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Return time sets and product recurrence

by

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Abstract. Let G be a countable infinite discrete group. We show that a subset F of G contains a return time set of some piecewise syndetic recurrent point x in a compact Hausdorff space X with a G-action if and only if F is a quasi-central set. As an application, we show that if a nonempty closed subsemigroup S of the Stone–Čech compactification βG contains the smallest ideal $K(\beta G)$ of βG then S-product recurrence is equivalent to distality, which partially answers a question of Auslander and Furstenberg [Trans. Amer. Math. Soc. 343 (1994), 221–232].

1. Introduction. By a topological dynamical system, we mean a pair (X,T), where X is a compact metric space with a metric d and $T\colon X\to X$ is a continuous map. The study of recurrence is one of the central topics in topological dynamics. For a point $x\in X$ and a subset U of X, the *return time set* of x to U (in this paper, "neighborhood" always signifies an open neighborhood) is

$$N(x,U)=\{n\in\mathbb{N}_0\colon T^n x\in U\},$$

where $\mathbb{N}_0$ denotes the collection of nonnegative integers. Recurrent time sets are closely associated with the combinatorial property of the sets of nonnegative integers. In the seminal monograph [11], Furstenberg characterized the return time sets of a recurrent point via IP-subsets of $\mathbb{N}_0$ which is defined combinatorially. Recall that a point $x\in X$ is called *recurrent* if for every neighborhood U of x, the recurrent time set $N(x,U)$ is infinite, and a subset F of $\mathbb{N}_0$ is called an *IP-set* if there exists a sequence $\{p_i\}_{i=1}^\infty$ in $\mathbb{N}_0$ such that the finite sum $FS(\{p_i\}_{i=1}^\infty)$ of $\{p_i\}_{i=1}^\infty$ is infinite and contained in F, where

$$FS(\{p_i\}_{i=1}^\infty)=\Big\{\sum_{i\in\alpha}p_i\colon \alpha \text{ is a nonempty finite subset of } \mathbb{N}\Big\}.$$

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THEOREM 1.1 ([11, Theorem 2.17]).

(1) *Given a topological dynamical system* (X, T)*, if a point* $x \in X$ *is recurrent, then for any neighborhood* U *of* x*,* $N(x, U)$ *is an IP-set.*
(2) *If a subset* F *of* $\mathbb{N}_0$ *is an IP-set, then there exists a topological dynamical system* (X, T)*, a recurrent point* $x \in X$ *and a neighborhood* U *of* x *such that* $N(x, U) \subset F \cup \{0\}$.

Furstenberg introduced the concept of central subsets of $\mathbb{N}_0$ and proved the so-called "central sets theorem" (see [11, Proposition 8.21]), which has many combinatorial consequences. For a recent survey on central sets, we refer the reader to [15]. In [16] Hindman et al. introduced the notion of quasi-central sets, and both concepts were further generalized to be applicable to arbitrary semigroups. Motivated by Theorem 1.1, we characterize the return time sets of a piecewise syndetic recurrent point via quasi-central subsets of $\mathbb{N}_0$.

THEOREM 1.2.

(1) *Given a topological dynamical system* (X, T)*, if a point* $x \in X$ *is piecewise syndetic recurrent, then for every neighborhood* U *of* x*,* $N(x, U)$ *is a quasi-central set.*
(2) *For any quasi-central subset* F *of* $\mathbb{N}_0$*, there exists a topological system* (X, T)*, a piecewise syndetic recurrent point* $x \in X$ *and a neighborhood* U *of* x *such that* $N(x, U) \subset F \cup \{0\}$.

The proof of Theorem 1.2 is presented in Section 2. In fact, we will show that a more general version of Theorem 1.2 also holds for G-systems and some special kinds of recurrence; see Theorem 5.7 for details. Recall that a *G-system* is a pair (X, G), where X is a compact Hausdorff space and G is a countable discrete group continuously acting on X. A key aspect of the proof of Theorem 5.7 is a "purely" combinatorial characterization of the recurrent time sets corresponding to certain specific types of recurrent points; see Theorem 4.4.

Let (X, T) be a topological dynamical system. Recall that two points $x, y \in X$ are called *proximal* if $\liminf_{k\to\infty} d(T^k x, T^k y) = 0$, and a point $x \in X$ is called *distal* if it is not proximal to any point in its orbit closure other than itself. By the well-known Auslander–Ellis theorem (see e.g. [11, Theorem 8.7]), any distal point is uniformly recurrent. In [11], Furstenberg also characterized distal points in terms of recurrent time sets and synchronized recurrence with certain types of recurrent points (see [9] and [7] for G-systems). Recall that a subset F of $\mathbb{N}_0$ is called an *IP*-set* if for any IP-subset F' of $\mathbb{N}_0$, $F \cap F' \neq \emptyset$.

THEOREM 1.3 ([11, Theorem 9.11]). *Let* (X, T) *be a topological dynamical system and* $x \in X$*. Then the following assertions are equivalent:*

(1) *x is distal;*
(2) *x is IP*-recurrent, that is, for any neighborhood U of x, $N(x,U)$ is an IP*-set;*
(3) *x is product recurrent, that is, for any topological dynamical system (Y,S) and any recurrent point $y \in Y$, (x,y) is recurrent in the product system $(X \times Y, T \times S)$;*
(4) *for any topological dynamical system (Y,S) and any uniformly recurrent point $y \in Y$, (x,y) is uniformly recurrent in the product system $(X \times Y, T \times S)$.*

In [2], Auslander and Furstenberg treated directly the action $E \times X \ni (p,x) \mapsto px \in X$ of a compact right topological semigroup E on a compact Hausdorff space X. It should be noticed that the maps $x \mapsto px$ are often discontinuous for such semigroup actions. Such an action is referred to as an Ellis action in [1]. Within this framework the authors of [1] investigated the relationships between dynamics of an action and an algebraic structure of E. For instance, they obtained several characterizations of distal, semidistal and almost-distal flows for an Ellis action. The Stone–Čech compactification βG of a discrete group G forms a compact right topological semigroup, and its action constitutes an important example of Ellis action (referred to as a βG-action).

Partially motivated by Theorem 1.3, Auslander and Furstenberg [2] introduced the concept of S-product recurrence for a closed subsemigroup S of E, and showed that under certain conditions, a point is S-product recurrent if and only if it is a distal point. In the end of [2], Auslander and Furstenberg asked the following two questions:

QUESTION 1.4. *How to characterize the closed subsemigroups S of a compact right topological semigroup for which an S-product recurrent point is distal?*

QUESTION 1.5. *If (x,y) is recurrent for any almost periodic point y, is x necessarily a distal point?*

Question 1.5 was answered negatively by Haddad and Ott [14] for topological dynamical systems. In fact, this question is related to dynamical systems which are disjoint from all minimal systems. In [8], Dong, Shao and Ye studied general product recurrence properties systematically and in [21] Oprocha and Zhang showed that if (x,y) is recurrent for any piecewise syndetic recurrent point y, then x is a distal point.

Recall that the Stone–Čech compactification βG of G has a smallest ideal $K(\beta G)$ which is the union of all minimal left ideals of βG. We consider βG-actions on compact Hausdorff spaces and obtain the following sufficient condition for the closed subsemigroups S of βG for which an S-product recurrent point is a distal point, partly answering Auslander and Furstenberg's Question 1.4.

Theorem 1.6. *Let $(X,\beta G)$ be a βG-action and S be a nonempty closed subsemigroup of $\beta G\setminus G$. If $K(\beta G)\subset S$, then a point $x\in X$ is distal if and only if x is S-product recurrent.*

As an application, we obtain a characterization of distal points in terms of product recurrence for G-systems on compact Hausdorff spaces. It should be noted that some special cases for a topological dynamical system (X,T) were established by Oprocha and Zhang [21].

Theorem 1.7. *Let G be a countable infinite discrete group and $\mathcal{F}\subset\mathcal{P}(G)$ be a Furstenberg family. If $\mathcal{F}$ has the Ramsey property and the hull of $\mathcal{F}$,*

$$h(\mathcal{F}):=\{p\in\beta G: p\subset\mathcal{F}\},$$

is a subsemigroup of βG and $\mathcal{F}\supset\mathcal{F}_{\mathrm{ps}}$, then for any G-system (X,G) and $x\in X$, the following assertions are equivalent:

(1) *x is distal;*
(2) *x is $\mathcal{F}$-product recurrent, that is, for any G-system (Y,G) and any $\mathcal{F}$-recurrent point $y\in Y$, (x,y) is recurrent in the product system $(X\times Y,G)$;*
(3) *for any G-system (Y,G) and any $\mathcal{F}$-recurrent point $y\in Y$, (x,y) is $\mathcal{F}$-recurrent in the product system $(X\times Y,G)$.*

The paper is organized as follows. To illustrate the core idea, in Section 2 we focus on topological dynamical systems and prove Theorem 1.2. The proof takes advantage of the order of natural numbers and is thus relatively straightforward. In the rest of this paper, we consider general group actions and Ellis actions. In Section 3, we investigate some properties of several collections of subsets in a countably infinite discrete group G. In Section 4, for compact metric G-systems we provide combinatorial characterizations of the return time sets of $\mathcal{F}$-recurrent points under the conditions (P1) and (P2) introduced in Section 3. We also present an application of product recurrence for G-systems. In Section 5, we recall some results about Stone–Čech compactification βG of G and prove the main result (Theorem 5.7) of this paper, which can be regarded as a generalization of Theorem 1.2. In Section 6, we study βG-actions on compact Hausdorff spaces and prove Theorems 1.6 and 1.7.

2. Proof of Theorem 1.2. In this section, we focus on continuous maps acting on a compact metric space and prove Theorem 1.2. It should be noted that the natural order of $\mathbb{N}_0$ plays a significant role in the proof of Theorem 1.2, whereas in the general case (G-system), the situation becomes more complicated. To illustrate the core idea of the construction, we prove

Theorem 1.2 in a separate section, which may be of independent interest. We will try our best to make this section self-contained to ensure that readers can understand it independently. Readers are referred to Theorems 4.4 and 5.7 for the general case.

In Section 2.1 we will discuss some equivalent definitions of quasi-central sets. For the proof of Theorem 1.2, readers may refer directly to Section 2.2.

2.1. Some equivalent definitions of quasi-central sets. First we introduce the structure of $\beta\mathbb{N}_0$. Denote by $\mathcal{P} = \mathcal{P}(\mathbb{N}_0)$ the collection of all subsets of $\mathbb{N}_0$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a *Furstenberg family* (or just a *family*) if it is upward hereditary, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called *proper* if it is neither empty nor all of $\mathcal{P}$. A family is called a *filter* when it is a proper family closed under intersection, i.e., if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$. A family is called an *ultrafilter* if it is a filter that is maximal with respect to inclusion.

Before proceeding, let us recall some notions. By a *compact right topological semigroup*, we mean a triple $(E, \cdot, \mathcal{T})$, where $(E, \cdot)$ is a semigroup, and $(E, \mathcal{T})$ is a compact Hausdorff space, and for every $p \in E$, the right translation $\rho_p \colon S \to S$, $q \mapsto q \cdot p$, is continuous. If there is no ambiguity, we will say that E, instead of the triple $(E, \cdot, \mathcal{T})$, is a compact right topological semigroup. A nonempty subset I of E is called a *left ideal* of E if $E \cdot I \subset I$, a *right ideal* of E if $I \cdot E \subset I$ and an *ideal* of E if it is both a left ideal and a right ideal of E. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. A *minimal right ideal* is the right ideal that does not contain any proper right ideal. An element $p \in E$ is called *idempotent* if $p \cdot p = p$. An idempotent $p \in E$ is called a *minimal idempotent* if there exists a minimal left ideal L of E such that $p \in L$. The Ellis–Numakura theorem states that every compact right topological semigroup must contain an idempotent; see e.g. [17, Theorem 2.5].

Endowing $\mathbb{N}_0$ with the discrete topology, we take the points of the Stone–Čech compactification $\beta\mathbb{N}_0$ of $\mathbb{N}_0$ to be the ultrafilters on $\mathbb{N}_0$. For $A \subset \mathbb{N}_0$, let $\overline{A} = \{p \in \beta\mathbb{N}_0 : A \in p\}$. Then the set $\{\overline{A} : A \subset \mathbb{N}_0\}$ forms a basis for the open sets (and a basis for the closed sets) of $\beta\mathbb{N}_0$. Since $(\mathbb{N}_0, +)$ is a semigroup, we can extend the operation $+$ to $\beta\mathbb{N}_0$ by

$$p + q = \{F \subset \mathbb{N}_0 : \{n \in \mathbb{N}_0 : -n + F \in q\} \in p\}.$$

Then $(\beta\mathbb{N}_0, +)$ is a compact Hausdorff right topological semigroup with $\mathbb{N}_0$ contained in the topological center of $\beta\mathbb{N}_0$. That is, for each $p \in \beta\mathbb{N}_0$ the map $\rho_p : \beta\mathbb{N}_0 \to \beta\mathbb{N}_0$, $q \mapsto q + p$, is continuous, and for each $n \in \mathbb{N}_0$ the map $\lambda_n : \beta\mathbb{N}_0 \to \beta\mathbb{N}_0$, $q \mapsto n + q$, is continuous. It is well-known that $\beta\mathbb{N}_0$ has a smallest ideal $K(\beta\mathbb{N}_0) = \bigcup\{L : L \text{ is a minimal left ideal of } \beta\mathbb{N}_0\} = \bigcup\{R : R \text{ is a minimal right ideal of } \beta\mathbb{N}_0\}$ [17, Theorem 2.8]. Let $p \in \beta\mathbb{N}_0$, $\{x_n\}_{n\in\mathbb{N}_0}$

be an indexed family in a compact Hausdorff space X and $y \in X$. If for every neighborhood U of y, $\{n \in \mathbb{N}_0 \colon x_n \in U\} \in p$, then we say that the *p-limit* of $\{x_n\}_{n\in\mathbb{N}_0}$ is y, denoted by $p\text{-}\lim_{n\in\mathbb{N}_0} x_n = y$. As X is a compact Hausdorff space, $p\text{-}\lim_{n\in\mathbb{N}_0} x_n$ exists and is unique.

According to [16, Definition 1.2], we introduce the following definition of quasi-central sets.

Definition 2.1. Let $F \subset \mathbb{N}_0$. Then F is *quasi-central* if there exists some idempotent $p \in \mathrm{cl}(K(\beta\mathbb{N}_0))$ with $F \in p$.

We recall some classes of subsets of $\mathbb{N}_0$.

Definition 2.2. Let A be a subset of $\mathbb{N}_0$.

(1) If for every $L \in \mathbb{N}$, there exists $n \in \mathbb{N}_0$ such that $\{n, n+1, \ldots, n+L\} \subset A$, then we say that A is *thick*.
(2) If there exists $L \in \mathbb{N}$ such that for any $n \in \mathbb{N}_0$, $\{n, n+1, \ldots, n+L\} \cap A \neq \emptyset$, then we say that A is *syndetic*.
(3) If there exists a thick set $B \subset \mathbb{N}_0$ and a syndetic $C \subset \mathbb{N}_0$ such that $A = B \cap C$, then we say that A is *piecewise syndetic*.

Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system defined in [5] where S is a semigroup. Note that when $S = \mathbb{N}_0$, the action is generated by a continuous evolution map T and we simply write the dynamical system as (X, T) in this section (the underlying space X is a compact metric space). By the proof of [5, Theorem 3.4], we have the following theorem, which is a dynamical characterization of quasi-central sets.

Theorem 2.3. *Let $F \subset \mathbb{N}_0$. Then F is quasi-central if and only if there exists a dynamical system (X, T), points x and y of X, and a neighborhood U of y such that*

(1) *for any neighborhood V of y, $N((x, y), V \times V)$ is piecewise syndetic;*
(2) *$N(x, U) = F$.*

We will need the following equivalent characterizations of quasi-central sets.

Proposition 2.4. *Let $F \subset \mathbb{N}_0$. Then the following assertions are equivalent:*

(1) *F is quasi-central;*
(2) *there exists a dynamical system (X, T), points x and y of X, and a neighborhood U of y such that*
    (a) *for any neighborhood V of y, $N((x, y), V \times V)$ is piecewise syndetic;*
    (b) *$N(x, U) \subset F \cup \{0\}$;*

(3) *there exists a dynamical system* (X,T)*, points* x *and* y *of* X*, and a neighborhood* U *of* y *such that*

(a) *for any neighborhood* V *of* y*,* $N((x,y),V\times V)$ *is piecewise syndetic;*
(b) $N((x,y),U\times U)\subset F\cup\{0\}$.

Proof. (1)⇒(2). This follows from Theorem 2.3.

(2)⇒(3). This follows from the fact that $N((x,y),U\times U)\subset N(x,U)$.

(3)⇒(1). By [5, Lemma 3.3] one can pick an idempotent $p\in \mathrm{cl}(K(\beta\mathbb{N}_0))$ such that $p\text{-}\lim_{n\in\mathbb{N}_0}T^n x = p\text{-}\lim_{n\in\mathbb{N}_0}T^n y = y$. For the neighborhood U of y, $N((x,y),U\times U)=N(x,U)\cap N(y,U)\in p$. Then $F\cup\{0\}\in p$. Since $K(\beta\mathbb{N}_0)\subset\beta\mathbb{N}_0\setminus\mathbb{N}_0$, $F\in p$. By definition F is quasi-central. ∎

2.2. Proof of Theorem 1.2. To prove Theorem 1.2, we need the following definition and lemma.

Definition 2.5. Let (X,T) be a topological dynamical system and $x\in X$. We say that x is *a piecewise syndetic recurrent point* if for any neighborhood U of x, $N(x,U):=\{n\in\mathbb{N}_0: T^n x\in U\}$ is a piecewise syndetic set.

Lemma 2.6. *Let* (X,T) *be a dynamical system, let* $x,y\in X$*, and assume that for every neighborhood* V *of* y*,* $N((x,y),V\times V)$ *is piecewise syndetic in* $\mathbb{N}_0$*. Let* U *be a neighborhood of* y *and let* $a\in\mathbb{N}$*. There are a set* H *which is thick in* $\mathbb{N}_0$ *and a set* S *which is syndetic in* $\mathbb{N}_0$ *such that* $H\cap S\subset N((x,y),U\times U)$ *and* $S\subset(a+1)\mathbb{N}$.

Proof. By Proposition 2.4, $N((x,y),U\times U)$ is a quasi-central set. Then by [17, Lemma 5.19.2] or [20, Proposition 6.7], $\frac{1}{a_1+1}N((x,y),U\times U)\cap\mathbb{N}$ is piecewise syndetic in $\mathbb{N}$, so also in $\mathbb{N}_0$. There exists a thick subsset H' of $\mathbb{N}$ and a syndetic set S' in $\mathbb{N}$ such that

$$H'\cap S'=\frac{1}{a_1+1}N((x,y),U\times U)\cap\mathbb{N}.$$

Let $H=\bigcup_{j=0}^{a_1}((a_1+1)H'+j)$ and $S=(a_1+1)S'$. Then H is thick, S is syndetic and

$$H\cap S\subset(a_1+1)(H'\cap S')\subset N((x,y),U\times U). \ ∎$$

Now we introduce the symbolic dynamical system (Σ_2,σ). Let

$$\Sigma_2=\{0,1\}^{\mathbb{N}_0}=\{x_0x_1\ldots: x_i\in\{0,1\},\ i\in\mathbb{N}_0\},$$

endowed with the product topology on $\{0,1\}^{\mathbb{N}_0}$, while $\{0,1\}$ is endowed with the discrete topology. A compatible metric d on Σ_2 is defined by

$$d(x,y)=\begin{cases}0, & x=y,\\ \frac{1}{2^k}, & k=\min\{i\in\mathbb{N}_0: x_i\neq y_i\},\end{cases}$$

for any $x, y \in \Sigma_2$. Then (Σ_2, d) is a compact metric space. Define the *shift map* as follows

$$\sigma\colon \Sigma_2 \to \Sigma_2,\ x_0x_1\ldots \mapsto x_1x_2\ldots$$

Then (Σ_2, σ) is a topological dynamical system. Besides infinite symbolic sequences we consider also finite symbolic sequences or *words* $u = u_0u_1\ldots u_{n-1}$ where $u_i \in \{0,1\}$ for $i = 0, \ldots, n-1$. If $u = u_0u_1\ldots u_{n-1}$ is a word of $\{0,1\}$, we define the *cylinder* of u by

$$[u] = \big\{v \in \Sigma_2\colon v_i = u_i \text{ for any } 0 \le i \le n-1\big\}.$$

Obviously $[u]$ is a clopen subset of Σ_2. Let $\{0,1\}^n = \{x_0x_1\ldots x_{n-1} : x_i \in \{0,1\}, 0 \le i \le n-1\}$ and $\{0,1\}^* = \bigcup_{n=1}^{\infty}\{0,1\}^n$. Then the collection $\{[u] : u \in \{0,1\}^*\}$ of all cylinders forms a topological basis of the topology of Σ_2. In particular, for any $x = x_0x_1\ldots \in \Sigma_2$, we denote by $x|_{[i,j]} = x_i\ldots x_j$ the word which occurs in x between coordinates i and j. Then we can consider the cylinder $[x|_{[i,j]}]$, i.e., $[x|_{[i,j]}] = \{v \in \Sigma_2\colon v_s = x_s \text{ for any } i \le s \le j\}$. For any $x, y \in \Sigma_2$, $x|_{[i,j]} = y|_{[i,j]}$ means that the two words are identical, i.e., for any $s \in \{i, \ldots, j\}$, $x_s = y_s$.

Proof of Theorem 1.2. (1) Since $x \in X$ is piecewise syndetic recurrent, for every neighborhood V of x, $N(x, V)$ is a piecewise syndetic set. Then for the system (X, T), $x \in X$ and a neighborhood U of x, it satisfies the following:

(i) for every neighborhood V of x, $N((x,x), V \times V) = N(x, V)$ is piecewise syndetic;
(ii) $N(x, U) = N(x, U) \cup \{0\}$.

Thus $N(x, U)$ is quasi-central.

(2) Let F be a quasi-central subset of $\mathbb{N}_0$. By Proposition 2.4, there exists a topological dynamical system (X, T), $x, y \in X$ and a neighborhood U of y such that

(i) for every neighborhood V of y, $N((x,y), V \times V)$ is piecewise syndetic in $\mathbb{N}_0$;
(ii) $N(x, U) \subset F \cup \{0\}$.

We shall show that for the symbolic dynamical system (Σ_2, σ), there exists a point $z \in \Sigma_2$ which is a piecewise syndetic recurrent point such that $[1]$ is a neighborhood of z and $N(z, [1]) \subset F \cup \{0\}$.

Let $U_1 = U$. Since $N((x,y), U_1 \times U_1)$ is piecewise syndetic in $\mathbb{N}_0$, pick a set H_1 which is thick in $\mathbb{N}_0$ and a set S_1 which is syndetic in $\mathbb{N}_0$ such that $H_1 \cap S_1 = N((x,y), U_1 \times U_1)$. Pick a finite integer interval $I_1^{(1)} \subset H_1$ such that $I_1^{(1)} \cap S_1 \neq \emptyset$, $\min I_1^{(1)} > 1$ and $|I_1^{(1)}| > 1$, where $|\cdot|$ denotes cardinality.

Define $z^{(1)} \in \Sigma_2$ as follows:

$$z^{(1)}(n) = \begin{cases} 1, & n = 0, \\ 1, & n \in I_1^{(1)} \cap S_1, \\ 0, & n \in \mathbb{N}_0 \setminus \{\{0\} \cup (I_1^{(1)} \cap S_1)\}. \end{cases}$$

Then $z^{(1)}(0) = 1$, $z^{(1)}(1) = 0$ and $N(z^{(1)}, [1]) = \{0\} \cup (I_1^{(1)} \cap S_1)$. Let $A_1 = N(z^{(1)}, [1])$ and let $a_1 = \max A_1$. Then A_1 is a finite subset of $\mathbb{N}_0$ and $A_1 \subset N((x, y), U_1 \times U_1) \cup \{0\}$.

Let $k \in \mathbb{N}$ and assume that we have chosen $\langle z^{(i)} \rangle_{i=1}^k$ in Σ_2, $\langle U_i \rangle_{i=1}^k$ neighborhoods of y in X, $\langle A_i \rangle_{i=1}^k$, $\langle a_i \rangle_{i=1}^k$, $\langle H_i \rangle_{i=1}^k$, $\langle S_i \rangle_{i=1}^k$ and $\langle \langle I_i^{(j)} \rangle_{j=1}^i \rangle_{i=1}^k$ satisfying the following hypotheses for $i \in \{1, \ldots, k\}$:

(1) $A_i = N(z^{(i)}, [1]) \subset N((x, y), U_1 \times U_1) \cup \{0\}$ and $a_i = \max A_i$;
(2) if $i > 1$, then $A_{i-1} \subset A_i$ and $a_{i-1} < a_i$;
(3) if $i > 1$, then $U_i = \bigcap_{j \in A_{i-1}} T^{-j} U_1$;
(4) H_i is thick in $\mathbb{N}_0$, S_i is syndetic in $\mathbb{N}_0$, and $H_i \cap S_i \subset N((x, y), U_i \times U_i)$;
(5) if $i > 1$, then $S_i \subset (a_{i-1} + 1)\mathbb{N}$;
(6) if $1 \le j \le i$, then $I_i^{(j)}$ is a finite interval, $|I_i^{(j)}| > i$, $I_i^{(j)} \subset H_j$ and $I_i^{(1)} \cap S_1 \neq \emptyset$;
(7) if $i > 1$, then $\min I_i^{(1)} > a_{i-1}$, and $\min I_i^{(2)} > \max I_i^{(1)}$;
(8) if $i > 2$, then $\min I_i^{(1)} > \max I_{i-1}^{(i-1)} + a_{i-2}$, $\min I_i^{(2)} > \max I_i^{(1)}$ and if $3 \le j \le i$, then $\min I_i^{(j)} > \max I_{i-1}^{(j-1)} + a_{j-2}$;
(9) if $i > 1$, then $z^{(i)}|_{[0, a_{i-1}]} = z^{(i-1)}|_{[0, a_{i-1}]}$;
(10) if $n \in I_i^{(1)} \cap S_1$, then $z^{(i)}(n) = 1$;
(11) if $2 \le j \le i$ and $n \in I_i^{(j)} \cap S_j$, $z^{(i)}|_{[n, n+a_{j-1}]} = z^{(j-1)}|_{[0, a_{j-1}]}$;
(12) if $i > 1$ and $t \in \mathbb{N}_0 \setminus ([0, a_{i-1}] \cup (I_i^{(1)} \cap S_1) \cup \bigcup_{j=2}^i \bigcup_{n \in I_i^{(j)} \cap S_j} [n, n+a_{j-1}])$, then $z^{(i)}(t) = 0$.

All hypotheses are satisfied for $i = 1$, all but (1), (4), (6) and (10) vacuously.

We now show that all hypotheses are satisfied for $i = k + 1$. Let $U_{k+1} = \bigcap_{j \in A_k} T^{-j} U_1$. By hypothesis (1), if $j \in A_k$, then $j \in N((x, y), U_1 \times U_1) \cup \{0\}$ so $T^j y \in U_1$. Therefore U_{k+1} is an open neighborhood of y. By Lemma 2.6, pick a thick subset H_{k+1} of $\mathbb{N}_0$ and a syndetic subset S_{k+1} of $\mathbb{N}_0$ such that $S_{k+1} \subset (a_k + 1)\mathbb{N}$ and $H_{k+1} \cap S_{k+1} \subset N((x, y), U_{k+1} \times U_{k+1})$.

Take a finite interval $I_{k+1}^{(1)}$ in H_1 with $\min I_{k+1}^{(1)} > a_k$ such that $I_{k+1}^{(1)} \cap S_1 \neq \emptyset$ and $\min I_{k+1}^{(1)} > \max I_k^{(k)} + u$ where

$$u = \begin{cases} 0 & \text{if } k = 1, \\ a_{k-1} & \text{if } k > 1. \end{cases}$$

For $j \in \{2,3,\ldots,k+1\}$ pick a finite interval $I^{(j)}_{k+1}$ in H_j such that $|I^{(j)}_{k+1}| > k+1$, $\min I^{(j)}_{k+1} > \max I^{(j-1)}_{k+1}$ and if $j \geq 3$, then $\min I^{(j)}_{k+1} > \max I^{(j-1)}_{k+1} + a_{j-2}$.

We claim that we can define $z^{(k+1)} \in \Sigma_2$ as required by hypotheses (9)–(12) for $i = k+1$. That is,

(9) $z^{(k+1)}|_{[0,a_k]} = z^{(k)}|_{[0,a_k]}$;

(10) if $n \in I^{(1)}_{k+1} \cap S_1$, then $z^{(k+1)}(n) = 1$;

(11) if $2 \leq j \leq k+1$ and $n \in I^{(j)}_{k+1} \cap S_j$, $z^{(k+1)}|_{[n,n+a_{j-1}]} = z^{(j-1)}|_{[0,a_{j-1}]}$;

(12) if $t \in \mathbb{N}_0 \setminus ([0,a_k] \cup (I^{(1)}_{k+1} \cap S_1) \cup \bigcup_{j=2}^{k+1} \bigcup_{n \in I^{(j)}_{k+1} \cap S_j} [n, n+a_{j-1}])$, then $z^{(k+1)}(t) = 0$.

By the construction of $I^{(j)}_{k+1}$, $j = 1,\ldots,k+1$, we have $\min I^{(j)}_{k+1} > \min I^{(1)}_{k+1} > a_k$ for $j \in \{1,\ldots,k+1\}$, which implies that (9) does not contradict (10) or (11).

To see that (10) does not contradict any part of (11), let $j \in \{1,2,\ldots, k+1\}$, let $m \in I^{(j)}_{k+1} \cap S_j$ and let $t \in [0,a_{j-1}]$. Then $m + t \geq \min I^{(j)}_{k+1} > \max I^{(1)}_{k+1} \geq n$.

Finally, we show that no parts of (11) contradict each other. Suppose we have $2 \leq j \leq l \leq k+1$, $n \in I^{(j)}_{k+1} \cap S_j$, $m \in I^{(l)}_{k+1} \cap S_l$, $t \in [0,a_{j-1}]$ and $s \in [0,a_{l-1}]$ such that $n+t = m+s$. Assume first that $j = l$. If $n = m$, then $t = s$ and there is no conflict. So suppose without loss of generality that $n < m$. Then $n,m \in (a_{j-1}+1)\mathbb{N}$ so pick $b < c$ in $\mathbb{N}$ such that $n = (a_{j-1}+1)b$ and $m = (a_{j-1}+1)c$. Then $n+t = (a_{j-1}+1)b + t < (a_{j-1}+1)c \leq m+s$, a contradiction. Thus we must have $j < l$ so $l \geq 3$. Then $m+s \geq \min I^{(l)}_{k+1} > \max I^{(l-1)}_{k+1} + a_{l-2} \geq \max I^{(j)}_{k+1} + a_{j-1} \geq n+t$, a contradiction.

Let $A_{k+1} = N(z^{k+1}, [1])$ and let $a_{k+1} = \max A_{k+1}$. All hypotheses are satisfied directly except for (1) and (2). To see that $A_k \subset A_{k+1}$, let $n \in A_k$. Then $n \leq a_k$ so by hypothesis (9), $z^{(k+1)}(n) = z^{(k)}(n) = 1$. Also by (6), $I^{(1)}_{k+1} \cap S_1 \neq \emptyset$ so by (10), $a_{k+1} \geq \min(I^{(1)}_{k+1} \cap S_1)$ and $\min(I^{(1)}_{k+1} \cap S_1) \geq \min I^{(1)}_{k+1} > a_k$ by (7). Thus, hypothesis (2) holds.

To verify hypothesis (1) we need to show that

$$N(z^{(k+1)}, [1]) \subset N((x,y), U_1 \times U_1) \cup \{0\}.$$

So let $m \in N(z^{(k+1)}, [1])$. If $m \in [0,a_k]$, then $m \in A_k \subset N((x,y), U_1 \times U_1) \cup \{0\}$. If $m \in I^{(1)}_{k+1} \cap S_1$, then $m \in H_1 \cap S_1 \subset N((x,y), U_1 \times U_1)$. So assume that we have $2 \leq j \leq k+1$, $n \in I^{(j)}_{k+1} \cap S_j$, and $t \in A_{j-1}$ such that $m = n+t$. By hypotheses (6) and (4), $n \in N((x,y), U_j \times U_j)$ so $T^n x \in U_j$ and $T^n y \in U_j$. By hypothesis (3), $T^t(T^n x) \in U_1$ and $T^t(T^n y) \in U_1$ so $m = n+t \in N((x,y), U_1 \times U_1)$. The inductive construction is complete.

We now establish some facts.

(a) *If* $1 \le r < j \le i$, *then for each* $n \in I_j^{(r+1)} \cap S_{r+1}$,

$$z^{(i)}|_{[n,n+a_r]} = z^{(j)}|_{[n,n+a_r]} = z^{(r)}|_{[0,a_r]}.$$

To establish (a), let $1 \le r < j \le i$, let $n \in I_j^{(r+1)} \cap S_{r+1}$ and let $t \in [0, a_r]$. By hypothesis (11), $z^{(j)}(n+t) = z^{(r)}(t)$. Now $z^{(j)}(n+a_r) = z^{(r)}(a_r) = 1$ so $n + a_r \in A_j$ and thus $n + a_r \le a_j$. Then by hypotheses (2) and (9), $z^{(i)}(n+t) = z^{(j)}(n+t) = z^{(r)}(t)$.

(b) *If* $1 \le r < j \le i$, *then* $I_j^{(r+1)} \cap S_{r+1} \subset N(z^{(i)}, [z^{(r)}|_{[0,a_r]}])$.

To establish (b), let $1 \le r < j \le i$ and let $n \in I_j^{(r+1)} \cap S_{r+1}$. Then by (a), for each $t \in [0, a_r]$, $\sigma^n(z^{(i)})(t) = z^{(i)}(n+t) = z^{(r)}(t)$ so $n \in N(z^{(i)}, [z^{(r)}|_{[0,a_r]}])$ as required.

Since $\langle z^{(i)} \rangle_{i=1}^{\infty}$ is a sequence in compact space Σ_2, we may pick a cluster point $z \in \Sigma_2$ of the sequence $\langle z^{(i)} \rangle_{i=1}^{\infty}$.

(c) *For each* $j \in \mathbb{N}$, $z|_{[0,a_j]} = z^{(j)}|_{[0,a_j]}$.

To establish (c), let $j \in \mathbb{N}$ and let $t \in [0, a_j]$. Since z is a cluster point of the sequence $\langle z^{(i)} \rangle_{i=1}^{\infty}$ and $[z|_{[0,a_j]}]$ is a neighborhood of z, we can pick $i > j$ such that $z^{(i)} \in [z|_{[0,a_j]}]$. Then $z^{(i)}|_{[0,a_j]} = z|_{[0,a_j]}$ and by hypotheses (2) and (9), $z^{(j)}|_{[0,a_j]} = z^{(i)}|_{[0,a_j]} = z|_{[0,a_j]}$.

As a consequence of (c), for each $r \in \mathbb{N}$, $[z^{(r)}|_{[0,a_r]}]$ is a neighborhood of z. So $\{[z^{(r)}|_{[0,a_r]}] : r \in \mathbb{N}\}$ is a neighborhood basis for z.

(d) *If* $1 \le r < i$. *Then* $N(z^{(i)}, [z^{(r)}|_{[0,a_r]}]) \subset N(z, [z^{(r)}|_{[0,a_r]}])$.

To establish (d), let $1 \le r < i$ and $n \in N(z^{(i)}, [z^{(r)}|_{[0,a_r]}])$. Then for any $t \in [0, a_r]$,

$$\sigma^n(z^{(i)})(t) = z^{(i)}(n+t) = z^{(r)}(t).$$

In particular, $z^{(i)}(n+a_r) = z^{(r)}(a_r) = 1$ so $n+a_r \in A_i$ and thus $n+a_r \le a_i$. By (c), $z|_{[0,a_i]} = z^{(i)}|_{[0,a_i]}$ so $\sigma^n(z)(t) = z(n+t) = z^{(i)}(n+t) = z^{(r)}(t)$. Thus, $n \in N(z, [z^{(r)}|_{[0,a_r]}])$ as claimed.

Now we claim that z is a piecewise syndetic recurrent point of Σ_2. To see this, let R be a neighborhood of z and pick $r \in \mathbb{N}$ such that

$$[z^{(r)}|_{[0,a_r]}] \subset R.$$

As S_{r+1} is syndetic and $\bigcup_{i=r+1}^{\infty} I_i^{(r+1)}$ is thick, $S_{r+1} \cap (\bigcup_{i=r+1}^{\infty} I_i^{(r+1)})$ is piecewise syndetic and

$$\begin{aligned} S_{r+1}\cap\Big(\bigcup_{i=r+1}^{\infty} I_i^{(r+1)}\Big) &= \bigcup_{i=r+1}^{\infty}(S_{r+1}\cap I_i^{(r+1)}) \\ &\subset \bigcup_{i=r+1}^{\infty} N(z^{(i)},[z^{(r)}|_{[0,a_r]}]) \subset N(z,[z^{(r)}|_{[0,a_r]}]), \end{aligned}$$

where the first inclusion holds by (b) and the second inclusion holds by (d). So z is a piecewise syndetic recurrent point of Σ_2.

By (c), [1] is a neighborhood of z. We conclude the proof by showing that $N(z,[1])\subset F\cup\{0\}$. If $n\in N(z,[1])$ and $a_i>n$ then by (c), $z(n)=z^{(i)}(n)$ so $N(z,[1])\subset\bigcup_{i=1}^{\infty}N(z^{(i)},[1])$. By hypothesis (1), for each $i\in\mathbb{N}$, $N(z^{(i)},[1])\subset N((x,y),U_1\times U_1)\cup\{0\}\subset N(x,U)\cup\{0\}$ so $N(z,[1])\subset F\cup\{0\}$. ∎

3. Subsets in a countable infinite group. In this section we investigate some classes of subsets in a countable infinite discrete group. We propose two abstract properties (P1) and (P2) for a Furstenberg family which we will use in Section 3 to characterize recurrent time sets. We will verify that the collection of all piecewise syndetic sets and the collection of all infinite sets satisfy the two abstract properties. If the group is amenable, the collection of all sets with positive upper density (with positive upper Banach density, respectively) also satisfies the two abstract properties.

Let G be a countable infinite discrete group with identity e. Denote by $\mathcal{P}(G)$ and $\mathcal{P}_{\mathrm{f}}(G)$ the collections of all subsets of G and all nonempty finite subsets of G respectively. Let $\mathcal{F}\subset\mathcal{P}(G)\setminus\{\emptyset\}$. If for any $F\in\mathcal{F}$, $F\subset H\subset G$ implies $H\in\mathcal{F}$, then we say that $\mathcal{F}$ is a *Furstenberg family* (or just a *family*). A Furstenberg family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}(G)\setminus\{\emptyset\}$. For a Furstenberg family $\mathcal{F}$, the *dual family* of $\mathcal{F}$, denoted by $\mathcal{F}^*$, is

$$\{F\in\mathcal{P}(G): F\cap F'\neq\emptyset \text{ for any } F'\in\mathcal{F}\}.$$

Note that $\mathcal{F}^*=\{F\in\mathcal{P}(G): G\setminus F\notin\mathcal{F}\}$. A Furstenberg family $\mathcal{F}$ is called a *filter* if $A,B\in\mathcal{F}$ imply $A\cap B\in\mathcal{F}$. An *ultrafilter* is a filter which is not properly contained in any other filter. A Furstenberg family $\mathcal{F}$ has the *Ramsey property* if whenever $A\in\mathcal{F}$ and $A=A_1\cup A_2$ there exists some $i\in\{1,2\}$ such that $A_i\in\mathcal{F}$. It is easy to see that a Furstenberg family $\mathcal{F}$ has the Ramsey property if and only if the dual family $\mathcal{F}^*$ is a filter.

Let A be a subset of G.

(1) If for every $K\in\mathcal{P}_{\mathrm{f}}(G)$, there exists $g\in G$ such that $Kg\subset A$, then we say that A is *thick*.
(2) If there exists $K\in\mathcal{P}_{\mathrm{f}}(G)$ such that for any $g\in G$, $Kg\cap A\neq\emptyset$ (i.e. $G=K^{-1}A$), then we say that A is *syndetic*.

(3) If there exists a thick set $B \subset G$ and a syndetic $C \subset G$ such that $A = B \cap C$, then we say that A is *piecewise syndetic*.

Denote by $\mathcal{F}_{\rm t}$, $\mathcal{F}_{\rm s}$, $\mathcal{F}_{\rm ps}$, $\mathcal{F}_{\rm inf}$ the collections of all thick, syndetic, piecewise syndetic and infinite subsets of G, respectively.

We say that a Furstenberg family $\mathcal{F}$ *satisfies* (P1) if for any $A \in \mathcal{F}$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{P}_{\rm f}(G)$ such that

(1) for every $n \in \mathbb{N}$, $A_n \subset A$;
(2) for any distinct $n, m \in \mathbb{N}$, $A_n \cap A_m = \emptyset$;
(3) for every strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{k=1}^{\infty} A_{n_k} \in \mathcal{F}$,

and $\mathcal{F}$ *satisfies* (P2) if for any $F \in \mathcal{F}$ and any $K \in \mathcal{P}_{\rm f}(G)$, there exists a subset F' of F such that $F' \in \mathcal{F}$ and for any distinct $f_1, f_2 \in F' \cup \{e\}$, $Kf_1 \cap Kf_2 = \emptyset$.

First we need the following lemma.

Lemma 3.1. *For every $F, H \in \mathcal{P}_{\rm f}(G)$ if $|H| > |F|^2$ there exists $h \in H$ such that $F \cap Fh = \emptyset$.*

Proof. For any $f_1, f_2 \in F$, let $B(f_1, f_2) = \{h \in H \colon f_1 = f_2 h\}$. As G is a group, each $B(f_1, f_2)$ is the empty set or a singleton. If for every $h \in H$, $F \cap Fh \neq \emptyset$, then $\bigcup_{f_1, f_2 \in F} B(f_1, f_2) = H$. As $|H| > |F|^2$, there exist $f_1, f_2 \in F$, such that $B(f_1, f_2)$ contains at least two points. This is a contradiction. ∎

The following result must be folklore. We provide a proof for the sake of completeness.

Lemma 3.2. *$\mathcal{F}_{\rm t}$ satisfies* (P1).

Proof. We need the following claim.

Claim. *Let F be a thick set. Fix $K \in \mathcal{P}_{\rm f}(G)$, then $\{g \in G \colon Kg \subset F\}$ is a thick set.*

Proof. For any $H \in \mathcal{P}_{\rm f}(G)$, $KH \in \mathcal{P}_{\rm f}(G)$. As F is thick, there exists $h \in G$ such that $KHh \subset F$. Then $Hh \subset \{g \in G \colon Kg \subset F\}$. So $\{g \in G : Kg \subset F\}$ is a thick set. ∎

Now fix a thick set A. As G is countable, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{P}_{\rm f}(G)$ such that $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$. As A is thick, there exists $g_1 \in G$ such that $G_1 g_1 \subset A$. Let $A_1 = G_1 g_1$. Let $B_2 = A_1 \cup G_2$. By the claim, $\{g \in G \colon B_2 g \subset A\}$ is thick. By Lemma 3.1, there exists $g_2 \in \{g \in G \colon B_2 g \subset A\}$ such that $B_2 \cap B_2 g_2 = \emptyset$. Let $A_2 = G_2 g_2$.

By induction, we construct two sequences $\{A_n\}$, $\{B_n\}$ in $\mathcal{P}_{\rm f}(G)$ and a sequence $\{g_n\}$ in G such that for any $n \geq 2$,

(1) $B_n = \bigcup_{i=1}^{n-1} A_i \cup G_n$;
(2) $B_n g_n \subset A$;

(3) $B_n \cap B_n g_n = \emptyset$;
(4) $A_n = G_n g_n$.

Then for any $n \in \mathbb{N}$, $A_n \subset B_n g_n \subset A$; for any $n, m \in \mathbb{N}$ with $n \neq m$, without loss of generality assume that $n > m$, $A_n \cap A_m \subset A_n \cap B_n \subset B_n g_n \cap B_n = \emptyset$. Since $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$, for every strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{k=1}^{\infty} A_{n_k} \in \mathcal{F}_t$. Thus $\mathcal{F}_{\mathrm{t}}$ satisfies (P1). ∎

In [22] Xu and Ye showed that $\mathcal{F}_{\mathrm{s}}$ satisfies (P2). Here we have the following sufficient condition for a Furstenberg family to satisfy (P2).

Proposition 3.3. *Let $\mathcal{F}$ be a proper Furstenberg family in $\mathcal{P}(G)\setminus\{\emptyset\}$. If $\mathcal{F}$ has the Ramsey property and for all $A \in \mathcal{F}$ and $g \in G$, we have $gA \in \mathcal{F}$, then $\mathcal{F}$ satisfies* (P2).

Proof. We first show the following Claim.

Claim. *For every $A \in \mathcal{F}$ and $K \in \mathcal{P}_{\mathrm{f}}(G)$, $A \setminus K \in \mathcal{F}$.*

Proof of the Claim. Let $A \in \mathcal{F}$ and $K \in \mathcal{P}_{\mathrm{f}}(G)$. As $\mathcal{F}$ has the Ramsey property and $A = (A \cap K) \cup (A \setminus K)$, either $A \cap K \in \mathcal{F}$ or $A \setminus K \in \mathcal{F}$. Now we assume that $A \cap K \in \mathcal{F}$ and write the finite $A \cap K$ as $\{k_1, \ldots, k_n\}$. By the Ramsey property of $\mathcal{F}$ again, there exists some $1 \leq i \leq n$ such that $\{k_i\} \in \mathcal{F}$. For every $g \in G$, $g\{k_i\} = \{gk_i\} \in \mathcal{F}$. As $\mathcal{F}$ is a Furstenberg family, $\mathcal{F} = \mathcal{P}(G)\setminus\{\emptyset\}$, which contradicts $\mathcal{F}$ being proper. Therefore, $A \setminus K \in \mathcal{F}$. ∎

Now fix $A \in \mathcal{F}$ and $K \in \mathcal{P}_{\mathrm{f}}(G)$. Let

$$\mathcal{B} = \{B \subset A\colon \text{for any distinct } b_1, b_2 \in B \cup \{e\},\ Kb_1 \cap Kb_2 = \emptyset\}.$$

By the Claim, A is infinite. By Lemma 3.1, there exists $h \in A\setminus\{e\}$ such that $K \cap Kh = \emptyset$, then $\{h\} \in \mathcal{B}$, which implies that $\mathcal{B}$ is not empty. By Zorn's Lemma, pick $B \in \mathcal{B}$ which is maximal with respect to inclusion. If $D \in \mathcal{B}$ then also $D \cup \{e\} \in \mathcal{B}$ and since $B \in \mathcal{B}$ is maximal with respect to inclusion, $e \in B$.

Now we will show that $B \in \mathcal{F}$. For any $a \in A$, there exists $b \in B$ such that $Ka \cap Kb \neq \emptyset$. (For otherwise there is $a \in A$ such that for any $b \in B$, we have $Ka \cap Kb = \emptyset$, so $a \notin B$, $B \subsetneq B \cup \{a\} \in \mathcal{B}$, contradicting the maximality of B.) Then $a \in K^{-1}Kb$. This shows that $A \subset K^{-1}KB$. Then $K^{-1}KB \in \mathcal{F}$ as $A \in \mathcal{F}$. As $\mathcal{F}$ has the Ramsey property and $K^{-1}K$ is finite, there exists some $g \in K^{-1}K$ such that $gB \in \mathcal{F}$. Then $B = g^{-1}(gB) \in \mathcal{F}$. ∎

It is easy to see that $\mathcal{F}_{\mathrm{inf}}$ satisfies the properties (P1) and (P2). Now we show that so does $\mathcal{F}_{\mathrm{ps}}$.

Lemma 3.4. *$\mathcal{F}_{\mathrm{ps}}$ satisfies* (P1) *and* (P2).

Proof. (1) To prove that $\mathcal{F}_{\mathrm{ps}}$ satisfies (P1), let $F \in \mathcal{F}_{\mathrm{ps}}$. By the definition of $\mathcal{F}_{\mathrm{ps}}$, there exists a thick set $A \subset G$ and a syndetic set $B \subset G$ such that

$F = A \cap B$. By Lemma 3.2, if $\mathcal{F}_t$ satisfies (P1), then there exists a sequence $\{A_n\}_{n=1}^\infty$ in $\mathcal{P}_{\mathrm{f}}(G)$ such that

- for every $n \in \mathbb{N}$, $A_n \subset A$;
- for all distinct $n, m \in \mathbb{N}$, $A_n \cap A_m = \emptyset$;
- for every strictly increasing sequence $\{n_k\}_{k=1}^\infty$ in $\mathbb{N}$, $\bigcup_{k=1}^\infty A_{n_k} \in \mathcal{F}_t$.

Let $F_n = A_n \cap B$ for $n \in \mathbb{N}$. Then $\{F_n\}_{n=1}^\infty$ is the sequence that satisfies (P1) for F. By the arbitrariness of F, $\mathcal{F}_{\mathrm{ps}}$ satisfies (P1).

(2) To prove that $\mathcal{F}_{\mathrm{ps}}$ satisfies (P2), let $F \in \mathcal{F}_{\mathrm{ps}}$. By the definition of $\mathcal{F}_{\mathrm{ps}}$, there exists a thick set $A \subset G$ and a syndetic set $B \subset G$ such that $F = A \cap B$. For any $K \in \mathcal{P}_{\mathrm{f}}(G)$, by [22, Lemma 2.7], $\mathcal{F}_s$ satisfies (P2). Then there exists a subset B' of B such that $B' \in \mathcal{F}_s$ and for any distinct $b_1, b_2 \in B' \cup \{e\}$, $Kb_1 \cap Kb_2 = \emptyset$. Let $F' = A \cap B'$. Then $F' \subset F$ and $F' \in \mathcal{F}_{\mathrm{ps}}$. For any distinct $f_1, f_2 \in F' \cup \{e\}$, we have $f_1, f_2 \in B' \cup \{e\}$, thus $Kf_1 \cap Kf_2 = \emptyset$. By the arbitrariness of F, $\mathcal{F}_{\mathrm{ps}}$ satisfies (P2). ∎

A Følner sequence of a group G can be used to define the density of a set $A \subset G$ in a way analogous to the definition given for a subset of nonnegative integers of natural density.

For any nonempty subsets A, B in G, denote $A \triangle B = (A \setminus B) \cup (B \setminus A)$. It is easy to verify that for any nonempty subsets A, B, C, D in G, $(A \setminus B) \triangle (C \setminus D) \subset (A \triangle C) \cup (B \triangle D)$.

Definition 3.5. Let G be a countable infinite discrete group and $\{F_n\}$ be a sequence of nonempty finite subsets of G. We say that $\{F_n\}$ is a *Følner sequence* if for any $g \in G$, we have

$$\lim_{n\to\infty} \frac{|(gF_n) \triangle F_n|}{|F_n|} = 0.$$

It is obvious that if $\{F_n\}$ is a Følner sequence, then $\lim_{n\to\infty} |F_n| = +\infty$.

A countable infinite discrete group G is called an *amenable group* if there exists some Følner sequence $\{F_n\}$ in G.

Definition 3.6. Let G be a countable infinite discrete amenable group and $\{F_n\}$ be a Følner sequence in G. For a subset A of G, the *upper density* of A with respect to the Følner sequence $\{F_n\}$ is defined by

$$\bar{d}_{\{F_n\}}(A) = \limsup_{n\to\infty} \frac{1}{|F_n|} |F_n \cap A|.$$

It is obvious that $0 \le \bar{d}_{\{F_n\}}(A) \le 1$. For a given Følner sequence $\{F_n\}$, denote

$$\mathcal{F}_{\mathrm{pud}}^{\{F_n\}} = \{A \subset G\colon \bar{d}_{\{F_n\}}(A) > 0\}.$$

The *upper Banach density* of A is defined by

$$d^*(A) = \sup\{\bar{d}_{\{F_n\}}(A)\colon \{F_n\} \text{ is a Følner sequence in } G\}.$$

It is obvious that $0 \le d^*(A) \le 1$. Denote $\mathcal{F}_{\mathrm{pubd}} = \{A \subset G\colon d^*(A) > 0\}$.

In the following we show that if G is an amenable group and $\{F_n\}$ is a Følner sequence in G, then $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ satisfy the properties (P1) and (P2).

LEMMA 3.7. *Let G be an amenable group and $\{F_n\}$ be a Følner sequence in G. Then $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ satisfy* (P1) *and* (P2).

Proof. (1) $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ *satisfies* (P1). Let $A \in \mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$, then

$$\bar{d}_{\{F_n\}}(A) = \limsup_{n\to\infty} \frac{1}{|F_n|}|F_n \cap A| > 0.$$

Then there exists a Følner subsequence $\{F_n'\} \subset \{F_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{|F_n'|}|F_n' \cap A| > 0.$$

Without loss of generality we assume that $|F_n'| > (n+1)(|F_1'| + \cdots + |F_{n-1}'|)$ for any $n \ge 2$. Define $E_1 := F_1'$ and $E_n := F_n' \setminus (F_1' \cup \cdots \cup F_{n-1}')$ for any $n \ge 2$. It is clear that $E_i \cap E_j = \emptyset$ for any distinct $i, j \in \mathbb{N}$.

CLAIM. *$\{E_n\}$ is a Følner sequence and $\bar{d}_{\{E_n\}}(A) = \bar{d}_{\{F_n'\}}(A)$.*

Proof of the Claim. Since

$$\begin{aligned}(gE_n) \triangle E_n &= \big((gF_n') \setminus g(F_1' \cup \cdots \cup F_{n-1}')\big) \triangle \big(F_n' \setminus (F_1' \cup \cdots \cup F_{n-1}')\big)\\ &\subset \big((gF_n') \triangle F_n'\big) \cup \big(g(F_1' \cup \cdots \cup F_{n-1}') \triangle (F_1' \cup \cdots \cup F_{n-1}')\big),\end{aligned}$$

we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{|(gE_n) \triangle E_n|}{|E_n|}\\ &\quad\le \lim_{n\to\infty} \frac{|(gF_n') \triangle F_n'|}{|E_n|} + \lim_{n\to\infty} \frac{|g(F_1' \cup \cdots \cup F_{n-1}') \triangle (F_1' \cup \cdots \cup F_{n-1}')|}{|E_n|}\\ &\quad\le \lim_{n\to\infty} \frac{|(gF_n') \triangle F_n'|}{|F_n'|} + \lim_{n\to\infty} \frac{2|F_1' \cup \cdots \cup F_{n-1}'|}{n(|F_1'| + \cdots + |F_{n-1}'|)} = 0.\end{aligned}$$

So by definition $\{E_n\}$ is a Følner sequence.

It is easy to verify that

$$\begin{aligned}\bar{d}_{\{E_n\}}(A) &= \limsup_{n\to\infty} \frac{|(F_n' \setminus (F_1' \cup \cdots \cup F_{n-1}')) \cap A|}{|F_n' \setminus (F_1' \cup \cdots \cup F_{n-1}')|}\\ &= \limsup_{n\to\infty} \frac{|F_n' \cap A|}{|F_n'|} = \bar{d}_{\{F_n'\}}(A). \quad ■\end{aligned}$$

Similarly, we can verify that for every strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, $\{E_{n_k}\}$ is a Følner sequence and $\bar{d}_{\{E_{n_k}\}}(A) = \bar{d}_{\{F'_{n_k}\}}(A)$.

Let $A_n := E_n \cap A$. Then $A_n \subset A$ and $A_n \cap A_m = \emptyset$ for every $n, m \in \mathbb{N}$ with $n \neq m$. For any strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$,

$$\begin{aligned}\bar{d}_{\{F_n\}}\Big(\bigcup_{k=1}^{\infty} A_{n_k}\Big) \geq \bar{d}_{\{F'_n\}}\Big(\bigcup_{k=1}^{\infty} A_{n_k}\Big) &= \limsup_{n\to\infty} \frac{1}{|F'_n|}\Big|F'_n \cap \bigcup_{k=1}^{\infty} A_{n_k}\Big| \\ &\geq \limsup_{k\to\infty} \frac{1}{|E_{n_k}|}|E_{n_k} \cap A| \\ &= \bar{d}_{\{E_{n_k}\}}(A).\end{aligned}$$

By the Claim, $\bar{d}_{\{E_{n_k}\}}(A) = \bar{d}_{\{F'_{n_k}\}}(A) = \bar{d}_{\{F'_n\}}(A) > 0$. So $\bigcup_{k=1}^{\infty} A_{n_k} \in \mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$. Thus $\{A_n\}$ is the sequence satisfying (P1) for A. By the arbitrariness of A, $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ satisfies (P1).

(2) $\mathcal{F}_{\mathrm{pubd}}$ *satisfies (P1).* Let $A \in \mathcal{F}_{\mathrm{pubd}}$. There exists a Følner sequence $\{F_n\}$ such that $\bar{d}_{\{F_n\}}(A) > 0$. Then the assertion follows from the proof that $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ satisfies (P1).

(3) It is easy to verify that $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ satisfy all the conditions in Proposition 3.3. Then $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ satisfy (P2).

4. Return time sets and product recurrence for G-systems on compact metric spaces. In this section we study recurrent time sets of points with some special recurrent property in a G-system (X, G). Note that in this section, we always assume that X is a compact metric space. Using the abstract properties (P1) and (P2) of Furstenberg families in Section 3 we give combinatorial characterizations of return time sets of $\mathcal{F}$-recurrent points. We also apply those results to the study of product recurrence.

First we introduce G-system and recall some definitions. By a compact (metric) *G-system*, we mean a triple (X, G, Π), where X is a compact (metric) space with a metric d, G is a countable infinite discrete group with an identity e and $\Pi : G \times X \to X$ is a continuous map satisfying $\Pi(e, x) = x$, for all $x \in X$ and $\Pi(h, \Pi(g, x)) = \Pi(hg, x)$, for all $x \in X$, $h, g \in G$. For convenience, we will use the pair (X, G) instead of (X, G, Π) to denote the G-system, and $gx := \Pi(g, x)$ if the map Π is unambiguous. For two systems (X, G) and (Y, G), there is a natural product system $(X \times Y, G)$ as $g(x, y) = (gx, gy)$ for every $g \in G$ and $(x, y) \in X \times Y$. A nonempty closed G-invariant subset $Y \subseteq X$ naturally defines a subsystem (Y, G) of (X, G). A /G-system (X, G) is called *minimal* if it contains no proper subsystem. Each point belonging to some minimal subsystem of (X, G) is called a *minimal point.* By Zorn's Lemma, every G-system has a minimal subsystem.

Let (X, G) be a G-system. For a point $x \in X$ and open subsets $U, V \subset X$, define

$$N(x, U) = \{g \in G \colon gx \in U\}, \qquad N(U, V) = \{g \in G \colon gU \cap V \neq \emptyset\}.$$

The *orbit* of a point $x \in X$ is the set $Gx = \{gx : g \in G\}$, and the *orbit closure* is $\overline{Gx}$. Any point with dense orbit is called *transitive*. It is easy to see that (X, G) is minimal if and only if every point in X is transitive. A G-system (X, G) is called *transitive* if for any nonempty open sets U and V of X, $N(U, V) \neq \emptyset$. A point $x \in X$ is called *recurrent* if for any neighborhood U of x, $N(x, U)$ is infinite, and *almost periodic* (it is also known as uniformly recurrent) if for any neighborhood U of x, $N(x, U)$ is a syndetic set. It is well known that a point x is almost periodic if and only if the system $(\overline{Gx}, G)$ is minimal.

Definition 4.1. Let G be a countable infinite discrete group. For a sequence $\{p_i\}_{i=1}^\infty$ in G, we define the *finite product* of $\{p_i\}_{i=1}^\infty$ by

$$FP(\{p_i\}_{i=1}^\infty) = \Big\{\prod_{i \in \alpha} p_i \colon \alpha \text{ is a nonempty finite subset of } \mathbb{N}\Big\},$$

where $\prod_{i\in\alpha} p_i$ is the product in increasing order of indices. A subset F of G is called an *IP-set* if there exists a sequence $\{p_i\}_{i=1}^\infty$ in G such that $FP(\{p_i\}_{i=1}^\infty)$ is infinite and $FP(\{p_i\}_{i=1}^\infty) \subset F$. Denote by $\mathcal{F}_{\mathrm{ip}}$ the collection of all IP-subsets of G.

Let (X, G) be a G-system, $x \in X$ and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family. We say that x is *$\mathcal{F}$-recurrent* if for every neighborhood U of x, $N(x, U) \in \mathcal{F}$. We also say that a $\mathcal{F}_{\mathrm{ps}}$-recurrent point is *piecewise syndetic recurrent* point. We will further study recurrent time sets of $\mathcal{F}$-recurrent points. First we introduce the Bernoulli shift (Σ_2, G) and symmetrically $\mathcal{F}$-sets which are closely related to the corresponding recurrent time sets.

For a countable infinite discrete group G with identity e, let $\Sigma_2 = \{0, 1\}^G$, endowed with the product topology on $\{0, 1\}^G$, while $\{0, 1\}$ is endowed with the discrete topology. An element of Σ_2 is a function $z : G \to \{0, 1\}$. Enumerate G as $\{g_i\}_{i=0}^\infty$ with $g_0 = e$. A compatible metric d on Σ_2 is defined by

$$d(z_1, z_2) = \begin{cases} 0, & z_1 = z_2, \\ 1/2^k, & k = \min\{i \in \mathbb{N}_0 \colon z_1(g_i) \neq z_2(g_i)\}, \end{cases}$$

for any $z_1, z_2 \in \Sigma_2$. Then (Σ_2, d) is a compact metric space.

For any $K \in \mathcal{P}_{\mathrm{f}}(G)$ and $u \in \{0, 1\}^K$, define a *cylinder* as follows:

$$[u] = \{z \in \Sigma_2 : z(g) = u(g) \text{ for } g \in K\}.$$

Then the collection of all cylinders $\{[u] : u \in \{0, 1\}^K \text{ for some } K \in \mathcal{P}_{\mathrm{f}}(G)\}$ forms a topological basis of the topology of Σ_2. For every $z \in \Sigma_2$ and

$K \in \mathcal{P}_{\mathrm{f}}(G)$, define $z|_K \in \{0,1\}^K$ by $z|_K(g) = z(g)$ for every $g \in K$, then we can consider the cylinder $[z|_K]$. For convenience, we denote $[1] = \{z \in \Sigma_2 \colon z(e) = 1\}$.

For $g \in G$, define $T_g : \Sigma_2 \to \Sigma_2$ by

$$T_g z(t) = z(tg) \quad \text{for any } t \in G.$$

Then $(\Sigma_2, (T_g)_{g\in G})$ is a G-system, which is called the *symbolic dynamical system over* G. We briefly denote $(\Sigma_2, (T_g)_{g\in G})$ by (Σ_2, G).

For a subset $F \subset G$, let $\mathbf{1}_F \in \Sigma_2$ be the characteristic function of F, that is,

$$\mathbf{1}_F(g) = \begin{cases} 1, & g \in F, \\ 0, & \text{otherwise.} \end{cases}$$

In [19] Kennedy et al. introduced the concept of symmetrically syndetic set and showed that the dual family of symmetrically syndetic sets is the family of dense orbit sets, which answered Question 9.6 in [13]. Recall that a subset $A \subset G$ is *symmetrically syndetic* if for every pair of nonempty finite subsets $F_1 \subset A$ and $F_2 \subset G \setminus A$, the set

$$\bigcap_{f_1 \in F_1} f_1^{-1} A \cap \bigcap_{f_2 \in F_2} f_2^{-1}(G \setminus A)$$

is syndetic. In [22] Xu and Ye showed a subset of G is symmetrically syndetic if and only if it is a return time set of an almost periodic point in the Bernoulli shift (Σ_2, G).

Similar to the symmetrically syndetic set, a general symmetrically set can be defined. Given a Furstenberg family $\mathcal{F}$ over G, a subset $A \subset G$ is a *symmetrically* $\mathcal{F}$*-set*, if for any nonempty finite subsets $F_1 \subset A$ and $F_2 \subset G \setminus A$,

$$\bigcap_{f_1 \in F_1} f_1^{-1} A \cap \bigcap_{f_2 \in F_2} f_2^{-1}(G \setminus A) \in \mathcal{F}.$$

We show that the family of sets containing a symmetrically $\mathcal{F}$-set coincides with the collection of the return time sets of $\mathcal{F}$-recurrent points.

PROPOSITION 4.2. *Let G be a countable infinite discrete group with identity e and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family. For a given subset F of G with $e \in F$, the following assertions are equivalent:*

(1) *F contains a symmetrically $\mathcal{F}$-set F' with $e \in F'$;*
(2) *there exists an $\mathcal{F}$-recurrent point $x \in \{0,1\}^G$ with $x \in [1]$ such that $N(x,[1]) \subset F$;*
(3) *there exists a G-system (X,G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x,U) \subset F$.*

Proof. (1)⇒(2). As G is countable, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{P}_{\rm f}(G)$ such that $e \in G_1$, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$. Consider the Bernoulli shift (Σ_2, G). Define

$$\mathbf{1}_{F'}(g) = \begin{cases} 1, & g \in F', \\ 0, & \text{otherwise.} \end{cases}$$

For any $n \in \mathbb{N}$, let

$$I_n = F' \cap G_n, \qquad J_n = G_n \setminus F'.$$

Then for any $n \in \mathbb{N}$, $I_n \sqcup J_n = G_n$, $[\mathbf{1}_{F'}|_{G_n}] = [\mathbf{1}_{F'}|_{I_n}] \cap [\mathbf{1}_{F'}|_{J_n}]$,

$$N(\mathbf{1}_{F'}, [\mathbf{1}_{F'}|_{G_n}]) = \bigcap_{f_1 \in I_n} f_1^{-1} F' \cap \bigcap_{f_2 \in J_n} f_2^{-1}(G \setminus F') \in \mathcal{F}.$$

Obviously $\{[\mathbf{1}_{F'}|_{G_n}] \colon n \in \mathbb{N}_0\}$ is a neighborhood basis of $\mathbf{1}_{F'}$. By the arbitrariness of n, this shows that $\mathbf{1}_{F'}$ is an $\mathcal{F}$-recurrent point in (Σ_2, G). It is clear that $N(\mathbf{1}_{F'}, [1]) = F' \subset F$.

(2)⇒(3). This is clear.

(3)⇒(1). As G is countable, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{P}_{\rm f}(G)$ such that $e \in G_1$, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$. According to (3), there exists a G-system (X, G), an $\mathcal{F}$-recurrent point x and a neighborhood U of x such that $F \supset N(x, U)$. Since G is countable, Gx is countable, we can choose a neighborhood V of x such that $\overline{V} \subset U$ and for any $g \in G$, either $gx \in V$ or $gx \in X \setminus \overline{V}$.

Let $F' := N(x, V)$. Then $e \in F' \subset N(x, U)$. Now it is sufficient to show that F' is a symmetrically $\mathcal{F}$-set. For any $g \in G$, we can choose a neighborhood W_g of x with $W_g \subset V$ such that if $gx \in V$ then $gW_g \subset V$ and if $gx \in X \setminus \overline{V}$ then $gW_g \subset X \setminus \overline{V}$. For any finite set G_n, $\bigcap_{g \in G_n} W_g$ is a neighborhood of x. Denote $W := \bigcap_{g \in G_n} W_g$. Then $N(x, W) \subset F$ and $N(x, W) \in \mathcal{F}$. Let $I_n = G_n \cap F'$, $J_n = G_n \setminus F'$. We have

$$N(x, W) \subset \bigcap_{f_1 \in I_n} f_1^{-1} F' \cap \bigcap_{f_2 \in J_n} f_2^{-1}(G \setminus F') \in \mathcal{F}.$$

Thus F' is a symmetrically $\mathcal{F}$-set. ∎

By the proof of Proposition 4.2, we have the following consequence.

Corollary 4.3. *Let G be a countable infinite discrete group with identity e and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family. For a given subset F of G with $e \in F$, the following assertions are equivalent:*

(1) *F is a symmetrically $\mathcal{F}$-set;*
(2) *there exists an $\mathcal{F}$-recurrent point $x \in \{0,1\}^G$ such that $N(x, [1]) = F$.*

Though Proposition 4.2 connects the recurrent time sets of $\mathcal{F}$-recurrent points with symmetrically $\mathcal{F}$-sets, usually it is not easy to verify whether a set is a symmetrically $\mathcal{F}$-set. Under the conditions (P1) and (P2) introduced in Section 3, we have the following combinatorial characterization of recurrent time sets of $\mathcal{F}$-recurrent points, which is the main result in this section.

THEOREM 4.4. *Let G be a countable infinite discrete group with identity e and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family satisfying* (P1) *and* (P2)*. For a given $F \in \mathcal{F}$ with $e \in F$, the following assertions are equivalent:*

(1) *there exists a G-system (X, G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F$;*
(2) *there exists a decreasing sequence $\{F_n\}$ of subsets of F in $\mathcal{F}$ such that for any $n \in \mathbb{N}$ and $f \in F_n$ there exists $m \in \mathbb{N}$ such that $fF_m \subset F_n$.*

Proof. (1)$\Rightarrow$(2). According to (1), there exists a G-system (X, G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F$. Then there exists $\delta > 0$, such that $B(x, \delta) \subset U$.

For $n \in \mathbb{N}$, define $F_n := N(x, B(x, \delta/n))$. It is clear that $F_{n+1} \subset F_n \subset F$ and $F_n \in \mathcal{F}$ for $n \in \mathbb{N}$. Now fix F_n and $f \in F_n$, then $fx \in B(x, \delta/n)$ and $x \in f^{-1}B(x, \delta/n)$. It is clear that $f^{-1}B(x, \delta/n)$ is a neighborhood of x, thus there exists $m \in \mathbb{N}$ such that $B(x, \delta/m) \subset f^{-1}B(x, \delta/n)$. Then we have $fN(x, B(x, \delta/m)) \subset N(x, B(x, \delta/n))$, i.e. $fF_m \subset F_n$.

(2)$\Rightarrow$(1). As G is countable, fix a sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{P}_{\mathrm{f}}(G)$ such that $G_1 = \{e\}$, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$. Without loss of generality assume that $e \in F_n$ for any $n \in \mathbb{N}$. Let $m_1 = 1$, $F_1' = F_1$ and $B_1 = \{e\}$. Since $\mathcal{F}$ satisfies the condition (P1), for $F_1 \in \mathcal{F}$, there exists a sequence $\{C_n^{(1)}\}_{n=1}^{\infty}$ in $\mathcal{P}_{\mathrm{f}}(G)$ such that

- for every $n \in \mathbb{N}$, $C_n^{(1)} \subset F_1$;
- for all $n, n' \in \mathbb{N}$ with $n \neq n'$, $C_n^{(1)} \cap C_{n'}^{(1)} = \emptyset$;
- for every strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{i=1}^{\infty} C_{n_i}^{(1)} \in \mathcal{F}$.

Let $A_1^{(1)} = C_1^{(1)}$. Consider the symbolic dynamical system (Σ_2, G). First, we define $z^{(1)} \in \Sigma_2$ as follows:

$$z^{(1)}(g) = \begin{cases} 1, & g = e, \\ 1, & g \in A_1^{(1)}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $k \in \mathbb{N}$ and assume that we have chosen $\{z^{(i)}\}_{i=1}^{k}$ in Σ_2, $\{F_{m_i}\}_{i=1}^{k}$ and $\{F'_{m_i}\}_{i=1}^{k}$ in $\mathcal{F}$, $\{B_n\}_{n=1}^{k}$, $\{C_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, \dots, k$ and $\{A_n^{(i)}\}_{n=1}^{k}$, $i = 1, \dots, k$ in $\mathcal{P}_{\mathrm{f}}(G)$, $\{\{t(j,i)\}_{j=1}^{i}\}_{i=2}^{k}$ in $\mathbb{N}$ satisfying the following hypotheses for $i \in \{1, \dots, k\}$:

(1) if $i > 1$, then $B_i = N(z^{(i-1)}, [1]) \cup G_i$;
(2) $N(z^{(i)}, [1]) \in \mathcal{P}_{\mathrm{f}}(F_1)$;

(3) if $i>1$, then $N(z^{(i)},[1])=N(z^{(i-1)},[1])\cup A_i^{(1)}\cup\bigcup_{j=1}^{i-1}(N(z^{(j)},[1])A_i^{(j+1)})$;
(4) $F'_{m_i}\subset F_{m_i}$;
(5) if $i>1$, then $N(z^{(i-1)},[1])F_{m_i}\subset F_1$;
(6) for any distinct $f_1,f_2\in F'_{m_i}$, $B_if_1\cap B_if_2=\emptyset$.
(7) for every $n\in\mathbb{N}$, $C_n^{(i)}\subset F'_{m_i}$;
(8) for any distinct $n,n'\in\mathbb{N}$, $C_n^{(i)}\cap C_{n'}^{(i)}=\emptyset$;
(9) for every strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{t=1}^{\infty}C_{n_t}^{(i)}\in\mathcal{F}$;
(10) $t(1,1)=1$;
(11) if $i\geq 2$ and $1\leq j\leq i-1$, then $t(j,i)>t(j,i-1)$;
(12) if $i\geq 2$, then $t(i,i)>i-1$;
(13) if $i\geq 2$ and $1\leq j\leq i-1$, then $A_j^{(i)}=C_j^{(i)}$;
(14) if $1\leq j\leq i$, then $A_i^{(j)}=C_{t(j,i)}^{(j)}$;
(15) if $i\geq 2$, $C_{t(1,i)}^{(1)}\cap B_i=\emptyset$,
$C_{t(2,i)}^{(2)}\cap(B_2^{-1}B_i\cup B_2^{-1}A_i^{(1)})=\emptyset$,
$\dots,$
$C_{t(i,i)}^{(i)}\cap(B_i^{-1}B_i\cup B_i^{-1}A_i^{(1)}\cup B_i^{-1}B_2A_i^{(2)}\cdots\cup B_i^{-1}B_{i-1}A_i^{(i-1)})=\emptyset$;
(16) if $i\geq 3$, $C_{t(1,i)}^{(1)}\cap(\bigcup_{t=2}^{i-1}(B_tA_t^{(t)}\cup\cdots\cup B_tA_{i-1}^{(t)}))=\emptyset$,
$C_{t(2,i)}^{(2)}\cap(\bigcup_{t=2}^{i-1}(B_2^{-1}B_tA_t^{(t)}\cup\cdots\cup B_2^{-1}B_tA_{i-1}^{(t)}))=\emptyset$,
$\dots,$
$C_{t(i,i)}^{(i)}\cap(\bigcup_{t=2}^{i-1}(B_i^{-1}B_tA_t^{(t)}\cup\cdots\cup B_i^{-1}B_tA_{i-1}^{(t)}))=\emptyset$;
(17) if $i>1$, then $z^{(i)}|_{B_i}=z^{(i-1)}|_{B_i}$;
(18) if $g\in A_i^{(1)}$, then $z^{(i)}(g)=1$;
(19) if $2\leq j\leq i$, $h\in B_j$ and $g\in hA_i^{(j)}$, then $z^{(i)}(g)=z^{(j-1)}(h)$;
(20) if $i>1$ and $g\in G\setminus(B_i\cup A_i^{(1)}\cup\bigcup_{j=2}^{i}B_jA_i^{(j)})$, then $z^{(i)}(g)=0$.

All hypotheses are satisfied for $i=1$, all but (2), (4), (6), (7), (8), (9), (14) and (18) vacuously.

We now show that all hypotheses are satisfied for $i=k+1$. By hypothesis (2), $N(z^{(k)},[1])\in\mathcal{P}_{\mathrm{f}}(F_1)$. For any $f\in N(z^{(k)},[1])$, by (2) there exists $m=m(f)\in\mathbb{N}$ such that $fF_m\subset F_1$. Let $m_{k+1}=\max\{m(f)\colon f\in N(z^{(k)},[1])\}$. Since $\{F_n\}$ is a decreasing sequence, $fF_{m_{k+1}}\subset F_1$ for every $f\in N(z^{(k)},[1])$.

Let $B_{k+1}=N(z^{(k)},[1])\cup G_{k+1}$. By condition (P2), for $F_{m_{k+1}}\in\mathcal{F}$ and $B_{k+1}\in\mathcal{P}_{\mathrm{f}}(G)$, there exists $F'_{m_{k+1}}\subset F_{m_{k+1}}$ with $F'_{m_{k+1}}\in\mathcal{F}$ such that for any distinct $f_1,f_2\in F'_{m_{k+1}}$, $B_{k+1}f_1\cap B_{k+1}f_2=\emptyset$. Since $F'_{m_{k+1}}\in\mathcal{F}$, again by condition (P1), there exists a sequence $\{C_n^{(k+1)}\}_{n=1}^{\infty}$ in $\mathcal{P}_{\mathrm{f}}(G)$ such that

- for every $n \in \mathbb{N}$, $C_n^{(k+1)} \subset F'_{m_{k+1}}$;
- for any distinct $n, n' \in \mathbb{N}$, $C_n^{(k+1)} \cap C_{n'}^{(k+1)} = \emptyset$;
- for every strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{t=1}^{\infty} C_{n_t}^{(k+1)} \in \mathcal{F}$.

Let $A_j^{(k+1)} = C_j^{(k+1)}$ for $1 \le j \le k$. Since $B_{k+1} \in \mathcal{P}_{\mathrm{f}}(G)$ and

$$B_j A_j^{(j)} \cup \cdots \cup B_j A_k^{(j)} \in \mathcal{P}_{\mathrm{f}}(G)$$

for $k \ge 2$, $j = 2, \ldots, k$ and the elements in $\{C_n^{(1)}\}_{n=1}^{\infty}$ are pairwise disjoint, there exists $t(1, k+1) > t(1, k)$ such that $C_{t(1,k+1)}^{(1)} \cap B_{k+1} = \emptyset$ and

$$C_{t(1,k+1)}^{(1)} \cap \Big(\bigcup_{j=2}^{k} (B_j A_j^{(j)} \cup \cdots \cup B_j A_k^{(j)}) \Big) = \emptyset$$

for $k \ge 2$. Let $A_{k+1}^{(1)} = C_{t(1,k+1)}^{(1)}$. Similarly there exists $t(j, k+1) > t(j, k)$ for $2 \le j \le k$ such that

$$C_{t(j,k+1)}^{(j)} \cap (B_j^{-1} B_{k+1} \cup B_j^{-1} B_1 A_{k+1}^{(1)} \cup \cdots \cup B_j^{-1} B_{j-1} A_{k+1}^{(j-1)}) = \emptyset$$

and

$$C_{t(j,k+1)}^{(j)} \cap \Big(\bigcup_{t=2}^{k} (B_j^{-1} B_t A_t^{(t)} \cup \cdots \cup B_j^{-1} B_t A_k^{(t)}) \Big) = \emptyset.$$

Let $A_{k+1}^{(j)} = C_{t(j,k+1)}^{(j)}$ for $2 \le j \le k$. And there exists $t(k+1, k+1) > k$ such that

$$C_{t(k+1,k+1)}^{(k+1)} \cap (B_{k+1}^{-1} B_{k+1} \cup B_{k+1}^{-1} B_1 A_{k+1}^{(1)} \cup \cdots \cup B_{k+1}^{-1} B_k A_{k+1}^{(k)}) = \emptyset$$

and

$$C_{t(k+1,k+1)}^{(k+1)} \cap \Big(\bigcup_{t=2}^{k} (B_{k+1}^{-1} B_t A_t^{(t)} \cup \cdots \cup B_{k+1}^{-1} B_t A_k^{(t)}) \Big) = \emptyset \quad \text{for } k \ge 2.$$

Let $A_{k+1}^{(k+1)} = C_{t(k+1,k+1)}^{(k+1)}$.

We claim that we can define $z^{(k+1)} \in \Sigma_2$ as required by hypotheses (17)–(20) for $i = k+1$. That is,

(17) $z^{(k+1)}|_{B_{k+1}} = z^{(k)}|_{B_{k+1}}$;
(18) if $g \in A_{k+1}^{(1)}$, then $z^{(k+1)}(g) = 1$;
(19) if $2 \le j \le k+1$, $h \in B_j$ and $g \in h A_{k+1}^{(j)}$, then $z^{(k+1)}(g) = z^{(j-1)}(h)$;
(20) if $g \in G \setminus \{B_{k+1} \cup A_{k+1}^{(1)} \cup \bigcup_{j=2}^{k+1} B_j A_{k+1}^{(j)}\}$, then $z^{(k+1)}(g) = 0$.

By the construction of B_{k+1}, $C_{t(1,k+1)}^{(1)}$ and $A_{k+1}^{(1)}$, we have

$$C_{t(1,k+1)}^{(1)} \cap B_{k+1} = \emptyset$$

and $A^{(1)}_{k+1} = C^{(1)}_{t(1,k+1)}$, thus $A^{(1)}_{k+1} \cap B_{k+1} = \emptyset$, which implies that (17) does not conflict with (18).

For $1 \le j \le k+1$, by the construction of B_{k+1}, $C^{(j)}_{t(j,k+1)}$ and $A^{(j)}_{k+1}$, we have $C^{(j)}_{t(j,k+1)} \cap B_j^{-1} B_{k+1} = \emptyset$ and $A^{(j)}_{k+1} = C^{(j)}_{t(j,k+1)}$, thus $B_{k+1} \cap B_j A^{(j)}_{k+1} = \emptyset$ for $2 \le j \le k+1$, which implies that (17) cannot conflict with (19).

For $1 \le j \le k+1$, by the construction of B_{k+1}, $C^{(j)}_{t(j,k+1)}$ and $A^{(j)}_{k+1}$, we have $C^{(j)}_{t(j,k+1)} \cap B_j^{-1} A^{(1)}_{k+1} = \emptyset$ for $2 \le j \le k+1$ and $A^{(j)}_{k+1} = C^{(j)}_{t(j,k+1)}$, thus $A^{(1)}_{k+1} \cap B_j A^{(j)}_{k+1} = \emptyset$ for $2 \le j \le k+1$, which implies that (18) does not conflict with any part of (19).

Finally, we show that no parts of (19) contradict each other. By the construction of B_{k+1}, $C^{(j)}_{t(j,k+1)}$ and $A^{(j)}_{k+1}$, we have

$$C^{(j)}_{t(j,k+1)} \cap (B_j^{-1} B_1 A^{(1)}_{k+1} \cup \cdots \cup B_j^{-1} B_{j-1} A^{(j-1)}_{k+1}) = \emptyset$$

for $2 \le j \le k+1$. Therefore for any $2 \le j \ne j' \le k+1$, $B_j A^{(j)}_{k+1} \cap B_{j'} A^{(j')}_{k+1} = \emptyset$.

Now all hypotheses are satisfied directly for $i = k+1$ except for (2) and (3). By the construction of $z^{(k+1)}$,

$$N(z^{(k+1)}, [1]) = N(z^{(k)}, [1]) \cup A^{(1)}_{k+1} \cup \bigcup_{j=1}^{k} N(z^{(j)}, [1]) A^{(j+1)}_{k+1},$$

which implies that (3) holds for $i = k+1$.

By (2) for $i = k$, $N(z^{(k)}, [1]) \in \mathcal{P}_{\mathrm{f}}(F_1)$. By (4), (7) and (14), $A^{(1)}_{k+1} = C^{(1)}_{t(1,k+1)} \subset F'_{m_1} \subset F_{m_1} = F_1$. Since $\{C^{(1)}_n\}_{n=1}^{\infty}$ is in $\mathcal{P}_{\mathrm{f}}(G)$, $A^{(1)}_{k+1} \in \mathcal{P}_{\mathrm{f}}(F_1)$. By (5), $N(z^{(j)}, [1]) F_{m_{j+1}} \subset F_1$ for $j = 1, \dots, k$. By (4), (7) and (14),

$$A^{(j+1)}_{k+1} = C^{(j+1)}_{t(j+1,k+1)} \subset F'_{m_{j+1}} \subset F_{m_{j+1}}$$

for $j = 1, \dots, k$. Thus, for $j = 1, \dots, k$, $N(z^{(j)}, [1]) A^{(j+1)}_{k+1} \subset F_1$. By (2) for $i = 1, \dots, k$ and since $A^{(j+1)}_{k+1}$, $j = 1, \dots, k$, is in $\mathcal{P}_{\mathrm{f}}(G)$, we have $N(z^{(j)}, [1]) A^{(j+1)}_{k+1} \in \mathcal{P}_{\mathrm{f}}(F_1)$ for $j = 1, \dots, k$. In conclusion,

$$N(z^{(k+1)}, [1]) = N(z^{(k)}, [1]) \cup A^{(1)}_{k+1} \cup \bigcup_{j=1}^{k} N(z^{(j)}, [1]) A^{(j+1)}_{k+1} \in \mathcal{P}_{\mathrm{f}}(F_1),$$

which implies that (2) holds for $i = k+1$.

We now establish some facts.

(i) *If $1 \le r < j$, then for each $h \in B_{r+1}$ and each $g \in A^{(r+1)}_{r+1} \cup A^{(r+1)}_{r+2} \cup \cdots \cup A^{(r+1)}_{j}$, $z^{(j)}(hg) = z^{(r)}(h)$.*

By (19), for each $h \in B_{r+1}$ and each $g \in A_j^{(r+1)}$, $z^{(j)}(hg) = z^{(r)}(h)$. If $j = r+1$, then the proof is finished. Otherwise $j > r+1 \geq 2$ and thus $j \geq 3$. To see that for each $h \in B_{r+1}$ and each $g \in A_{j-1}^{(r+1)}$, $z^{(j)}(hg) = z^{(r)}(h)$, we will first show that for each $h \in B_{r+1}$ and each $g \in A_{j-1}^{(r+1)}$, $z^{(j)}(hg) = z^{(j-1)}(hg)$. By (17), $z^{(j)}|_{B_j} = z^{(j-1)}|_{B_j}$. By (1), $B_j = N(z^{(j-1)}, [1]) \cup G_j$. So $z^{(j-1)}(hg) = 1$ implies $z^{(j)}(hg) = 1$ for $g \in A_{j-1}^{(r+1)}$ and $h \in B_{r+1}$. It is sufficient to show that $z^{(j-1)}(hg) = 0$ implies $z^{(j)}(hg) = 0$ for $g \in A_{j-1}^{(r+1)}$ and $h \in B_{r+1}$. To prove this we note that by (14) and (16), $A_j^{(1)} \cap B_{r+1} A_{j-1}^{(r+1)} = \emptyset$, $B_t A_j^{(t)} \cap B_{r+1} A_{j-1}^{(r+1)} = \emptyset$ for $2 \leq t \leq j$. Now by (19), for each $h \in B_{r+1}$ and each $g \in A_{j-1}^{(r+1)}$, we have $z^{(j)}(hg) = z^{(j-1)}(hg) = z^{(r)}(h)$. If $j-1 = r+1$ then the proof is finished. Otherwise $j-1 > r+1 \geq 2$ and thus $j \geq 4$, and again we can see that for each $h \in B_{r+1}$ and each $g \in A_{j-2}^{(r+1)}$, $z^{(j)}(hg) = z^{(r)}(h)$. By induction the proof is finished.

Since $\{z^{(i)}\}_{i=1}^{\infty}$ is a sequence in compact space Σ_2, we may pick a cluster point $z \in \Sigma_2$ of the sequence $\{z^{(i)}\}_{i=1}^{\infty}$.

(ii) *For each $j \in \mathbb{N}$, $z|_{B_{j+1}} = z^{(j)}|_{B_{j+1}}$.*

To establish (ii), let $j \in \mathbb{N}$ and let $g \in B_{j+1}$. Since z is a cluster point of the sequence $\{z^{(i)}\}_{i=1}^{\infty}$ and $[z|_{B_{j+1}}]$ is a neighborhood of z, we can pick $i > j$ such that $z^{(i)} \in [z|_{B_{j+1}}]$. Then $z^{(i)}|_{B_{j+1}} = z|_{B_{j+1}}$. By construction we have $B_n \subset B_{n+1}$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} B_n \supset \bigcup_{n=1}^{\infty} G_n = G$. So by (17), $z^{(j)}|_{B_{j+1}} = z^{(i)}|_{B_{j+1}} = z|_{B_{j+1}}$.

As a consequence of (ii), for each $r \in \mathbb{N}$, $[z^{(r)}|_{B_{r+1}}]$ is a neighborhood of z so $\{[z^{(r)}|_{B_{r+1}}] : r \in \mathbb{N}\}$ is a neighborhood basis for z.

(iii) *If $1 \leq r < i$, then $A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)} \subset N(z, [z^{(r)}|_{B_{r+1}}])$.*

To establish (iii), for any $g \in A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)}$ and for any $h \in B_{r+1}$, if $z^{(i)}(hg) = 1$, then $hg \in N(z^{(i)}, [1]) \subset B_{i+1}$. By (i),

$$A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)} \subset N(z^{(i)}, [z^{(r)}|_{B_{r+1}}]),$$

then $z^{(i)}(hg) = z^{(r)}(h)$. By (ii), $z|_{B_{i+1}} = z^{(i)}|_{B_{i+1}}$, thus we have $z(hg) = z^{(i)}(hg) = z^{(r)}(h)$.

If $z^{(i)}(hg) = 0$ and $hg \in B_{i+1}$, then we still have $z(hg) = z^{(i)}(hg) = z^{(r)}(h)$. If $z^{(i)}(hg) = 0$ and $hg \notin B_{i+1}$, then since $B_n \subset B_{n+1}$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} B_n \supset \bigcup_{n=1}^{\infty} G_n = G$, we have $hg \in B_t$ for some $t > i+1$. Noting that $1 \leq r < i < i+1 < t$, by (i),

$$N(z^{(t)}, [z^{(r)}|_{B_{r+1}}]) \supset A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_t^{(r+1)} \supset A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)}.$$

By (ii), $z|_{B_t} = z^{(t)}|_{B_t}$, so we have $z(hg) = z^{(t)}(hg) = z^{(r)}(h)$ for $g \in A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)}$ and $h \in B_{r+1}$.

In conclusion, for any $g \in A_{r+1}^{(r+1)} \cup A_{r+2}^{(r+1)} \cup \cdots \cup A_i^{(r+1)}$ and for any $h \in B_{r+1}$, we have $T_g(z)(h) = z(hg) = z^{(i)}(hg) = z^{(r)}(h)$, which implies that $g \in N(z, [z^{(r)}|_{B_{r+1}}])$.

Now we claim that z is an $\mathcal{F}$-recurrent point of Σ_2. To see this, let R be a neighborhood of z and pick $r \in \mathbb{N}$ such that $[z^{(r)}|_{B_{r+1}}] \subset R$. Thus we have

$$N\big(z, R\big) \supset N(z, [z^{(r)}|_{B_{r+1}}]) \supset \bigcup_{i=r+1}^{\infty} \bigcup_{j=r+1}^{i} A_j^{(r+1)} = \bigcup_{i=r+1}^{\infty} A_i^{(r+1)},$$

where the second inclusion holds by (iii). By the construction of $\{A_n^{(r+1)}\}_{n=1}^{\infty}$,

$$\bigcup_{i=r+1}^{\infty} A_i^{(r+1)} \in \mathcal{F}.$$

So z is an $\mathcal{F}$-recurrent point of Σ_2.

By (ii), $[1] = \{z \in \Sigma_2 \colon z(e) = 1\}$ is a neighborhood of z. We conclude the proof by showing that $N(z, [1]) \subset F$. Note that $N(z, [1]) = \{g \in G : T_g z \in [1]\} = \{g \in G : z(g) = 1\}$. By the construction $B_n \subset B_{n+1}$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} B_n \supset \bigcup_{n=1}^{\infty} G_n = G$. Thus for any $g \in N(z, [1])$, there exists $r \in \mathbb{N}$ such that $g \in B_{r+1}$; then by (ii), $z(g) = z^{(r)}(g) = 1$, which implies that $g \in N(z^r, [1])$. So $N(z, [1]) \subset \bigcup_{r=1}^{\infty} N(z^{(r)}, [1])$. By (1), for each $r \in \mathbb{N}$, $N(z^{(r)}, [1]) \subset F_1$ so $N(z, [1]) \subset F_1 \subset F$. ∎

REMARK 4.5. In Section 3, we showed that $\mathcal{F}_{\mathrm{ps}}$ and $\mathcal{F}_{\mathrm{inf}}$ satisfy the properties (P1) and (P2). If G is amenable and $\{F_n\}$ is a Følner sequence in G, $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ also satisfy the properties (P1) and (P2). So we can apply Theorem 4.4 to Furstenberg families $\mathcal{F}_{\mathrm{ps}}$, $\mathcal{F}_{\mathrm{inf}}$, $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$.

DEFINITION 4.6. Let (X, G) be a G-system. A pair $(x_1, x_2) \in X \times X$ is said to be *proximal* if $\inf_{g \in G} d(gx_1, gx_2) = 0$, and *distal* if it is not proximal. A point $x \in X$ is called *distal* if for any $y \in \overline{Gx}$ with $y \neq x$, (x, y) is distal.

DEFINITION 4.7. If for any G-system (Y, G) and any recurrent point $y \in Y$, (x, y) is recurrent in the product system $(X \times Y, G)$, then we say that x is *product recurrent*.

DEFINITION 4.8. Let G be a countable infinite discrete group. A subset $F \subset G$ is called *central* if there exists a G-system (X, G), a point $x \in X$, an almost periodic point $y \in X$ and a neighborhood U of y such that (x, y) is proximal and $N(x, U) \subset F$. Denote by $\mathcal{F}_{\mathrm{cen}}$ the collection of all central subsets of G.

A subset $A \subset G$ is called *IP*-set* (resp. *central*-set*) if for any IP-subset (reps. central subset) F of G, $A \cap F \neq \emptyset$. Denote by $\mathcal{F}_{\mathrm{ip}}^*$ and $\mathcal{F}_{\mathrm{cen}}^*$ the collection of all IP*-subsets and central*-subsets of G. It is not hard to see that $\mathcal{F}_{\mathrm{t}} \subset \mathcal{F}_{\mathrm{cen}} \subset \mathcal{F}_{\mathrm{ip}}$ and $\mathcal{F}_{\mathrm{ip}}^* \subset \mathcal{F}_{\mathrm{cen}}^* \subset \mathcal{F}_{\mathrm{s}}$; see e.g. [17].

The following characterizations of distal points were proved by Furstenberg in [11] for topological dynamical systems and [9] for G-systems (see [9, Corollaries 5.30 and 5.36]).

THEOREM 4.9. *Let (X, G) be a G-system and $x \in X$. Then the following assertions are equivalent:*

(1) *x is a distal point;*
(2) *x is an $\mathcal{F}_{\mathrm{ip}}^*$-recurrent point;*
(3) *x is an $\mathcal{F}_{\mathrm{cen}}^*$-recurrent point;*
(4) *x is a product recurrent point.*

The notion of weak product recurrence was first introduced by Haddad and Ott [14] for topological dynamical systems. Let (X, G) be a G-system and $x \in X$. If for any G-system (Y, G) and any almost periodic point $y \in Y$, (x, y) is recurrent in the product system $(X \times Y, G)$, then we say that x is *weak product recurrent.*

In [2] Auslander and Furstenberg asked whether weak product recurrent point is product recurrent. It is answered by Haddad and Ott [14] negatively for topological dynamical systems. In [8], Dong, Shao and Ye related product recurrence with disjointness, which was introduced by Furstenberg in his seminal paper [10], and proved that if a nontrivial transitive system is disjoint from any minimal system, then every transitive point is weak product recurrent but not minimal. Here we generalize this result to G-systems.

DEFINITION 4.10. Let (X, G) and (Y, G) be two G-systems. We say that a nonempty closed subset $J \subset X \times Y$ is a *joining* of (X, G) and (Y, G) if it is G-invariant and its projections onto the first and second coordinates are X and Y respectively.

If every joining is equal to $X \times Y$, then we say that (X, G) and (Y, G) are *disjoint.*

In [13], Glasner et al. showed that for any infinite discrete group G, the Bernoulli shift is disjoint from any minimal system. Recently, Xu and Ye [22] gave a necessary and sufficient condition for a transitive system (X, G) to be disjoint from any minimal system when G is a countable discrete group. In the following we show that any transitive point in such a nontrivial transitive system is weak product recurrent but not product recurrent, which shows that Question 1.5 is also negative for G-systems.

In [8, Theorem 4.3] the authors proved the following result for a topological dynamical system (X,T); we generalize the result to G-systems.

THEOREM 4.11. *Let (X,G) be a nontrivial transitive system. If (X,G) is disjoint from any minimal system, then every transitive point $x \in X$ is weak product recurrent but not product recurrent.*

Proof. Let x be a transitive point in (X,G). First we show that x is weak product recurrent. Given any almost periodic point y in a G-system (Y,G), we need to show that (x,y) is recurrent. Since x is transitive, $\overline{G(x,y)}$ is a joining of X and $\overline{Gy}$. Since (X,G) is disjoint from any minimal system, in particular (X,G) and $(\overline{Gy},G)$ are disjoint, thus $\overline{G(x,y)} = X \times \overline{Gy}$. Then for any neighborhood $U \times V$ of (x,y) in $X \times Y$, $G(x,y) \cap (U \times (V \cap \overline{Gy}))$ is an infinite set, i.e. (x,y) is recurrent.

Now we show that x is not product recurrent. Since $\mathcal{F}_{\rm cen}^* \subset \mathcal{F}_{\rm s}$, by Theorem 4.9, it is sufficient to show that x is not almost periodic. Assume on the contrary that x is an almost periodic point. Then (X,G) is a minimal system. By assumption, (X,G) is disjoint from itself. It is clear that $\{(z,z) : z \in X\}$ is a joining of (X,G) and (X,G). Since (X,G) is nontrivial, $\{(z,z) : z \in X\} \neq X \times X$. This is a contradiction. ∎

In [21], Oprocha and Zhang showed that the intersection of a dynamical syndetic set and a thick set contains a recurrent time set of a piecewise syndetic recurrent point for topological dynamical systems. In fact, a subset of $\mathbb{N}_0$ is the intersection of a dynamical syndetic set and a thick set if and only if it is central; see e.g. [18, Theorem 3.7]. Using Theorem 4.4, we generalize Oprocha and Zhang's result to G-systems.

LEMMA 4.12. *Let G be a countable infinite discrete group with identity e and $F \subset G$. If F is a central set with $e \in F$, then there exists a G-system (X,G), an $\mathcal{F}_{\rm ps}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x,U) \subset F$.*

Proof. It is sufficient to show that F satisfies Theorem 4.4(2) for the case $\mathcal{F} = \mathcal{F}_{\rm ps}$. That is, there exists a decreasing sequence $\{F_n\}$ of subsets of F in $\mathcal{F}_{\rm ps}$ such that for any $n \in \mathbb{N}$ and $f \in F_n$ there exists $m \in \mathbb{N}$ such that $fF_m \subset F_n$.

Since F is a central set, by the definition, there exists a G-system (X,G), a point $x \in X$, an almost periodic point $y \in X$ and a neighborhood U of y such that (x,y) is proximal and $N(x,U) \subset F$. Since U is a neighborhood of y, there exists $\epsilon > 0$ such that $B(y,\epsilon) \subset U$.

For $n \in \mathbb{N}$, define $F_n := N((x,y), B(y,\epsilon/n) \times B(y,\epsilon/n))$. It is clear that $F_n \subset F$ and $F_{n+1} \subset F_n$ for $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We will show that $F_n \in \mathcal{F}_{\rm ps}$.

Let $A := N(y, B(y, \epsilon/(2n)))$ and $B := \{g \in G : d(gx, gy) < \epsilon/(2n)\}$. Since y is an almost periodic point, A is a syndetic set. Since (x, y) is proximal, B is a thick set. For any $g \in A \cap B$, $d(gx, y) \leq d(gx, gy) + d(gy, y) < \epsilon/n$, then $gx \in B(y, \epsilon/n)$. Thus $A \cap B \subset N((x, y), B(y, \epsilon/n) \times B(y, \epsilon/n)) = F_n$ and $F_n \in \mathcal{F}_{\mathrm{ps}}$.

Now fix F_n and $f \in F_n$. Note that $f(x, y) \in B(y, \epsilon/n) \times B(y, \epsilon/n)$ and $y \in f^{-1}B(y, \epsilon/n)$. It is clear that $f^{-1}B(y, \epsilon/n)$ is a neighborhood of y, thus there exists $m \in \mathbb{N}$ such that $B(y, \epsilon/m) \subset f^{-1}B(y, \epsilon/n)$. Then

$$fN((x, y), B(y, \epsilon/m) \times B(y, \epsilon/m)) \subset N((x, y), B(y, \epsilon/n) \times B(y, \epsilon/n)),$$

i.e. $fF_m \subset F_n$. ■

In [8], Dong, Shao and Ye further studied product recurrent properties via Furstenberg families. Let $\mathcal{F}$ be a Furstenberg family and (X, G) be a G-system. We say that a point $x \in X$ is *$\mathcal{F}$-product recurrent* if for any given $\mathcal{F}$-recurrent point y in any G-system (Y, G), (x, y) is recurrent in the product system $(X \times Y, G)$. Dong, Shao and Ye [8] asked a question that if x is $\mathcal{F}_{\mathrm{ps}}$-product recurrent, is x necessarily a distal point? In [21] Oprocha and Zhang gave an affirmative answer to this question for topological dynamical systems. In the following result we will answer this question for G-systems.

THEOREM 4.13. *Let (X, G) be a G-system and $x \in X$. Then the following assertions are equivalent:*

(1) *x is distal;*
(2) *x is $\mathcal{F}_{\mathrm{ps}}$-product recurrent;*
(3) *for every $\mathcal{F}_{\mathrm{ps}}$-recurrent point y in the Bernoulli shift (Σ_2, G), (x, y) is recurrent in the product system $(X \times \Sigma_2, G)$.*

Proof. (1)⇒(2). This follows from Theorem 4.9.

(2)⇒(3). This is clear.

(3)⇒(1). By Theorem 4.9 it is sufficient to show that x is an $\mathcal{F}_{\mathrm{cen}}^*$-recurrent point. For any neighborhood U of x and any central subset A of G, by Lemma 4.12 there exists a G-system (Y, G), an $\mathcal{F}_{\mathrm{ps}}$-recurrent point $y \in Y$ and a neighborhood V of y such that $N(y, V) \subset A \cup \{e\}$. Then by Proposition 4.2, there exists an $\mathcal{F}_{\mathrm{ps}}$-recurrent point $z \in \Sigma_2$ with $z \in [1]$ such that $N(z, [1]) \subset A \cup \{e\}$. By (3), (x, z) is recurrent. Thus

$$N(x, U) \cap N(z, [1]) = N((x, z), U \times [1])$$

is an infinite set. Then we have $N(x, U) \cap A \neq \emptyset$, which implies that $N(x, U) \in \mathcal{F}_{\mathrm{cen}}^*$. ■

5. Return time sets for G-systems on compact Hausdorff spaces. In this section, by virtue of the algebraic properties of the Stone–Čech compactification βG of G, we investigate return time sets for general G-systems on compact Hausdorff spaces.

First, we briefly introduce the concept of a compact right topological semigroup and its basic properties. By a *compact right topological semigroup*, we mean a triple $(E,\cdot,\mathcal{T})$, where $(E,\cdot)$ is a semigroup and $(E,\mathcal{T})$ is a compact Hausdorff space, and for every $p\in E$, the right translation $\rho_p\colon S\to S$, $q\mapsto q\cdot p$, is continuous. If there is no ambiguity, we will say that E, instead of the triple $(E,\cdot,\mathcal{T})$, is a compact right topological semigroup. A nonempty subset L of E is called a *left ideal* of E if $E\cdot L\subset L$, and a *right ideal* of E if $L\cdot E\subset L$. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. A subset I of E is called an *ideal* of E if I is both a left ideal and a right ideal of E. It is well known that E has a smallest ideal, denoted by $K(E)$, which is the union of all minimal left ideals of E; see e.g. [17, Theorem 2.8]. An element $p\in E$ is called *idempotent* if $p\cdot p=p$. An idempotent $p\in E$ is called a *minimal idempotent* if there exists a minimal left ideal L of E such that $p\in L$. The following celebrated Ellis–Numakura theorem states that every compact right topological semigroup must contain an idempotent; see e.g. [17, Theorem 2.5].

THEOREM 5.1. *Let E be a compact right topological semigroup. Then there exists $p\in E$ such that $p\cdot p=p$.*

Now we recall the definition and algebraic structure of Stone–Čech compactification of a countable infinite discrete group. For further details on this topic, we refer the reader to [17]. Let G be a countable infinite discrete group and βG the collection of ultrafilters on G. By [17, Theorem 3.6], we know that each ultrafilter has the Ramsey property. Given $A\subset G$, let $\widehat{A}:=\{p\in\beta G: A\in p\}$. If $g\in G$, then $\mathfrak{e}(g):=\{A\in\mathcal{P}(G): g\in A\}$ is easily seen to be an ultrafilter on G, which is called the *principal ultrafilter* defined by g. Once we have identified $g\in G$ with $\mathfrak{e}(g)\in\beta G$, we shall suppose that $G\subset\beta G$. In fact, the set $\{\widehat{A}: A\subset G\}$ forms a basis of a topology $\mathcal{T}$ on βG (see [17, Section 3.2]). Then $(\beta G,\mathcal{T})$ is the Stone–Čech compactification of G (see [17, Section 3.3]), that is, for any compact Hausdorff space Y and any function $\varphi\colon G\to Y$ there exists a continuous function $\widetilde{\varphi}\colon\beta G\to Y$ such that $\widetilde{\varphi}|_G=\varphi$. The operation $\cdot$ on G can be uniquely extended to an operation $\cdot$ on βG such that for any $p,q\in\beta G$,

$$p\cdot q=\{A\subset G:\{x\in G: x^{-1}A\in q\}\in p\}.$$

Then $(\beta G,\cdot,\mathcal{T})$ is a compact Hausdorff right topological semigroup.

Recall that we introduced the definition of central set in Section 4. In [4] Bergelson and Hindman obtained the following characterization of central sets via the algebra properties of βG.

THEOREM 5.2. *Let G be a countable infinite discrete group. A subset F of G is central if and only if there exists a minimal idempotent $p\in\beta G$ such that $F\in p$.*

The extension of the operation $\cdot$ on G can be expressed by p-limits. We refer to [17, Section 3.5] for more about p-limits.

Definition 5.3. Let $p \in \beta G$, $\{x_g\}_{g\in G}$ be an indexed family in a compact Hausdorff space X, and $y \in X$. If for every neighborhood U of y, $\{g \in G : x_g \in U\}$ is in p, then we say that the *p-limit* of $\{x_g\}_{g\in G}$ is y and denote it by $p\text{-}\lim_{g\in G} x_g = y$. As X is a compact Hausdorff space, $p\text{-}\lim_{g\in G} x_g$ exists and is unique.

If we view $\{g\}_{g\in G}$ as an indexed family in βG, then $p\text{-}\lim_{g\in G} g = p$.

For a Furstenberg family $\mathcal{F} \subset \mathcal{P}(G)$, the *hull* of $\mathcal{F}$ is defined by

$$h(\mathcal{F}) = \{p \in \beta G : p \subset \mathcal{F}\}.$$

If $\mathcal{F}$ has the Ramsey property, then $h(\mathcal{F})$ is a nonempty closed subset of βG. For further details on this notion, we refer to [12], which in fact establishes a one-to-one correspondence between the set of Furstenberg families with the Ramsey property and the set of nonempty closed subsets of βG.

A Furstenberg family $\mathcal{F} \subset \mathcal{P}(G)$ is called *left shift-invariant* if for any $A \in \mathcal{F}$ and $g \in G$, we have $gA \in \mathcal{F}$. We have the following equivalent condition for $h(\mathcal{F})$ to be a nonempty closed left ideal; see [20, Lemma 3.4] for the case $\mathbb{N}$ and [6, Theorem 5.1.2] for a general discrete group.

Lemma 5.4. *Let G be a countable infinite discrete group and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family with the Ramsey property. Then $h(\mathcal{F})$ is a nonempty closed left ideal of βG if and only if $\mathcal{F}$ is left shift-invariant.*

The following lemma is folklore; see e.g. [20, Theorem 4.4] or [6, Lemma 5.2.2].

Lemma 5.5. *Let G be a countable infinite discrete group and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family with the Ramsey property. If $h(\mathcal{F})$ is a nonempty closed subsemigroup of βG, then for any G-system (X, G) on a compact Hausdorff space X, a point $x \in X$ is $\mathcal{F}$-recurrent if and only if there exists an idempotent $p \in h(\mathcal{F})$ such that $p\text{-}\lim_{g\in G} gx = x$.*

We say a subset F of G is an *essential $\mathcal{F}$-set* if there exists an idempotent $p \in h(\mathcal{F})$ such that $F \in p$. We present the following combinatorial characterization of essential $\mathcal{F}$-sets, which was proved in [20, Proposition 4.13] for the case of $\mathbb{N}$; however, it is routine to verify the proof extends to a general countably infinite discrete group G.

Proposition 5.6. *Let G be a countable infinite discrete group and $\mathcal{F}$ be a Furstenberg family with the Ramsey property. If $h(\mathcal{F})$ is a nonempty closed subsemigroup of βG, then a subset F of G is an essential $\mathcal{F}$-set if and only if there exists a decreasing sequence $\{F_n\}$ of subsets of F in $\mathcal{F}$ such that for any $n \in \mathbb{N}$ and $f \in F_n$ there exists $m \in \mathbb{N}$ such that $fF_m \subset F_n$.*

Now we have the following main result of this section, which characterizes the recurrent time sets of $\mathcal{F}$-recurrent points in a G-system on a compact Hausdorff space.

THEOREM 5.7. *Let G be a countable infinite discrete group with identity e and $\mathcal{F} \subset \mathcal{P}(G)$ be a Furstenberg family with the Ramsey property. If $\mathcal{F}$ satisfies* (P1) *and* (P2) *and $h(\mathcal{F})$ is a nonempty closed subsemigroup of βG, then*

(1) *for any G-system (X, G) on a compact Hausdorff space X, if a point $x \in X$ is $\mathcal{F}$-recurrent, then for every neighborhood U of x, $N(x, U)$ is an essential $\mathcal{F}$-set;*
(2) *for any essential $\mathcal{F}$-subset F of G, there exists a G-system (X, G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F \cup \{e\}$.*

Proof. (1) Let (X, G) be a G-system and $x \in X$ be an $\mathcal{F}$-recurrent point. As $h(\mathcal{F})$ is a nonempty closed subsemigroup of βG, by Lemma 5.5 there exists an idempotent $p \in h(\mathcal{F})$ such that $p\text{-}\lim_{g\in G} gx = x$. For every neighborhood U of x, $N(x, U) = \{g \in G\colon gx \in U\} \in p$. So $N(x, U)$ is an essential $\mathcal{F}$-set.

(2) Let $F \subset G$ be an essential $\mathcal{F}$-set. As $h(\mathcal{F})$ is a nonempty closed subsemigroup of βG, by Proposition 5.6 there exists a decreasing sequence $\{F_n\}$ of subsets of F in $\mathcal{F}$ such that for any $n \in \mathbb{N}$ and $f \in F_n$ there exists $m \in \mathbb{N}$ such that $fF_m \subset F_n$. As $\mathcal{F}$ satisfies (P1) and (P2), by Theorem 4.4 there exists a G-system (X, G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F \cup \{e\}$. ■

The following examples show that some Furstenberg families introduced in Section 3 satisfy the conditions of Theorem 5.7.

EXAMPLE 5.8. Recall that $\mathcal{F}_{\text{inf}}$ is the collection of all infinite subsets of G. It is easy to verify that $\mathcal{F}_{\text{inf}}$ satisfies the properties (P1) and (P2) and has the Ramsey property. Note that $h(\mathcal{F}_{\text{inf}}) = \beta G \setminus G$. Then $h(\mathcal{F}_{\text{inf}})$ is a closed ideal of βG. Therefore, all the conditions of Theorem 5.7 are satisfied for $\mathcal{F}_{\text{inf}}$. By [17, Theorem 5.12] a subset F of G is an essential $\mathcal{F}_{\text{inf}}$-set if and only if it is an IP-set. Notice that the IP-set defined in this paper must be an infinite subset of G. So Theorem 5.7 for the Furstenberg family $\mathcal{F}_{\text{inf}}$ characterizes the recurrent time sets of recurrent points via IP-sets.

EXAMPLE 5.9. Recall that $\mathcal{F}_{\text{ps}}$ is the collection of all piecewise syndetic subsets of G. Then $\mathcal{F}_{\text{ps}}$ has the Ramsey property and by Lemma 3.4 $\mathcal{F}_{\text{ps}}$ satisfies (P1) and (P2). We know that $h(\mathcal{F}_{\text{ps}}) = \text{cl}_{\beta G} K(\beta G)$ (see e.g. [17, Corollary 4.41]) and $\text{cl}_{\beta G} K(\beta G)$ is a closed ideal of βG; see e.g. [17, Theoerem 4.44]. Therefore, all conditions of Theorem 5.7 are satisfied for $\mathcal{F}_{\text{ps}}$. Following [16], we say that a subset A of G is *quasi-central* if there exists an idempotent $p \in \mathrm{cl}_{\beta G} K(\beta G)$ such that $A \in p$. So Theorem 5.7 for the Furstenberg family $\mathcal{F}_{\mathrm{ps}}$ characterizes the recurrent time sets of $\mathcal{F}_{\mathrm{ps}}$-recurrent points via quasi-central sets, which is similar to Theorem 1.2 in the introduction.

EXAMPLE 5.10. Let G be a countable infinite discrete amenable group and $\{F_n\}$ be a Følner sequence in G. Recall that $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ are the collection of all subset of G with positive upper density with respect to $\{F_n\}$, and the collection of all subsets of G with positive upper Banach density, respectively. By Lemma 3.7, $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$ satisfy (P1) and (P2). By Lemma 5.4, $h(\mathcal{F}_{\mathrm{pud}}^{\{F_n\}})$ and $h(\mathcal{F}_{\mathrm{pubd}})$ are closed left ideals of βG. Therefore, all conditions of Theorem 5.7 are satisfied for $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ and $\mathcal{F}_{\mathrm{pubd}}$.

Following [3], we say that a subset A of G is a *D-set* if there exists an idempotent $p \in h(\mathcal{F}_{\mathrm{pubd}})$ such that $A \in p$. So Theorem 5.7 for the Furstenberg family $\mathcal{F}_{\mathrm{pubd}}$ characterizes the recurrent time sets of $\mathcal{F}_{\mathrm{pubd}}$-recurrent points via D-sets.

6. βG-actions and product recurrence. In [2] Auslander and Furstenberg initiated the study of the action of a compact right topological semigroup on a compact Hausdorff space. In this section, we will focus on the βG-action and give a sufficient condition for the closed semigroups S of βG for which an S-product recurrent point is a distal point.

DEFINITION 6.1. Let G be a countable infinite discrete group and βG be the Stone–Čech compactification of G. By an *action* of βG on a compact Hausdorff space X, we mean a map $\Phi\colon \beta G \times X \to X$, $(p, x) \mapsto px$, such that $p(qx) = (pq)x$, for all $p, q \in \beta G$ and $x \in X$, and such that for each $x \in X$ the map $\Phi_x : \beta G \to X$, $p \mapsto px$, is continuous. For convenience, we denote such an action of βG on X by $(X, \beta G)$. Notice that it is not assumed that for each $p \in \beta G$, the map $X \to X$, $x \mapsto px$, is continuous.

For two actions $(X, \beta G)$ and $(Y, \beta G)$, define a map $\Psi\colon \beta G \times (X \times Y) \to X \times Y$, $(p, (x, y)) \mapsto (px, py)$. Then it is an action on $X \times Y$, which we denote by $(X \times Y, \beta G)$.

REMARK 6.2. Let $(X, \beta G)$ be a βG-action. By definition, for each $x \in X$, $\Phi_x : p \mapsto px$ is a continuous map from βG to X. For every neighborhood V of px, there exists some $A \in p$ such that $\Phi_x(\widehat{A}) \subset V$. Since $p\text{-}\lim_{g \in G} g = p$, we have $\{g \in G : g \in \widehat{A}\} \in p$. Note that

$$\{g \in G : g \in \widehat{A}\} \subset \{g \in G : gx \in V\},$$

so we have $\{g \in G : gx \in V\} \in p$. By the uniqueness of p-limit,

$$p\text{-}\lim_{g \in G} gx = px.$$

REMARK 6.3. When (X, G) is a G-system with X being a compact Hausdorff space, there is a naturally induced action of βG on X. For every $g \in G$, we view g as a continuous map from X to X. Define $\theta : G \to X^X$ by $\theta(g) = g$. As βG is the Stone–Čech compactification of G, θ has a continuous extension $\widetilde{\theta}\colon \beta G \to X^X$. By the map $\widetilde{\theta}$, βG acts on X.

Now we recall some basic dynamical concepts in the context of βG-actions.

DEFINITION 6.4. Let $(X, \beta G)$ be a βG-action. We say that a pair (x, y) of points in X is *proximal* if there exists some $p \in \beta G$ such that $px = py$. If (x, y) is not proximal, then (x, y) is said to be *distal*. A point $x \in X$ is called *distal* if for any $y \in \beta G x$ with $y \neq x$, (x, y) is distal.

DEFINITION 6.5. Let $(X, \beta G)$ be a βG-action. We say that a point $x \in X$ is *recurrent* if there exists some $p \in \beta G \setminus G$ such that $px = x$, and *almost periodic* if there exists some minimal idempotent p in βG such that $px = x$.

REMARK 6.6. It should be noticed that the notation $(X, \beta G)$ denotes the action of βG on X as in Definition 6.1. In general, $(X, \beta G)$ is not a dynamical system since it is not assumed that the map $\Phi\colon \beta G \times X \to X$ is continuous in Definition 6.1. Here we define the notions "proximal", "distal", "recurrent" and "almost periodic" for $(X, \beta G)$. It is not hard to see that if the βG-action is induced by a G-system (see Remark 6.3) then the notions of "proximal", "distal", "recurrent" and "almost periodic" introduced here agree with the corresponding notions for G-systems.

Let S be a nonempty closed subsemigroup of $\beta G \setminus G$. A point $x \in X$ is said to be *S-recurrent* if there exists some $p \in S$ such that $px = x$.

It is easy to see that a point x is recurrent in $(X, \beta G)$ if and only if there exists an idempotent $p \in \beta G \setminus G$ such that $px = x$, and a point is almost periodic in $(X, \beta G)$ if and only if it is L-recurrent for some minimal left ideal L of βG. If $x \in X$ and u is a minimal idempotent in βG, then (x, ux) is proximal in $(X, \beta G)$ as $u(ux) = ux$. It follows that a distal point of $(X, \beta G)$ is almost periodic in $(X, \beta G)$.

In [2] Auslander and Furstenberg generalized the characterization of distal points to general compact right topological semigroup actions.

THEOREM 6.7 ([2, Theorem 1]). *Let $(X, \beta G)$ be a βG-action and $x \in X$. Then the following are equivalent:*

(1) *x is a distal point;*
(2) *for any almost periodic point $y \in X$, (x, y) is almost periodic in $(X \times X, \beta G)$;*

(3) *for any βG-action $(Y, \beta G)$ and any almost periodic point $y \in Y$, (x, y) is an almost periodic point in $(X \times Y, \beta G)$;*
(4) *for any idempotent $p \in \beta G$, $px = x$;*
(5) *for any minimal idempotent $p \in \beta G$, $px = x$;*
(6) *there is a minimal left ideal L in βG such that for any idempotent p in L, $px = x$.*

DEFINITION 6.8. Let $(X, \beta G)$ be a βG-action and S be a nonempty closed subsemigroup of $\beta G \setminus G$. A point $x \in X$ is said to be *S-product recurrent* if for any βG-action $(Y, \beta G)$ and any S-recurrent point $y \in Y$, (x, y) is an S-recurrent point in $(X \times Y, \beta G)$, and *weakly S-product recurrent* if for any βG-action $(Y, \beta G)$ and any S-recurrent point $y \in Y$, (x, y) is a recurrent point in $(X \times Y, \beta G)$.

By Theorem 6.7, if L is a minimal left ideal in βG, then L-product recurrence coincides with distality.

In [2], Auslander and Furstenberg studied the general compact right topological semigroup E actions on a compact Hausdorff space X. They introduced the cancellation semigroup condition and showed that if a nonempty closed subsemigroup $S \subset E$ satisfies the cancellation semigroup condition and contains a minimal left ideal of E, then S-product recurrence coincides with distality; see [2, Corollary 4 and Theorem 4]. This inspired Auslander and Furstenberg to ask Question 1.4.

We obtain the following sufficient conditions on the closed subsemigroup S of βG for which S-product recurrence coincides with distality, which partly answers Question 1.4 for βG-actions. Note that Theorem 1.6 is a direct consequence of the following result.

THEOREM 6.9. *Let $(X, \beta G)$ be a βG-action and $x \in X$. If S be a nonempty closed subsemigroup of $\beta G \setminus G$ with $K(\beta G) \subset S$, then the following assertions are equivalent:*

(1) *x is distal;*
(2) *x is S-product recurrent;*
(3) *x is weakly S-product recurrent.*

Proof. (1)⇒(2). Assume that x is a distal point. Given any S-recurrent point y in any action $(Y, \beta G)$, there exists $p \in S$ such that $py = y$. Let $L := \{q \in S : qy = y\}$. Then L is a nonempty closed subsemigroup of βG. By Ellis–Numakura Theorem (Theorem 5.1) there exists an idempotent $u \in L$. That is, there exists an idempotent $u \in S$ such that $uy = y$. Since x is a distal point, by Theorem 6.7, $ux = x$, and then $u(x, y) = (x, y)$, so that (x, y) is S-recurrent in $(X \times Y, \beta G)$.

(2)⇒(3). This is clear.

(3)⇒(1). Assume for contradiction that x is not distal. Then by Theorem 6.7, there exists a minimal idempotent $p \in \beta G$ such that $px \neq x$. By Remark 6.2 and the Ramsey property of the ultrafilter, there exists a neighborhood U of x such that $\{g \in G\colon gx \in X \setminus U\} \in p$. By Theorem 5.2,

$$\{g \in G\colon gx \in X \setminus U\}$$

is a central set. Now by Lemma 4.12 and Proposition 4.2, there exists an $\mathcal{F}_{\mathrm{ps}}$-recurrent point y with $y \in [1]$ in the Bernoulli shift (Σ_2, G) such that

$$N(y, [1]) \subset \{g \in G\colon gx \in X \setminus U\} \cup \{e\}.$$

Let $(\Sigma_2, \beta G)$ be the action of βG on Σ_2 induced by (Σ_2, G). Since $h(\mathcal{F}_{\mathrm{ps}}) = \mathrm{cl}_{\beta G} K(\beta G)$, by Lemma 5.5, Remark 6.2 and $\mathrm{cl}_{\beta S} K(\beta G) \subset S$, y is S-recurrent in $(\Sigma_2, \beta G)$. As x is weakly S-product recurrent, (x, y) is recurrent in $(X \times \Sigma_2, \beta G)$. But

$$\begin{aligned}\{g \in G\colon (gx, gy) \in U \times [1]\}& \\ &\subset \{g \in G\colon gx \in U\} \cap (\{g \in G\colon gx \in X \setminus U\} \cup \{e\}) \\ &= \{e\},\end{aligned}$$

which is a contradiction. ∎

Applying Theorem 6.9, we prove Theorem 1.7 as follows.

Proof of Theorem 1.7. For a Furstenberg family $\mathcal{F} \subset \mathcal{P}(G)$, if $\mathcal{F}$ has the Ramsey property, then the hull $h(\mathcal{F})$ of $\mathcal{F}$ is a nonempty closed subset of $\beta G \setminus G$. If $\mathcal{F} \supset \mathcal{F}_{\mathrm{ps}}$, then $h(\mathcal{F}) \supset h(\mathcal{F}_{\mathrm{ps}}) = \mathrm{cl}_{\beta G} K(\beta G)$. Let (X, G) be a G-system. Consider the action βG of G induced by (X, G). By Lemma 5.5 and Remark 6.2, the result is an immediate consequence of Theorem 6.9. ∎

Remark 6.10. Notice that Theorem 1.7 holds for the Furstenberg families $\mathcal{F}_{\mathrm{ps}}$ and $\mathcal{F}_{\mathrm{inf}}$, and if in addition G is amenable, then it holds for the Furstenberg family $\mathcal{F}_{\mathrm{pubd}}$.

Let G be a countable infinite discrete amenable group and $\{F_n\}$ be a Følner sequence in G. Recall that $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$ is the collection of all subsets of G with positive upper density with respect to $\{F_n\}$. We know that $h(\mathcal{F}_{\mathrm{pud}}^{\{F_n\}})$ is a nonempty closed left ideal of βG. As $\mathcal{F}_{\mathrm{ps}} \not\subset \mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$, we cannot apply Theorem 1.7. So we have the following natural question:

Question 6.11. *Is $\mathcal{F}_{\mathrm{pud}}^{\{F_n\}}$-product recurrence equivalent to distality?*

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