BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Online First version

MEASURE AND INTEGRATION

Nontotal boundedness of the Fréchet–Nikodym space of a nonatomic quasi-measure

by

Zbigniew LIPECKI

Presented by Feliks PRZYTYCKI

Summary. Let A be a Boolean algebra and let μ be a nonatomic probability quasi-measure on A. Then there exists an infinite sequence (a_n) in A such that $\mu(a_i \triangle a_j) > \frac{1}{2}$ for all $i \neq j$. The constant $\frac{1}{2}$ is best possible. A similar problem concerning finite sequences in A is also considered and its connection with the theory of binary codes is established.

1. Introduction. Let X be a nonempty set and let d be a pseudometric in X. (By definition, a pseudometric differs from a metric by the absence of the condition: $d(x, y) = 0$ implies $x = y$ for all $x, y \in X$.) Among the measures of noncompactness or rather nontotal boundedness of (X, d) at least three are well known. One of them goes back to F. Hausdorff (1914), another one was introduced by K. Kuratowski (1930). We shall deal below with the third one in the special case indicated in the title of the paper.

Let (X, d) be a pseudometric space. Given $x_1, \dots, x_n \in X$, where $n \geqslant 2$, we set

$$\beta(x_1, \dots, x_n) = \min_{1 \leqslant i < j \leqslant n} d(x_i, x_j).$$

Moreover, we set

$$\beta_n = \sup \{\beta(x_1, \dots, x_n) : x_1, \dots, x_n \in X\}, \quad n \geqslant 2.$$

As easily seen, $\beta_2 \geqslant \beta_3 \geqslant \cdots$, and $\beta_n \to 0$ if and only if (X, d) is totally bounded. In the opposite case there exists $s > 0$ and an infinite sequence

2020 *Mathematics Subject Classification*: Primary 28A10; Secondary 54E25, 54E35, 05B40, 94B99.
Key words and phrases: Boolean algebra, quasi-measure, finitely additive measure, charge, nonatomic, strongly continuous, Darboux property, Fréchet–Nikodym pseudometric, non-totally bounded, Kottman constant, binary code, Hadamard matrix, sphere packing.
Received 12 June 2025; revised 5 March 2026.
Published online 31 March 2026.

(x_n) in X such that $d(x_i, x_j) > s$ for all $i \neq j$. Thus, the supremum of such s, call it β_0, is a measure of nontotal boundedness of (X, d). In the case where d is a metric, β_0 is considered, with different notation, in [14] and [6]. In the case where X is the unit sphere of an infinite-dimensional Banach space the condition above holds with $s = 1$. This is a classical result due to C. A. Kottman (1975). It was improved by J. Elton and E. Odell (1981) to the effect that "1" can be replaced by "$1 + \varepsilon$", where $\varepsilon > 0$ depends on the Banach space in question. See [7, pp. 7 and 241, respectively]; see also [10, Section 10.1] and [12, Section 40.2], where more recent relevant results are discussed.

The author does not know whether the coefficients β_n introduced above have already been investigated in the literature either in general or in some special cases. Similarly, he does not know of any results of the Kottman–Elton–Odell type concerning pseudometric spaces other than the unit sphere of a Banach space. On the other hand, the dual sphere-packing problem is a classical subject in geometry.

The pseudometric space studied in this paper is a Boolean algebra A equipped with the Fréchet–Nikodym pseudometric d_μ, which is defined by means of a probability quasi-measure μ on A (see the beginning of Section 2). Not much seems to be known about this space from the geometric point of view even in the special case where A is σ-complete and μ is σ-additive. According to classical results, (A, d_μ) is then complete, and convex provided μ is nonatomic (see, e.g., [13, p. 169, (8)]). Otherwise some geometric properties of the ring of Lebesgue measurable subsets of $\mathbb{R}$ of finite Lebesgue measure equipped with the corresponding pseudometric were established by Engelking [8], [9].

The main results of the paper are obtained under the assumption that μ be nonatomic. In this connection recall that (A, d_μ) is totally bounded if and only if μ does not dominate a non-zero nonatomic quasi-measure on A ([22, (4.7)]; cf. also [4, Corollary 7.23]). We prove a Kottman type result with $s = \frac{1}{2}$ (Theorem 1 in Section 4). This result is, in some sense, best possible, due to a theorem of G. Plebanek (Theorem 3 in Section 6). We also estimate β_n from below for all $n \geqslant 2$ (Proposition 5 in Section 4). Those estimates are improved for small n by using appropriate binary codes (Proposition 6(a) in Section 5).

We also show that the latter estimates are precise for $n = 3, 4$. (That $\beta_2 = 1$ is obvious.) Moreover, we define, in terms of binary codes composed of n words, a coefficient γ_n and prove that $\beta_n = \gamma_n$ for all $n \geqslant 2$ (Theorem 2 in Section 5). This allows us to get the improved estimates of β_n from below by using Hadamard matrices of order n.

We note that the main results of the paper seem new already in the case where μ is the Lebesgue or Peano–Jordan measure on the unit interval.

The measure-theoretic notation and terminology used below are explained at the beginning of Sections 2–4. Sections 2 and 3 also contain some auxiliary results. The main results are presented in Sections 4 and 5. The penultimate Section 6 is independent of the rest of the paper as far as the proofs are concerned. It discusses the theorem of Plebanek mentioned above (Theorem 3) and a finitely additive version of it (Theorem 4).

Thanks are due to Grzegorz Plebanek for answering a question of the author, which resulted in the material presented in Section 6.

The author is much indebted to the referee for his/her contribution to the paper which is a part of the final Section 7. Theorem B thereof gives an estimate of β_n from above for all $n \geqslant 2$ and an arbitrary quasi-measure μ. Theorem C shows, in turn, that that estimate cannot be improved, in general. As a consequence of Theorem B, we infer that $\beta_n \to 1/2$ in the case where μ is nonatomic (Corollary F).

2. Preliminaries. *Throughout the paper A stands for a Boolean algebra*, with the operations of join, meet, difference and symmetric difference denoted by $\vee$, $\wedge$, $\setminus$ and $\triangle$, respectively. The natural ordering of A is denoted by $\leqslant$ and its minimal and maximal elements by 0 and 1, respectively. For every $a \in A$ we denote by a^c the complement of a and by $[0, a]$ the set $\{b \in A : b \leqslant a\}$.

We call a function $\mu\colon A \to \mathbb{R}$ a *quasi-measure* if it is additive and positive. In an alternative terminology, μ is a (positive) finitely additive measure or a charge. *Throughout the paper we assume that* $\mu(1) = 1$, i.e., μ is a probability quasi-measure.

We call $\mu\colon A \to \mathbb{R}$ a *measure* if A is a Boolean σ-algebra and μ is positive and σ-additive.

We denote by d_μ the *Fréchet–Nikodym pseudometric* on A generated by a quasi-measure μ on A. It is defined by the formula

$$d_\mu(a, b) = \mu(a \triangle b), \qquad a, b \in A.$$

Clearly, d_μ is a metric if and only if μ is strictly positive, i.e., $\mu(a) > 0$ whenever $a \in A$ and $a \neq 0$.

The following proposition will be applied in the proofs of Lemma 5 and Theorem 1 in Section 4.

PROPOSITION 1. *Let μ be a strictly positive quasi-measure on A. Then there exists a Boolean algebra $\tilde{A}$, an injective homomorphism $h\colon A \to \tilde{A}$ and a strictly positive quasi-measure $\tilde{\mu}$ on $\tilde{A}$ such that*

(1) *$\tilde{\mu}(h(a)) = \mu(a)$ for all $a \in A$;*
(2) *$(\tilde{A}, d_{\tilde{\mu}})$ is complete;*
(3) *$h(A)$ is dense in $(\tilde{A}, d_{\tilde{\mu}})$.*

This is Theorem 2.1 in [15]. The proof is an adaptation of the standard construction of the completion of a metric space. Proposition 1 is also a direct consequence of [4, Lemma 1.1], which is proved by another method. Conversely, in view of Remark 1 below, Lemma 1.1 of [4] follows from Proposition 1.

REMARK 1. In connection with Proposition 1 we note that if μ is strictly positive and (A, d_μ) is complete, then A is σ-complete and μ is a measure (see [15, Theorems 2.3 and 2.2]; see also [4, Theorem 3.3] or [3, Proposition 2]. The converse implication is, of course, a standard result.

3. Four lemmas on quasi-measures. *Throughout this section μ denotes an arbitrary quasi-measure on A.*

Given $a_1, \ldots, a_n \in A$, where $n \geqslant 2$, we set

$$\beta(\mu)(a_1, \ldots, a_n) = \min_{1 \leqslant i < j \leqslant n} \mu(a_i \bigtriangleup a_j).$$

Moreover, we set

$$\beta_n(\mu) = \sup\{\beta(\mu)(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A\}, \quad n \geqslant 2.$$

In fact, we shall simplify this notation and write $\beta(a_1, \ldots, a_n)$ and β_n when a single quasi-measure μ is under consideration.

Clearly, $\beta_2 = 1$ and $\beta_2 \geqslant \beta_3 \geqslant \cdots$ As easily seen, $\beta_n \to 0$ if and only if (A, d_μ) is totally bounded.

We note that when evaluating β_n we may always assume that a_1 is a fixed element of A, e.g., $a_1 = 0$ or 1. This is because $(A, \bigtriangleup)$ is an Abelian group whose every nonzero element is of order 2, Moreover, we may confine ourselves to considering sequences $a_1, \ldots, a_n$ with $a_1 \vee \cdots \vee a_n = 1$.

The following simple lemma will be applied in the proofs of Lemma 5 in Section 4 and Lemma 8 in Section 5.

LEMMA 1. *For all $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in A we have*

$$|\beta(a_1, \ldots, a_n) - \beta(b_1, \ldots, b_n)| \leqslant 2 \max_{1 \leqslant 1 \leqslant n} \mu(a_i \bigtriangleup b_i).$$

Proof. We first show that the assertion holds for $n = 2$. This is seen because we have

$$\begin{aligned}\mu(a_1 \bigtriangleup a_2) - \mu(b_1 \bigtriangleup b_2) &\leqslant \mu(a_1 \bigtriangleup a_2 \bigtriangleup b_1 \bigtriangleup b_2)\\ &\leqslant \mu(a_1 \bigtriangleup b_1) + \mu(a_2 \bigtriangleup b_2) \leqslant 2 \max_{i=1,2} \mu(a_i \bigtriangleup b_i)\end{aligned}$$

Applying the assertion for $n = 2$, we get

$$\beta(a_1, \ldots, a_n) - \mu(b_j \bigtriangleup b_k) \leqslant \mu(a_j \bigtriangleup a_k) - \mu(b_j \bigtriangleup b_k) \leqslant 2 \max_{1 \leqslant i \leqslant n} \mu(a_i \bigtriangleup b_i)$$

for all $1 \leqslant j, k \leqslant n$. ■

With the purpose of estimating β_3 and β_5 from above we shall establish the following three lemmas.

Lemma 2. *Let a, $b \in A$ and $s \in \mathbb{R}$ be such that $\mu(a), \mu(b), \mu(a \vartriangle b) \geqslant s$. Then*

(a) $2s - 1 \leqslant \mu(a \wedge b) \leqslant 1 - s$;
(b) $\mu(a \vee b) \geqslant 3s - 1$;
(c) $\mu(a^c \wedge b^c) \leqslant 2 - 3s$.

Proof. (a): Since $a \wedge b \leqslant (a \vartriangle b)^c$, we have $\mu(a \wedge b) \leqslant 1 - s$. Since

$$\mu(a \wedge b) = \mu(a) + \mu(b) - \mu(a \vartriangle b),$$

it follows that $\mu(a \wedge b) \geqslant 2s - 1$.

(b): Since $\mu(a \vee b) = \mu(a \vartriangle b) + \mu(a \wedge b)$, the assertion follows from (a).

(c): We have $\mu(a^c \wedge b^c) = 1 - \mu(a \vee b)$, and so the assertion follows from (b). ∎

Lemma 3. *Let a, b and s be as in Lemma 2 and let $c \in A$ be such that $\mu(c), \mu(a \vartriangle c), \mu(b \vartriangle c) \geqslant s$. Then*

$$\mu((a \vartriangle b) \setminus c) \leqslant 4 - 6s.$$

Proof. By Lemma 2(c), we have

$$\mu(a^c \wedge c^c), \mu(b^c \wedge c^c) \leqslant 2 - 3s.$$

Since $(a \vartriangle b) \setminus c \leqslant (a^c \wedge c^c) \vee (b^c \wedge c^c)$, the assertion follows. ∎

Lemma 4. *Let a, b, c and s be as in Lemma 3 and let $d \in A$ be such that*

$$\mu(d), \mu(a \vartriangle d), \mu(b \vartriangle d), \mu(c \vartriangle d) \geqslant s.$$

Then $s \leqslant \frac{9}{14}$.

Proof. By Lemma 3,

$$\mu((a \vartriangle b) \setminus c), \mu((a \vartriangle b) \setminus d) \leqslant 4 - 6s.$$

Hence

$$\mu((a \vartriangle b) \setminus (c \wedge d)) \leqslant 8 - 12s.$$

Since

$$c \vartriangle d \leqslant [(a \vartriangle b) \setminus (c \wedge d)] \vee (a \vartriangle b)^c,$$

it follows that

$$\mu(c \vartriangle d) \leqslant (8 - 12s) + (1 - s).$$

This implies the assertion. ∎

Proposition 2. *We have*

(a) $\beta_3 \leqslant \frac{2}{3}$;
(b) $\beta_5 \leqslant \frac{9}{14}$.

Part (a) follows from Lemma 2(a), while part (b) is a direct consequence of Lemma 4.

Part (a) of Proposition 2 is optimal (see Proposition 6(b) in Section 4). For an improvement of part (b) see Theorem B in Section 7.

4. Quasi-measures with Darboux type properties. Let again μ be a quasi-measure on A. We say that μ *has the Darboux θ-property*, where $0 < \theta < 1$, if, for every $a \in A$, there exists $b \in [0, a]$ with $\mu(b) = \theta\mu(a)$. In the case where μ has this property for every $0 < \theta < 1$ we say that it *has the Darboux property*. The latter terminology coincides with that of [20], while in [5, Definition 5.1.5] μ is then called *strongly nonatomic* and in [18, p. 45] the term *full valued* is used.

We say that μ is *nonatomic* if, for every $\varepsilon > 0$, there exist $a_1, \ldots, a_n \in A$ such that

$$a_1 \vee \cdots \vee a_n = 1 \quad \text{and} \quad \mu(a_i) \leqslant \varepsilon \text{ for all } i.$$

In [5, Definition 5.1.4] such quasi-measures are called *strongly continuous*, while in [21, p. 841] they are called *continuous*.

The implications between conditions (i), (iii) and (iv) of the next proposition have been studied by many authors. See [21, Theorem 5.1], [18, Theorem 2], [4, Theorem 5.10], [20, Theorem 5 and Corollary], [2, Lemma 2–1], [11, (2.5) and (2.6)], [23, Theorem 1 and Example 1]. The implication (iii)⇒(i) of this proposition in the case where (A, d_μ) is assumed complete will be used in the proofs of Theorem 1 and Lemma 5 below.

PROPOSITION 3. *Let μ be a quasi-measure on A. Then each of the following conditions implies the next one:*

(i) *μ has the Darboux property;*
(ii) *μ has the Darboux θ-property for some θ with $0 < \theta < 1$;*
(iii) *μ is nonatomic;*
(iv) *$\mu([0, a])$ is dense in $[0, \mu(a)]$ for every $a \in A$.*

Moreover, (iv) *implies* (iii)*, and if (A, d_μ) is complete, then conditions* (i)–(iv) *are all equivalent.*

Proof. The implication (i)⇒(ii) is obvious, while the implication (iii)⇒(iv) is straightforward. We shall show that (ii) implies (iii) and so does (iv). Next, to complete the proof, we shall establish the final assertion.

Suppose (ii) holds. Fix $0 < \varepsilon < \theta$, and then $m \in \mathbb{N}$ with $\theta^m < \varepsilon$. Define, by induction, a sequence (a_n) in A with the following properties:

$$\mu(a_1) = \theta^m, \ a_{n+1} \in [0, (a_1 \vee \cdots \vee a_n)^c] \text{ and } \mu(a_{n+1}) = \theta^m \mu((a_1 \vee \cdots \vee a_n)^c).$$

Since a_n are pairwise disjoint, we have $\mu(a_{n_0+1}) < \theta^{2m}$ for some n_0, and so

$\mu((a_1 \vee \dots \vee a_n)^c) < \theta^m$. It follows that

$$a_1, \dots, a_{n_0}, (a_1 \vee \dots \vee a_{n_0})^c$$

form a partition of 1 into elements of μ-quasi-measure $< \varepsilon$. Thus, (iii) holds.

Suppose (iv) holds, and fix $0 < \varepsilon < 1$. Then there exists $a_1 \in A$ with $\varepsilon/2 < \mu(a_1) < \varepsilon$. If $\mu(a_1^c) < \varepsilon$, we are done. Otherwise there exists $a_2 \in [0, a_1^c]$ with $\varepsilon/2 < \mu(a_2) < \varepsilon$. Continuing this way, we get a sequence $a_1, \dots, a_n$ of pairwise disjoint elements of A such that

$$\frac{\varepsilon}{2} < \mu(a_i) < \varepsilon \text{ for all } i \quad \text{ and } \quad \mu((a_1 \vee \dots \vee a_n)^c) < \varepsilon,$$

where $n < 2/\varepsilon$. This shows that (iii) holds.

To establish the final assertion, suppose again that (iv) holds. Fix θ with $0 < \theta < 1$, and then $0 < \theta_1 < \theta_2 < \dots < \theta$ with $\theta_n \to \theta$. We shall define an increasing sequence (a_n) in A such that $\theta_n \leqslant \mu(a_n) < \theta$ for all n. The existence of a_1 is a direct consequence of (iv). Suppose $a_1, \dots, a_n$ with the desired property have already been defined. If $\mu(a_n) \geqslant \theta_{n+1}$, we set $a_{n+1} = a_n$. Otherwise we have

$$0 < \theta_{n+1} - \mu(a_n) < \theta - \mu(a_n) < \mu(a_n^c).$$

By (iv) again, there exists

$$b \in [0, a_n^c] \quad \text{with} \quad \theta_{n+1} - \mu(a_n) < \mu(b) < \theta - \mu(a_n).$$

Set $a_{n+1} = a_n \vee b$ to complete the induction procedure. Clearly, (a_n) is a Cauchy sequence in (A, d_μ). Let a be a d_μ-limit of (a_n). We then have $\mu(a) = \theta$. ■

Remark 2. In connection with the final part of Proposition 3 we note that a quasi-measure μ on A can have the Darboux property even if neither is (A, d_μ) complete nor is A σ-complete. Indeed, let A be the algebra of subsets of $[0, 1)$ generated by the intervals $[0, t)$, where $0 < t < 1$, and let μ be the restriction of the Lebesgue measure to A.

Remark 3 (cf. [5, Remark 5.1.7(ii)]). The implication (i)⇒(ii) of Proposition 3 cannot be reversed, in general. Indeed, let A_0 be the algebra of subsets of $[0, 1)$ generated by the intervals $[0, t)$, where $0 < t < 1$ and $t \in \mathbb{Q}$, and let μ_0 be the restriction of the Lebesgue measure to A_0. Clearly, μ_0 is nonatomic. On the other hand, μ_0 has the Darboux θ-property, where $0 < \theta < 1$, if and only if $\theta \in \mathbb{Q}$.

Remark 4. The implication (ii)⇒(iii) of Proposition 3 cannot be reversed in the following strong sense. There exists a Boolean algebra A and a nonatomic quasi-measure μ on A such that μ does not have the Darboux θ-property for any $0 < \theta < 1$. To see this, consider first a fixed $\theta \in \mathbb{Q}$ with $0 < \theta < 1$. Let $p, q \in \mathbb{N}$ be such that $\theta = p/q$ and $(p, q) = 1$. Let q_1 be a prime number with $q_1 \mid q$. Choose $r \in \mathbb{N}$ with $r \geqslant 2$ and $q_1 \nmid r$. We then have $\theta \neq m/r^n$ for all $m, n \in \mathbb{N}$. Let A_θ be the algebra of subsets of $[0,1)$ generated by the intervals $[(m-1)/r^n, m/r^n)$, where $n \in \mathbb{N}$ and $m = 1, \ldots, r^n$. Denote by μ_θ the restriction of the Lebesgue measure to A_θ. Clearly, μ_θ is nonatomic. Moreover, $\theta \notin \mu(A_\theta)$, and so μ_θ does not have the Darboux θ-property. Putting together (A_0, μ_0) of Remark 3 and (A_θ, μ_θ), where $\theta \in \mathbb{Q}$ and $0 < \theta < 1$, just defined, we can easily construct the desired example.

The following lemma will be instrumental in establishing Propositions 5 and 6, and Theorem 2 below.

LEMMA 5. *Let μ be a nonatomic quasi-measure on A. Then there exists a Boolean algebra $\tilde{A}$ and a quasi-measure $\tilde{\mu}$ on $\tilde{A}$ such that $\tilde{\mu}$ has the Darboux property and $\beta_n(\tilde{\mu}) = \beta_n(\mu)$ for $n \geqslant 2$.*

Proof. We may assume that μ is strictly positive. Indeed, let B be the quotient Boolean algebra of A modulo the ideal of μ-null elements and let ν be the corresponding quasi-measure on B. Clearly, ν is also nonatomic and $\beta_n(\nu) = \beta_n(\mu)$ for $n \geqslant 2$.

Let then μ be strictly positive and let $\tilde{A}$ and $\tilde{\mu}$ be given by Proposition 1. It follows from Lemma 1 that $\beta_n(\tilde{\mu}) = \beta_n(\mu)$ for $n \geqslant 2$. An application of the final assertion of Proposition 3 shows that $\tilde{\mu}$ has the Darboux property. ∎

The following proposition will be used in the proofs of Theorem 1 and Lemma 6 below. Its proof applies a well-known argument (see, e.g., [19, p. 119] or [4, proof of Lemma 2.3]).

PROPOSITION 4. *Let μ be a quasi-measure on A with the Darboux $\frac{1}{2}$-property and let $a_1, \ldots, a_n$ be in A. Then, for $\theta = \frac{1}{2}$ or $\frac{3}{4}$, there exists $a_{n+1} \in A$ such that*

(1) $\mu(a_1 \vee \cdots \vee a_{n+1}) = \mu(a_1 \vee \cdots \vee a_n) + \theta\mu(a_1^c \wedge \cdots \wedge a_n^c)$;
(2) $\mu(a_1 \mathbin{\triangle} a_{n+1}) = \frac{1}{2}\mu(a_1 \vee \cdots \vee a_n) + \theta\mu(a_1^c \wedge \cdots \wedge a_n^c)$ *for* $1 \leqslant i \leqslant n$.

In particular,

(3) *if* $\mu(a_1^c \wedge \cdots \wedge a_n^c) = \frac{1}{2^n}$, *then there exists* $a_{n+1} \in A$ *with* $\mu(a_1^c \wedge \cdots \wedge a_{n+1}^c) = \frac{1}{2^{n+1}}$ *and* $\mu(a_i \mathbin{\triangle} a_{n+1}) = \frac{1}{2}$ *for* $1 \leqslant i \leqslant n$;
(4) *if* $\mu(a_1^c \wedge \cdots \wedge a_n^c) > 0$, *then there exists* $a_{n+1} \in A$ *with* $\mu(a_1^c \wedge \cdots \wedge a_{n+1}^c) > 0$ *and* $\mu(a_i \mathbin{\triangle} a_{n+1}) > \frac{1}{2}$ *for* $1 \leqslant i \leqslant n$.

Proof. For $a \in A$ we set $a^\varepsilon = a$ or a^c according as $\varepsilon = 0$ or 1. By assumption, there exists $b_{\varepsilon_1 \ldots \varepsilon_n} \in [0, a_1^{\varepsilon_1} \wedge \cdots \wedge a_n^{\varepsilon_n}]$ such that $\mu(b_{\varepsilon_1 \ldots \varepsilon_n}) = \frac{1}{2}\mu(a_1^{\varepsilon_1} \wedge \cdots \wedge a_n^{\varepsilon_n})$ for $(\varepsilon_1 \ldots \varepsilon_n) \in \{0,1\}^n$ with

$$(\varepsilon_1 \ldots \varepsilon_n) \neq (1 \ldots 1).$$

Similarly, there exists $b_{1\ldots 1} \in [0, a_1^1 \wedge \cdots \wedge a_n^1]$ with $\mu(b_{1\ldots 1}) = \theta\mu(a_1^1 \wedge \cdots \wedge a_n^1)$. Set

$$a_{n+1} = \bigvee_{(\varepsilon_1 \ldots \varepsilon_n) \in \{0,1\}^n} b_{\varepsilon_1 \ldots \varepsilon_n}.$$

As easily seen, a_{n+1} satisfies (1) and (2). By a joint application of (1) and (2) with $\theta = \frac{1}{2}$ and $\theta = \frac{3}{4}$, we get (3) and (4), respectively. ∎

REMARK 5. The assumption $\mu(a_1^c \wedge \ldots \wedge a_n^c) > 0$ in (4) of Proposition 4 is, clearly, necessary for the validity of its first assertion. On the other hand, this assumption is not necessary for the validity of its second assertion. Indeed, let A and μ be as in Remark 2. Set

$$a_1 = [0, 2/3), \quad a_2 = [1/3, 1) \quad \text{and} \quad a_3 = [0, 1/3) \cup [2/3, 1).$$

We then have $a_1 \vee a_2 = [0, 1)$ and $\mu(a_1 \vartriangle a_3) = \mu(a_2 \vartriangle a_3) = \frac{2}{3}$.

THEOREM 1. *If μ is a nonatomic quasi-measure on A, then there exists a sequence (a_n) in A with $\mu(a_i \vartriangle a_j) > \frac{1}{2}$ for all $i \neq j$.*

Proof. Without loss of generality, we assume that μ is strictly positive. We then consider a Boolean algebra $\tilde{A}$ and a quasi-measure $\tilde{\mu}$ on it such that $\tilde{\mu}$ extends μ, $(\tilde{A}, d_{\tilde{\mu}})$ is complete and A is dense in $(\tilde{A}, d_{\tilde{\mu}})$ (see Proposition 1). By Proposition 3, $\tilde{\mu}$ has the Darboux property.

We construct the sequence (a_n) by induction. We choose $a_1 \in A$ arbitrarily with $\mu(a_1) < 1$. Suppose $a_1, \ldots, a_n$ in A such that

$$\mu(a_i \vartriangle a_j) > \frac{1}{2} \text{ for } 1 < i < j \leqslant n \quad \text{and} \quad \mu(a_1 \vee \cdots \vee a_n) < 1$$

have already been constructed. By Proposition 4(4), there exists $\tilde{a} \in \tilde{A}$ such that

$$\tilde{\mu}(a_i \vartriangle \tilde{a}) > \frac{1}{2} \text{ for } 1 \leqslant i \leqslant n \quad \text{and} \quad \tilde{\mu}(a_1 \vee \cdots \vee a_n \vee \tilde{a}) < 1.$$

Let $a_{n+1} \in A$ be such that

$$\tilde{\mu}(\tilde{a} \vartriangle a_{n+1}) < \min_{1 \leqslant i \leqslant n} \tilde{\mu}(a_i \vartriangle \tilde{a}) - \frac{1}{2}, 1 - \tilde{\mu}(a_1 \vee \cdots \vee a_n \vee \tilde{a}).$$

Then, as easily seen, $a_1, \ldots, a_{n+1}$ have the desired properties. ∎

The constant $\frac{1}{2}$ in Theorem 1 is optimal (see Theorem 4 in Section 6 or Theorem B in Section 7).

LEMMA 6. *If μ is a quasi-measure on A with the Darboux $\frac{1}{2}$-property, then, for every $n \geqslant 2$ and $k = 1, 2, \ldots,$ there exist $a_{1k}, \ldots, a_{nk}$ in A such that*

$$\mu(a_{ik} \vartriangle a_{jk}) = \frac{1}{2}\sum_{l=1}^{k} \frac{1}{2^{n(l-1)}} \qquad \textit{for all } i \neq j,$$

$$\mu(a_{1k}^c \wedge \cdots \wedge a_{nk}^c) = \frac{1}{2^{nk}}.$$

Proof. We argue by induction on k. The existence of $a_{11}, \ldots, a_{n1}$ is a direct consequence of (3) of Proposition 4. To pass from k to $k+1$, we consider the Boolean algebra $B = [0, a_{1k}^c \wedge \cdots \wedge a_{nk}^c]$ and the quasi-measure ν on B defined by the formula $\nu(b) = 2^{nk}\mu(b)$ for $b \in B$. Applying (3) of Proposition 4 again, this time to ν, we find $b_1, \ldots, b_n$ in B with

$$\mu(b_i \vartriangle b_j) = \frac{1}{2^{nk+1}} \text{ for } i \neq j \quad \text{and} \quad \mu(b_1^c \wedge \cdots \wedge b_n^c) = \frac{1}{2^{n(k+1)}}.$$

Setting $a_{i,k+1} = a_{ik} \vee b_i$, $i = 1, \ldots, n$, we are done. ■

REMARK 6. If A is σ-complete and μ is a nonatomic measure on A, then, for every $n \geqslant 2$, there exist $a_1, \ldots, a_n$ in A such that

$$\mu(a_i \vartriangle a_j) = \frac{2^{n-1}}{2^n - 1} \qquad \text{for all } i \neq j.$$

Indeed, let a_{ik} be as in the proof of Lemma 6 and set

$$a_i = \bigvee_{k=1}^{\infty} a_{ik}, \qquad i = 1, \ldots, n.$$

PROPOSITION 5. *If μ is a nonatomic quasi-measure on A, then*

$$\beta_n \geqslant \frac{2^{n-1}}{2^n - 1} \qquad \textit{for all } n \geqslant 2.$$

In the special case where μ has the Darboux $\frac{1}{2}$-property the assertion is an immediate consequence of Lemma 6. The general case follows by an application of Lemma 5.

The estimate of Proposition 5 is far from optimal, at least for $n = 2$, 3 and 4 and for those n for which a Hadamard matrix of order n exists (see Proposition 6 in Section 5). The precise value of β_n in the nonatomic case is given in Theorem E in Section 7.

The author does not know whether $\beta_n \to \frac{1}{2}$ if μ is a nonatomic quasi-measure. Proposition 5 above and Theorem 4 in Section 6 suggest that this might be the case. As shown subsequently by the referee, this is, indeed, the case (see Corollary F in Section 7).

5. β_n and binary codes. Let $E_1, \ldots, E_n$, where $n \geqslant 2$, be finite sets not all empty. We set

$$E = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad \gamma(E_1, \ldots, E_n) = \frac{1}{|E|} \min\{|E_i \vartriangle E_j| : 1 \leqslant i < j \leqslant n\}.$$

Moreover, we set

$$\gamma_n = \sup\{\gamma(E_1,\ldots,E_n) : E_1,\ldots,E_n \text{ are finite sets}\}.$$

Clearly, in the definition of γ_n it is enough to take into account only sequences $E_1,\ldots,E_n$ with $E_i \neq E_j$ for all $i \neq j$. In this case $1_{E_1},\ldots,1_{E_n}$, where the characteristic functions 1_{E_i} are defined on E, form a *binary code*. The cardinal $|E|$ is called its length, while the cardinal $|E_i \triangle E_j|$ is called the (Hamming) distance between E_i and E_j. The minimal distance of the code $1_{E_1},\ldots,1_{E_n}$ is

$$d := \min\{|E_i \triangle E_j| : 1 \leqslant i < j \leqslant n\}.$$

In the notation of [17, p. 38] this is a $(|E|, n, d)$ code.

LEMMA 7. *Let μ is a quasi-measure on A with the Darboux property and let $E_1,\ldots,E_n$, where $n \geqslant 2$, be finite sets, not all empty, with union E. Then there exist $a_1,\ldots,a_n$ in A such that*

$$\mu(a_i \triangle a_j) = \frac{1}{|E|}|E_i \triangle E_j| \quad \textit{for all } i,j.$$

In particular, $\beta(a_1,\ldots,a_n) = \gamma(E_1,\ldots,E_n)$.

Proof. Let $r = |E|$. By assumption, there exist pairwise disjoint $b_1,\ldots,b_r$ in A with $\mu(b_k) = 1/r$ for all $k \in E$. Setting

$$a_i = \bigvee_{k\in E_i} b_k,$$

we are done. ∎

In connection with the next proposition, recall that a necessary condition for the existence of a Hadamard matrix of order $n > 2$ is that n be a multiple of 4 (see [17, Chapter 2, Theorem 5]).

PROPOSITION 6. *Let μ be a nonatomic quasi-measure on A. Then*

(a) $\beta_n \geqslant \frac{n}{2n-2}$, *where $n \geqslant 2$ is such that a Hadamard matrix of order n exists;*
(b) $\beta_3 = \beta_4 = \frac{2}{3}$.

Proof. We assume that μ has the Darboux property. The general case follows by Lemma 5.

For n as in (a), there exists a binary code $\left(n-1, n, \frac{1}{2}n\right)$ (see [17, p. 49]). Therefore, $\gamma_n \geqslant \frac{n}{2n-2}$, and so (a) holds by Lemma 7.

As for (b), we have $\beta_4 \leqslant \beta_3 \leqslant \frac{2}{3}$ by Proposition 2(a). On the other hand, $\beta_4 \geqslant \frac{2}{3}$ by (a) (see [17, p. 44]). More explicitly, setting $E_i = \{i\}$, where $i = 1,2,3$, and $E_4 = \{1,2,3\}$, we have $|E_i \triangle E_j| = 2$ for $i \neq j$, and so $\gamma(E_1,E_2,E_3,E_4) = \frac{2}{3}$. (The corresponding standard binary code appears in [17, p. 40].) This yields the desired estimate by Lemma 7. ∎

As in the proof of Proposition 6, we get, for μ nonatomic, $\beta_8 \geqslant \frac{4}{7}$ and $\beta_{12} \geqslant \frac{6}{11}$. For the corresponding binary codes see [17, Figures 2.7 and 2.1, respectively]. In fact, these inequalities turn out to be equalities (see Theorem E in Section 7).

Lemma 8. *Let μ is a quasi-measure on A with the Darboux property, let $a_1, \dots, a_n$, where $n \geqslant 2$, be in A, and let $\varepsilon > 0$. Then there exist finite sets $E_1, \dots, E_n$ such that*

$$\beta(a_1, \dots, a_n) \leqslant \gamma(E_1, \dots, E_n) + \varepsilon.$$

Proof. Without loss of generality, we assume that $a_1 \vee \dots \vee a_n = 1$. We first show the assertion under the additional assumption that there exist $b_1, \dots, b_m$ in A and $E_1, \dots, E_n \subset [1, m]$ such that

$$b_i \wedge b_j = 0 \text{ for all } i \neq j, \quad \mu(b_i) = \frac{1}{m} \quad \text{and} \quad a_i = \bigvee_{j \in E_i} b_j \text{ for all } i.$$

Indeed, we then have

$$\bigcup_{i=1}^{n} E_i = [1, m] \quad \text{and} \quad \mu(a_i \vartriangle a_j) = \frac{1}{m} |E_i \vartriangle E_j| \text{ for all } i, j,$$

and so

$$\beta(a_1, \dots, a_n) = \gamma(E_1, \dots, E_n).$$

In the general case, let nonzero $c_1, \dots, c_p$ in A with $c_i \wedge c_j = 0$ for all $i \neq j$ and $F_1, \dots, F_n \subset [1, p]$ be such that

$$a_i = \bigvee_{j \in F_i} c_j \quad \text{for all } i.$$

We then have $\bigcup_{i=1}^{n} F_i = [1, p]$. By a standard argument, we can find $b'_1, \dots, b'_m$ in A and $G_1, \dots, G_p \subset [1, m]$ such that

$$b'_i \wedge b'_j = 0 \text{ for all } i \neq j, \quad \mu(b'_i) = \frac{1}{m} \text{ for all } i, \quad \bigcup_{r=1}^{p} G_r = [1, m]$$

and

$$\sum_{r=1}^{p} \mu\Big(c_r \vartriangle \bigvee_{s \in G_r} b'_s\Big) < \frac{\varepsilon}{2}.$$

Set

$$a'_i = \bigvee_{j \in F_i} \bigvee_{s \in G_j} b'_s \quad \text{for all } i.$$

We then have $\mu(a_i \vartriangle a'_i) < \frac{\varepsilon}{2}$. An application of the assertion established in the first part of the proof to

$$a'_1, \dots, a'_n, b'_1, \dots, b'_m \quad \text{and} \quad \bigcup_{j \in F_1} G_j, \dots, \bigcup_{j \in F_n} G_j$$

and Lemma 1 completes the proof. ∎

THEOREM 2. *If μ is nonatomic, then $\beta_n = \gamma_n$ for $n \geqslant 2$.*

In the case where μ has the Darboux property this is a direct consequence of Lemmas 7 and 8. The general case follows by Lemma 5.

Theorem 2 shows that the coefficients β_n are the same for all nonatomic (probability) quasi-measures. Therefore, when trying to calculate or estimate them it is worth while to consider quasi-measures as simple as possible, for example that of Remark 3.

6. Appendix. The following result was obtained by Grzegorz Plebanek in answer to a question of the author.

THEOREM 3. *Let $(\Omega, \mathfrak{M}, \lambda)$ be a probability measure space and let $s > 0$. If (f_n) is a sequence in $L_1(\lambda)$ with $0 \leqslant f_n \leqslant 1$, $n = 1, 2, \ldots$, and $\|f_i - f_j\| \geqslant s$ whenever $i \neq j$, then $s \leqslant \frac{1}{2}$.*

Proof. We shall use the following elementary inequalities:

$$t + s - 2ts \geqslant |t - s| \quad \text{for all } t, s \in [0, 1];$$

$$t(1 - t) \leqslant \frac{1}{4} \quad \text{for all } t \in \mathbb{R}.$$

The order interval $[0, 1_\Omega]$ in $L_2(\lambda)$ is weakly compact, by the Banach–Alaoglu theorem (see, e.g., [1, Theorem 9.20]). Therefore, a subsequence of (f_n) is weakly convergent, since the weak topology of the unit ball of $L_2(\lambda)$ is metrizable provided $L_2(\lambda)$ is separable (see, e.g., [1, Theorem 10.7]), which we may assume.

Without loss of generality, we then assume that $f_n \to f$ weakly in $L_2(\lambda)$. By the first inequality above,

$$s \leqslant \int (f_i + f_j - 2f_i f_j)\, d\lambda \quad \text{whenever } i \neq j.$$

Passing to the limit when $i \to \infty$ and then when $j \to \infty$, we get, taking into account the second inequality above,

$$s \leqslant \int (2f - 2f^2)\, d\lambda \leqslant \frac{1}{2}. \ \blacksquare$$

THEOREM 4. *Let μ be a probability quasi-measure on a Boolean algebra A and let $s > 0$. If (a_n) is a sequence in A such that $\mu(a_i \triangle a_j) \geqslant s$ whenever $i \neq j$, then $s \leqslant \frac{1}{2}$.*

This can be derived from Theorem 3 by a standard technique (Stone representation theorem for Boolean algebras combined with the basic measure extension theorem). Indeed, we may assume that A is a subalgebra of a σ-algebra $\mathfrak{M}$ of sets and μ extends to a measure λ on $\mathfrak{M}$. Then it is enough to apply Theorem 3 with $f_n = 1_{a_n}$ to get the assertion.

It would be desirable to find a more direct proof of Theorem 3, which might result in a version of it for finite sequences of functions.

As for Theorem 4, this has been done subsequently by the referee (see Theorem B in Section 7 and its proof).

7. Postscript. Theorems B and C below and their proofs are due to the referee.

LEMMA A. *Let $a_1, \dots, a_p$, where $p \geqslant 2$, be subsets of a set X and let $x \in X$. Set*

$$E = \{\{i,j\} : 1 \leqslant i,j \leqslant p \text{ and } x \in a_i \triangle a_j\}.$$

We have $|E| \leqslant n^2$ or $|E| \leqslant n(n+1)$ according as $p = 2n$ or $p = 2n+1$.

Proof. Set

$$k = |\{i : 1 \leqslant i \leqslant p \text{ and } x \in a_i\}|.$$

We have $|E| = k(p-k)$ and the assertion follows. ∎

THEOREM B. *For every quasi-measure μ on A we have*

$$\beta_{2n}(\mu) \leqslant \frac{n}{2n-1} \quad \text{and} \quad \beta_{2n+1}(\mu) \leqslant \frac{n+1}{2n+1}, \quad n = 1, 2, \dots$$

Proof. Without loss of generality, we assume that A is finite, and so it is (isomorphic to) an algebra of subsets of a set X. Given $a_1, \dots, a_p$ in A, where $p \geqslant 2$, we have, by Lemma A,

$$\sum_{1 \leqslant i < j \leqslant p} 1_{a_i \triangle a_j} \leqslant n^2 \text{ or } n(n+1)$$

according as $p = 2n$ or $p = 2n+1$. Integration of both sides of this inequality with respect to μ yields

$$\frac{p(p-1)}{2} \min_{1 \leqslant i < j \leqslant p} \mu(a_i \triangle a_j) \leqslant n^2 \text{ or } n(n+1)$$

for p as above (cf. [16, proof of Proposition 1]). This yields the assertion. ∎

Theorem 4 is an obvious consequence of Theorem B.

THEOREM C. *For every $p \geqslant 2$ there exists a (strictly positive) quasi-measure μ on a finite Boolean algebra A such that*

$$\beta_p(\mu) = \frac{n}{2n-1} \quad \text{or} \quad \frac{n+1}{2n+1}$$

according as $p = 2n$ or $p = 2n+1$.

Proof. We consider the case where $p = 2n$. The other case can be established analogously. Set

$$X = \Big\{x \in \{0,1\}^p : \sum_{i=1}^{p} x(i) = n\Big\},$$

$$a_i = \{x \in X : x(i) = 1\}, \quad i = 1, \dots, p.$$

We then have

$$|a_i \setminus a_j| = \binom{2n-2}{n-1} \quad \text{whenever } i \neq j.$$

Let the quasi-measure μ on $A = 2^X$ be defined by the condition $\mu(\{x\}) = |X|^{-1}$ for all $x \in X$. It follows that

$$\mu(a_i \bigtriangleup a_j) = \frac{n}{2n-1} \quad \text{whenever } i \neq j.$$

Combined with Theorem B, this yields the assertion for $p = 2n$. ∎

Remark D. The functions $1_{a_1}, \ldots, 1_{a_p}$ of the proof of Theorem C, which are defined on X, form a binary code

$$\left(\binom{2n}{n},\ 2n,\ 2\binom{2n-2}{n-1}\right).$$

In the case where $p = 2n+1$ the corresponding binary code is

$$\left(\binom{2n+1}{n},\ 2n+1,\ 2\binom{2n-1}{n-1}\right).$$

The author does not know whether those codes have already been considered in the literature.

Theorem E. *For a nonatomic quasi-measure μ on A we have*

$$\beta_{2n}(\mu) = \frac{n}{2n-1} \quad \textit{and} \quad \beta_{2n+1}(\mu) = \frac{n+1}{2n+1}, \quad n = 1, 2, \ldots$$

Proof. Without loss of generality, we assume that μ has the Darboux property (see Lemma 5). Let ν be a strictly positive quasi-measure on a finite Boolean algebra B. Then there exists an isomorphic embedding φ of B into A such that $\nu(b) = \mu(\varphi(b))$ for all $b \in B$. Therefore, it follows from Theorem C that in the asserted equalities the inequalities "$\geqslant$" hold. The converse inequalities follow from Theorem B. ∎

Corollary F. *If μ is a nonatomic quasi-measure on A, then $\beta_n(\mu) \to \frac{1}{2}$.*

This is a direct consequance of Theorem E. It also follows from Theorem 1 and Theorem B.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985.

[2] T. E. Armstrong and K. Prikry, *Liapounoff's theorem for nonatomic, finite-dimensional vector-valued measures*, Trans. Amer. Mat. Soc. 266 (1981), 499–514.

[3] T. E. Armstrong and K. Prikry, *On the semimetric on a Boolean algebra induced by a finitely additive probability measure*, Pacific J. Math. 99 (1982), 249–264.

[4] M. Bhaskara Rao and K. P. S. Bhaskara Rao, *Topological properties of charge algebras*, Rev. Roumaine Math. Pures Appl. 22 (1977), 363–375.

[5] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges. A Study of Finitely Additive Measures*, Academic Press, London, 1983.
[6] J. Daneš, *On the Istrăţescu's measure of noncompactness*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 16 (1972), 403–406.
[7] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, New York, 1984.
[8] R. Engelking, *On the space of measurable sets of real numbers*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 (1961), 75–76.
[9] R. Engelking, *Four theorems on the space of measurable sets of real numbers*, Prace Mat. 7 (1962), 43–54 (in Polish).
[10] K. Goebel and S. Prus, *Elements of Geometry of Balls in Banach Spaces*, Oxford Univ. Press, Oxford, 2018.
[11] G. H. Greco and M. P. Moschen, *Algebre d'insiemi e misure finitamente additive*, Boll. Un. Mat. Ital. B (6) 1 (1982), 103–117.
[12] A. J. Guirao, V. Montesinos and V. Zizler, *Renormings in Banach Spaces*, Monografie Mat. 75, Birkhäuser, 2022.
[13] P. R. Halmos, *Measure Theory*, Grad. Texts in Math. 18, Springer, New York, 1974.
[14] V. I. Istrăţescu, *On a measure of noncompactness*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 16 (1972), 195–197.
[15] D. A. Kappos, *Probability Algebras and Stochastic Spaces*, Academic Press, New York and London, 1969.
[16] J. L. Kelley, *Measures on Boolean algebras*, Pacific J. Math. 9 (1959), 1165–1177.
[17] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Math. Library 16, North-Holland, Amsterdam, 1977.
[18] D. Maharam, *Finitely additive measures on the integers*, Sankhyā Ser. A 38 (1976), 44–59.
[19] E. Marczewski, *Remarks on the convergence of measurable sets and measurable functions*, Colloq. Math. 3 (1955), 118–124; reprinted in: E. Marczewski, Collected Mathematical Papers, Inst. Math., Polish Acad. Sci., Warszawa, 1996, 475–481.
[20] V. Olejček, *Ultrafilters and Darboux property of finitely additive measure*, Math. Slovaca 31 (1981), 263–276.
[21] A. Sobczyk and P. C. Hammer, *A decomposition of additive set functions*, Duke Math. J. 11 (1944), 839–846.
[22] H. Weber, *Die atomare Struktur topologischer Boolescher Ringe und s-beschränkter Inhalte*, Studia Math. 74 (1982), 57–81.
[23] R. Wituła, *Continuity and the Darboux property of nonatomic finitely additive measures*, in: Generalized Functions and Convergence, P. Antosik and A. Kamiński (eds.), World Sci., Singapore, 1990, 377–386.

Zbigniew Lipecki
Institute of Mathematics
Polish Academy of Sciences, Wrocław Branch
Kopernika 18
51-617 Wrocław, Poland
E-mail: lipecki@impan.pl