

COMPLEMENTED IDEALS OF ℓ_∞

BY

MICHAEL HRUŠÁK and LUIS SÁENZ

Abstract. Answering questions raised by Leonetti and by Rincón-Villamizar and Uzcátegui-Aylwin we characterize ideals $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that $c_{0,\mathcal{I}}$ is complemented in ℓ_∞ as exactly those ideals for which the space $K_{\mathcal{I}} = \text{Stone}(\mathcal{P}(\omega)/\mathcal{I})$ is approximable, i.e., the unit ball of the space $M(K_{\mathcal{I}})$ of signed Radon measures on $K_{\mathcal{I}}$ is separable in the weak* topology.

Introduction. This note concerns the structure of the ideals of ℓ_∞ , the Banach lattice of all bounded sequences of reals equipped with the supremum norm and pointwise order. It is important to distinguish between the two kinds of ideals we deal with.

Recall that a *Banach lattice* is a Banach space equipped with a lattice order relation such that if $|x| = x \vee -x$, then for any $x, y \in X$, if $|x| \leq |y|$, then $\|x\| \leq \|y\|$. An *ideal* Y of a Banach lattice X is a linear subspace of X , such that if $y \in Y$, $x \in X$ and $|x| \leq |y|$, then $x \in Y$.

Further, given a nonempty set X , $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal* if $A, B \in \mathcal{I}$ implies that $A \cup B \in \mathcal{I}$, and if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$. An ideal is *proper* if $X \notin \mathcal{I}$. All ideals we consider consist of subsets of ω (the set of all natural numbers) and contain all finite subsets of ω .

Closed Banach ideals of the Banach lattice ℓ_∞ and set theoretic ideals of $\mathcal{P}(\omega)$ are closely related, as we now explain. Given an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, for a sequence $(x_n)_{n \in \omega} \in \ell_\infty$ we say its \mathcal{I} -*limit* is $r \in \mathbb{R}$ if for any $\epsilon > 0$, $\{n \in \omega : |x_n - r| \geq \epsilon\} \in \mathcal{I}$; we then write $\mathcal{I}\text{-lim } x_n = r$. So we may define

$$c_{0,\mathcal{I}} = \{(x_n)_{n \in \omega} \in \ell_\infty : \mathcal{I}\text{-lim } x_n = 0\}.$$

As noted by Uzcátegui and Rincón-Villamizar [12], every closed ideal (in the Banach lattice sense) $X \subseteq \ell_\infty$ that contains c_0 is of the form $c_{0,\mathcal{I}}$ for some (set-theoretic) ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$.

There has been interest in the question: *When is $c_{0,\mathcal{I}}$ complemented in ℓ_∞ ?* The history of this question started at the problem session of the

2020 *Mathematics Subject Classification*: Primary 46E05; Secondary 46E27, 03E05.

Key words and phrases: ideal, ℓ_∞ , complemented subspace.

Received 28 August 2025; revised 19 November 2025.

Published online 6 March 2026.

45th Winter School in Abstract Analysis (Czech Republic, 2017). Tommaso Russo communicated this question to Paolo Leonetti [9], who showed that if $\mathcal{P}(\omega)/\mathcal{I}$ is not ccc, then the answer is negative. Not long after, Tomasz Kania [7] noticed that a little more can be proved: if $\mathcal{P}(\omega)/\mathcal{I}$ is not ccc, then $C(K_{\mathcal{I}})$ cannot be embedded into ℓ_{∞} (we clarify the notation in the following section). As a consequence of our main result, both answers are equivalent, i.e., $c_{0,\mathcal{I}}$ is complemented in ℓ_{∞} if and only if $C(K_{\mathcal{I}})$ can be embedded into ℓ_{∞} . Recently, Carlos Uzcátegui and Michael A. Rincón-Villamizar [13] proposed a series of properties to analyze the question. In the last section, we provide the pertinent (counter-)examples.

Our main result seems to give a definitive answer to these questions:

THEOREM 0.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal. Then the following are equivalent:*

- (1) $c_{0,\mathcal{I}}$ is complemented in ℓ_{∞} .
- (2) $K_{\mathcal{I}}$ is approximable.
- (3) $C(K_{\mathcal{I}})$ is isometric to a subspace of ℓ_{∞} .
- (4) $C(K_{\mathcal{I}})$ is isomorphic to a subspace of ℓ_{∞} .
- (5) There is an $\epsilon \in (0, 1)$ such that $(\mathcal{P}(\omega)/\mathcal{I})^+$ can be covered by a countable family of subsets each of them with intersection number bigger than $1 - \epsilon$.

In particular, ideals \mathcal{I} satisfying the conditions of the theorem have to be such that $\mathcal{P}(\omega)/\mathcal{I}$ admits a strictly positive finitely additive measure, while ideals \mathcal{I} such that $\mathcal{P}(\omega)/\mathcal{I}$ is σ -centered, i.e. $K_{\mathcal{I}}$ is separable, provide examples.

1. Preliminaries. The following are standard notions in their respective areas.

Given an ideal \mathcal{I} , we denote by \mathcal{I}^* its dual filter; i.e., $\mathcal{I}^* = \{\omega \setminus I : I \in \mathcal{I}\}$. We denote by $K_{\mathcal{I}} = \{\mathcal{U} \in \beta\omega : \mathcal{I}^* \subseteq \mathcal{U}\}$ the corresponding compact subspace of $\beta\omega$.

We say that \mathbb{B} is *ccc* if every antichain in \mathbb{B} is countable, and \mathbb{B} is *σ -centered* if \mathbb{B}^+ can be covered by countably many ultrafilters. By $\text{Stone}(\mathbb{B})$ we denote the *Stone space* of the algebra \mathbb{B} (i.e the unique compact space such that $\mathbb{B} \simeq \text{Clopen}(\mathbb{K})$, the Boolean algebra of clopen sets.) A compact space K is *extremally disconnected* if the closure of any open set is open, equivalently if the Boolean algebra of clopen subsets of K is complete.

We say that a Boolean algebra \mathbb{B} *admits a strictly positive measure* if there is a finitely additive measure μ on \mathbb{B} such that for any $b \in \mathbb{B}^+ = \mathbb{B} \setminus \{\mathbf{0}\}$, $\mu(b) > 0$. If $s \in \mathbb{B}^{<\omega}$, we let

$$i(s) = \max \left\{ |F| : F \subseteq \text{dom}(s) \wedge \bigcap_{i \in F} s(i) \neq \emptyset \right\}.$$

Given $\mathcal{A} \subseteq \mathbb{B}$, the *intersection number* of \mathcal{A} is defined as

$$I(\mathcal{A}) = \inf \left\{ \frac{i(s)}{|s|} : s \in \mathcal{A}^{<\omega} \right\}.$$

Recall Kelley's fundamental result:

THEOREM 1.1 (Kelley [8]). *Let \mathbb{B} be a Boolean algebra. Then \mathbb{B} admits a strictly positive measure if and only if \mathbb{B}^+ can be covered by a countable family of subsets with positive intersection number.*

Given a compact space K , a *Radon measure* over K is a regular finite measure over the Borel algebra of K . We denote by $M(K)$ the set of all Radon signed measures over a compact space K . This set, equipped with the total variation norm, is a Banach space. By the Riesz representation theorem, $M(K)$ can be naturally regarded as $C(K)^*$, the dual Banach space of the space of all continuous functions over K . Following Plebanek and Džamonja [4], we say K is *approximable* if the unit ball of $M(K)$ (the set of all measures of norm at most 1) is separable in the w^* -topology. This property may be characterized strictly in terms of the Boolean algebra:

THEOREM 1.2 (Mägerl–Namioka [10]). *Let \mathbb{B} be a Boolean algebra. Then the space $\text{Stone}(\mathbb{B})$ is approximable if and only if there is $\epsilon \in (0, 1)$ such that \mathbb{B}^+ can be covered by a countable family of subsets each of them with intersection number greater than $1 - \epsilon$.*

We call a Banach space X *injective* if for any Banach spaces $Y \subseteq Z$ and any operator $T : Y \rightarrow X$ there exists an operator $\hat{T} : Z \rightarrow X$ that extends T (i.e. $\hat{T}|_Y = T$). Furthermore, we call X *1-injective* if \hat{T} can be found such that $\|\hat{T}\|/\|T\| = 1$.

THEOREM 1.3 (Goodner–Nachbin, see [1]). *Let K be a compact space. Then $C(K)$ is 1-injective if and only if K is extremally disconnected.*

PROPOSITION 1.4 ([1]). *Let X be an injective Banach space, and let Y be any Banach space such that X embeds into Y . Then X is isomorphic to a complemented subspace of Y .*

Given two Banach spaces X and Y , recall that X *embeds into* Y if there is a linear injective operator with closed range $T : X \rightarrow Y$, and that X *embeds isometrically into* Y if there is a linear operator $T : X \rightarrow Y$ such that $\|T(x)\| = \|x\|$. In this context, we will use the following result by Talagrand:

THEOREM 1.5 (Talagrand [14, Theorem II.1]). *Let K be a compact space. Then $C(K)$ embeds into ℓ_∞ if and only if $C(K)$ embeds isometrically into ℓ_∞ .*

Finally, recall fundamental result of Lindenstrauss:

THEOREM 1.6 (Lindenstrauss [1]). *Let X be an infinite-dimensional Banach subspace of ℓ_∞ . Then X is complemented if and only if $X \simeq \ell_\infty$.*

2. The main result. In order to prove Theorem 0.1, we require the following folklore result. We provide its short proof.

PROPOSITION 2.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal such that $\mathcal{P}(\omega)/\mathcal{I}$ is ccc. Then $\mathcal{P}(\omega)/\mathcal{I}$ is a complete Boolean algebra.*

Proof. It is enough to prove that if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega)/\mathcal{I}$ are disjoint such that $\mathcal{A} \cup \mathcal{B}$ is a maximal antichain, then they can be *separated*, i.e., there is a $c \in \mathbb{B}$ such that $a \leq c$ for every $a \in \mathcal{A}$ while $b \wedge c = \mathbf{0}$ for every $b \in \mathcal{B}$.

By ccc, both families are countable, since in $\mathcal{P}(\omega)/[\omega]^{<\omega}$ there are no (ω, ω) -gaps, so there is $C \in [\omega]^\omega$ that separates $(\mathcal{A}, \mathcal{B})$.

If $\mathcal{A} = \emptyset$, then C separates. Otherwise, if $A \in \mathcal{A}$, then $C \in \mathcal{I}^+$, because $A \in \mathcal{I}^+$ and $A \subseteq^* C$. ■

The next result was originally announced in [14] without proof. We provide its easy proof.

PROPOSITION 2.2. *Let K be a compact space. Then $C(K)$ is isometric to a subspace of ℓ_∞ if and only if K is approximable.*

Proof. Assume that there is an isometry $T : C(K) \rightarrow \ell_\infty$. Let $(x_n^*)_{n \in \omega}$ be any w^* -dense subset of $B_{\ell_\infty^*}$. Consider the (signed) measure μ_n defined by $\mu_n = T^*(x_n^*)$. As T is an isometry, T^* is also an isometry. Therefore, for every n , $\|\mu_n\| \leq 1$. The w^* -density of $(x_n^*)_{n \in \omega}$ implies that $(\mu_n)_{n \in \omega}$ is w^* -dense in $B_{M(K)}$.

Assume that K is approximable. Let $(\mu_n)_{n \in \omega}$ be a w^* -dense sequence in $B_{M(K)}$ (the set of measures with norm at most 1), and let $T : C(K) \rightarrow \ell_\infty$ be defined by $T(f)(n) = \int_K f d\mu_n$. Linearity is clear. Notice that

$$\|f\| = \sup_{n \in \omega} |\mu_n(f)| = \sup_{n \in \omega} \left| \int_K f d\mu_n \right| = \|T(f)\|. \quad \blacksquare$$

We also need a result mentioned by Kania without proof.

PROPOSITION 2.3 (Kania [7]). *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal. Then $\ell_\infty/c_{0,\mathcal{I}}$ is isometric to $C(K_{\mathcal{I}})$.*

Proof. Consider $T : \ell_\infty \rightarrow C(\beta\omega)$ given by $T((x_n)_{n \in \omega})(\mathcal{U}) = \mathcal{U}\text{-lim } x_n$. This is clearly a linear isometry. We have $f \in T[c_{0,\mathcal{I}}]$ if and only if, for any $\mathcal{U} \in K_{\mathcal{I}}$, $f(\mathcal{U}) = 0$. Indeed, $T((x_n)_{n \in \omega}) \in T[c_{0,\mathcal{I}}]$ if and only if $\mathcal{I}\text{-lim } x_n = 0$, if and only if, for any $\mathcal{U} \in K_{\mathcal{I}}$, $\mathcal{U}\text{-lim } x_n = 0$. This establishes that if $U : \ell_\infty/c_{0,\mathcal{I}} \rightarrow C(K_{\mathcal{I}})$ is defined by $U((x_n)_{n \in \omega}) + c_{0,\mathcal{I}}(\mathcal{U}) = \mathcal{U}\text{-lim } x_n$, then the map is well defined and linear.

Now consider the canonical quotient map $\pi : C(\beta\omega) \rightarrow C(\beta\omega)/T[c_{0,\mathcal{I}}]$. It is clear that $\|\pi(f)\| = \sup \{|f(\mathcal{U})| : \mathcal{U} \in K_{\mathcal{I}}\}$. As T is an isometry, this establishes that U is an isometry. Finally, an appeal to Tietze's extension theorem shows U is surjective. ■

Proof of Theorem 0.1. (2) \leftrightarrow (3) \leftrightarrow (4) \leftrightarrow (5) These follow from Proposition 2.2, Theorem 1.2, and Theorem 1.5.

(1) \rightarrow (3) Assume that $c_{0,\mathcal{I}}$ is complemented in ℓ_∞ . Then there is a closed subspace $Y \subseteq \ell_\infty$ such that $Y \oplus c_{0,\mathcal{I}} = \ell_\infty$. By Proposition 2.3 we deduce that $C(K_{\mathcal{I}})$ is isometric to Y .

(2) \rightarrow (1) If $K_{\mathcal{I}}$ is approximable then by Theorems 1.2 and 1.1, \mathcal{I}^+ admits a strictly positive measure. Therefore, $\mathcal{P}(\omega)/\mathcal{I}$ is ccc. By Proposition 2.1 it is extremally disconnected. By Proposition 2.2, $C(K_{\mathcal{I}})$ is isometric to a subspace of ℓ_∞ . As $K_{\mathcal{I}}$ is extremally disconnected, $C(K_{\mathcal{I}})$ is 1-injective and so it is embedded as a complemented subspace, and therefore $c_{0,\mathcal{I}}$ is complemented. ■

REMARK. It is worth noting that for compact spaces of the form $K_{\mathcal{I}}$, $C(K_{\mathcal{I}})$ being complemented in ℓ_∞ is equivalent to $C(K_{\mathcal{I}})$ being (isometrically) isomorphic to a subspace of ℓ_∞ , a fact that is not true for arbitrary compact spaces. As a direct corollary of Theorems 0.1 and 1.6 we get:

COROLLARY 2.4. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal such that $C(K_{\mathcal{I}})$ is infinite-dimensional. Then we can add to Theorem 0.1 equivalent condition that $C(K_{\mathcal{I}}) \simeq \ell_\infty$.*

Next we shall provide illuminating examples that together with Theorem 0.1 answer several questions from [13].

The following observation appears in [3, Corollary 1.2] attributed to Alan Dow:

PROPOSITION 2.5 ([3]). *Let \mathbb{B} be a complete Boolean algebra of size at most \mathfrak{c} . Then there is an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{B} \simeq \mathcal{P}(\omega)/\mathcal{I}$.*

Theorem 1.2 provides a combinatorial characterization of the fact that $C(\text{Stone}(B))$ embeds into ℓ_∞ .

PROPOSITION 2.6 ([11]). *Let \mathbb{B} be a Boolean algebra with $P(\text{Stone}(\mathbb{B}))$ approximable. Then \mathbb{B} admits a strictly positive measure.*

As the standard measure algebra $\mathbb{B}(\omega_1)$ over 2^{ω_1} to add ω_1 -many random reals does not fulfill the hypothesis of Theorem 1.2, this exemplifies why the converse implication does not follow.

Recall that the *Maharam type* of a measure is the minimal size of a subset of the σ -algebra of measurable sets dense in the metric $d(A, B) = \mu(A \Delta B)$.

PROPOSITION 2.7 ([11]). *Let K be a compact space and μ be a positive Radon measure over K of countable Maharam type. Then K is approximable.*

The authors of [4] provide a ZFC example of a Boolean algebra whose Stone space is approximable but admits no strictly positive measure with countable Maharam type (a result that improves an earlier construction of [14] under CH).

EXAMPLE 1. *There is an ideal \mathcal{I} such that $\mathcal{P}(\omega)/\mathcal{I}$ is ccc but $C(K_{\mathcal{I}})$ does not embed into ℓ_{∞} .*

For instance, a classical example in [5] presents a Boolean algebra \mathbb{B}' that is ccc but does not admit a strictly positive measure. Let \mathbb{B} be its completion. Find, using Proposition 2.5, an ideal such that $\mathcal{P}(\omega)/\mathcal{I} \simeq \mathbb{B}$. Then $K_{\mathcal{I}}$ cannot be approximable, so $C(K_{\mathcal{I}})$ cannot be embedded into ℓ_{∞} .

EXAMPLE 2. *There is an ideal \mathcal{I} such that $c_{0,\mathcal{I}}$ is complemented but $\mathcal{P}(\omega)/\mathcal{I}$ is not σ -centered (ω -maximal in the language of [13]).*

Indeed, consider the *random forcing* $\mathbb{B}(\omega) = \text{Borel}(2^{\omega})/\text{null}$. This is a complete Boolean algebra whose space of probability measures is w^* -separable (as the Maharam type of the Lebesgue measure is countable) but is not σ -centered. Use Proposition 2.5 once again to find the desired example.

We point out that our main result sheds some light on Problem 9 of [6]: *Characterize those ideals \mathcal{I} for which $c_{0,\mathcal{I}}$ is a Grothendieck space* ⁽¹⁾. It is known that any complemented subspace of a Grothendieck space is itself Grothendieck. As ℓ_{∞} is Grothendieck, any ideal \mathcal{I} that fulfills any (and therefore all) of the properties of our main theorem is such that $c_{0,\mathcal{I}}$ is Grothendieck.

Finally, we would like to know the answers to the following questions:

- (a) *Is it possible to construct a ZFC example of an extremally disconnected compact space K such that it is approximable but does not admit a strictly positive measure of countable Maharam type?*
- (b) *Is it possible to give a ZFC example of an extremally disconnected compact space K such that the ball of $M(K)$ is w^* -separable but $M(K)$ is not?* ⁽²⁾

We find these questions interesting because *negative* answers would allow us to add more properties to the list of equivalent conditions of our main theorem: (a) there is a measure on K of countable Maharam type, or/and (b) $M(K)$ is w^* -separable.

Acknowledgements. We would like to thank the anonymous referee for making several suggestions that greatly improved the quality of the paper.

Funding. The authors gratefully acknowledge support from a DGAPA-PAPIIT grant IN101323 and a SECIHTI grant CBF-2025-I-898.

⁽¹⁾ Recall that a Banach space X is *Grothendieck* if every sequence of elements of X^* that is w^* -convergent is w -convergent.

⁽²⁾ Counterexamples to both properties but without the condition of being extremally disconnected can be found in [4] and [2], respectively.

REFERENCES

- [1] F. Albiac and N. Kalton, *Topics in Banach space theory*, Springer, New York, 2006.
- [2] A. Avilés, G. Plebanek, and J. Rodríguez, *A weak* separable $C(K)^*$ space whose unit ball is not weak* separable*, Trans. Amer. Math. Soc. 366 (2011), 4733–4752.
- [3] G. Campero-Arena, O. Guzmán-González, M. Hrušák, and D. Meza-Alcántara, *Forcing properties of Boolean algebras of the type $\mathcal{P}(\omega)/I$* , Fund. Math. 272 (2026), 1–24.
- [4] M. Džamonja and G. Plebanek, *Strictly positive measures on Boolean algebras*, J. Symbolic Logic 73 (2008), 1416–1432.
- [5] H. Gaifman, *Concerning measures on Boolean algebras*, Pacific J. Math. 14 (1964), 61–73.
- [6] M. González and T. Kania, *Grothendieck spaces: the landscape and perspectives*, Japan. J. Math. 16 (2021), 247–313.
- [7] Tomasz Kania, *A letter concerning Leonetti’s paper ‘Continuous projections onto ideal convergent sequences’*, Results Math. 74 (2018), no. 1, art. 12, 4 pp.
- [8] J. L. Kelley, *Measures on Boolean algebras*, Pacific J. Math. 9 (1959), 1165–1177.
- [9] P. Leonetti, *Continuous projections onto ideal convergent sequences*, Results Math. 73 (2018), no. 3, art. 114, 5 pp.
- [10] G. Mägerl and I. Namioka, *Intersection numbers and weak* separability of spaces of measures*, Math. Ann 249 (1980), 273–279.
- [11] G. Plebanek, *A survey on topological properties of $P(K)$ spaces*, Japan. J. Math. 19 (2024), 1–38.
- [12] M. A. Rincón-Villamizar and C. Uzcátegui-Aylwin, *Banach spaces of \mathcal{I} -convergent sequences*, J. Math. Anal. Appl. 536 (2024), no. 2, art. 128271, 19 pp.
- [13] M. A. Rincón-Villamizar and C. Uzcátegui-Aylwin, *On the complementation of spaces of \mathcal{I} -null sequences*, arXiv:2507.13866v1 (2025).
- [14] M. Talagrand, *Séparabilité vague dans l’espace des mesures sur un compact*, Israel J. Math. 37 (1980), 171–180.

Michael Hrušák, Luis Sáenz
Centro de Ciencias Matemáticas
UNAM
Morelia, Michoacán, 58089, Mexico
E-mail: michael@matmor.unam.mx
luisdavidr@ciencias.unam.mx