

Counting paths in directed graphs

by

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Summary. We consider the class of directed graphs with $N \geq 1$ edges and without loops shorter than $k \geq 1$. Using the concept of a labelled graph, we determine graphs from this class that maximize the number of paths of length k . Then we show an R -labelled version of this result for semirings R contained in the semiring of non-negative real numbers and containing the semiring of non-negative rational numbers. We end by posing a related open problem concerning the maximal dimension of the path algebra of a connected acyclic directed graph with $N \geq 1$ edges.

1. Introduction. Graph theory is considered one of the oldest and most accessible branches of combinatorics and has numerous natural connections to other areas of mathematics. In particular, directed graphs, or quivers, are fundamental tools in representation theory [1] as well as in noncommutative geometry [6] and topology [4, 3]. In this paper, we focus entirely on the combinatorics of directed graphs: optimization and counting problems are common throughout combinatorics, and our paper solves one of them (cf. [7]).

Our goal is to extend the known Theorem 2.2 counting fixed-positive-length paths in acyclic directed graphs to the realm of directed graphs admitting loops. Without any assumptions on which loops are admissible, the counting problem of Theorem 2.2 trivializes as the maximal number of paths of length k in a directed graph with N edges is N^k , and the directed graph realizing this number is the directed graph with one vertex and N edges (a Hawaiian earring). To overcome this problem, we propose that to count paths of length k we should only allow loops of length at least k . This leads to our main result, which is Theorem 3.3.

2020 *Mathematics Subject Classification*: Primary 05C20; Secondary 05C30.

Key words and phrases: labelled graph, optimization.

Received 7 May 2025.

Published online 20 March 2026.

In the next section, we recall the basic definitions and results needed for our paper. Some of them are standard, some of them come from [2], and there is the inspirational Theorem 2.2. Then, in the subsequent section, first we adapt the results of Alexandru Chirvasitu to the setting of paths of length k in finite R -labelled directed graphs whose loops are of length at least k . This allows us to prove our main theorem. We follow with a bonus theorem, which is an R -labelled version of Theorem 3.3. Finally, we apply the R -labelling to find an exponential bound on the number of positive-length paths in an acyclic directed graph in terms of the number of edges. As we consider only directed graphs, in the remaining part of the paper we will often refer to directed graphs as graphs.

2. Preliminaries

DEFINITION 2.1. Let R be a commutative semiring. An R -labelled directed graph

$$E := (E^0, E^1, s, t, l)$$

is a quintuple consisting of

- (1) the set of vertices E^0 and the set of edges E^1 ,
- (2) the source and target maps $s, t: E^1 \rightarrow E^0$ assigning to each edge its source vertex and target vertex, respectively,
- (3) the label map $l: E^1 \rightarrow R \setminus \{0\}$,

satisfying the uniqueness condition (no repeating edges)

$$(s(e) = s(f) \text{ and } t(e) = t(f)) \implies e = f.$$

For $R = \mathbb{N}$, we recover the usual definition of a graph with finitely many edges between any two vertices by replacing an edge e with label $n \in \mathbb{N}$ by n edges all starting at $s(e)$ and ending at $t(e)$.

A *finite path* in an R -labelled graph E is a vertex or a finite sequence $p_n := (e_1, \dots, e_n)$ of edges satisfying

$$(2.1) \quad t(e_1) = s(e_2), \quad t(e_2) = s(e_3), \quad \dots, \quad t(e_{n-1}) = s(e_n).$$

The beginning and the end of a vertex is the vertex itself. The beginning $s(p_n)$ of p_n is $s(e_1)$ and the end $t(p_n)$ of p_n is $t(e_n)$. If $s(p_n) = t(p_n)$, we call p_n a *loop*, or a *cycle*. An R -labelled graph without loops is called *acyclic*. The length of a path is the length of the sequence. Every edge is a path of length 1, and vertices are considered as finite paths of length 0. The set of all paths in E of length k is denoted by $FP_k(E)$, and the set of all finite paths in E by $FP(E)$. *Infinite paths* are sequences of edges satisfying $t(e) = s(f)$ whenever the edge f follows the edge e , and going to infinity from the initial edge, arriving from infinity to the final edge, or being infinite in both directions. *Undirected paths* are paths in which the condition $t(e) = s(f)$

whenever the edge f follows the edge e is replaced by the requirement that $\{s(e), t(e)\} \cap \{s(f), t(f)\} \neq \emptyset$ whenever the edge f follows the edge e . We say that an R -labelled graph is *connected* when any two vertices are connected by a finite (in the sense above) undirected path.

THEOREM 2.2 ([5, 2]). *Let E be an acyclic directed graph with $N \geq 1$ edges, and let*

$$1 \leq k \leq N =: nk + r, \quad 0 \leq r \leq k - 1.$$

Then there are at most

$$P_k^N := (n + 1)^r n^{k-r}$$

different paths of length k , and the bound is optimal.

Since the number P_k^N corresponds to all possible combinations of dealing the deck of N cards to k players, we call it the *card-deck* number.

LEMMA 2.3. *Let E be an R -labelled directed graph. If any edge repeats itself in a path (e_1, \dots, e_k) in E , or if $|\{t(e_i)\}_{i=1}^k| < k$, then there is a loop of length less than k .*

Proof. Let $p := (e_1, \dots, e_k)$ be a path of length k . If $e_i = e_j$ for $i < j$, then

$$(2.2) \quad s(e_i) = s(e_j) = t(e_{j-1}),$$

so the path $p_{ij} := (e_i, \dots, e_{j-1})$ is a loop:

$$(2.3) \quad s(p_{ij}) = s(e_i) = t(e_{j-1}) = t(p_{ij}).$$

The smallest possible i is 1 and the largest possible j is k , so the maximal length of the above loop is $k - 1 < k$. Much in the same way, if $t(e_i) = t(e_j)$ for $i < j$, then (e_{i+1}, \dots, e_j) is a loop of length $j - i \leq k - 1$. ■

THEOREM 2.4. *Let E be an R -labelled directed graph. If there exists a path p that is finite, or infinite in one direction, and whose edges can be non-trivially rearranged (permuted) into a path, then there exists a loop in E .*

Proof. For starters, note that a path infinite in one direction in E is a path infinite in the opposite direction in the opposite graph E^{op} , i.e. the graph in which the source and the target maps are interchanged. Furthermore, a non-trivial permutation of edges in a path in E is tantamount to a non-trivial permutation of edges in this path considered as a path in E^{op} . Also, any loop in E is a loop in E^{op} and vice versa. Assume now that having a path $(e_i)_{i \in \mathbb{N}}$ in E that can be non-trivially permuted into a path in E implies that there is a loop in E . Then having a path $(f_j)_{j \in -\mathbb{N}}$ in E^{op} that can be non-trivially permuted into a path in E^{op} implies that there is a loop in E^{op} . Therefore, as for any graph F we have $F = (F^{\text{op}})^{\text{op}}$, we can assume without loss of generality that our path is finite or has edges indexed by \mathbb{N} .

Let S be a subset of \mathbb{N} containing at least two elements, and let $\sigma : S \rightarrow S$ be a bijection that is not the identity. Since $\sigma \neq \text{id}$, there exists the smallest $j \in S$ such that $\sigma(j) \neq j$. As σ is bijective, $\sigma(j) > j$. Indeed, if j is the smallest element of S , we are done. If there is $i < j$, then $\sigma(j) \neq \sigma(i) = i$, so $\sigma(j) > j$. Furthermore, $\sigma^{-1}(j) \neq j$. If $\sigma^{-1}(j) < j$, then we get a contradiction: $j = \sigma(\sigma^{-1}(j)) = \sigma^{-1}(j) < j$. Therefore, also $\sigma^{-1}(j) > j$.

Next, let $p := (e_1, \dots, e_n, \dots)$ or $p := (e_1, \dots, e_n)$ with $n \geq 2$. Now, let $S := \mathbb{N}$ or $S := \{1, \dots, n\}$, respectively. Suppose $p_\sigma := (e_{\sigma(1)}, \dots, e_{\sigma(n)}, \dots)$ or $p_\sigma := (e_{\sigma(1)}, \dots, e_{\sigma(n)})$ is again a path for a bijection $\sigma \neq \text{id}$. Then $(e_j, \dots, e_{\sigma^{-1}(j)})$ is a subpath of p , so $(e_{\sigma(j)}, \dots, e_j)$ is a subpath of p_σ . If $\sigma(j) = j + 1$, then the path $(e_{\sigma(j)}, \dots, e_j) = (e_{j+1}, \dots, e_j)$ is already a loop. If $\sigma(j) > j + 1$, then, following this path with the path $(e_{j+1}, \dots, e_{\sigma(j)-1})$, we obtain a loop: $(e_{\sigma(j)}, \dots, e_j, e_{j+1}, \dots, e_{\sigma(j)-1})$. ■

DEFINITION 2.5 ([2]). Let E be a finite R -labelled directed graph. The elements of R given by

$$ct(E) := \prod_{e \in E^1} l(e), \quad wt(E) := \sum_{e \in E^1} l(e)$$

are called the *content* and the *weight* of E , respectively. Furthermore, for $S \subseteq E^1$, the *S -exclusive content* of E is

$$ct_{\cancel{S}}(E) := \prod_{e \in E^1 \setminus S} l(e).$$

DEFINITION 2.6 ([2]). Let E be a finite R -labelled directed graph and Γ be a fixed finite directed graph without repeated edges. We define

$$ct_R^\Gamma(E) := \sum_{\gamma} ct(\gamma),$$

with γ ranging over the R -labelled subgraphs of E isomorphic to Γ as directed graphs.

DEFINITION 2.7. Let E be a finite R -labelled directed graph, and $k \in \mathbb{N} \setminus \{0\}$. We define

$$(2.4) \quad ct_R^k(E) := \sum_{p \in FP_k(E)} ct(p), \quad ct((e_1, \dots, e_k)) := \prod_{i=1}^k l(e_i),$$

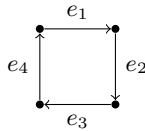
$$ct_{\cancel{e_j}}((e_1, \dots, e_k)) := \prod_{i \in \{1, \dots, k\} \setminus \{j\}} l(e_i).$$

Note that even if a finite R -labelled graph E is not acyclic, so $FP(E)$ is infinite, $FP_k(E)$ is still finite for any k , so the sum in (2.4) is finite.

Furthermore, to maximize $ct_R^\Gamma(E)$ for finite R -labelled graphs E with a fixed weight N as in [2, Theorem 3.2] (which should be formulated more precisely than it is), we need the following fundamental concept:

DEFINITION 2.8 ([2]). Let Γ be a fixed finite directed graph without repeated edges and containing at least one edge, and let E be an R -labelled directed graph. We say that E has the *Chirvasitu property* for Γ when any two edges of E belong to some common embedded copy of $\Gamma \subseteq E$.

As we are interested in maximizing $ct_R^k(E)$ rather than $ct_R^\Gamma(E)$, let us first point out that, if an R -labelled graph is not acyclic, then paths are not necessarily subgraphs. Indeed, in the graph



the two different paths $(e_1, \dots, e_4, e_1, \dots, e_4)$ and (e_1, \dots, e_4) have the same underlying subgraph.

However, due to Theorem 2.4, any finite path in an acyclic R -labelled graph can be identified with a subgraph because then the order of edges is uniquely fixed. In this setting, we have the following result (which is a special case of Lemma 3.2):

LEMMA 2.9. *Let E be a finite acyclic R -labelled directed graph such that*

$$s(E^1) \cup t(E^1) = E^0 \neq \emptyset$$

and p be a path in E of length $k \geq 1$. Then the only such graphs satisfying the Chirvasitu property for p with its labelling omitted are paths (e_1, \dots, e_k) .

3. Counting paths in non-acyclic graphs. First, we need to prove [2, Theorem 3.2] and Lemma 2.9 in the desired setting. Note that our path version of [2, Theorem 3.2] enjoys basically the same proof as the original theorem.

LEMMA 3.1. *Let R be a subsemiring of $\mathbb{R}_{\geq 0}$, $N \in R$, and FG_R^N be the set of all finite R -labelled directed graphs of weight N . Then*

$$\sup_{E \in FG_R^N} ct_R^k(E), \quad \text{for any } k \in \mathbb{N} \setminus \{0\},$$

is zero or is achieved over graphs E with the following property (the Chirvasitu property for paths):

(3.1) *Any two edges of E belong to the same path of length k , i.e.*

$$e, f \in E^1 \implies \exists \text{ a path } p \text{ of length } k \text{ such that } e \text{ and } f \text{ belong to } p.$$

Proof. The case $N = 0$ is trivial, so we assume that $N > 0$. Next, the supremum is zero if and only if there are no graphs with paths of length k , so we can further assume that E has at least one path of length k . Now, let $e, f \in E^1$, and let us suppose that they do not belong to the same path of length k . Next, let S_e and S_f be the sets of all k -paths containing e and f , respectively. Then $S_e \cap S_f = \emptyset$. Furthermore, observe that

$$(3.2) \quad \begin{aligned} ct_R^k(E) &= \sum_{\alpha \in S_e} ct(\alpha) + \sum_{\beta \in S_f} ct(\beta) + \sum_{\gamma \notin S_e \cup S_f} ct(\gamma) \\ &= l(e) \sum_{\alpha \in S_e} ct_{\neq}(\alpha) + l(f) \sum_{\beta \in S_f} ct_{\neq}(\beta) + \sum_{\gamma \notin S_e \cup S_f} ct(\gamma) \end{aligned}$$

and assume, without loss of generality, that

$$(3.3) \quad \sum_{\alpha \in S_e} ct_{\neq}(\alpha) \geq \sum_{\beta \in S_f} ct_{\neq}(\beta).$$

Now we produce a graph E' of the same weight N by eliminating the edge f and adding its label to the label of the edge e , so that the label of e is now $l(e) + l(f)$. The k -content of E' is

$$(3.4) \quad ct_R^k(E') = (l(e) + l(f)) \sum_{\alpha \in S_e} ct_{\neq}(\alpha) + \sum_{\gamma \notin S_e \cup S_f} ct(\gamma).$$

Finally, we compute

$$(3.5) \quad \begin{aligned} ct_R^k(E') - ct_R^k(E) &= (l(e) + l(f)) \sum_{\alpha \in S_e} ct_{\neq}(\alpha) - l(e) \sum_{\alpha \in S_e} ct_{\neq}(\alpha) - l(f) \sum_{\beta \in S_f} ct_{\neq}(\beta) \\ &= l(f) \left(\sum_{\alpha \in S_e} ct_{\neq}(\alpha) - \sum_{\beta \in S_f} ct_{\neq}(\beta) \right) \geq 0. \end{aligned}$$

Hence, whenever we have a finite R -labelled graph with a pair of edges that do not belong to the same path of length k , we can always eliminate one of these two edges to create a new finite R -labelled graph with at least the same k -content. Since the number of edges is finite, we finally arrive at a finite R -labelled graph satisfying (3.1). ■

Also the following lemma is just a variation of [2, Lemma 3.3] adapted to our goals.

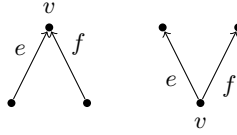
LEMMA 3.2. *Let $k \in \mathbb{N} \setminus \{0\}$ and let E be a finite R -labelled directed graph without loops shorter than k and such that*

$$(3.6) \quad s(E^1) \cup t(E^1) = E^0 \neq \emptyset.$$

Then the only such graphs satisfying (3.1) (the Chirvasitu property for paths of length k) are path (e_1, \dots, e_k) directed graphs such that $|\{t(e_i)\}_{i=1}^k| = k$,

or loop (f_1, \dots, f_m) directed graphs such that $k \leq m \leq 2k-1$ and $|\{t(f_i)\}_{i=1}^m| = m$.

Proof. Suppose that E contains one of the following subgraphs:



We call these subgraphs A and V , respectively. For any of them, to have both e and f in the same path of length k we must have a loop based at v of length at most $k-1$, which contradicts our assumption. Therefore, E cannot contain such subgraphs.

Next, the combination of (3.1) with (3.6) implies that E must be a non-empty connected graph. Furthermore, as $E^1 \neq \emptyset$, by (3.1) E must have at least one path of length k . Hence, it must have at least k edges and its paths of the form (e_1, \dots, e_k) must satisfy $|\{t(e_i)\}_{i=1}^k| = k$ by Lemma 2.3 and the assumption that there are no loops shorter than k . It follows that, if $|E^1| = k$, then E is the graph underlying a path (e_1, \dots, e_k) , which is either an open path of length k without repeated edges or a loop of length k satisfying $|\{t(e_i)\}_{i=1}^k| = k$.

If $|E^1| = m > k$, then our path (e_1, \dots, e_k) cannot be a loop as then there would be no way to attach any of the remaining $m-k$ edges without creating a forbidden A or V subgraph. Hence, (e_1, \dots, e_k) is an open path, and the only way to attach an edge to it is at the beginning or the end. Now, the loose end of the attached edge can either remain loose or be identified with the opposite end of (e_1, \dots, e_k) . Thus, either we have an open path of length $k+1$ without repeated edges or vertices, or a loop of length $k+1$ with $k+1$ different edges and $k+1$ different vertices. Next, if $m = k+1$, then only the latter case satisfies (3.1). If $m > k+1$, then only the former case allows us to attach one of the remaining $m-k-1$ edges. We repeat this reasoning inductively until we reach $m = 2k$. Then neither of the cases satisfies (3.1), and the procedure halts exhausting all graphs claimed in the lemma. ■

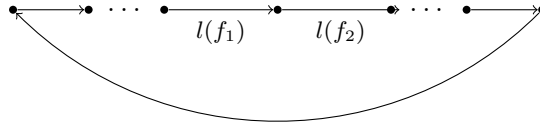
We are now ready to prove our main result:

THEOREM 3.3. *Let E be a directed graph with $N \geq 1$ edges, and let $1 \leq k \leq N =: nk + r$, $0 \leq r \leq k-1$. Assume also that there are no loops shorter than k . Then there are at most kP_k^N different paths of length k , and the bound is optimal.*

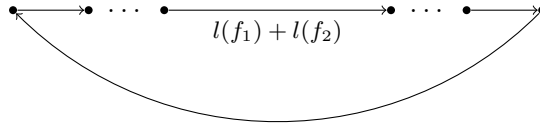
Proof. The claim is trivial for $k = 1$, so we assume that $k \geq 2$. Next, we put $R = \mathbb{N}$ in Lemmas 3.1 and 3.2. Furthermore, without loss of generality,

we can assume that E satisfies (3.6) as $|E^1| \geq 1$ and disconnected vertices have no bearing on the number of paths of length k . Now we can treat E as a finite \mathbb{N} -labelled graph satisfying (3.6), without loops shorter than k , and of weight N . Thus, all assumptions of both lemmas are fulfilled, and we can conclude that graphs maximizing the number of all paths of length k are of the form indicated in Lemma 3.2. It is clear that an open \mathbb{N} -labelled path of length k has fewer paths of length k than an \mathbb{N} -labelled loop of length k . Now we will show that shrinking \mathbb{N} -labelled loops of length m , with $k < m < 2k$, increases the number of k -paths. To this end, we must have $k \geq 2$.

Let (f_1, \dots, f_m) be an \mathbb{N} -labelled loop as in Lemma 3.2. Without the loss of generality, we can assume that $l(f_1)$ is the lowest label: $l(f_1) := \min\{l(f_i)\}_{i=1}^m$. Now we can shrink our graph E



by eliminating the edge f_1 and adding its label to the label of the edge f_2 . Thus, we obtain a graph E'



without loops shorter than k and with the same weight (number of edges) N , which is an \mathbb{N} -labelled loop that is 1 shorter. It remains to show $|FP_k(E')| > |FP_k(E)|$.

Let $S_1 \subseteq FP_k(E)$ be the set of all k -paths containing f_1 , and let $S_2 \subseteq FP_k(E)$ and $S'_2 \subseteq FP_k(E')$ be the sets of all k -paths containing f_2 . To shorten notation, let us put $\prod_{i=x}^y l(f_i) =: \Pi_x^y$ and $\Pi_x^y := 1$ when $y < x$. Then

$$\begin{aligned}
 (3.7) \quad ct_{\mathbb{N}}^k(E) &= \sum_{j=2}^{k+1} \Pi_{m-k+j}^m \Pi_1^{j-1} + \Pi_2^{k+1} + \sum_{\gamma \in FP_k(E) \setminus (S_1 \cup S_2)} ct(\gamma) \\
 &= l(f_1) \Pi_{m-k+2}^m + \sum_{j=3}^{k+1} \Pi_{m-k+j}^m \Pi_1^{j-1} + \Pi_2^{k+1} + \sum_{\gamma \in FP_k(E) \setminus (S_1 \cup S_2)} ct(\gamma) \\
 &= l(f_1) \Pi_{m-k+2}^m + \sum_{j=2}^k \Pi_{m-k+j+1}^m \Pi_1^j + \Pi_2^{k+1} + \sum_{\gamma \in FP_k(E) \setminus (S_1 \cup S_2)} ct(\gamma)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.8) \quad ct_{\mathbb{N}}^k(E') &= (l(f_1) + l(f_2)) \sum_{j=2}^{k+1} \Pi_{m-k+j}^m \Pi_3^j + \sum_{\gamma \in F P_k(E') \setminus S_2'} ct(\gamma) \\
 &= l(f_1) \sum_{j=2}^{k+1} \Pi_{m-k+j}^m \Pi_3^j + \sum_{j=2}^{k+1} \Pi_{m-k+j}^m \Pi_2^j + \sum_{\gamma \in F P_k(E') \setminus S_2'} ct(\gamma) \\
 &= l(f_1) \Pi_{m-k+2}^m + l(f_1) \sum_{j=3}^{k+1} \Pi_{m-k+j}^m \Pi_3^j + \sum_{j=2}^k \Pi_{m-k+j}^m \Pi_2^j \\
 &\quad + \Pi_2^{k+1} + \sum_{\gamma \in F P_k(E') \setminus S_2'} ct(\gamma).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (3.9) \quad ct_{\mathbb{N}}^k(E') - ct_{\mathbb{N}}^k(E) &= l(f_1) \sum_{j=3}^{k+1} \Pi_{m-k+j}^m \Pi_3^j + \sum_{j=2}^k \Pi_{m-k+j}^m \Pi_2^j - \sum_{j=2}^k \Pi_{m-k+j+1}^m \Pi_1^j \\
 &= l(f_1) \sum_{j=3}^{k+1} \Pi_{m-k+j}^m \Pi_3^j + \sum_{j=2}^k \Pi_{m-k+j+1}^m (l(f_{m-k+j}) - l(f_1)) \Pi_2^j \\
 &> 0.
 \end{aligned}$$

Here the last step follows from the minimality of $l(f_1)$ and the fact that $k \geq 2$.

Now we can conclude that the number of paths of length k is maximized by \mathbb{N} -labelled loops of length k with k different vertices. It is clear that the number of k -paths in such a loop equals k times the number of k -paths in an \mathbb{N} -labelled open path obtained by cutting the loop at any vertex. As the latter number is maximized by $P_k^{\mathbb{N}}$ by Theorem 2.2, we infer that the maximal number of k -paths in our case is $k P_k^{\mathbb{N}}$, and this maximum is realized by \mathbb{N} -labelled loops obtained by identifying the beginning and the end of an \mathbb{N} -labelled k -path maximizing the number of k -paths in the acyclic case. ■

Furthermore, observe that the proof of the above theorem does not depend on labels being natural numbers. Therefore, repeating the proof for a semiring R such that $\mathbb{Q}_{\geq 0} \subseteq R \subseteq \mathbb{R}_{\geq 0}$ and using [2, Theorem 6], we arrive at the following bonus result:

THEOREM 3.4. *Let R be a subsemiring of $\mathbb{R}_{\geq 0}$ containing $\mathbb{Q}_{\geq 0}$, $N \in R \setminus \{0\}$, $k \in \mathbb{N} \setminus \{0\}$ and let $k\text{-FG}_R^N$ be the set of all finite R -labelled directed*

graphs of weight N without loops shorter than k . Then

$$\sup_{E \in k\text{-FG}_R^N} ct_R^k(E) = k \left(\frac{N}{k} \right)^k.$$

The supremum is achieved by the R -labelled directed graph given by the loop of length k with k vertices whose edges all have the same label N/k .

To end the article, let us state the following open problem. We already know the maximal number of fixed-positive-length paths in acyclic graphs with N edges (Theorem 2.2). However, the question of what is the maximal number of arbitrary paths in a connected acyclic graph with N edges remains wide open. It is particularly interesting because it is equivalent to the question of what is the biggest finite dimension of a path algebra with N grade-one generators [1]. Alas, all we can prove thus far is:

PROPOSITION 3.5. *Let E be an acyclic directed graph with $N \geq 1$ edges. Then the number of all positive-length paths is not greater than $N(\sqrt[k]{e})^N$.*

Proof. As the arithmetic mean is always greater than or equal to the geometric mean, first we observe that $(N/k)^k \geq P_k^N$. Combining now Theorem 2.2 with elementary calculus to maximize the function

$$(3.10) \quad [1, N] \ni x \mapsto x^{N/x} \in \mathbb{R}$$

yields the claim when substituting $x = N/k$. ■

Funding. This work is part of the project “Applications of graph algebras and higher-rank graph algebras in noncommutative geometry” partially supported by NCN grant UMO-2021/41/B/ST1/03387.

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