\mathsf{C}_{\alpha,\beta}(t) \cong \mathsf{C}_{\mathrm{D}}(t)|_{\mathscr{Y}_{\alpha,\beta}}, \qquad t \in \mathbb{R}.$$

The explicit form of the underlying isomorphism, provided in (4.1), is closely related to another important result. Specifically, for $f = (f_i)_{i\in K} \in \mathscr{X}_0$, we have

$$(\mathsf{C}_{\alpha,\beta}(t)f)_i = C_{\min}(t)(f_i - \overline{f}) + C_{\mathrm{st}}(t)\overline{f}, \qquad i \in K,\ t \in \mathbb{R}, \tag{2.5}$$

where the right-hand side involves the cosine operators of the minimal and the sticky Brownian motions on $\mathbb{R}_+$, as defined in Sections 1 and 3, respectively. Here, $\overline{f}$ is also given by (2.3), but in this context, it represents an element of $C[0,\infty]$.

To discuss the meaning of (2.5), we look at the foundational definition of Walsh's process semigroup $\{\mathsf{T}_\beta(t) : t \geq 0\}$ without stickiness, i.e., when $\alpha \to \infty$ in the transmission condition (2.2). By [3, (2.2)], for $f \in \mathscr{X}_0$, we have

$$(\mathsf{T}_\beta(t)f)_i = T_{\min}(t)(f_i - \overline{f}) + T_{\mathrm{ref}}(t)\overline{f}, \quad i \in K,\ t \geq 0. \tag{2.6}$$

Here, $T_{\min}(t)$ and $T_{\mathrm{ref}}(t)$ are operators forming semigroups of, respectively, the minimal and the reflecting Brownian motions on $\mathbb{R}_+$. Now, let us recall that the cosine family is a more fundamental object than the semigroup. In particular, each cosine family generator automatically generates a strongly continuous semigroup, although the converse is not always true. From the *Weierstrass formula* (see, e.g., [1, p. 219], it follows that the semigroup generated by the cosine generator $\mathsf{A}_{\alpha,\beta}$ is given by $\mathsf{T}_{\alpha,\beta}(0)f = f$ and

$$\mathsf{T}_{\alpha,\beta}(t)f = \frac{1}{\sqrt{\pi t}} \int_0^\infty \mathrm{e}^{-\frac{s^2}{4t}} \mathsf{C}_{\alpha,\beta}(s)f\,\mathrm{d}s, \qquad t > 0,\ f \in \mathscr{X}_0.$$

Thus, many properties of the semigroup are rooted in the properties of the cosine family. For example, representation (2.6) is a direct consequence of (2.5), because in the limit as $\alpha \to \infty$, the sticky boundary condition transitions to the Neumann condition, and the operators $C_{\mathrm{st}}(t)$ must be replaced by $C_{\mathrm{ref}}(t)$, $t \in \mathbb{R}$.

The isomorphism established in Section 4 can also be used to derive equations for the transition kernels of Walsh's spider process on S_k in a straightforward manner (similar to the approach taken for Brownian motions on $\mathbb{R}_+$ in [5]). However, we will pursue a different approach, as these formulas have already been obtained by alternative methods in [20].

2.1. Complementary transmission conditions. Our next goal, motivated by (1.4) and achieved in Section 5, is to identify a natural operator $\mathsf{P}_{\alpha,\beta}$ that projects $\mathscr{Y}$ onto the subspace $\mathscr{Y}_{\alpha,\beta}$, thereby yielding a natural decomposition of $\mathscr{Y}$. We discover that the subspace complementing $\mathscr{Y}_{\alpha,\beta}$ is

$$\mathscr{Y}_{\alpha,\beta}^{\perp} := \{f \in \mathscr{Y} : \varSigma f^{o} = \alpha e_{\alpha} * \varSigma f,\ \beta_i^{-1} f_i^{o} = \beta_j^{-1} f_j^{o} \text{ for all } i, j \in K\}, \tag{2.7}$$

where

$$\varSigma f := \sum_{i \in K} f_i, \qquad f = (f_i)_{i \in K} \in \mathscr{Y}.$$

This leads to an analogue of the decomposition (1.3) for the Cartesian product of k copies of $C[-\infty, \infty]$:

$$\mathscr{Y} = \mathscr{Y}_{\alpha,\beta} \oplus \mathscr{Y}_{\alpha,\beta}^{\perp}. \tag{2.8}$$

We emphasize that (2.8) generalizes the main result of [7], as the definition of the projection $\mathsf{P}_{\alpha,\beta}$, given in (5.3), coincides with (1.4) for $k = 1$.

The subspace $\mathscr{Y}_{\alpha,\beta}$ is composed of extensions of members of $\mathscr{X}_0$, and the form of these extensions is uniquely determined by the transmission condition (2.2) that describes Walsh's spider process. The subspace $\mathscr{Y}_{\alpha,\beta}^{\perp}$ is investigated in Section 6. As it turns out, it is also invariant under the basic Cartesian product cosine family. Moreover, it is isomorphic to the space $\mathscr{X}$ and is intimately connected with the following transmission conditions:

$$\alpha^{-1} \varSigma f'(0) - \varSigma f(0) = 0 \quad \text{and} \quad \beta_i^{-1} f_i'(0) = \beta_j^{-1} f_j'(0), \qquad i, j \in K. \tag{2.9}$$

In other words, the transmission condition (2.2) forms a complementary pair with (2.9). Finally, we establish a representation similar to (2.5) for the cosine family $\{\mathsf{C}_{\alpha,\beta}^{\perp}(t) : t \in \mathbb{R}\}$ in $\mathscr{X}$ associated with (2.9). Specifically, for $f \in \mathscr{X}$, we obtain

$$(\mathsf{C}_{\alpha,\beta}^{\perp}(t) f)_i = C_{\mathrm{ref}}(t)(f_i - \beta_i \varSigma f) + \beta_i C_{\mathrm{el}}(t) \varSigma f, \qquad i \in K,\ t \in \mathbb{R}.$$

2.2. The non-sticky case. In this paper, we focus on the sticky Brownian motion on the graph S_k, but our results remain valid in the limit as $\alpha \to \infty$. In particular, the subspace determined by the transmission conditions

$$\sum_{i\in K}\beta_i f_i'(0)=0 \tag{2.10}$$

takes a remarkably elegant form

$$\mathscr{Y}_\beta := \{f\in\mathscr{Y} : \overline{f}^o = 0,\ f_i^e = f_j^e \text{ for all } i,j\in K\}. \tag{2.11}$$

Furthermore, there exists a corresponding family of direct sum decompositions

$$\mathscr{Y} = \mathscr{Y}_\beta \oplus \mathscr{Y}_\beta^\perp, \tag{2.12}$$

where the subspace

$$\mathscr{Y}_\beta^\perp := \{f\in\mathscr{Y} : \Sigma f^e = 0,\ \beta_i^{-1}f_i^o = \beta_j^{-1}f_j^o \text{ for all } i,j\in K\} \tag{2.13}$$

is uniquely determined by conditions (7.1).

3. Elaboration on the elastic and sticky boundary conditions on $\mathbb{R}_+$. As the correspondence (1.2) will be important in our analysis of transmission conditions (2.2) and (2.9), we will examine it in greater detail in this section.

The invariance of the subspace $C_{\mathrm{el}}[-\infty,\infty]$ under the basic cosine family is a direct consequence of the fact that this family commutes with the operation of taking the odd part, combined with the following lemma (compare [8, Section 2.1]).

Lemma 3.1. *For $f\in C[-\infty,\infty]$, the following conditions are equivalent:*

(a) $f\in C_{\mathrm{el}}[-\infty,\infty]$;
(b) $f^o = \alpha e_\alpha * f$ *on* $\mathbb{R}_+$;
(c) *for all* $t\in\mathbb{R}$, $C(t)(e_\alpha * f) = e_\alpha * C(t)f$ *on* $\mathbb{R}$.

Next, by l'Hospital's Rule, for $f\in C[0,\infty]$ we have

$$\lim_{x\to\infty} e_\alpha * f(x) = \alpha^{-1}f(\infty). \tag{3.1}$$

This, combined with point (b) of Lemma 3.1, implies that $C_{\mathrm{el}}[-\infty,\infty]$ is isomorphic to $C[0,\infty]$: there exists an isomorphism E_{el} mapping $f\in C[0,\infty]$ to its extension $E_{\mathrm{el}}f$ specified by

$$E_{\mathrm{el}}f(-x) = f(x) - 2\alpha e_\alpha * f(x), \quad x>0.$$

It follows (see also [6, Section 5.2]) that the formula

$$C_{\mathrm{el}}(t) = RC(t)E_{\mathrm{el}}, \quad t\in\mathbb{R}, \tag{3.2}$$

with R being as in (1.1), defines a cosine family of operators in $C[0,\infty]$. The generator of this cosine family is the Laplace operator with domain consisting of twice differentiable $f\in C[0,\infty]$ whose second derivative also belongs to $C[0,\infty]$ and that satisfies the elastic boundary condition $f'(0)=\alpha f(0)$. In fact, only for such f does the extension $E_{\mathrm{el}}f$ belong to the domain of the generator of $\{C(t) : t \in \mathbb{R}\}$, i.e., only for such f, $E_{\mathrm{el}} f$ is twice differentiable with its second derivative in $C[-\infty, \infty]$.

Turning to the sticky boundary condition, guided by [8, Section 4.1], we give two more characterizations of the space $C_{\mathrm{st}}[-\infty, \infty]$.

LEMMA 3.2. *For $f \in C[-\infty, \infty]$, the following conditions are equivalent:*

(a) $f \in C_{\mathrm{st}}[-\infty, \infty]$;
(b) $f^e = f(0)e_\alpha + \alpha e_\alpha * f$ *on* $\mathbb{R}_+$;
(c) *for all* $t \in \mathbb{R}$, *we have*

$$C(t)(f(0)e_\alpha + \alpha e_\alpha * f) = [C(t)f(0)]e_\alpha + \alpha e_\alpha * C(t)f \qquad \text{on } \mathbb{R}.$$

Accordingly, given $f \in C[0, \infty]$, we define its extension $E_{\mathrm{st}} f \in C[-\infty, \infty]$ by

$$E_{\mathrm{st}} f(-x) = -f(x) + 2f(0)e_\alpha(x) + 2\alpha e_\alpha * f(x), \qquad x > 0.$$

The operator E_{st} is an isomorphism between $C[0, \infty]$ and the subspace $C_{\mathrm{st}}[-\infty, \infty]$, which is invariant under the basic cosine family by Lemma 3.2. Then, it is easy to see that the cosine family of operators in $C[0, \infty]$ given by

$$C_{\mathrm{st}}(t) = RC(t)E_{\mathrm{st}}, \qquad t \in \mathbb{R}, \tag{3.3}$$

is generated by the Laplace operator with domain described by the sticky boundary condition $f''(0) = \alpha f'(0)$.

4. Generation theorem. In this section, we prove that the operator $\mathsf{A}_{\alpha,\beta}$, introduced in Section 2, is the generator of a cosine family, and we establish the connection between the transmission condition (2.2) and the subspace $\mathscr{Y}_{\alpha,\beta}$.

THEOREM 4.1.

(a) *The operators $\mathsf{C}_{\alpha,\beta}(t)$, $t \in \mathbb{R}$, defined in* (2.5) *form a strongly continuous cosine family in $\mathscr{X}_0$. The generator of this family is $\mathsf{A}_{\alpha,\beta}$.*
(b) *Let the extension operator*

$$\mathsf{E}_{\alpha,\beta} \colon \mathscr{X}_0 \ni (f_i)_{i\in K} \mapsto (g_i)_{i\in K} \in \mathscr{Y}$$

be defined by

$$\begin{aligned} g_i(x) &= f_i(x), \\ g_i(-x) &= -f_i(x) + 2\overline{f}(0)e_\alpha(x) + 2\alpha e_\alpha * \overline{f}(x), \end{aligned} \qquad x \ge 0,\ i \in K, \tag{4.1}$$

and let R be the restriction operator $\mathsf{R} \colon \mathscr{Y} \to \mathscr{X}$ that assigns to each $(f_i)_{i\in K} \in \mathscr{Y}$ the member $(f_i|_{\mathbb{R}_+})_{i\in K}$ of $\mathscr{X}$. Then

$$\mathsf{C}_{\alpha,\beta}(t) = \mathsf{R}\mathsf{C}_{\mathrm{D}}(t)\mathsf{E}_{\alpha,\beta}, \qquad t \in \mathbb{R}.$$

Proof. (a) It is clear that $\mathsf{C}_{\alpha,\beta}(0)$ is the identity operator. Thus we need to show that d'Alembert's functional equation is satisfied, i.e.,

$$\mathsf{C}_{\alpha,\beta}(t)\mathsf{C}_{\alpha,\beta}(s) = \tfrac{1}{2}\mathsf{C}_{\alpha,\beta}(t+s) + \tfrac{1}{2}\mathsf{C}_{\alpha,\beta}(t-s), \qquad s,t \in \mathbb{R}. \tag{4.2}$$

Let $s,t \in \mathbb{R}$ and $f = (f_i)_{i\in K} \in \mathscr{X}_0$. By definition, the ith entry of the sequence $\mathsf{C}_{\alpha,\beta}(t)\mathsf{C}_{\alpha,\beta}(s)f$ equals

$$C_{\min}(t)\big(C_{\min}(s)f_i + (C_{\mathrm{st}}(s) - C_{\min}(s))\overline{f}\big) \\ + (C_{\mathrm{st}}(t) - C_{\min}(t))\big[\overline{(C_{\min}(s)f_i + (C_{\mathrm{st}}(s) - C_{\min}(s))\overline{f})_{i\in K}}\big].$$

The term in square brackets simplifies to $C_{\mathrm{st}}(s)\overline{f}$, so we are left with

$$C_{\min}(t)C_{\min}(s)(f_i - \overline{f}) + C_{\mathrm{st}}(t)C_{\mathrm{st}}(s)\overline{f}.$$

To complete the proof of (4.2), we use d'Alembert's equations for cosine families of minimal and sticky Brownian motions. The strong continuity of these families also implies an analogous property of $\{\mathsf{C}_{\alpha,\beta}(t) : t \in \mathbb{R}\}$.

Turning to its generator, say G, we prove that it extends the operator $\mathsf{A}_{\alpha,\beta}$. Indeed, let $f = (f_i)_{i\in K}$ be a member of the domain of $\mathsf{A}_{\alpha,\beta}$. For each $i \in K$, $f_i - \overline{f}$ satisfies the Dirichlet boundary condition and belongs to the domain of the generator of $\{C_{\min}(t) : t \in \mathbb{R}\}$. Moreover, by (2.1) and (2.2), $\overline{f}$ satisfies the sticky boundary condition and is an element of the domain of the generator of $\{C_{\mathrm{st}}(t) : t \in \mathbb{R}\}$. Hence

$$\lim_{t\to 0} 2t^{-2}(\mathsf{C}_{\alpha,\beta}(t)f - f) = ((f_i - \overline{f})'' + (\overline{f})'')_{i\in K} = (f_i'')_{i\in K} = \mathsf{A}_{\alpha,\beta}f.$$

This implies that $\mathsf{G} = \mathsf{A}_{\alpha,\beta}$, since it is impossible to have two generators with one being a proper extension of the other (see e.g. [10, p. 267]), and $\mathsf{A}_{\alpha,\beta}$ is known to be the generator of a Feller semigroup.

(b) To check that $\mathsf{E}_{\alpha,\beta}$ is indeed a member of $\mathscr{Y}$, we apply (3.1) to the function $\overline{f} \in C[0,\infty]$. Combining formula (2.5) with (1.1) and (3.3) yields

$$\mathsf{C}_{\alpha,\beta}(t)f = \big(RC(t)[E_{\min}(f_i - \overline{f}) + E_{\mathrm{st}}\overline{f}]\big)_{i\in K} = \mathsf{R}\mathsf{C}_{\mathrm{D}}(t)\mathsf{E}_{\alpha,\beta}f$$

for any $f \in \mathscr{X}_0$ and $t \in \mathbb{R}$, as desired. ∎

The next result allows us to view $\mathscr{Y}_{\alpha,\beta}$ as the image of the space $\mathscr{X}_0$ under the extension mapping $\mathsf{E}_{\alpha,\beta}$.

LEMMA 4.2. *For $f = (f_i)_{i\in K} \in \mathscr{Y}$, the following conditions are equivalent:*

(a) $f \in \mathscr{Y}_{\alpha,\beta}$;
(b) *for all $i \in K$, $f_i^e = \overline{f}(0)e_\alpha + \alpha e_\alpha * \overline{f}$ on $\mathbb{R}$;*
(c) *for all $i \in K$, $f_i^e = \overline{f}(0)e_\alpha + \alpha e_\alpha * \overline{f}$ on $\mathbb{R}_+$.*

Proof. Let $f \in \mathscr{Y}_{\alpha,\beta}$. Using (2.1), we obtain $\overline{f}^e = f_i^e$ for all $i \in K$, and thus (a) implies (b).

Now, assume that (c) holds. We claim that $f_i^e = f_j^e$ on $\mathbb{R}$ for any $i, j \in K$. This follows from the fact that even functions that agree on $\mathbb{R}_+$ must coincide on $\mathbb{R}$. Furthermore, by (2.1), $\overline{f}$ satisfies condition (b) of Lemma 3.2. Hence $\overline{f} \in C_{\mathrm{st}}[-\infty, \infty]$ and the relation $\overline{f}^e = \overline{f}(0)e_\alpha + \alpha e_\alpha * \overline{f}$ holds on the entire $\mathbb{R}$. Therefore, (c) implies (a).

Finally, since (b) obviously implies (c), the proof is complete. ∎

Theorem 4.3. *The extension operator* $\mathsf{E}_{\alpha,\beta}$ *is an isomorphism from* $\mathscr{X}_0$ *onto* $\mathscr{Y}_{\alpha,\beta}$.

Proof. We begin by noting that each g_i in (4.1) is continuous and, by (3.1), has a finite limit at negative infinity. Consequently, each g_i belongs to $C[-\infty, \infty]$. Using Lemma 4.2, it is straightforward to verify that $\mathsf{E}_{\alpha,\beta}$ is a bijection from $\mathscr{X}_0$ onto $\mathscr{Y}_{\alpha,\beta}$. Indeed, $\mathsf{E}_{\alpha,\beta}$ is surjective because every member $(f_i)_{i\in K} \in \mathscr{Y}_{\alpha,\beta}$ satisfies $f_i(0) = f_j(0)$ for all $i, j \in K$ and is uniquely determined by the values of f_i on the positive half-axis for $i \in K$. Moreover, $\mathsf{E}_{\alpha,\beta}$ is linear and bounded, with $\|\mathsf{E}_{\alpha,\beta}\| \le 5$ when $\mathscr{Y}_{\alpha,\beta}$ is equipped with the norm $\|(f_i)_{i\in K}\| = \max_{i\in K} \|f_i\|$. ∎

Remark 4.4. The space $\mathscr{Y}_{\alpha,\beta}$ contains invariant subspaces that are related to Walsh's spider processes on subgraphs of S_k. Specifically, let m be a positive integer less than k, and let σ be a surjection from K onto $M := \{1, \dots, m\}$. The function σ describes how the edges of the graph S_k are merged to form the star-like graph S_m with m infinitely long edges.

It is straightforward to see that the subspace

$$\mathscr{Y}^\sigma := \{f \in \mathscr{Y} : f_i = f_j \text{ for all } i, j \in K \text{ such that } \sigma(i) = \sigma(j)\}$$

is invariant under the basic Cartesian product cosine family. Consequently, the intersection $\mathscr{Y}^\sigma_{\alpha,\beta} := \mathscr{Y}^\sigma \cap \mathscr{Y}_{\alpha,\beta}$ is also invariant. Moreover, $\mathscr{Y}^\sigma_{\alpha,\beta}$ is isometrically isomorphic to the space

$$\{f \in (C[-\infty, \infty])^m : \widetilde{f}^e = \widetilde{f}(0)e_\alpha + \alpha e_\alpha * \widetilde{f}^e,\ f_i^e = f_j^e \text{ for all } i, j \in M\},$$

where

$$\widetilde{f} = \sum_{i\in M} \widetilde{\beta}_i f_i \quad \text{and} \quad \widetilde{\beta}_i = \sum_{j\in\sigma^{-1}(i)} \beta_j, \qquad i \in M,$$

which is directly associated with the transmission condition

$$\alpha^{-1} f''(0) - \sum_{i\in M} \widetilde{\beta}_i f_i'(0) = 0$$

for the Laplace operator on S_m.

5. Natural projections onto the spaces $\mathscr{Y}_{\alpha,\beta}$ and $\mathscr{Y}^\perp_{\alpha,\beta}$. The ultimate goal of this section is to establish the direct sum decomposition (2.8).

For notational convenience, we first introduce a constant

$$\gamma := \alpha \cdot \sqrt{k \sum_{i\in K} \beta_i^2}$$

and two functions

$$\sinh_\gamma(x) := \sinh(\gamma x) \quad \text{and} \quad \cosh_\gamma(x) := \cosh(\gamma x), \qquad x \in \mathbb{R}.$$

The following identities will also be useful.

LEMMA 5.1. *Let $a > 0$. Then*

$$\begin{aligned} e_a * [a \sinh_\gamma + \gamma \cosh_\gamma] &= \sinh_\gamma, \\ e_a * [\gamma \sinh_\gamma + a \cosh_\gamma] &= \cosh_\gamma - e_a. \end{aligned}$$

LEMMA 5.2. *For $f \in C[-\infty, \infty]$, we have*

$$\begin{aligned} \lim_{x\to\pm\infty} \int_{-\infty}^{x} \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y &= \gamma^{-1} f(\pm\infty), \\ \lim_{x\to\pm\infty} \int_{x}^{\infty} \mathrm{e}^{\gamma(x-y)} f(y)\,\mathrm{d}y &= \gamma^{-1} f(\pm\infty). \end{aligned}$$

THEOREM 5.3. *Let the map*

$$\mathsf{P}_{\alpha,\beta} \colon \mathscr{Y} \ni (f_i)_{i\in K} \mapsto (g_i)_{i\in K} \in \mathscr{Y}$$

be defined by

$$\begin{aligned} g_i = f_i^o + c(\gamma \cosh_\gamma + \alpha\beta_i k \sinh_\gamma) + \alpha(\overline{f}^o - \beta_i \varSigma f^e) * \cosh_\gamma \\ + \left(\frac{\alpha^2 \beta_i k}{\gamma} \overline{f}^o - \frac{\gamma}{k} \varSigma f^e \right) * \sinh_\gamma, \end{aligned} \tag{5.1}$$

where

$$c := \int_0^\infty e^{-\gamma x} \left[\frac{1}{k} \varSigma f^e(x) - \frac{\alpha}{\gamma} \overline{f}^o(x) \right] \mathrm{d}x. \tag{5.2}$$

Then $\mathsf{P}_{\alpha,\beta}$ is a projection from $\mathscr{Y}$ onto $\mathscr{Y}_{\alpha,\beta}$.

Proof. Let $f = (f_i)_{i\in K} \in \mathscr{Y}$. Our task is to show that

(a) $(g_i)_{i\in K}$ of (5.1) is a member of $\mathscr{Y}$;
(b) we have $g_i^e = g_j^e$ for any $i, j \in K$ and $\overline{g}^e = \overline{g}(0) e_\alpha + \alpha e_\alpha * \overline{g}$;
(c) if $f \in \mathscr{Y}_{\alpha,\beta}$, then $g_i = f_i$ for all $i \in K$.

(a) Let $i \in K$. Using the standard identities between hyperbolic and exponential functions, together with the integral expression for c given in (5.2), the definition of g_i can be rewritten as

$$g_i(x) = f_i^o(x) + \tfrac{1}{2}(\gamma - \alpha\beta_i k) \int_{-\infty}^{x} \mathrm{e}^{-\gamma(x-y)} \left[\frac{1}{k}\Sigma f^e(y) + \frac{\alpha}{\gamma}\overline{f}^o(y)\right] \mathrm{d}y \\ + \tfrac{1}{2}(\gamma + \alpha\beta_i k) \int_{x}^{\infty} \mathrm{e}^{\gamma(x-y)} \left[\frac{1}{k}\Sigma f^e(y) - \frac{\alpha}{\gamma}\overline{f}^o(y)\right] \mathrm{d}y, \quad x \in \mathbb{R}. \tag{5.3}$$

Since $f_i^o, \Sigma f^e$ and $\overline{f}^o$ all belong to $C[-\infty,\infty]$, Lemma 5.2 guarantees that g_i also belongs to $C[-\infty,\infty]$.

(b) It is easy to check that if f is even and g is odd, then $f * g$ is even. Similarly, the convolution of two even functions or two odd functions is odd. Therefore, (5.1) yields

$$g_i^e = c\gamma\cosh_\gamma + \alpha\overline{f}^o * \cosh_\gamma - \frac{\gamma}{k}\Sigma f^e * \sinh_\gamma, \qquad i \in K.$$

The right-hand side does not depend on i, which proves the first assertion.

Turning to the second assertion, using (5.1) once more, we obtain

$$\overline{g} = \overline{f}^o + c\left(\gamma\cosh_\gamma + \frac{\gamma^2}{\alpha}\sinh_\gamma\right) + \left(\alpha\overline{f}^o - \frac{\gamma^2}{\alpha k}\Sigma f^e\right) * \cosh_\gamma \\ + \left(\gamma\overline{f}^o - \frac{\gamma}{k}\Sigma f^e\right) * \sinh_\gamma.$$

Therefore $\overline{g}(0) = c\gamma$, and, by the second equality in Lemma 5.1, it remains to show that

$$\alpha\overline{f}^o * [\cosh_\gamma - e_\alpha - e_\alpha * (\alpha\cosh_\gamma + \gamma\sinh_\gamma)] \\ = \frac{\gamma}{k}\Sigma f^e * [\sinh_\gamma - e_\alpha * (\gamma\cosh_\gamma + \alpha\sinh_\gamma)].$$

Since, again by Lemma 5.1, both expressions in square brackets vanish, the proof is complete.

(c) Let $f = (f_i)_{i\in K} \in \mathscr{Y}_{\alpha,\beta}$ and fix an index $i \in K$. From (2.1) and the fact that $f_i^e = f_j^e$ for all $j \in K$, it follows that $\Sigma f^e = k f_i^e$. Additionally, we have $\overline{f}^o = \overline{f} - \overline{f}^e = \overline{f} - f_i^e$. Substituting these expressions into (5.3) and then applying condition (b) of Lemma 4.2, we obtain

$$g_i = f_i - \overline{f}(0)e_\alpha - \alpha e_\alpha * \overline{f} + \overline{f}(0)I_1 + I_2,$$

where

$$I_1(x) = \frac{\gamma-\alpha}{2\gamma}(\gamma - \alpha\beta_i k)\mathrm{e}^{-\gamma x} \int_{-\infty}^{x} \mathrm{e}^{(\gamma-\alpha)y}\,\mathrm{d}y \\ + \frac{\gamma+\alpha}{2\gamma}(\gamma + \alpha\beta_i k)\mathrm{e}^{\gamma x} \int_{x}^{\infty} \mathrm{e}^{-(\gamma+\alpha)y}\,\mathrm{d}y,$$

$$I_2(x) = \frac{\alpha}{2\gamma}(\gamma - \alpha\beta_i k) \int\limits_{-\infty}^{x} \mathrm{e}^{-\gamma(x-y)}[(\gamma-\alpha)e_\alpha * \overline{f}(y) + \overline{f}(y)]\,\mathrm{d}y$$
$$+ \frac{\alpha}{2\gamma}(\gamma + \alpha\beta_i k) \int\limits_{x}^{\infty} \mathrm{e}^{\gamma(x-y)}[(\gamma+\alpha)e_\alpha * \overline{f}(y) - \overline{f}(y)]\,\mathrm{d}y,$$

for $x \in \mathbb{R}$. A direct calculation shows that $I_1 = e_\alpha$. After integrating both convolution terms by parts, the expression for I_2 simplifies to

$$I_2 = \frac{\alpha}{2\gamma}(\gamma - \alpha\beta_i k)e_\alpha * \overline{f} + \frac{\alpha}{2\gamma}(\gamma + \alpha\beta_i k)e_\alpha * \overline{f} = \alpha e_\alpha * \overline{f}.$$

As a result, we can conclude that $g_i = f_i$, as desired. ■

THEOREM 5.4. *The map* $\mathsf{Q}_{\alpha,\beta} := \mathsf{I} - \mathsf{P}_{\alpha,\beta}$ *is a projection from* $\mathscr{Y}$ *onto the subspace* $\mathscr{Y}^{\perp}_{\alpha,\beta}$ *introduced in* (2.7).

Proof. Let $f = (f_i)_{i\in K} \in \mathscr{Y}$. We define

$$g_i = f_i^e - c(\gamma \cosh_\gamma + \alpha\beta_i k \sinh_\gamma) - \alpha(\overline{f}^o - \beta_i \varSigma f^e) * \cosh_\gamma - \left(\frac{\alpha^2\beta_i k}{\gamma}\overline{f}^o - \frac{\gamma}{k}\varSigma f^e\right) * \sinh_\gamma, \quad i \in K, \tag{5.4}$$

and we aim to prove the following:

(a) $(g_i)_{i\in K} \in \mathscr{Y}^{\perp}_{\alpha,\beta}$;
(b) if additionally $f \in \mathscr{Y}^{\perp}_{\alpha,\beta}$, then $g_i = f_i$ for all $i \in K$.

(a) The fact that $(g_i)_{i\in K}$ is a member of $\mathscr{Y}$ follows directly from Theorem 5.3. In a similar manner to the proof of point (b) of that theorem, we obtain

$$g_i^o = -c\alpha\beta_i k \sinh_\gamma + \alpha\beta_i \varSigma f^e * \cosh_\gamma - \frac{\alpha^2\beta_i k}{\gamma}\overline{f}^o * \sinh_\gamma, \quad i \in K.$$

Since the right-hand side, when divided by β_i, does not depend on i, the second equality in the definition of $\mathscr{Y}^{\perp}_{\alpha,\beta}$ is satisfied. Moving on to the first equality, that is,

$$\varSigma g^o = \alpha e_\alpha * \varSigma g, \tag{5.5}$$

we find, by (5.4), that

$$\varSigma g = \varSigma f^e - ck(\gamma\cosh_\gamma + \alpha\sinh_\gamma) - \alpha(k\overline{f}^o - \varSigma f^e) * \cosh_\gamma - \left(\frac{\alpha^2 k}{\gamma}\overline{f}^o - \gamma\varSigma f^e\right) * \sinh_\gamma.$$

Using the first equality in Lemma 5.1, we verify that all terms in (5.5) involving c cancel out, reducing (5.5) to

$$\begin{aligned}\alpha\varSigma f^e * [\cosh_\gamma - e_\alpha - e_\alpha * (\alpha\cosh_\gamma + \gamma\sinh_\gamma)]&\\ = \frac{\alpha^2 k}{\gamma}\overline{f}^{o} * [\sinh_\gamma - e_\alpha * (\gamma\cosh_\gamma + \alpha\sinh_\gamma)]&.\end{aligned}$$

Finally, again by Lemma 5.1, we observe that both expressions in square brackets are identically zero.

(b) Let $i \in K$. We start by noting that g_i may be written as (compare the equivalent formula for $\mathsf{P}_{\alpha,\beta} f$ given in (5.3))

$$\begin{aligned}(5.6)\quad g_i(x) = f_i^e(x) - \tfrac{1}{2}(\gamma - \alpha\beta_i k)\int\limits_{-\infty}^{x} \mathrm{e}^{-\gamma(x-y)}\left[\frac{1}{k}\varSigma f^e(y) + \frac{\alpha}{\gamma}\overline{f}^{o}(y)\right]\mathrm{d}y&\\ - \tfrac{1}{2}(\gamma + \alpha\beta_i k)\int\limits_{x}^{\infty} \mathrm{e}^{\gamma(x-y)}\left[\frac{1}{k}\varSigma f^e(y) - \frac{\alpha}{\gamma}\overline{f}^{o}(y)\right]\mathrm{d}y,\qquad x \in \mathbb{R}&.\end{aligned}$$

Then, we proceed as in part (c) of the proof of Theorem 5.4. Namely, assume that $f \in \mathscr{Y}_{\alpha,\beta}^{\perp}$. From the definition of this space, it follows that $\overline{f}^{o} = \frac{\gamma^2}{\alpha k} e_\alpha * \varSigma f$, $\varSigma f^e = \varSigma f - \alpha e_\alpha * \varSigma f$, and $f_i^e = f_i - \alpha\beta_i e_\alpha * \varSigma f$. Therefore, (5.6) becomes

$$g_i = f_i - \alpha\beta_i e_\alpha * \varSigma f - J,$$

where

$$\begin{aligned}J(x) = \frac{1}{2k}(\gamma - \alpha\beta_i k)\int\limits_{-\infty}^{x} \mathrm{e}^{-\gamma(x-y)}[\varSigma f(y) + (\gamma - \alpha)e_\alpha * \varSigma f(y)]\,\mathrm{d}y&\\ + \frac{1}{2k}(\gamma + \alpha\beta_i k)\int\limits_{x}^{\infty} \mathrm{e}^{\gamma(x-y)}[\varSigma f(y) - (\gamma + \alpha)e_\alpha * \varSigma f(y)]\,\mathrm{d}y,\qquad x \in \mathbb{R}&.\end{aligned}$$

After integrating both convolution terms in J by parts, we obtain

$$J = \frac{1}{2k}(\gamma - \alpha\beta_i k)e_\alpha * \varSigma f - \frac{1}{2k}(\gamma + \alpha\beta_i k)e_\alpha * \varSigma f = -\alpha\beta_i e_\alpha * \varSigma f.$$

This clearly yields $g_i = f_i$, as desired. ■

Now, the direct sum decomposition (2.8) follows immediately from Theorems 5.3 and 5.4, thereby accomplishing the main goal of this section.

6. $\mathscr{Y}_{\alpha,\beta}^{\perp}$ as an invariant subspace shaped by the boundary conditions (2.9).

This section focuses on the subspace $\mathscr{Y}_{\alpha,\beta}^{\perp}$ and explores various mathematical entities and properties associated with it.

THEOREM 6.1. *The subspace $\mathscr{Y}_{\alpha,\beta}^{\perp}$ is invariant under the basic Cartesian product cosine family defined in* (2.4)

Proof. The property that each component function has finite limits at both infinities is evidently preserved by the basic Cartesian product cosine family. It is also clear that

$$\varSigma(\mathsf{C}_{\mathrm{D}}(t)f) = C(t)\varSigma f, \qquad f \in \mathscr{Y},\, t \in \mathbb{R}.$$

Combining this fact with the invariance of the space $C_{\mathrm{el}}[-\infty,\infty]$ (defined in (1.2)) under the basic cosine family establishes the preservation of the first condition in the definition of $\mathscr{Y}_{\alpha,\beta}^{\perp}$. To complete the proof, for any $f \in \mathscr{Y}_{\alpha,\beta}$, $i, j \in K$, and $t \in \mathbb{R}$, we verify that

$$\beta_i{}^{-1}(C_D(t)f)_i^o = C(t)(\beta_i{}^{-1}f_i^o) = C(t)(\beta_j{}^{-1}f_j^o) = \beta_j{}^{-1}(C_D(t)f)_j^o,$$

where the first and the last equalities hold because the basic cosine family commutes with the componentwise operation of taking the odd part of a function. ∎

Lemma 6.2. *For $f = (f_i)_{i\in K} \in \mathscr{Y}$ the following conditions are equivalent:*

(a) $f \in \mathscr{Y}_{\alpha,\beta}^{\perp}$;
(b) $\beta_i^{-1}f_i^o = \alpha e_\alpha * \varSigma f$ *on* $\mathbb{R}$ *for all* $i \in K$;
(c) $\beta_i^{-1}f_i^o = \alpha e_\alpha * \varSigma f$ *on* $\mathbb{R}_+$ *for all* $i \in K$.

Proof. Let $f \in \mathscr{Y}_{\alpha,\beta}^{\perp}$. By (2.1), we have $\varSigma f^o = \beta_i^{-1}f_i^o$ for each $i \in K$, which shows that (a) implies (b).

Next, assume that (c) holds. Then $\varSigma f^o = \alpha e_\alpha * \varSigma f$ on $\mathbb{R}_+$, and by Lemma 3.1, this equality extends to $\mathbb{R}$. Moreover, the second equation in the definition of $\mathscr{Y}_{\alpha,\beta}^{\perp}$ is also fulfilled, as the functions $\beta_i^{-1}f_i^o$ for $i \in K$ are odd and coincide on $\mathbb{R}_+$ by the definition of $\mathscr{Y}_{\alpha,\beta}^{\perp}$. This establishes (a).

Since it is clear that (b) implies (c), the proof is complete. ∎

Guided by condition (c) of Lemma 6.2, we define the extension operator

$$\mathsf{E}_{\alpha,\beta}^{\perp}\colon \mathscr{X} \ni (f_i)_{i\in K} \mapsto (g_i)_{i\in K} \in \mathscr{Y}$$

by

$$\begin{aligned} g_i(x) &= f_i(x), \\ g_i(-x) &= f_i(x) - 2\beta_i \alpha e_\alpha * \varSigma f(x), \end{aligned} \qquad x \geq 0. \tag{6.1}$$

We note that each g_i is a member of $C[-\infty,\infty]$ and it has a finite limit at $-\infty$ by (3.1).

Proposition 6.3. *The operator $\mathsf{E}_{\alpha,\beta}^{\perp}$ is an isomorphism from $\mathscr{X}$ onto $\mathscr{Y}_{\alpha,\beta}^{\perp}$.*

Proof. By definition, we have $\frac{1}{\beta_i}g_i^o = \alpha e_\alpha * \varSigma f$ on $\mathbb{R}_+$. Combining this with Lemma 6.2 implies that the image of $\mathscr{X}$ is contained in $\mathscr{Y}_{\alpha,\beta}^{\perp}$. The surjectivity of $\mathsf{E}_{\alpha,\beta}^{\perp}$ follows from the fact that each member f of $\mathscr{Y}_{\alpha,\beta}^{\perp}$ satisfies $f_i(-x) = f_i(x) - 2\beta_i\alpha e_\alpha * \varSigma f(x)$ for $x \geq 0$. Moreover, $\mathsf{E}_{\alpha,\beta}^{\perp}$ is linear and bounded, with norm not exceeding 5. Its inverse is the operator R (introduced in Theorem 4.1) restricted to the domain $\mathscr{Y}_{\alpha,\beta}^{\perp}$. ∎

The invariance of $\mathscr{Y}^{\perp}_{\alpha,\beta}$ together with the fact that $\mathsf{E}^{\perp}_{\alpha,\beta}$ is an isomorphism allows us to define a strongly continuous cosine family in $\mathscr{X}$ via the abstract Kelvin formula

$$\mathsf{C}^{\perp}_{\alpha,\beta}(t) := \mathsf{R} C_D(t) \mathsf{E}^{\perp}_{\alpha,\beta}, \qquad t \in \mathbb{R}. \tag{6.2}$$

THEOREM 6.4. *Let* $\mathsf{A}^{\perp}_{\alpha,\beta}$ *be the operator in* $\mathscr{X}$ *defined by*

$$\mathsf{A}^{\perp}_{\alpha,\beta} f = \big(f_i''\big)_{i\in K}$$

on the domain $D(\mathsf{A}^{\perp}_{\alpha,\beta})$ *consisting of all functions* $f = (f_i)_{i\in K} \in \mathscr{X}$ *such that each* f_i *is twice continuously differentiable with* $f_i'' \in C[-\infty,\infty]$ *and condition* (2.9) *holds. Then* $\mathsf{A}^{\perp}_{\alpha,\beta}$ *is the generator of the cosine family defined by* (6.2).

Proof. Let G be the generator of $\{\mathsf{C}^{\perp}_{\alpha,\beta}(t) : t \in \mathbb{R}\}$. We need to show that $D(\mathsf{G}) = D(\mathsf{A}^{\perp}_{\alpha,\beta})$ and that the operators G and $\mathsf{A}^{\perp}_{\alpha,\beta}$ coincide.

STEP 1. By definition, $f = (f_i)_{i\in K} \in \mathscr{X}$ belongs to $D(\mathsf{G})$ if and only if each g_i of (6.1) is twice continuously differentiable on the entire $\mathbb{R}$, with $g_i'' \in C[-\infty,\infty]$. Equivalently, each f_i is twice continuously differentiable with $f_i'' \in C[0,\infty]$ and each g_i has derivative of second order at 0.

Assume that $f \in D(\mathsf{G})$. Our goal is to show that condition (2.9) holds. To this end, we first use (6.1) to write out the formula for the difference quotient and see that the left-hand derivative of g_i at 0 coincides with its right-hand derivative at 0 provided that $\frac{1}{\beta_i} f_i'(0) = \alpha \varSigma f(0)$. This implies the second equality in (2.9). Similarly, the left-hand second derivative of g_i at 0 is equal to its right-hand second derivative at 0 plus $2\beta_i\alpha(\varSigma f'(0) - \alpha\varSigma f(0))$. Since they coincide by assumption and the derivatives of g_i and f_i coincide on $\mathbb{R}_+$, we get $\varSigma f'(0) - \alpha \varSigma f(0) = 0$. This proves $D(\mathsf{G}) \subset D(\mathsf{A}^{\perp}_{\alpha,\beta})$.

To establish the converse inclusion, assume that $f \in D(\mathsf{A}^{\perp}_{\alpha,\beta})$. Then, writing out the formula for the difference quotient and using l'Hospital's Rule, we check that the left-hand derivative of g_i at 0 exists and equals $-f_i'(0) + 2\beta_i\alpha\varSigma f(0)$. On the other hand, combining (2.9) with (2.1), we see that $f_i'(0) = \beta_i\alpha\varSigma f(0)$ and that the one-sided derivatives of g_i coincide. Next, we note that

$$(g_i)'(-x) = -f_i'(x) + 2\beta_i\alpha(\varSigma f(0) - \alpha e_\alpha * \varSigma f)(x), \qquad x > 0,$$

and argue as above, using l'Hospital's Rule, that the left-hand second order derivative of g_i at 0 exists and by (2.9) agrees with $f_i''(0)$. Therefore $f \in D(\mathsf{G})$. Thus $D(\mathsf{A}^{\perp}_{\alpha,\beta}) \subset D(\mathsf{G})$ and finally $D(\mathsf{A}^{\perp}_{\alpha,\beta}) = D(\mathsf{G})$.

STEP 2. Let $f \in D(\mathsf{A}^{\perp}_{\alpha,\beta})$. From Step 1, $g = (g_i)_{i\in K}$ of (6.1) is a vector of twice continuously differentiable functions, and therefore

$$\lim_{t\to 0} 2t^{-2}[\mathsf{C}_{\mathrm{D}}(t)g - g] = (g_i'')_{i\in K}.$$

It follows that $\mathsf{G}f = \mathsf{R}((g_i'')_{i\in K}) = (f_i'')_{i\in K} = \mathsf{A}^{\perp}_{\alpha,\beta}f$, which completes the proof. ■

Theorem 6.5. *For any* $f = (f_i)_{i\in K} \in \mathscr{X}$, *we have*

$$(\mathsf{C}^{\perp}_{\alpha,\beta}(t)f)_i = C_{\mathrm{ref}}(t)(f_i - \beta_i \Sigma f) + \beta_i C_{\mathrm{el}}(t)\Sigma f, \qquad i \in K,\ t \in \mathbb{R}.$$

Proof. This follows directly from the fact that (6.2) can be rewritten using (1.1) and (3.2) as

$$\mathsf{R}\mathsf{C}_{\mathrm{D}}(t)\mathsf{E}^{\perp}_{\alpha,\beta}f = (RC(t)[E_{\mathrm{ref}}(f_i - \beta_i\Sigma f) + \beta_i E_{\mathrm{el}}\Sigma f])_{i\in K}, \qquad f \in \mathscr{X},\ t \in \mathbb{R}. \ ■$$

7. Non-sticky Walsh's Brownian motion. In this section, we gather results concerning the transmission condition (2.2) in the limit as $\alpha \to \infty$. We begin by defining an operator A_β as the Laplace operator on S_k with the domain described by the transmission condition (2.10).

Theorem 7.1. *The operator* A_β *is the generator of a cosine family* $\{\mathsf{C}_\beta(t) : t \in \mathbb{R}\}$ *which is isomorphic to the basic Cartesian product cosine family restricted to the invariant subspace* $\mathscr{Y}_\beta$ *defined in* (2.11),

$$\mathsf{C}_\beta(t) \cong \mathsf{C}_{\mathrm{D}}(t)\big|_{\mathscr{Y}_\beta}, \qquad t \in \mathbb{R}.$$

The cosine family $\{\mathsf{C}_\beta(t) : t \in \mathbb{R}\}$ *is given via the abstract Kelvin formula*

$$\mathsf{C}_\beta(t) = \mathsf{R}\mathsf{C}_{\mathrm{D}}(t)\mathsf{E}_\beta, \qquad t \in \mathbb{R},$$

where the extension operator $\mathsf{E}_\beta \colon \mathscr{X}_0 \to \mathscr{Y}$ *is defined by*

$$\begin{aligned} (\mathsf{E}_\beta f)_i(x) &= f_i(x), \\ (\mathsf{E}_\beta f)_i(-x) &= -f_i(x) + 2\overline{f}(x), \end{aligned} \qquad x \geq 0,\ i \in K,$$

and is an isomorphism from $\mathscr{X}_0$ *onto* $\mathscr{Y}_\beta$.

Lemma 7.2. *Let* $f \in C[-\infty,\infty]$. *Then*

$$\lim_{a\to\infty} a \int_{-\infty}^{x} \mathrm{e}^{-a(x-y)} f(y)\,\mathrm{d}y = \lim_{a\to\infty} a \int_{x}^{\infty} \mathrm{e}^{a(x-y)} f(y)\,\mathrm{d}y = f(x)$$

and the convergence in both limits is uniform in $x \in \mathbb{R}$.

Lemma 7.2 implies that, for any $f \in \mathscr{Y}$, $\mathsf{P}_{\alpha,\beta}f$ and $\mathsf{Q}_{\alpha,\beta}f$ (defined as $(g_i)_{i\in K}$ with the aid of (5.3) and (5.6), respectively) converge strongly to

$$\mathsf{P}_\beta f = f^o + \frac{\Sigma f^e}{k} - \frac{\beta_i}{\sum_{j\in K}\beta_j^2}\overline{f}^o \quad \text{and} \quad \mathsf{Q}_\beta f = (\mathsf{I} - \mathsf{P}_\beta)f$$

as $\alpha \to \infty$. Furthermore, it turns out that the limit operators P_β and Q_β are projections onto the subspaces defined by the transmission conditions (2.10) and (7.1), respectively. More specifically, we have the following result.

Theorem 7.3. P_β *is a projection on* $\mathscr{Y}_\beta$, Q_β *is a projection on* $\mathscr{Y}_\beta^{\perp}$ *and the direct sum decomposition* (2.12) *holds.*

As a complementary remark, we note that for $f \in \mathscr{Y}_\beta$ and $g \in \mathscr{Y}_\beta^\perp$, a straightforward computation show that $\Sigma(fg)$ is odd. Consequently, the following orthogonality property holds:

$$\int_{-a}^{a} \Big(\sum_{i\in K} f_i(x) g_i(x) \Big)\, \mathrm{d}x = 0, \quad a > 0.$$

THEOREM 7.4.

(a) *The subspace* $\mathscr{Y}_\beta^\perp$ *defined in* (2.13) *is invariant under the basic Cartesian product cosine family.*

(b) *Let* $\mathscr{X}_1 := \{f \in \mathscr{X} : \Sigma f(0) = 0\}$. *The extension operator* $\mathsf{E}_\beta^\perp \colon \mathscr{X}_1 \to \mathscr{Y}$ *defined by*

$$\begin{aligned} (\mathsf{E}_\beta^\perp f)_i(x) &= f_i(x), \\ (\mathsf{E}_\beta^\perp f)_i(-x) &= f_i(x) - 2\beta_i \Sigma f(x), \end{aligned} \qquad f \in \mathscr{X}_1,\ x \ge 0,\ i \in K,$$

is an isomorphism from $\mathscr{X}_1$ *onto* $\mathscr{Y}_\beta^\perp$.

(c) *The cosine family*

$$\mathsf{C}_\beta^\perp(t) := \mathsf{R} C_D(t) \mathsf{E}_\beta^\perp, \qquad t \in \mathbb{R},$$

is generated by the operator $\mathsf{A}_\beta^\perp$ *in* $\mathscr{X}_1$ *defined by* $\mathsf{A}_\beta^\perp f = (f_i'')_{i\in K}$ *on the domain consisting of functions* $f = (f_i)_{i\in K} \in \mathscr{X}$ *such that each* f_i *is twice differentiable with* $f_i'' \in C[-\infty, \infty]$,

$$\Sigma f''(0) = 0 \quad \text{and} \quad \beta_i^{-1} f_i'(0) = \beta_j^{-1} f_j'(0), \qquad i, j \in K. \tag{7.1}$$

Finally, we note that for any $i \in K$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} (\mathsf{C}_\beta(t) f)_i &= C_{\min}(t)(f_i - \overline{f}) + C_{\mathrm{ref}}(t)\overline{f}, && f \in \mathscr{X}_0, \\ (\mathsf{C}_\beta^\perp(t) f)_i &= C_{\mathrm{ref}}(t)(f_i - \beta_i \Sigma f) + \beta_i C_{\min}(t) \Sigma f, && f \in \mathscr{X}_1. \end{aligned}$$

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