

Matrix-weighted inequalities for fractional type integrals associated to operators with new classes of weights

by

YONGMING WEN and HUOXIONG WU

Abstract. Let e^{-tL} be the analytic semigroup generated by $-L$, where L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^d)$. Assume that the kernels of e^{-tL} , denoted by $p_t(x, y)$, only satisfy the upper bound: for all $N > 0$, there are constants $c, C > 0$ such that

$$|p_t(x, y)| \leq \frac{C}{t^{d/2}} e^{-\frac{|x-y|^2}{ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

holds for all $x, y \in \mathbb{R}^d$ and $t > 0$. We first establish the quantitative matrix-weighted inequalities for fractional type integrals associated to L with new classes of matrix weights, which are a nontrivial extension of the results established by Li, Rahm and Wick (2019). Next, we give new two-weight bump conditions with Young functions satisfying wider conditions for fractional type integrals associated to L , which cover the result obtained by Cruz-Uribe, Isralowitz and Moen (2018). The new classes of matrix weights and bump conditions are larger and weaker than the classical ones. Our results can be applied to the settings of the magnetic Schrödinger operator, Laguerre operators, etc.

1. Introduction and main results

1.1. Background. In recent years, the problem of quantitative weighted estimates for singular integrals and related operators has attracted the attention of many mathematicians. Buckley [10] first studied sharp weighted inequalities

$$\|Mf\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{1/(p-1)} \|f\|_{L^p(\omega)}, \quad 1 < p < \infty,$$

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for the Hardy–Littlewood maximal function M , where the A_p constant $[\omega]_{A_p}$ is defined by

$$(1.1) \quad [\omega]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx \right)^{p-1} < \infty.$$

Later, Petermichl [40, 41] considered the sharp A_p estimates for the Hilbert and Riesz transforms. For general Calderón–Zygmund operators T , Hytönen [24] proved the dyadic representation theorem for T , which leads to the full picture of the sharp A_p bounds for T . Subsequently, Lacey, Moen, Pérez and Torres [29] established the following sharp weighted estimates for the classical fractional integral operator I_α and the fractional maximal operator M_α in terms of the $A_{p,q}$ constant.

THEOREM 1.1 ([29]). *Suppose that $0 < \alpha < d$, $1 < p < d/\alpha$ and q satisfies $1/q = 1/p - \alpha/d$. Then*

$$\|I_\alpha f\|_{L^q(\omega)} \lesssim [\omega]_{A_{p,q}}^{(1-\alpha/d) \max\{1, p'/q\}} \|f\|_{L^p(\omega^{p/q})}$$

and

$$\|M_\alpha f\|_{L^q(\omega)} \lesssim [\omega]_{A_{p,q}}^{(1-\alpha/d)p'/q} \|f\|_{L^p(\omega^{p/q})}.$$

Furthermore, both the results above are sharp, where

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad M_\alpha f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |f(y)| dy,$$

and the $A_{p,q}$ constant $[\omega]_{A_{p,q}}$ is defined by

$$[\omega]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'/q} dx \right)^{q/p'} < \infty.$$

In the Schrödinger setting, Bongioanni, Harboure and Salinas [8] introduced a new class $A_p^{\rho, \theta}$ of weights, which is defined as follows:

$$[\omega]_{A_p^{\rho, \theta}} := \sup_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q \omega(x)^{-p'/p} dx \right)^{p/p'} < \infty,$$

where $\psi_\theta(Q) := (1 + l_Q/\rho(x_Q))^\theta$, $\theta \geq 0$, ρ is the critical radius function (see Section 2 for a precise definition), and x_Q , l_Q are the center of the cube Q and the side-length of Q , respectively. In [8], the authors established weighted estimates for this class of weights of the semigroup maximal function, Riesz transforms, fractional integrals and the g-function associated to the Schrödinger operator. For more works on the weighted theory of $A_p^{\rho, \theta}$ in the Schrödinger setting, we refer the readers to [1, 3, 4, 5, 6, 7, 8, 9, 11, 14, 44] and the references therein. In 2019, Li, Rahm and Wick [33] first investigated the quantitative weighted estimates of the fractional integral $L^{-\alpha/2}$ and the

fractional maximal operator $M_\alpha^{\rho,\theta}$ in the Schrödinger setting, where $L^{-\alpha/2}$ and $M_\alpha^{\rho,\theta}$ are defined by

$$(1.2) \quad L^{-\alpha/2} f(x) := \int_0^\infty e^{-tL} f(x) t^{\alpha/2-1} dt, \quad 0 < \alpha < d,$$

$$M_\alpha^{\rho,\theta} f(x) := \sup_{Q \ni x} \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |f(y)| dy, \quad \theta \geq 0.$$

The main results of Li, Rahm and Wick [33] are as follows.

THEOREM 1.2 ([33]). *Suppose that $0 < \alpha < d$. Let $1 \leq p < d/\alpha$ and q satisfy $1/q = 1/p - \alpha/d$. Let $\theta \geq 0$, $\gamma = \theta/(1 + p'/q)$ and let K be defined by the equation $(1/K + q/(Kp'))(1 - \alpha/d) \max\{1, p'/q\} = 1/2$. Then*

$$\|L^{-\alpha/2} f\|_{L^q(\omega)} \lesssim [\omega]_{A_{p,q}^{\rho,\theta/(3K)}}^{(1-\alpha/d) \max\{1, p'/q\}} \|f\|_{L^p(\omega^{p/q})}, \quad \omega \in A_{p,q}^{\rho,\theta/(3K)},$$

and

$$\|M_\alpha^{\rho,\theta} f\|_{L^q(\omega)} \lesssim [\omega]_{A_{p,q}^{\rho,\gamma/3}}^{(1-\alpha/d)p'/q} \|f\|_{L^p(\omega^{p/q})}, \quad \omega \in A_{p,q}^{\rho,\gamma/3},$$

where $A_{p,q}^{\rho,\theta}$ is the class of weights ω such that

$$[\omega]_{A_{p,q}^{\rho,\theta}} := \sup_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q \omega(x)^{-p'/q} dx \right)^{q/p'} < \infty.$$

Recall that a *matrix weight* $W : \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$ is a self-adjoint matrix function with locally integrable entries such that $W(x)$ is positive definite almost everywhere. In the setting of matrix weights, Bickel, Petermichl and Wick [2] proved the following quantitative weighted estimates for the Hilbert transform H :

$$\|H\|_{L^2(W) \rightarrow L^2(W)} \lesssim [W]_{A_2}^{3/2} \log([W]_{A_2}),$$

where $L^p(W)$ is the collection of measurable vector-valued functions $\vec{f} : \mathbb{R}^d \rightarrow \mathbb{C}^n$ with

$$\|\vec{f}\|_{L^p(W)} := \left(\int_{\mathbb{R}^d} |W(x)^{1/p} \vec{f}(x)|^p dx \right)^{1/p} < \infty,$$

and W^r for any $r \in \mathbb{R}$ is defined by setting $W^r := OD(\lambda_i^r)O^T$ for some measurable orthogonal matrix function O (here $D(\lambda_i)$ is the diagonal matrix and λ_i ($i = 1, \dots, n$) are the positive eigenvalues of W). The result above was improved by Nazarov et al. [36] and Culiuc et al. [20] by extending it to all Calderón–Zygmund operators T and eliminating $\log([W]_{A_2})$. It was also extended to all $p \in (1, \infty)$ by Cruz-Uribe, Isralowitz and Moen [16]. However, only a few sharp quantitative matrix-weighted estimates have been known until now [25, 27, 28, 32]. Let W be a matrix weight and $1 < p, q < \infty$.

Isralowitz and Moen [27] introduced matrix $\mathcal{A}_{p,q}$ weights: we write $W \in \mathcal{A}_{p,q}$ if

$$[W]_{\mathcal{A}_{p,q}} := \sup_Q \frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx < \infty,$$

where

$$|W(x)|_{\text{op}} := \sup_{\substack{\vec{e} \in \mathbb{C}^n \\ |\vec{e}|=1}} |W(x)\vec{e}|.$$

Isralowitz and Moen [27] proved the following quantitative matrix-weighted estimates for I_α and the matrix-weighted fractional maximal function $M_{W,\alpha}$.

THEOREM 1.3 ([27]). *Suppose $0 < \alpha < d$, $1 < p < d/\alpha$ and $1/q = 1/p - \alpha/d$. If $W \in \mathcal{A}_{p,q}$, then*

$$\begin{aligned} \|M_{W,\alpha}\|_{L^p \rightarrow L^q} &\lesssim [W]_{\mathcal{A}_{p,q}}^{p'(1-\alpha/d)/q}, \\ \|I_\alpha\|_{L^p(W^{p/q}) \rightarrow L^q(W)} &\lesssim [W]_{\mathcal{A}_{p,q}}^{p'(1-\alpha/d)/q+1/q'}, \end{aligned}$$

where

$$M_{W,\alpha} \vec{f}(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |W(x)^{1/q} W(y)^{-1/q} \vec{f}(y)| dy.$$

For more historical works on matrix-weighted estimates and quantitative matrix-weighted estimates for harmonic analysis operators, we refer the readers to [21, 22, 23, 26, 35, 45, 46, 47], and the references therein.

On the other hand, it is well-known that an A_p weight ω can be characterized in terms of the $L^p(\omega)$ -boundedness of the Hardy–Littlewood maximal operator M or the Hilbert transform H . Muckenhoupt [34] raised the question of what type of a condition can be used to characterize M or H : $L^p(\nu) \rightarrow L^p(\mu)$. Analogously to the one-weight case, it is very natural to consider the condition

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \mu(x) dx \right) \left(\frac{1}{|Q|} \int_Q \nu(x)^{1-p'} dx \right)^{p-1} < \infty.$$

Unfortunately, this condition is not sufficient for singular integral operators to be bounded from $L^p(\nu)$ to $L^p(\mu)$ (see [17]). To solve this problem, Sawyer [43] first introduced the testing condition, which is sufficient and necessary for M to be bounded from $L^p(\nu)$ to $L^p(\mu)$. However, due to the operator M itself being involved in the testing condition, researchers attempted to find some sufficient conditions that are easy to verify and close to (1.1) in some sense. Neugebauer [37] first introduced the “power bump” condition for

$M: L^p(\nu) \rightarrow L^p(\mu)$ as follows:

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \mu(x)^r dx \right)^{1/r} \left(\frac{1}{|Q|} \int_Q \nu(x)^{r(1-p')} dx \right)^{(p-1)/r} < \infty, \quad r > 1.$$

Pérez [38] improved the “power bump” condition to the “Orlicz bump” condition

$$\sup_Q \|\mu^{1/p}\|_{p,Q} \|\nu^{-1/p}\|_{\Phi,Q} < \infty, \quad 1 < p < \infty,$$

where Φ is a Young function and $\bar{\Phi} \in B_p$. Here, we recall some standard notions. A *Young function* $\Phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous, increasing, convex function that satisfies $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \Phi(t)/t = \infty$. The corresponding complementary function of Φ , denoted by $\bar{\Phi}$, is given by

$$\bar{\Phi}(t) = \sup_{s>0} \{st - \Phi(s)\}.$$

We say that $\Phi \in B_p$ if

$$\int_1^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty, \quad 1 < p < \infty.$$

Let Φ be a Young function. The localized Orlicz norm $\|f\|_{\Phi,Q}$ is defined by

$$\|f\|_{\Phi,Q} := \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

Cruz-Urbe and Pérez [19] conjectured that if a pair (μ, ν) of weights satisfies

$$\sup_Q \|\mu^{1/p}\|_{\Psi,Q} \|\nu^{-1/p}\|_{\Phi,Q} < \infty, \quad 1 < p < \infty,$$

with $\bar{\Phi} \in B_p$ and $\bar{\Psi} \in B_{p'}$, then every Calderón–Zygmund operator is bounded from $L^p(\nu)$ to $L^p(\mu)$. This conjecture was finally solved by Lerner [30]. For fractional integrals, Pérez [39] proved that the Orlicz bump condition

$$\sup_Q |Q|^{\alpha/d+1/q-1/p} \|\mu^{1/q}\|_{\Psi,Q} \|\nu^{-1/p}\|_{\Phi,Q} < \infty, \quad \bar{\Psi} \in B_p, \bar{\Phi} \in B_{q'}$$

is sufficient for $I_\alpha: L^p(\nu) \rightarrow L^q(\mu)$. Cruz-Urbe and Moen [18] improved the conditions in [39] to weaker $B_{p,q}$ conditions: $\bar{\Psi} \in B_{p,q}$, $\bar{\Phi} \in B_{q',p'}$, where $\bar{\Psi} \in B_{p,q}$ if

$$\int_1^\infty \frac{\bar{\Psi}(t)^{q/p}}{t^q} \frac{dt}{t} < \infty.$$

In 2018, Cruz-Urbe, Isralowitz and Moen [16] extended the theory of two-weight, Orlicz bump conditions to the setting of matrix weights. For the purposes of this paper, we only state their results concerning fractional type integrals.

THEOREM 1.4 ([16]). *Let $0 \leq \alpha < d$, $1 < p \leq q < \infty$ with $1/p - 1/q \leq \alpha/d$. Assume that Φ is a Young function satisfying $\bar{\Phi} \in B_{p,q}$. If a pair (U, V) of matrix weights satisfies*

$$\sup_Q |Q|^{\alpha/d+1/q-1/p} \left(\int_Q \|U(x)^{1/q} V^{-1/q}\|_{\Phi, Q}^q dx \right)^{1/q} < \infty,$$

then $M_{U,V,\alpha} : L^p(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L^q(\mathbb{R}^d, \mathbb{C}^n)$, where

$$M_{U,V,\alpha} \vec{f}(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy.$$

THEOREM 1.5 ([16]). *Let $0 < \alpha < d$, $1 < p \leq q < \infty$ with $1/p - 1/q \leq \alpha/d$. Assume that Φ, Ψ are Young functions satisfying $\bar{\Phi} \in B_{p,q}$ and $\bar{\Psi} \in B_{q'}$. If a pair (U, V) of matrix weights satisfies*

$$\sup_Q |Q|^{\alpha/d+1/q-1/p} \left\| \|U(x)^{1/q} V(y)^{-1/q}\|_{\Phi_y, Q} \right\|_{\Psi_x, Q} < \infty,$$

then $I_\alpha : L^p(V^{p/q}) \rightarrow L^q(U)$.

REMARK 1.6. In the scalar case, since Theorem 1.5 does not capture the weaker hypothesis $\bar{\Psi} \in B_{q',p'}$ [18], the authors in [16] conjectured that Theorem 1.5 remains true if $\bar{\Psi} \in B_{q'}$ is replaced by $\bar{\Psi} \in B_{q',p'}$.

1.2. Aims and questions. Inspired by the works [16, 27, 33], this paper is devoted to studying the quantitative matrix-weighted estimates and two-weight matrix bounds for fractional type integrals associated with general differential operators.

To be more precise, let L be a non-negative self-adjoint operator defined on $L^2(\mathbb{R}^d)$. Thus, L generates an analytic heat semigroup e^{-tL} . Denote the kernel of e^{-tL} by $p_t(x, y)$, which merely satisfies the estimate in the abstract. Let $\theta \geq 0$, U, V be matrix weights and $\mathcal{U}_Q^{q,q}$ be a positive definite matrix, usually called a *reducing operator* of $U^{1/q}$, such that

$$|\mathcal{U}_Q^{q,q} \vec{e}| \sim \|U^{1/q} \vec{e}\|_{q,Q}, \quad 1 < q < \infty$$

(see [16, 23] for more details).

We will consider the fractional integrals $L^{-\alpha/2}$ defined as in (1.2), the matrix-weighted fractional maximal function $M_{U,V,\psi_\theta,\alpha}$ associated with a critical radius function, the auxiliary matrix-weighted fractional maximal function $M_{\mathcal{U},V,\psi_\theta,\alpha}$ associated with a critical radius function, the averaging operator $A_Q^{\alpha,\theta}$ associated with a critical radius function, and the auxiliary averaging operator $\mathcal{A}_Q^{\alpha,\theta}$ associated with a critical radius function, which are

defined by

$$\begin{aligned}
M_{U,V,\psi_\theta,\alpha}\vec{f}(x) &:= \sup_{Q\ni x} \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy, \\
M_{\mathcal{U},V,\psi_\theta,\alpha}\vec{f}(x) &:= \sup_{Q\ni x} \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy, \\
A_Q^{\alpha,\theta}\vec{f}(x) &:= \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q \vec{f}(y) dy \chi_Q(x), \\
\mathcal{A}_Q^{\alpha,\theta}\vec{f}(x) &:= \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy \chi_Q(x),
\end{aligned}$$

respectively. If $U = V = W$, we simply write $M_{W,\psi_\theta,\alpha}$ for $M_{U,V,\psi_\theta,\alpha}$.

To obtain the quantitative matrix-weighted estimates for fractional type integrals associated with L , it is natural to adopt the idea of [12, 13] to dominate the local part by a sparse operator, and dominate the global part by a maximal operator. However, compared to the scalar case, matrix products of self-adjoint matrices do not commute. Moreover, the technique of comparing objects and dominating one by another is lost in the vector case; this prevents us from utilizing the technique of [12, 13]. Further, there is an additional critical radius function factor in the operator. Hence, the following question is natural.

QUESTION 1. So far, there have never been quantitative matrix-weighted estimates for harmonic analysis operators associated to L , even when L is the Schrödinger operator. Is it possible to extend Theorem 1.2 to quantitative matrix-weighted estimates in the more general setting of differential operators?

Recently, Bongioanni, Harboure, and Quijano [7] proved that the bump condition

$$\sup_Q \psi_\theta(Q)^{-1} \|\mu^{1/p}\|_{p,Q} \|\nu^{-1/p}\|_{\bar{\Phi},Q} < \infty, \quad \bar{\Phi} \in B_p, \quad 1 < p < \infty,$$

is sufficient for the two-weight estimate of $M_0^{\rho,\theta}$. In light of the classes of weights in Theorem 1.2 and the bump condition for the two-weight inequality of $M_0^{\rho,\theta}$, it is likely that new classes of matrix weights and a new bump condition adapted to matrix weights will be needed.

QUESTION 2. In the setting of L , what types of matrix weights and bump conditions are appropriate for quantitative matrix-weighted estimates and two-weight inequalities for fractional type integrals, respectively? If new classes of weights exist, how to deal with those classes of matrix weights to achieve the desired conclusions? Can we find any characterizations of these classes of matrix weights?

Finally, concerning Remark 1.6, we also conjecture this conclusion is still true for fractional integrals associated with L . However, we believe that there is still a long way to go to prove this conjecture.

QUESTION 3. In our new setting, can we take a step forward on the path of proving this conjecture?

1.3. Main results. Our first results concern quantitative matrix-weighted estimates for fractional type integrals. The definitions of new classes of matrix weights are given in Section 2.

THEOREM 1.7. *Let $\theta \geq 0$, $0 < \alpha < d$, $1 < p < d/\alpha$ and $1/q = 1/p - \alpha/d$. Let K satisfy $(\frac{1}{K} + \frac{q}{K^p})[(1 - \frac{\alpha}{d})\frac{p'}{q} + \frac{1}{q'}] = \frac{1}{2}$. If $W \in \mathcal{A}_{p,q}^{\rho,\theta/(3K)}$, then*

$$\|L^{-\alpha/2}\|_{L^p(W^{p/q}) \rightarrow L^q(W)} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta/(3K)}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}}.$$

THEOREM 1.8. *Let $\theta \geq 0$, $0 \leq \alpha < d$, $1 < p < d/\alpha$ and $1/q = 1/p - \alpha/d$.*

- (1) *If $M_{W,\psi_\theta,\alpha}$ is bounded from L^p to L^q , then $W \in \mathcal{A}_{p,q}^{\rho,\theta}$.*
- (2) *Let K be defined by the equation $p'/(Kq) + 1/K = 1/2$. If $W \in \mathcal{A}_{p,q}^{\rho,\theta/(3K)}$, then*

$$\|M_{W,\psi_\theta,\alpha}\|_{L^p \rightarrow L^q} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta/(3K)}}^{p'(1-\alpha/d)/q}.$$

Next, we present the two-weight inequalities for fractional type integrals associated to L . Notice that the conditions on Young functions in Theorem 1.9 cover the ones in Theorem 1.5 since $B_{q',s'} = B_{q'}$ when $s = q$.

THEOREM 1.9. *Let $0 < \alpha < d$, $p < q$, $1 < p \leq s \leq q < \infty$. Assume that Φ and Ψ are Young functions such that $\bar{\Phi} \in B_{p,s}$ and $\bar{\Psi} \in B_{q',s'}$. If a pair (U, V) of weights satisfies*

$$[U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} := \sup_Q \frac{|Q|^{\alpha/d+1/q-1/p}}{\psi_{\theta/3}(Q)} \left\| \|U(x)^{1/q} V(y)^{-1/q}\|_{\Phi_y, Q} \right\|_{\Psi_x, Q} < \infty,$$

where $\|\cdot\|_{\Phi_y, Q}$ means the Orlicz norm is taken with respect to the y variable and $\|U\|$ means $\|U\|_{\text{op}}$, then

$$\|L^{-\alpha/2} \vec{f}\|_{L^q(U)} \lesssim [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \|\vec{f}\|_{L^p(V^{p/q})}.$$

THEOREM 1.10. *Let $\theta > 0$, $0 \leq \alpha < d$, $1 < p \leq q < \infty$ and $1/p - 1/q \leq \alpha/d$. Assume that $\bar{\Phi} \in B_{p,q}$. If a pair (U, V) of matrix weights satisfies*

$$[U, V]_{p,q,\alpha,\psi_{\theta'},\Phi} := \sup_Q \frac{|Q|^{\alpha/d+1/q-1/p}}{\psi_{\theta'/3}(Q)^{1-\alpha/d}} \left(\int_Q \|U(x)^{1/q} V^{-1/q}\|_{\bar{\Phi}, Q}^q dx \right)^{1/q} < \infty, \quad 0 < \theta' < \theta,$$

then $\|M_{U,V,\psi_\theta,\alpha} \vec{f}\|_{L^q} \lesssim [U, V]_{p,q,\alpha,\psi_{\theta'},\Phi} \|\vec{f}\|_{L^p}$.

REMARK 1.11. It is obvious that the classes $\mathcal{A}_{p,q}^\rho$ of matrix weights are larger than $\mathcal{A}_{p,q}$. Moreover, the bump conditions are weaker than the ones in Theorems 1.4 and 1.5. Hence, our results give better quantitative constants for fractional type integrals.

THEOREM 1.12. *Let $\theta \geq 0$, $0 \leq \alpha < d$, $1 \leq p \leq q < \infty$ with $1/p - 1/q = \alpha/d$. Let (U, V) be a pair of matrix weights.*

- (1) *If $(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta/3}$, then*

$$\max \{ \|M_{U,V,\psi_\theta,\alpha} \vec{f}\|_{L^{q,\infty}}, \|A_Q^{\alpha,\theta} \vec{f}\|_{L^{q,\infty}} \} \lesssim [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta/3}}^{1/q} \|\vec{f}\|_{L^p},$$

where the implicit constant is independent of Q .

- (2) *If $M_{U,V,\psi_\theta,\alpha}$ or $A_Q^{\alpha,\theta}$ is bounded from L^p to $L^{q,\infty}$ with norm independent of Q , then $(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta}$.*

REMARK 1.13. The original purpose of Theorems 1.8 and 1.12 was to give a characterization of $\mathcal{A}_{p,q}^{\rho,\theta}$ in terms of the boundedness of $M_{W,\psi_\theta,\alpha}$ and $M_{U,V,\psi_\theta,\alpha}$, respectively, as in [16, 27]. However, we do not expect that we can do it due to the annoying critical radius function in the operators.

The last theorem is related to the characterization of matrix weights.

THEOREM 1.14. *Let $\theta \geq 0$, $0 \leq \alpha < d$, $1 \leq p \leq q < \infty$ with $1/p - 1/q = \alpha/d$. Let (U, V) be a pair of matrix weights. The following statements are equivalent:*

- (1) *$(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta}$;*
 (2) *$\|A_Q^{\alpha,\theta} \vec{f}\|_{L^q(U)} \lesssim \|\vec{f}\|_{L^p(V^{p/q})}$ with norm independent of Q .*

Moreover, we have the estimate

$$\|A_Q^{\alpha,\theta} \vec{f}\|_{L^q(U)} \lesssim [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta}}^{1/q} \|\vec{f}\|_{L^p(V^{p/q})}.$$

REMARK 1.15. Our results have applications to the settings of the magnetic Schrödinger operator, Laguerre operators, etc.; see [12].

The structure of the paper is as follows. In Section 2, we introduce some new classes of matrix weights and some of their properties. In Section 3, we prove Theorems 1.7 and 1.8. The proofs of Theorems 1.9, 1.10 and 1.12 will be given in Section 4. Finally, we give the proof of Theorem 1.14 in Section 5.

Throughout the rest of the paper, we write $f \lesssim g$, $f \sim g$ if $f \leq Cg$ and $f \lesssim g \lesssim f$, respectively. For any cube $Q := Q(x_Q, r_Q) \subset \mathbb{R}^d$, denote the side-length of Q by l_Q ; x_Q represents the center of Q ; and $r_Q = l_Q/2$. By χ_Q we denote the characteristic function of Q , and $\sigma Q := Q(x_Q, \sigma r_Q)$.

2. New classes of matrix weights associated to critical radius functions. In this section, we introduce some new classes of matrix weights

associated to critical radius functions. Recall that a function $\rho : \mathbb{R}^d \rightarrow (0, \infty)$ is called a *critical radius function* if there are positive constants C_0 and N_0 such that for arbitrary $x, y \in \mathbb{R}^d$,

$$(2.1) \quad C_0^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq C_0 \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}.$$

DEFINITION 2.1. Let $\theta \geq 0$, $1 \leq p \leq q < \infty$ and ρ be a critical radius function. We say that a pair (U, V) of matrix weights is in $\mathcal{A}_{p,q}^{\rho,\theta}$ with $1 < p \leq q < \infty$ if

$$[U, V]_{\mathcal{A}_{p,q}^{\rho,\theta}} := \sup_Q \frac{1}{\psi_\theta(Q)|Q|} \int_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q |U(x)^{1/q} V(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx < \infty.$$

When $p = 1$ and $1 \leq q < \infty$, we write $(U, V) \in \mathcal{A}_{1,q}^{\rho,\theta}$ if

$$[U, V]_{\mathcal{A}_{1,q}^{\rho,\theta}} := \sup_Q \text{ess sup}_{y \in Q} \frac{1}{\psi_\theta(Q)|Q|} \int_Q |U(x)^{1/q} V(y)^{-1/q}|_{\text{op}}^q dx < \infty.$$

In particular, if $U = V = W$, we write $W \in \mathcal{A}_{p,q}^{\rho,\theta}$ for $(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta}$. To give some properties of this new class of matrix weights, we recall two important lemmas related to reducing operators.

LEMMA 2.2 ([23]). *Given a matrix weight A , a Young function Ψ , and a cube Q , there exists a matrix $\mathcal{A}_Q^\Psi$, called a reducing operator of A , such that for all $\vec{e} \in \mathbb{C}^n$,*

$$|\mathcal{A}_Q^\Psi \vec{e}| \sim \|A \vec{e}\|_{\Psi, Q}.$$

LEMMA 2.3 ([16]). *Consider matrix weights A and B , Young functions Φ and Ψ , a cube Q , and reducing operators $\mathcal{A}_Q^\Psi$ and $\mathcal{B}_Q^\Phi$. Then*

$$|\mathcal{A}_Q^\Psi \mathcal{B}_Q^\Phi|_{\text{op}} \sim \|A(x) \mathcal{B}_Q^\Phi\|_{\Psi_x, Q} \sim \|\|A(x) B(y)\|_{\Phi_y, Q}\|_{\Psi_x, Q}.$$

Let $\mathcal{W}_Q^{q,q}$ and $\widetilde{\mathcal{W}}_Q^{q,p'}$ be the reducing operators such that

$$|\mathcal{W}_Q^{q,q} \vec{e}| \sim \|W^{1/q} \vec{e}\|_{q, Q}, \quad |\widetilde{\mathcal{W}}_Q^{q,p'} \vec{e}| \sim \|W^{-1/q} \vec{e}\|_{p', Q}.$$

From Lemma 2.3, we have

$$(2.2) \quad \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx \right)^{p'/q} \sim |\mathcal{W}_Q^{q,q} \widetilde{\mathcal{W}}_Q^{q,p'}|_{\text{op}}^{p'}$$

and

$$(2.3) \quad \frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W(x)^{-1/q} W(y)^{1/q}|_{\text{op}}^q dy \right)^{p'/q} dx \sim |\widetilde{\mathcal{W}}_Q^{q,p'} \mathcal{W}_Q^{q,q}|_{\text{op}}^{p'}.$$

THEOREM 2.4. Let $\theta \geq 0$, $1 \leq p \leq q < \infty$ and $\mathcal{A}_{p,q}^{\rho,\infty} := \bigcup_{\theta>0} \mathcal{A}_{p,q}^{\rho,\theta}$. Then $\mathcal{A}_{p,q} \subsetneq \mathcal{A}_{p,q}^{\rho}$.

Proof. It is easy to see that $[W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \leq [W]_{\mathcal{A}_{p,q}}$, so $\mathcal{A}_{p,q} \subset \mathcal{A}_{p,q}^{\rho,\infty}$. In the scalar case, the classes $\mathcal{A}_{p,q}^{\rho,\theta}$ and $\mathcal{A}_{p,q}$ reduce to $A_{p,q}^{\rho,\theta}$ and $A_{p,q}$, respectively. It is known that $A_{p,q} \subsetneq A_{p,q}^{\rho,\infty}$. This proves the assertion. ■

THEOREM 2.5. Let $\theta \geq 0$ and $1 < p \leq q < \infty$. Then

$$[W^{-p'/q}]_{\mathcal{A}_{q',p'}^{\rho,\theta}} \sim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}^{p'/q}.$$

Proof. According to (2.2) and (2.3), we get

$$\begin{aligned} & [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}^{p'/q} \\ &= \sup_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx \right)^{p'/q} \\ &\sim \sup_Q \psi_\theta(Q)^{-p'/q-1} |\mathcal{W}_Q^{q,q} \widetilde{\mathcal{W}}_Q^{q,p'}|_{\text{op}}^{p'} \end{aligned}$$

and

$$\begin{aligned} & [W^{-p'/q}]_{\mathcal{A}_{q',p'}^{\rho,\theta}} \\ &= \sup_Q \frac{1}{\psi_\theta(Q)|Q|} \int_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q |W(x)^{-1/q} W(y)^{1/q}|_{\text{op}}^q dy \right)^{p'/q} dx \\ &= \sup_Q \psi_\theta(Q)^{-p'/q-1} |\widetilde{\mathcal{W}}_Q^{q,p'} \mathcal{W}_Q^{q,q}|_{\text{op}}^{p'}. \end{aligned}$$

The conclusion follows immediately by using $|AB|_{\text{op}} = |BA|_{\text{op}}$ for any two positive definite matrices A and B . ■

THEOREM 2.6. Let $\theta \geq 0$, $1 \leq p \leq q < \infty$ with $1/p - 1/q = \alpha/d$. If $W \in \mathcal{A}_{p,q}^{\rho,\theta}$, then for any unit vector $\vec{e}$, we have $|W^{1/q} \vec{e}|^q \in A_{p,q}^{\rho,\theta}$ and

$$[|W^{1/q} \vec{e}|^q]_{A_{p,q}^{\rho,\theta}} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}.$$

Proof. The case $\theta = 0$ is proved in [27]. Now we consider $\theta > 0$. Let ϕ be an arbitrary scalar function and $\vec{f} = \phi \vec{e}$. Note that $A_Q^{\alpha,\theta}(\phi \vec{e}) = A_{Q,\text{scalar}}^{\alpha,\theta}(\phi) \vec{e}$, where

$$A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)(x) := \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q \phi(y) dy \chi_Q(x)$$

is a scalar function. Then

$$\begin{aligned}
 (2.4) \quad \|A_Q^{\alpha,\theta}(\phi\vec{e})\|_{L^q(W)}^q &= \int_{\mathbb{R}^d} |W(x)^{1/q} A_Q^{\alpha,\theta}(\phi\vec{e})(x)|^q dx \\
 &= \int_{\mathbb{R}^d} |W(x)^{1/q} A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)(x)\vec{e}|^q dx \\
 &= \int_{\mathbb{R}^d} |A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)(x)W(x)^{1/q}\vec{e}|^q dx \\
 &= \int_{\mathbb{R}^d} |A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)(x)|^q |W(x)^{1/q}\vec{e}|^q dx
 \end{aligned}$$

On the other hand, by Theorem 1.14 and $W \in \mathcal{A}_{p,q}^{\rho,\theta}$, we have

$$\begin{aligned}
 (2.5) \quad \|A_Q^{\alpha,\theta}(\phi\vec{e})\|_{L^q(W)}^q &\lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \|\phi\vec{e}\|_{L^p(W^{p/q})}^q \\
 &= [W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \left(\int_{\mathbb{R}^d} |W(x)^{1/q} \phi(x)\vec{e}|^p dx \right)^{q/p} \\
 &= [W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \left(\int_{\mathbb{R}^d} |\phi(x)|^p |W(x)^{1/q}\vec{e}|^p dx \right)^{q/p}.
 \end{aligned}$$

Combining (2.4) and (2.5), we deduce that

$$\int_{\mathbb{R}^d} |A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)(x)|^q |W(x)^{1/q}\vec{e}|^q dx \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \left(\int_{\mathbb{R}^d} |\phi(x)|^p |W(x)^{1/q}\vec{e}|^p dx \right)^{q/p}.$$

That is, $A_{Q,\text{scalar}}^{\alpha,\theta}$ is bounded from the scalar-weighted $L^p(|W^{1/q}\vec{e}|^p)$ to the scalar-weighted $L^q(|W^{1/q}\vec{e}|^q)$, which shows that $|W^{1/q}\vec{e}|^q \in A_{p,q}^{\rho,\theta}$ by using (2) $\Rightarrow$ (1) of Theorem 1.14 in the scalar setting. Moreover, from the proof of Theorem 1.14, we get

$$[|W^{1/q}\vec{e}|^q]_{A_{p,q}^{\rho,\theta}} \sim \sup_{\|\phi\|_{L^p(|W^{1/q}\vec{e}|^p)}=1} \|A_{Q,\text{scalar}}^{\alpha,\theta}(\phi)\|_{L^q(|W^{1/q}\vec{e}|^q)}^q,$$

which yields $[|W^{1/q}\vec{e}|^q]_{A_{p,q}^{\rho,\theta}} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}$ by (2.5). ■

We also need the following class of matrix weights, which is an extension of that in [33] to the setting of matrix weights.

DEFINITION 2.7. Let $\theta \geq 0$, $1 \leq p \leq q < \infty$ and ρ be a critical radius function. We write $W \in \tilde{\mathcal{A}}_{p,q}^{\rho,\theta}$ with $1 < p \leq q < \infty$ if

$$\begin{aligned}
 [W]_{\tilde{\mathcal{A}}_{p,q}^{\rho,\theta}} &:= \sup_Q \frac{1}{\psi_\theta(Q)|Q|} \int_Q \left(\frac{1}{\widetilde{\psi}_\theta(Q)|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx \\
 &< \infty,
 \end{aligned}$$

When $p = 1$ and $1 \leq q < \infty$, we write $W \in \tilde{\mathcal{A}}_{1,q}^{\rho,\theta}$ if

$$[W]_{\tilde{\mathcal{A}}_{1,q}^{\rho,\theta}} := \sup_Q \operatorname{ess\,sup}_{y \in Q} \frac{1}{\tilde{\psi}_\theta(Q)|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_q^q dx < \infty,$$

where

$$\tilde{\rho}(Q) := \sup_{x \in Q} \rho(x), \quad \tilde{\psi}_\theta(Q) := \left(1 + \frac{l_Q}{\tilde{\rho}(Q)}\right)^\theta.$$

Denote $\tilde{\mathcal{A}}_{p,q}^{\rho,\infty} := \bigcup_{\theta > 0} \tilde{\mathcal{A}}_{p,q}^{\rho,\theta}$. The following lemma exhibits the relationship between $\mathcal{A}_{p,q}^{\rho,\infty}$ and $\tilde{\mathcal{A}}_{p,q}^{\rho,\infty}$.

LEMMA 2.8. *Let $\theta > 0$ and $1 \leq p \leq q < \infty$. Then $[W]_{\mathcal{A}_{p,q}^{\rho,\theta}} \leq [W]_{\tilde{\mathcal{A}}_{p,q}^{\rho,\theta}}$, $[W]_{\tilde{\mathcal{A}}_{p,q}^{\rho,3\theta}} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}$ and $\tilde{\mathcal{A}}_{p,q}^{\rho,\infty} = \mathcal{A}_{p,q}^{\rho,\infty}$.*

Proof. Observe that $\tilde{\psi}_\theta(Q) \leq \psi_\theta(Q)$, so $[W]_{\tilde{\mathcal{A}}_{p,q}^{\rho,\theta}} \geq [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}$. On the other hand, it was proved in [33] that

$$(2.6) \quad \frac{1}{\tilde{\psi}_{3\theta}(Q)} \lesssim \frac{1}{\psi_\theta(Q)},$$

which implies that $[W]_{\tilde{\mathcal{A}}_{p,q}^{\rho,3\theta}} \lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta}}$. Thus, $\tilde{\mathcal{A}}_{p,q}^{\rho,\infty} = \mathcal{A}_{p,q}^{\rho,\infty}$. ■

3. Quantitative weighted estimates for fractional type integrals associated with operators. This section is devoted to proving Theorems 1.7 and 1.8. The following upper bound for the kernel of $L^{-\alpha/2}$ is known when L is the Schrödinger operator (see [44]), but no proof is given. For the readers' convenience, we provide the proof in the setting of more general differential operators.

LEMMA 3.1. *Let $0 < \alpha < d$, $K_t(x, y)$ be the kernel of $L^{-\alpha/2}$ and ρ be critical radius function. Then for all $x, y \in \mathbb{R}^d$ and $N > 0$, there is a positive constant C such that*

$$|K_t(x, y)| \leq \frac{C}{\left(1 + |x - y| \left(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}\right)\right)^N} \frac{1}{|x - y|^{d-\alpha}}.$$

Proof. It is immediate that

$$(3.1) \quad \begin{aligned} |K_t(x, y)| &\leq \int_0^\infty |p_t(x, y)| t^{\alpha/2-1} dt \\ &= \int_0^{|x-y|^2} |p_t(x, y)| t^{\alpha/2-1} dt + \int_{|x-y|^2}^\infty |p_t(x, y)| t^{\alpha/2-1} dt. \end{aligned}$$

Applying the upper estimate of $p_t(x, y)$ and $e^{-s} \lesssim s^{-M/2}$ for any $M > 0$, we have

$$\begin{aligned}
 (3.2) \quad & \int_0^{|x-y|^2} |p_t(x, y)| t^{\alpha/2-1} dt \\
 & \lesssim \int_0^{|x-y|^2} t^{\alpha/2-d/2-1} e^{-\frac{|x-y|^2}{ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} dt \\
 & \lesssim \int_0^{|x-y|^2} t^{\alpha/2-d/2-1} \frac{t^{N/2+d/2-\alpha/2+1}}{|x-y|^{N+d-\alpha+2}} \left(\frac{\sqrt{t}}{|x-y|} + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} dt \\
 & = \frac{1}{(1 + |x-y|(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}))^N} \frac{1}{|x-y|^{d-\alpha}}.
 \end{aligned}$$

In addition, using the upper bound of $p_t(x, y)$ again, we get

$$\begin{aligned}
 \int_{|x-y|^2}^{\infty} |p_t(x, y)| t^{\alpha/2-1} dt & \lesssim \int_{|x-y|^2}^{\infty} t^{\alpha/2-d/2-1} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} dt \\
 & \leq \frac{1}{(1 + |x-y|(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}))^N} \int_{|x-y|^2}^{\infty} t^{\alpha/2-d/2-1} dt \\
 & \sim \frac{1}{(1 + |x-y|(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}))^N} \frac{1}{|x-y|^{d-\alpha}}.
 \end{aligned}$$

Combining this with (3.1) and (3.2), we get the conclusion. ■

We will prove Theorem 1.7 through approximating $L^{-\alpha/2}$ by a dyadic operator, so let us recall some facts about dyadic grids. We say that a collection of cubes $\mathcal{D}$ in $\mathbb{R}^d$ is a *dyadic grid* if it satisfies:

- (1) for any $Q \in \mathcal{D}$, $l(Q) = 2^k$ for some $k \in \mathbb{Z}$;
- (2) for any $Q_1, Q_2 \in \mathcal{D}$, $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$;
- (3) for any $k \in \mathbb{Z}$, the family $\mathcal{D}_k = \{Q \in \mathcal{D} : l(Q) = 2^k\}$ forms a partition of $\mathbb{R}^d$.

An important subcollection of dyadic grids is formed by sparse families.

DEFINITION 3.2 ([31]). Let $\eta \in (0, 1)$. A subset $\mathcal{S} \subset \mathcal{D}$ is said to be an η -sparse family if for any cube $Q \in \mathcal{S}$, there exists a measurable subset $E_Q \subset Q$ such that $\eta|Q| \leq |E_Q|$, and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are mutually disjoint.

The following lemma shows that any cube in $\mathbb{R}^d$ can be approximated by cubes from a finite collection of dyadic grids.

LEMMA 3.3 ([30]). For $t \in \{0, 1/3\}^d$, let

$$\mathcal{D}^t = \{2^{-k}([0, 1]^d + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}.$$

Then each $\mathcal{D}^t$ is a dyadic grid, and given any cube $Q \subset \mathbb{R}^d$, there exist t and $Q^t \in \mathcal{D}^t$ such that

$$Q \subset Q^t \quad \text{and} \quad l_{Q^t} \leq 6l_Q.$$

Based on Lemma 3.3, we show that $L^{-\alpha/2}$ can be approximated by a dyadic operator.

PROPOSITION 3.4. Let $\theta \geq 0$, $0 < \alpha < d$, $1 < q < \infty$ and let U, V be matrix weights. Then

$$\begin{aligned} & |\langle U^{1/q} L^{-\alpha/2} V^{-1/q} \vec{f}, \vec{g} \rangle_{L^2} | \\ & \lesssim \sum_{t \in \{0, 1/3\}^d} \sum_{Q \in \mathcal{D}^t} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int_Q \int_Q |\langle V(y)^{-1/q} f(y), U(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy. \end{aligned}$$

Proof. Taking into account that U is a self-adjoint matrix, and by Lemma 3.1, we have

$$\begin{aligned} & |\langle U^{1/q} L^{-\alpha/2} V^{-1/q} \vec{f}, \vec{g} \rangle_{L^2} | \\ & \lesssim \int \sum_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \int_{Q(x, 2^k) \setminus Q(x, 2^{k-1})} \frac{|\langle V(y)^{-1/q} \vec{f}(y), U(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}|}{(1 + |x - y|/\rho(y))^\theta |x - y|^{d-\alpha}} dy dx. \end{aligned}$$

For $y \in Q(x, 2^k) \setminus Q(x, 2^{k-1})$, note that $|x - y| \geq 2^{k-1}$ and $\rho(y) \leq \tilde{\rho}(Q(x, 2^k))$, and so

$$\frac{1}{(1 + |x - y|/\rho(y))^\theta |x - y|^{d-\alpha}} \lesssim \frac{1}{2^{(k-1)(d-\alpha)} \tilde{\psi}_\theta(Q(x, 2^k))}.$$

Now for each $k \in \mathbb{Z}$, Lemma 3.3 shows that there are $t \in \{0, 1/3\}^d$ and $Q_t \in \mathcal{D}^t$ such that $Q(x, 2^k) \subset Q^t$ and

$$2^{k+1} = l_{Q(x, 2^k)} \leq l_{Q^t} \leq 6l_{Q(x, 2^k)} = 12 \cdot 2^k,$$

which implies that

$$\frac{2^{-5(d-\alpha)}}{2^{(k-1)(d-\alpha)}} = \frac{1}{2^{(k+4)(d-\alpha)}} \leq \frac{1}{|Q^t|^{1-\frac{\alpha}{d}}},$$

and

$$\frac{1}{\tilde{\psi}_\theta(Q(x, 2^k))} \lesssim \frac{1}{\tilde{\psi}_\theta(Q^t)}.$$

Then

$$\begin{aligned}
& |\langle U^{1/q} L^{-\alpha/2} V^{-1/q} \vec{f}, \vec{g} \rangle_{L^2}| \\
& \lesssim \int \sum_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \frac{1}{\tilde{\psi}_\theta(Q(x, 2^k)) 2^{(k-1)(d-\alpha)}} \int_{Q(x, 2^k)} |\langle V(y)^{-1/q} \vec{f}(y), U(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dy dx \\
& \lesssim \int \sum_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_{t \in \{0, 1/3\}^d} \sum_{\substack{Q \in \mathcal{D}^t \\ 2^{k+1} \leq l_Q \leq 2^{k+4}}} \int_{Q(x, 2^k)} |\langle V(y)^{-1/q} \vec{f}(y), U(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dy \\
& \quad \times \frac{\chi_Q(x)}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} dx \\
& \lesssim \sum_{t \in \{0, 1/3\}^d} \sum_{Q \in \mathcal{D}^t} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int \int_Q |\langle V(y)^{-1/q} \vec{f}(y), U(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy. \blacksquare
\end{aligned}$$

In order to prove Theorem 1.7, we define the following class of weights. Let $\mathcal{Q}$ be a collection of dyadic cubes. We say that $W \in \mathcal{A}_{p,q}^{\mathcal{Q}}$ if

$$[W]_{\mathcal{A}_{p,q}^{\mathcal{Q}}} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q}|_{p'_{\text{op}}}^{p'} dy \right)^{q/p'} dx < \infty.$$

Proof of Theorem 1.7. Given a matrix weight W , observe that $L^{-\alpha/2} : L^p(W^{p/q}) \rightarrow L^q(W)$ if and only if

$$W^{1/q} L^{-\alpha/2} W^{-1/q} : L^p(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L^p(\mathbb{R}^d, \mathbb{C}^n).$$

Hence, it suffices to show that

$$|\langle W^{1/q} L^{-\alpha/2} W^{-1/q} \vec{f}, \vec{g} \rangle_{L^2}| \lesssim [W]_{\mathcal{A}_{p,q}^{\rho, \theta/(3K)}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}.$$

By Proposition 3.4, fixing a dyadic grid $\mathcal{D}$, it is enough to prove

$$\begin{aligned}
& \sum_{Q \in \mathcal{D}} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\
& \lesssim [W]_{\mathcal{A}_{p,q}^{\rho, \theta/(3K)}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}.
\end{aligned}$$

For $r \geq 0$, here and in the following, we denote $\mathcal{Q}_r := \{Q \in \mathcal{D} : \tilde{\psi}_\theta(Q) \sim 2^{r\theta}\}$. Then

$$\begin{aligned}
& \sum_{Q \in \mathcal{D}} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\
& = \sum_{r \geq 0} \sum_{Q \in \mathcal{Q}_r} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\
& \sim \sum_{r \geq 0} 2^{-r\theta} \sum_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy.
\end{aligned}$$

The operator

$$\sum_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int_Q \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy$$

is very similar to the operator

$$\sum_{Q \in \mathcal{D}} \frac{1}{|Q|^{1-\alpha/d}} \int_Q \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy,$$

given in [27] (except that $\mathcal{Q}_r = \mathcal{D}$), where the authors use it to obtain the quantitative matrix-weighted estimates in Theorem 1.3. Now for $W \in \mathcal{A}_{p,q}^{\mathcal{Q}_r}$, by carefully following a similar scheme to the proof of Theorem 1.3, we obtain

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int_Q \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\ \lesssim [W]_{\mathcal{A}_{p,q}^{\mathcal{Q}_r}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}. \end{aligned}$$

Note that for $Q \in \mathcal{Q}_r$,

$$\frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W(x)^{1/q} W(y)^{-1/q} |_{\text{op}}^{p'} dy \right)^{q/p'} dx \leq [W]_{\tilde{\mathcal{A}}_{p,q}}^{\rho, \theta/K} 2^{r(\theta/K + \theta q/(Kp'))}.$$

Thus, $[W]_{\mathcal{A}_{p,q}^{\mathcal{Q}_r}} \leq [W]_{\tilde{\mathcal{A}}_{p,q}^{\rho, \theta/K}} 2^{r(\theta/K + \theta q/(Kp'))}$. It follows that

$$\begin{aligned} \sum_{Q \in \mathcal{D}} \frac{1}{\tilde{\psi}_\theta(Q) |Q|^{1-\alpha/d}} \int_Q \int_Q |\langle W(y)^{-1/q} \vec{f}(y), W(x)^{1/q} \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\ \lesssim \sum_{r \geq 0} 2^{-r\theta} ([W]_{\tilde{\mathcal{A}}_{p,q}^{\rho, \theta/K}} 2^{r(\theta/K + \theta q/(Kp'))})^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}} \\ \lesssim \sum_{r \geq 0} 2^{-r\theta/2} [W]_{\mathcal{A}_{p,q}^{\rho, \theta/(3K)}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}} \\ \sim [W]_{\mathcal{A}_{p,q}^{\rho, \theta/(3K)}}^{\frac{p'}{q}(1-\frac{\alpha}{d})+\frac{1}{q'}} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}. \blacksquare \end{aligned}$$

PROPOSITION 3.5. *Let $\theta \geq 0$, $0 \leq \alpha < d$, $1 < q < \infty$ and U, V be matrix weights. Then for any $x \in \mathbb{R}^d$,*

$$M_{U,V,\psi_\theta,\alpha} \vec{f}(x) \lesssim \sum_{t \in \{0,1/3\}^d} M_{U,V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}^t} \vec{f}(x),$$

where

$$M_{U,V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} \vec{f}(x) := \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \frac{1}{(\tilde{\psi}_\theta(Q) |Q|)^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy.$$

Proof. We keep the notations given in the proof of Proposition 3.4. For any cube $Q := Q(x_Q, r_Q)$, by Lemma 3.3, there exist $t \in \{0, 1/3\}^d$ and $Q^t \in \mathcal{D}^t$ such that $Q \subset Q^t$ with $l_{Q^t} \leq 12r_Q$. Observe that $\rho(x_Q) \leq \tilde{\rho}(Q^t)$; then

$$\begin{aligned} & \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy \\ & \lesssim \frac{1}{(\tilde{\psi}_\theta(Q^t)|Q^t|)^{1-\alpha/d}} \int_{Q^t} |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy \\ & \leq M_{U,V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}^t} \vec{f}(x). \end{aligned}$$

Thus,

$$M_{U,V,\psi_\theta,\alpha} \vec{f}(x) \lesssim \sum_{t \in \{0,1/3\}^d} M_{U,V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}^t} \vec{f}(x). \quad \blacksquare$$

Proof of Theorem 1.8. (1) For $x \in \mathbb{R}^d$, fix a cube $Q \ni x$. It is immediate that

$$\begin{aligned} \left| \frac{\chi_Q(x)}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q W(x)^{1/q} \vec{f}(y) dy \right| & \leq \frac{\chi_Q(x)}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |W(x)^{1/q} \vec{f}(y)| dy \\ & \leq M_{W,\psi_\theta,\alpha} (W^{1/q} \vec{f}). \end{aligned}$$

By the assumption, we have

$$\sup_Q \left\| \frac{\chi_Q}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q \vec{f}(y) dy \right\|_{L^q(W)} \lesssim \|\vec{f}\|_{L^p(W^{p/q})}.$$

Thus, (1) follows by Theorem 1.14, which will be proved in Section 5.

(2) By Proposition 3.5, fixing a dyadic grid $\mathcal{D}$, it suffices to prove that

$$\|M_{W,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} \vec{f}\|_{L^q} \lesssim [W]_{A_{p,q}^{p',\theta/(3K)}}^{p'(1-\alpha/d)/q} \|\vec{f}\|_{L^p},$$

where $M_{W,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} := M_{W,W,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}}$. Note that

$$M_{W,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} \vec{f}(x) \leq \sum_{r \geq 0} 2^{-r\theta(1-\alpha/d)} \sup_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |W(x)^{1/q} W(y)^{-1/q} \vec{f}(y)| dy.$$

Following a similar scheme to the proof of Theorem 1.3, we have

$$\begin{aligned} & \left\| \sup_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |W(x)^{1/q} W(y)^{-1/q} \vec{f}(y)| dy \right\|_{L^q} \\ & \lesssim [W]_{A_{p,q}^{p',\theta/(3K)}}^{p'(1-\alpha/d)/q} \|\vec{f}\|_{L^p} \\ & \leq ([W]_{\tilde{A}_{p,q}^{p',\theta/K}} 2^{r(\theta/K + \theta q/(Kp'))})^{p'(1-\alpha/d)/q} \|\vec{f}\|_{L^p} \\ & \lesssim [W]_{A_{p,q}^{p',\theta/(3K)}}^{p'(1-\alpha/d)/q} 2^{r\theta(p'/(Kq) + 1/K)(1-\alpha/d)} \|\vec{f}\|_{L^p}. \end{aligned}$$

Hence,

$$\begin{aligned} \|M_{W,\psi_\theta,\alpha}^{\mathcal{D}} \vec{f}\|_{L^q} &\lesssim [W]_{\mathcal{A}_{p,q}^{\rho,\theta/(3K)}}^{p'(1-\alpha/d)/q} \|\vec{f}\|_{L^p} \sum_{r \geq 0} 2^{-r\theta(1-\alpha/d)/2} \\ &\leq [W]_{\mathcal{A}_{p,q}^{\rho,\theta/(3K)}}^{p'(1-\alpha/d)/q} \|\vec{f}\|_{L^p}. \blacksquare \end{aligned}$$

4. Two-weight estimates for fractional type integrals associated with operators. The goal of this section is to prove Theorems 1.9, 1.10 and 1.12.

Proof of Theorem 1.9. By Fatou's lemma, we only need to prove that

$$|\langle U^{1/q} L^{-\alpha/2} V^{-1/q} \vec{f}, \vec{g} \rangle_{L^2}| \lesssim [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}},$$

where $\vec{f}, \vec{g}$ are bounded functions of compact support. Applying Proposition 3.4, we have

$$\begin{aligned} &|\langle U^{1/q} L^{-\alpha/2} V^{-1/q} \vec{f}, \vec{g} \rangle_{L^2}| \\ &\lesssim \sum_{t \in \{0,1/3\}^d} \sum_{Q \in \mathcal{D}^t} \frac{|Q|^{\alpha/d-1}}{\tilde{\psi}_\theta(Q)} \int_Q \int_Q |\langle U(x)^{1/q} V(y)^{-1/q} \vec{f}(y), \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy. \end{aligned}$$

Now, fix a dyadic grid $\mathcal{D}$. By the generalized Hölder inequality and (2.6), we deduce that

$$\begin{aligned} &\sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/d-1}}{\tilde{\psi}_\theta(Q)} \int_Q \int_Q |\langle U(x)^{1/q} V(y)^{-1/q} \vec{f}(y), \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\ &\lesssim \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/d+1}}{\tilde{\psi}_\theta(Q)} \left\| \|U(x)^{1/q} V(y)^{-1/q}\|_{\Phi_y, Q} \right\|_{\Psi_x, Q} \|\vec{f}\|_{\bar{\Phi}, Q} \|\vec{g}\|_{\bar{\Psi}, Q} \\ &= \sum_{Q \in \mathcal{D}} \frac{1}{\tilde{\psi}_\theta(Q)} |Q|^{\alpha/d+1/q-1/p} \left\| \|U(x)^{1/q} V(y)^{-1/q}\|_{\Phi_y, Q} \right\|_{\Psi_x, Q} \\ &\quad \times |Q|^{1/p-1/q+1} \|\vec{f}\|_{\bar{\Phi}, Q} \|\vec{g}\|_{\bar{\Psi}, Q} \\ &\lesssim [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \sum_{Q \in \mathcal{D}} |Q|^{1/p-1/q+1} \|\vec{f}\|_{\bar{\Phi}, Q} \|\vec{g}\|_{\bar{\Psi}, Q}. \end{aligned}$$

Let

$$\mathcal{Q}^k := \{Q \in \mathcal{D} : a^k < \|\vec{f}\|_{\bar{\Phi}, Q} \leq a^{k+1}\},$$

where a is to be determined later. Let also $\mathcal{S}^k$ be the maximal disjoint collection of cubes $Q \in \mathcal{D}$ such that $a^k < \|\vec{f}\|_{\bar{\Phi}, Q}$. Then

$$\begin{aligned}
\sum_{Q \in \mathcal{D}} |Q|^{1/p-1/q+1} \|\vec{f}\|_{\vec{\Phi}, Q} \|\vec{g}\|_{\vec{\Psi}, Q} &\leq \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{P \in Q^k} |P|^{1/p-1/q+1} \|\vec{g}\|_{\vec{\Psi}, P} \\
&\leq \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{Q \in \mathcal{S}^k} \sum_{P \in \mathcal{D}(Q)} |P|^{1/p-1/q+1} \|\vec{g}\|_{\vec{\Psi}, P},
\end{aligned}$$

where $\mathcal{D}(Q)$ is the set of all dyadic cubes with respect to Q . By making use of the equivalence between the Orlicz average norm and the Amemiya norm (see [42, p. 69]),

$$\|\vec{f}\|_{\vec{\Phi}, Q} \leq \|\vec{f}\|'_{\vec{\Phi}, Q} \leq 2\|\vec{f}\|_{\vec{\Phi}, Q},$$

where

$$\|\vec{f}\|'_{\vec{\Phi}, Q} := \inf \left\{ \lambda + \frac{\lambda}{|Q|} \int_Q \Phi \left(\frac{|\vec{f}(y)|}{\lambda} \right) dy \right\},$$

for $Q \in \mathcal{S}^k$, we have

$$\begin{aligned}
&\sum_{P \in \mathcal{D}(Q)} |P|^{1/p-1/q+1} \|\vec{g}\|_{\vec{\Psi}, P} \\
&\leq \sum_{P \in \mathcal{D}(Q)} |P|^{1/p-1/q+1} \left(\lambda + \frac{\lambda}{|P|} \int_P \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right) \\
&= \lambda \sum_{r=0}^{\infty} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r} l_Q}} \left[2^{-(1/p-1/q+1)rd} |Q|^{1/p-1/q+1} \right. \\
&\quad \left. + 2^{-(1/p-1/q)rd} |Q|^{1/p-1/q} \int_P \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right].
\end{aligned}$$

Fixing $r > 0$, note that

$$\sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r} l_Q}} 1 = 2^{rd}.$$

Then

$$\begin{aligned}
(4.1) \quad &\lambda \sum_{r=0}^{\infty} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r} l_Q}} 2^{-(1/p-1/q+1)rd} |Q|^{1/p-1/q+1} \\
&= \lambda |Q|^{1/p-1/q+1} \sum_{r=0}^{\infty} 2^{-(1/p-1/q+1)rd} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r} l_Q}} 1 \\
&= \lambda |Q|^{1/p-1/q+1} \sum_{r=0}^{\infty} 2^{-(1/p-1/q)rd}.
\end{aligned}$$

On the other hand, since the collection $\{P\}_{P \in \mathcal{D}(Q), l_P = 2^{-r} l_Q}$ is mutually dis-

joint and $\bigcup_{P \in \mathcal{D}(Q), l_P = 2^{-r}l_Q} P = Q$, we infer that

$$\begin{aligned}
 (4.2) \quad & \lambda \sum_{r=0}^{\infty} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r}l_Q}} 2^{-(1/p-1/q)rd} |Q|^{1/p-1/q} \int_P \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \\
 &= \lambda |Q|^{1/p-1/q} \sum_{r=0}^{\infty} 2^{-(1/p-1/q)rd} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r}l_Q}} \int_P \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \\
 &= \lambda |Q|^{1/p-1/q} \int_Q \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \sum_{r=0}^{\infty} 2^{-(1/p-1/q)rd}.
 \end{aligned}$$

Hence, (4.1) and (4.2) imply that

$$\begin{aligned}
 & \lambda \sum_{r=0}^{\infty} \sum_{\substack{P \in \mathcal{D}(Q) \\ l_P = 2^{-r}l_Q}} \left[2^{-(1/p-1/q+1)rd} |Q|^{1/p-1/q+1} \right. \\
 & \quad \left. + 2^{-(1/p-1/q)rd} |Q|^{1/p-1/q} \int_P \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right] \\
 &= \lambda \sum_{r=0}^{\infty} 2^{-(1/p-1/q)rd} \left(|Q|^{1/p-1/q+1} + |Q|^{1/p-1/q} \int_Q \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \sum_{P \in \mathcal{D}(Q)} |P|^{1/p-1/q+1} \|\vec{g}\|_{\bar{\Psi}, P} \\
 & \leq \lambda \sum_{r=0}^{\infty} 2^{-(1/p-1/q)rd} \left(|Q|^{1/p-1/q+1} + |Q|^{1/p-1/q} \int_Q \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right) \\
 & \sim |Q|^{1/p-1/q+1} \left(\lambda + \frac{\lambda}{|Q|} \int_Q \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx \right).
 \end{aligned}$$

Choose λ such that

$$\lambda + \frac{\lambda}{|Q|} \int_Q \bar{\Psi} \left(\frac{|\vec{g}(x)|}{\lambda} \right) dx < 2 \|\vec{g}\|'_{\bar{\Psi}, Q} \leq 4 \|\vec{g}\|_{\bar{\Psi}, Q}.$$

We arrive at

$$\begin{aligned}
 & \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/d-1}}{\tilde{\psi}_{\theta}(Q)} \int_Q |\langle U(x)^{1/q} V(y)^{-1/q} \vec{f}(y), \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\
 & \lesssim [U, V]_{p, q, \alpha, \psi_{\theta}, \Phi, \Psi} \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{Q \in \mathcal{S}^k} |Q|^{1/p-1/q+1} \|\vec{g}\|_{\bar{\Psi}, Q}.
 \end{aligned}$$

Now we temporarily stop the proof of the conclusion, and turn to showing that $\mathcal{S} = \bigcup_k \mathcal{S}^k$ is sparse. We adapt the idea of [17]. Denote

$$\Omega_k = \bigcup_{Q \in \mathcal{S}^k} Q.$$

We are done if

$$(4.3) \quad |Q \cap \Omega_{k+1}| \leq |Q|/2.$$

To prove (4.3) note that

$$|Q \cap \Omega_{k+1}| = \sum_{\substack{P \in \Omega_{k+1} \\ P \subset Q}} |P|, \quad Q \in \mathcal{S}^k,$$

and

$$a^{k+1} \leq \lambda + \frac{\lambda}{|P|} \int_P \bar{\Phi} \left(\frac{|\vec{f}(x)|}{\lambda} \right) dx, \quad P \in \Omega_{k+1}.$$

Choose λ_0 such that

$$\lambda_0 + \frac{\lambda_0}{|Q|} \int_Q \bar{\Phi} \left(\frac{|\vec{f}(x)|}{\lambda_0} \right) dx < 4 \|\vec{f}\|_{\bar{\Phi}, Q}.$$

Hence

$$\begin{aligned} \sum_{\substack{P \in \Omega_{k+1} \\ P \subset Q}} |P| &\leq \sum_{\substack{P \in \Omega_{k+1} \\ P \subset Q}} a^{-k-1} \left(\lambda_0 |P| + \lambda_0 \int_P \bar{\Phi} \left(\frac{|\vec{f}(x)|}{\lambda_0} \right) dx \right) \\ &\leq a^{-k-1} \left(\lambda_0 |Q| + \lambda_0 \int_Q \bar{\Phi} \left(\frac{|\vec{f}(x)|}{\lambda_0} \right) dx \right) \\ &\leq \frac{2^{d+2} |Q|}{a^{k+1}} \|\vec{f}\|_{\bar{\Phi}, \hat{Q}} \leq |Q|/2, \end{aligned}$$

where $\hat{Q}$ is the dyadic parent of Q and $a = 2^{d+3}$. This proves (4.3).

We return to the proof of the desired conclusion. Let

$$\begin{aligned} \beta/d &= 1/p - 1/s, \\ \gamma/d &= 1/q' - 1/s'. \end{aligned}$$

Define

$$M_A^\beta f(x) := \sup_{Q \ni x} |Q|^{\beta/d} \|f\|_{A, Q},$$

where A is a Young function. It is known (see [18]) that

$$M_A^\beta : L^p(\mathbb{R}^d) \rightarrow L^s(\mathbb{R}^d), \quad A \in B_{p, s}.$$

Since $\mathcal{S}$ is sparse, we have

$$\begin{aligned}
& \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/d-1}}{\tilde{\psi}_\theta(Q)} \int_Q |\langle U(x)^{1/q} V(y)^{-1/q} \vec{f}(y), \vec{g}(x) \rangle_{\mathbb{C}^n}| dx dy \\
& \lesssim a[U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \sum_{Q \in \mathcal{S}} |Q|^{\beta/d+1/s} \|\vec{f}\|_{\vec{\Phi},Q} |Q|^{\gamma/d+1/s'} \|\vec{g}\|_{\vec{\Psi},Q} \\
& \lesssim [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \sum_{Q \in \mathcal{S}} |Q|^{\beta/d} \|\vec{f}\|_{\vec{\Phi},Q} |Q|^{\gamma/d} \|\vec{g}\|_{\vec{\Psi},Q} |E_Q| \\
& \leq [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \int_{\mathbb{R}^d} M_\Phi^\beta(\vec{f})(x) M_\Psi^\gamma(\vec{g})(x) dx \\
& \leq [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \|M_\Phi^\beta\|_{L^s} \|M_\Psi^\gamma\|_{L^{s'}} \\
& \lesssim [U, V]_{p,q,\alpha,\psi_\theta,\Phi,\Psi} \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}. \blacksquare
\end{aligned}$$

To prove Theorem 1.10, we first show that the case $1/p - 1/q > \alpha/d$ is trivial.

PROPOSITION 4.1. *Let $\theta \geq 0$, $0 < \alpha < d$, $1 < p < q < \infty$ and $1/p - 1/q > \alpha/d$. Assume that $M_{U,V,\psi_\theta,\alpha} : L^p \rightarrow L^q$ and $V^{1/q} \in L^p_{\text{loc}}$. Then $U(x) = 0$ for a.e. $x \in \mathbb{R}^d$.*

Proof. Fix a cube Q and a vector $\vec{e}$. Set $\vec{f}(y) = V(y)^{1/q} \vec{e} \chi_Q(y)$. From the definition of $M_{U,V,\psi_\theta,\alpha}$, for $x \in Q$,

$$M_{U,V,\psi_\theta,\alpha} \vec{f}(x) \geq \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |U(x)^{1/q} \vec{e}| dy = \frac{|Q|^{\alpha/d}}{\psi_\theta(Q)^{1-\alpha/d}} |U(x)^{1/q} \vec{e}|.$$

Then the $L^p \rightarrow L^q$ boundedness of $M_{U,V,\psi_\theta,\alpha}$ yields

$$\begin{aligned}
& \left(\frac{|Q|^{\alpha q/d}}{\psi_\theta(Q)^{q(1-\alpha/d)}} \int_Q |U(x)^{1/q} \vec{e}|^q dx \right)^{1/q} \\
& \leq \|M_{U,V,\psi_\theta,\alpha} \vec{f}\|_{L^q} \lesssim \|\vec{f}\|_{L^p} = \left(\int_Q |V(x)^{1/q} \vec{e}|^p dx \right)^{1/p}.
\end{aligned}$$

That is,

$$\begin{aligned}
(4.4) \quad & \left(\frac{1}{|Q|} \int_Q |U(x)^{1/q} \vec{e}|^q \right)^{1/q} \\
& \lesssim |Q|^{1/p-1/q-\alpha/d} \psi_\theta(Q)^{1-\alpha/d} \left(\frac{1}{|Q|} \int_Q |V(x)^{1/q} \vec{e}|^p \right)^{1/p}.
\end{aligned}$$

Let x_0 be any Lebesgue point of $|U(x)^{1/q} \vec{e}|^q$ and $|V(x)^{1/q} \vec{e}|^p$. Let Q_k be a sequence of cubes that are centered at x_0 and shrink to x_0 . Combining $1/p - 1/q - \alpha/d > 0$, $1 - \alpha/d > 0$ and the Lebesgue differentiation theorem

shows that the right-hand side of (4.4) tends to 0. So $|U(x_0)^{1/q}\vec{e}|^q = 0$ for any $\vec{e}$. Therefore $U(x_0) = 0$. ■

Proof of Theorem 1.10. Fix a dyadic grid $\mathcal{D}$. In the proof of Theorem 1.4, it was essentially proved that

$$\|M_{U,V,\alpha}^{\mathcal{D}}\|_{L^p \rightarrow L^q} \lesssim [U, V]_{p,q,\Phi}^{\mathcal{D}}$$

provided that $[U, V]_{p,q,\Phi}^{\mathcal{D}} < \infty$, where

$$M_{U,V,\alpha}^{\mathcal{D}} \vec{f}(x) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy$$

and

$$[U, V]_{p,q,\Phi}^{\mathcal{D}} := \sup_{Q \in \mathcal{D}} |Q|^{\alpha/d+1/q-1/p} \left(\int_Q \|U(x)^{1/q} V^{-1/q}\|_{\Phi,Q}^q dx \right)^{1/q}, \quad \bar{\Phi} \in B_{p,q}.$$

Denote

$$[U, V]_{p,q,\alpha,\tilde{\psi}_{\theta'},\Phi} := \sup_Q \frac{|Q|^{\alpha/d+1/q-1/p}}{\tilde{\psi}_{\theta'}(Q)^{1-\alpha/d}} \left(\int_Q \|U(x)^{1/q} V^{-1/q}\|_{\Phi,Q}^q dx \right)^{1/q}.$$

For any $Q \in \mathcal{Q}_r$, one can check that

$$|Q|^{\alpha/d+1/q-1/p} \left(\int_Q \|U(x)^{1/q} V^{-1/p}\|_{\Phi,Q}^q dx \right)^{1/q} \leq 2^{r\theta'(1-\alpha/d)} [U, V]_{p,q,\alpha,\tilde{\psi}_{\theta'},\Phi},$$

which implies that

$$[U, V]_{p,q,\Phi}^{\mathcal{Q}_r} \leq 2^{r\theta'(1-\alpha/d)} [U, V]_{p,q,\alpha,\tilde{\psi}_{\theta'},\Phi},$$

where

$$[U, V]_{p,q,\Phi}^{\mathcal{Q}_r} := \sup_{Q \in \mathcal{Q}_r} |Q|^{\alpha/d+1/q-1/p} \left(\int_Q \|U(x)^{1/q} V^{-1/p}\|_{\Phi,Q}^q dx \right)^{1/q}.$$

Mimicking the scheme of the proof of Theorem 1.4 and using $\theta > \theta'$, we have

$$\begin{aligned} & \|M_{U,V,\tilde{\psi}_{\theta},\alpha}^{\mathcal{D}} \vec{f}\|_{L^q} \\ & \leq \sum_{r \geq 0} 2^{-r\theta(1-\alpha/d)} \left\| \sup_{Q \in \mathcal{Q}_r} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |U(x)^{1/q} V(y)^{-1/q} \vec{f}(y)| dy \right\|_{L^q} \\ & \lesssim \sum_{r \geq 0} 2^{-r\theta(1-\alpha/d)} [U, V]_{p,q,\Phi}^{\mathcal{Q}_r} \|\vec{f}\|_{L^p} \\ & \lesssim [U, V]_{p,q,\alpha,\tilde{\psi}_{\theta'},\Phi} \|\vec{f}\|_{L^p}. \quad \blacksquare \end{aligned}$$

Finally, we prove Theorem 1.12. We first establish the following necessary lemma.

LEMMA 4.2. Let $1/p - 1/q = \alpha/d$ and $(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta/3}$. Then for each $\lambda > 0$, there is a disjoint collection $\{Q_j\}$ of maximal dyadic cubes such that

$$E_\lambda := \{x \in \mathbb{R}^d : M_{\mathcal{U},V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} \vec{f}(x) > \lambda\} = \bigcup_j Q_j$$

and for each j ,

$$\lambda < \frac{1}{(\tilde{\psi}_\theta(Q_j)|Q_j|)^{1-\alpha/d}} \int_{Q_j} |\mathcal{U}_{Q_j}^{q,q} V(y)^{-1/q} \vec{f}(y)| dy.$$

Proof. Assume that $E_\lambda \neq \emptyset$, otherwise there is nothing to prove. Let $\bar{E}_\lambda$ be the family of dyadic cubes such that

$$\lambda < \frac{1}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy.$$

It is easy to see that the set $\bar{E}_\lambda$ is non-empty. For any $Q \in \bar{E}_\lambda$, by our assumption, Hölder's inequality, $\alpha/d - 1/p = -1/q$, Lemma 2.3 and (2.6), one can check that

$$\begin{aligned} & \frac{1}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy \\ & \leq \frac{|Q|^{\alpha/d}}{\tilde{\psi}_\theta(Q)^{1-\alpha/d}} \left(\frac{1}{|Q|} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q}|_{\text{op}}^{p'} \right)^{1/p'} \left(\frac{1}{|Q|} \int_Q |\vec{f}(y)|^p dy \right)^{1/p} \\ & \lesssim |Q|^{-1/q} \tilde{\psi}_\theta(Q)^{\alpha/d-1} |\mathcal{U}_Q^{q,q} \tilde{\mathcal{V}}_Q^{q,p'}|_{\text{op}} \|\vec{f}\|_{L^p} \lesssim |Q|^{-1/q} [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta/3}}^{1/q} \|\vec{f}\|_{L^p}. \end{aligned}$$

It follows that

$$\lambda < \frac{1}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy \rightarrow 0$$

as $|Q| \rightarrow \infty$. Now denote the family of such maximal cubes by $\{Q_j\}$ (the subset of $\bar{E}_\lambda$). It is clear that these cubes are pairwise disjoint.

Finally, let us prove $E_\lambda = \bigcup_j Q_j$. Let $x \in E_\lambda$. There is a dyadic cube $Q \ni x$ such that

$$\lambda < \frac{1}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy.$$

So $E_\lambda \subset Q_j$ for some j .

On the other hand, let $x \in Q_j$. Since

$$\lambda < \frac{1}{(\tilde{\psi}_\theta(Q_j)|Q_j|)^{1-\alpha/d}} \int_{Q_j} |\mathcal{U}_{Q_j}^{q,q} V(y)^{-1/q} \vec{f}(y)| dy,$$

we have $M_{\alpha,\tilde{\psi}_\theta,\mathcal{U},V}^{\mathcal{D}} \vec{f}(x) > \lambda$. That is, $\bigcup_j Q_j \subset E_\lambda$. Hence, $E_\lambda = \bigcup_j Q_j$. ■

Proof of Theorem 1.12. (1) By $\mathcal{A}_Q^{\alpha,\theta} \vec{f} \leq M_{U,V,\psi,\alpha} \vec{f}$, it suffices to prove the result for $M_{U,V,\psi,\alpha}$. Similarly to the proof of Proposition 3.5, it suffices to prove $M_{\alpha,\tilde{\psi},\mathcal{U},V}^{\mathcal{D}}$ is bounded from L^p to $L^{q,\infty}$, where

$$M_{\alpha,\tilde{\psi},\mathcal{U},V}^{\mathcal{D}} \vec{f}(x) := \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \frac{1}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q |\mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y)| dy.$$

By Lemmas 2.3 and 4.2 and since $1/p - 1/q = \alpha/d$,

$$\begin{aligned} & |\{x \in \mathbb{R}^d : M_{\mathcal{U},V,\tilde{\psi}_\theta,\alpha}^{\mathcal{D}} \vec{f}(x) > \lambda\}| \\ & \leq \lambda^{-q} \sum_j |Q_j| \left(\frac{1}{(\tilde{\psi}_\theta(Q_j)|Q_j|)^{1-\alpha/d}} \int_{Q_j} |\mathcal{U}_{Q_j}^{q,q} V(y)^{-1/q} \vec{f}(y)| dy \right)^q \\ & \leq \lambda^{-q} \sum_j \tilde{\psi}_\theta(Q_j)^{q\alpha/d-q} \left(\int_{Q_j} |\mathcal{U}_{Q_j}^{q,q} V(y)^{-1/q}|_{\text{op}}^{p'} \right)^{q/p'} \left(\int_{Q_j} |\vec{f}(y)|^p dy \right)^{q/p} \\ & \sim \lambda^{-q} \sum_j \tilde{\psi}_\theta(Q_j)^{q\alpha/d-q} |\mathcal{U}_{Q_j}^{q,q} \tilde{\mathcal{V}}_{Q_j}^{q,p'}|_{\text{op}}^q \left(\int_{Q_j} |\vec{f}(y)|^p dy \right)^{q/p} \\ & \lesssim \lambda^{-q} [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta/3}} \|\vec{f}\|_{L^p}^q, \end{aligned}$$

where we use $p \leq q$ and the cubes $\{Q_j\}_j$ are disjoint in the last inequality.

(2) We only prove the conclusion in the case $p > 1$, since the case $p = 1$ is almost the same, except that we take the operator norm of the matrix and use Lemma 2.3 instead of using duality to obtain the $L^{p'}$ norm. Note that for any $\vec{e} \in \mathbb{C}^n$,

$$\|\chi_Q \vec{e}\|_{L^{q,\infty}} = |Q|^{1/q} |\vec{e}|.$$

This, together with duality, yields

$$\begin{aligned} & \sup_Q \sup_{\|\vec{f}\|_{L^p}=1} \left\| \chi_Q (\psi_\theta(Q)|Q|)^{\alpha/d-1} \int_Q \mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y) dy \right\|_{L^{q,\infty}} \\ & = \sup_Q \sup_{\|\vec{f}\|_{L^p}=1} \left| |Q|^{\alpha/d+1/q} \psi_\theta(Q)^{\alpha/d-1} \int_Q \mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y) dy \right| \\ & = \sup_Q \sup_{\|\vec{f}\|_{L^p}=1} \sup_{|\vec{e}|=1} |Q|^{\alpha/d+1/q} \psi_\theta(Q)^{\alpha/d-1} \int_Q \langle \mathcal{U}_Q^{q,q} V(y)^{-1/q} \vec{f}(y), \vec{e} \rangle_{\mathbb{C}^n} dy \\ & = \sup_Q \sup_{|\vec{e}|=1} \sup_{\|\vec{f}\|_{L^p}=1} |Q|^{\alpha/d+1/q-1} \psi_\theta(Q)^{\alpha/d-1} \int_{\mathbb{R}^d} \langle \vec{f}(y), \chi_Q V(y)^{-1/q} \mathcal{U}_Q^{q,q} \vec{e} \rangle_{\mathbb{C}^n} dy \\ & = \sup_Q \sup_{|\vec{e}|=1} |Q|^{\alpha/d+1/q-1} \psi_\theta(Q)^{\alpha/d-1} \|\chi_Q V(y)^{-1/q} \mathcal{U}_Q^{q,q} \vec{e}\|_{L^{p'}} \end{aligned}$$

$$\begin{aligned}
&= \sup_Q \sup_{|\vec{e}|=1} \psi_\theta(Q)^{\alpha/d-1} \left(\frac{1}{|Q|} \int_Q |V(y)^{-1/q} \mathcal{U}_Q^{q,q} \vec{e}|^{p'} dy \right)^{1/p'} \\
&\sim \sup_Q \psi_\theta(Q)^{\alpha/d-1} |\tilde{\mathcal{V}}_Q^{q,p'} \mathcal{U}_Q^{q,q}|_{\text{op}} \sim [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta}}^{1/q}.
\end{aligned}$$

Then (2) follows immediately by the assumption $\mathcal{A}_Q^{\alpha,\theta} : L^p \rightarrow L^{q,\infty}$ and $\mathcal{A}_Q^{\alpha,\theta} \vec{f} \leq M_{U,V,\psi_\theta,\alpha} \vec{f}$. ■

5. A characterization of matrix weights. In this section, we prove Theorem 1.14. The idea originates from [16], however, due to the additional critical radius function factor in the definitions of the average operator associated with the critical radius function, we get narrow ranges of p, q .

Proof of Theorem 1.14. (1) $\Rightarrow$ (2): CASE 1: $p > 1$. By Hölder's inequality, $(U, V) \in \mathcal{A}_{p,q}^{\rho,\theta}$ and $1 + q/p' = q - q\alpha/d$, we have

$$\begin{aligned}
&\|A_Q^{\alpha,\theta} \vec{f}\|_{L^q(U)}^q \\
&= \int_{\mathbb{R}^d} \left| \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q \chi_Q(x) U(x)^{1/q} V(y)^{-1/q} \vec{f}(y) dy \right|^q dx \\
&\leq \int_Q \frac{|Q|^{\alpha q/d}}{\psi_\theta(Q)^{q-q\alpha/d}} \left(\int_Q |U(x)^{1/q} V(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} \\
&\quad \times \left(\int_Q |V(y)^{1/q} \vec{f}(y)|^p dy \right)^{q/p} dx \\
&= \frac{1}{\psi_\theta(Q)|Q|} \int_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q |U(x)^{1/q} V(y)^{-1/q}|_{\text{op}}^{p'} dy \right)^{q/p'} dx \\
&\quad \times \left(\int_Q |V(y)^{1/q} \vec{f}(y)|^p dy \right)^{q/p} \\
&\leq [U, V]_{\mathcal{A}_{p,q}^{\rho,\theta}} \|\vec{f}\|_{L^p(V^{p/q})}^q.
\end{aligned}$$

CASE 2: $p = 1$. With the aid of Minkowski's inequality and $1 - \alpha/d = 1/q$, we obtain

$$\begin{aligned}
&\|A_Q^{\alpha,\theta} \vec{f}\|_{L^q(U)} \\
&\leq \int_Q \left(\frac{1}{\psi_\theta(Q)|Q|} \int_Q |U(x)^{1/q} V(y)^{-1/q}|_{\text{op}}^q dx \right)^{1/q} |V(y)^{1/q} \vec{f}(y)| dy \\
&\leq [U, V]_{\mathcal{A}_{1,q}^{\alpha,\theta}}^{1/q} \int_Q |V(y)^{1/q} \vec{f}(y)| dy \leq [U, V]_{\mathcal{A}_{1,q}^{\alpha,\theta}}^{1/q} \|\vec{f}\|_{L^1(V^{1/q})}.
\end{aligned}$$

(2) $\Rightarrow$ (1): CASE 1: $p > 1$. Lemma 2.3 and $1/q + 1/p' = 1 - \alpha/d$ show that

$$[U, V]_{\mathcal{A}_{p,q}^{\alpha,\theta}}^{1/q} \sim \sup_Q \psi_\theta(Q)^{\alpha/d-1} |\tilde{\mathcal{V}}_Q^{q,p'} \mathcal{U}_Q^{q,q}|_{\text{op}}.$$

Hence, we only need to prove that

$$\sup_Q \psi_\theta(Q)^{\alpha/d-1} |\tilde{\mathcal{V}}_Q^{q,p'} \mathcal{U}_Q^{q,q}|_{\text{op}} < \infty.$$

Now we fix a cube Q and let $\vec{e} \in \mathbb{C}^n$ with $|\vec{e}| = 1$. By duality, there exists $\vec{g} \in L^p(V^{p/q})$ with $\|\vec{g}\|_{L^p(V^{p/q})} = 1$ such that

$$\begin{aligned} |\tilde{\mathcal{V}}_Q^{q,p'} \mathcal{U}_Q^{q,q} \vec{e}| &\sim \left(\int_Q |V(y)^{-1/q} \mathcal{U}_Q^{q,q} \vec{e}|^{p'} dy \right)^{1/p'} = |Q|^{-1/p'} \|\chi_Q \mathcal{U}_Q^{q,q} \vec{e}\|_{L^{p'}(V^{-p'/q})} \\ &= |Q|^{-1/p'} \int_Q \langle \mathcal{U}_Q^{q,q} \vec{e}, \vec{g}(x) \rangle_{\mathbb{C}^n} dx = |Q|^{1/p} \left\langle \vec{e}, \mathcal{U}_Q^{q,q} \int_Q \vec{g}(x) dx \right\rangle_{\mathbb{C}^n} \\ &\leq |Q|^{1/p} \left| \mathcal{U}_Q^{q,q} \int_Q \vec{g}(x) dx \right| \\ &= |Q|^{1/p-\alpha/d} \psi_\theta(Q)^{1-\alpha/d} \left| \mathcal{U}_Q^{q,q} \frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_Q \vec{g}(x) dx \right| \\ &\sim |Q|^0 \psi_\theta(Q)^{1-\alpha/d} \left(\int_Q |U(y)^{1/q} A_Q^{\alpha,\theta} \vec{g}(y)|^q dy \right)^{1/q} \lesssim \psi_\theta(Q)^{1-\alpha/d}. \end{aligned}$$

Therefore, the desired result follows by rearranging the above terms.

CASE 2: $p = 1$. Fix a cube Q . Given any $S \subset Q$, where $|S| > 0$. Let $\vec{f} = \chi_S(x) V(x)^{-1/q} \vec{e}$ with $\vec{e} \in \mathbb{C}^n$ and $|\vec{e}| = 1$. By (2) and $q(1 - \alpha/d) = 1$, we have

$$\begin{aligned} |S| &\gtrsim \|A_Q^{\alpha,\theta} \vec{f}\|_{L^q(U)} \\ &= \left(\int_Q \left| U(x)^{1/q} \left(\frac{1}{(\psi_\theta(Q)|Q|)^{1-\alpha/d}} \int_S V(y)^{-1/q} \vec{e} dy \right) \right|^q dx \right)^{1/q} \\ &= |S| \psi_\theta(Q)^{\alpha/d-1} \left(\int_Q \left| U(x)^{1/q} \int_S V(y)^{-1/q} \vec{e} dy \right|^q dx \right)^{1/q}. \end{aligned}$$

Thus,

$$\psi_\theta(Q)^{\alpha/d-1} \left| \mathcal{U}_Q^{q,q} \left(\int_S V(y)^{-1/q} \vec{e} dy \right) \right| \lesssim 1.$$

Applying the Lebesgue differentiation theorem, we arrive at

$$\text{ess sup}_{y \in Q} \psi_\theta(Q)^{\alpha/d-1} |\mathcal{U}_Q^{q,q} V(y)^{-1/q}|_{\text{op}} \lesssim 1.$$

Hence, the desired conclusion follows by Lemma 2.3. ■

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Yongming Wen (corresponding author)
School of Mathematics and Statistics
Minnan Normal University
363000 Zhangzhou, China
E-mail: wenyongmingxmu@163.com

Huoxiong Wu
School of Mathematical Sciences
Xiamen University
361005 Xiamen, China
E-mail: huoxwu@xmu.edu.cn