COLLOQUIUM MATHEMATICUM

Online First version

ON LOCAL CONVEXITIES OF ORLICZ SPACES ENDOWED WITH s-NORMS

BY

ESRA BAŞAR and SERAP ÖZTOP

Abstract. Let Φ be an Orlicz function and $L^{\Phi}(X, \Sigma, \mu)$ be the corresponding Orlicz space on a non-atomic, σ-finite, complete measure space (X, Σ, μ). We describe the local uniform convexity of Orlicz spaces endowed with the s-norm and discuss the weak and compact variants of this property. Also, we derive some results in approximation theory, concerning best approximations and farthest points. Thus, our study provides a comprehensive generalization of several results that have been obtained for Orlicz spaces with the Orlicz norm and the Luxemburg norm.

1. Introduction. The notions of strict convexity and uniform convexity were introduced by J. A. Clarkson in his pivotal paper [9] on geometric aspects of Banach spaces. Between these two notions, a number of intermediate geometric properties have been investigated. Among them, local uniform convexity, introduced by A. R. Lovaglia [16], is one of the most significant. Also, some variants of local uniform convexity such as weak local uniform convexity and compact local uniform convexity have been subject of research [16, 23]. Applications of local convexities are found in various branches of mathematics such as approximation theory and probability theory. On the other hand, Orlicz spaces are an important class of Banach spaces that are a kind of generalization of Lebesgue spaces which have a richer geometric structure due to the variety of norms they admit. Local convexities of Orlicz function spaces endowed with the Orlicz norm and the Luxemburg norm have been investigated in various papers [6, 8, 13, 24].

In [25], using the concept of an outer function, M. Wisła introduced s-norms which are a general and universal method of introducing norms in Orlicz spaces and which cover the classical Orlicz, Luxemburg, and p-Amemiya norms ($1 \le p \le \infty$). Several geometric properties of Orlicz spaces endowed with the s-norm have been studied [2, 3, 4, 11, 22]. Our aim in this

2020 *Mathematics Subject Classification*: Primary 46E30; Secondary 46B20.

Key words and phrases: local uniform convexity, weak local uniform convexity, compact local uniform convexity, Orlicz space, s-norm, Fréchet differentiability, best approximation.

Received 30 August 2025.

Published online 1 April 2026.

DOI: 10.4064/cm9702-10-2025

work is to describe the local uniform convexity of Orlicz spaces endowed with the s-norm along with its generalizations in the weak and compact senses.

Furthermore, local convexities of Banach spaces lead to notable consequences in norm differentiability as well as approximation theory. One significant difference between Hilbert spaces and Banach spaces is that, in the latter, best approximations may fail to exist. In this connection, we provide some results on the existence and uniqueness of best approximations in Orlicz spaces endowed with s-norms in Section 5. Further, the continuity of the metric projection operator have found applications in various branches such as fixed point theory and optimization, so we shall give a result on the continuity of the metric projection on an Orlicz space endowed with the s-norm. Moreover, in [2], an extremal problem in Orlicz spaces was examined and we consider another such problem related to farthest points. As for norm-differentiability, the Gâteaux differentiability of the s-norm was discussed in [2]. In this paper, we deduce a stronger result thanks to local uniform convexity.

The structure of this is paper as follows. In Section 2, we provide the necessary definitions. In Section 3, we provide a sufficient condition for a sequence to be equi-continuous with respect to the s-norm (Theorem 3.3), and we provide a relation between convergence in measure and convergence in s-norm (Theorem 3.4). In Section 4, we describe local uniform convexity, weak local uniform convexity, and compact local uniform convexity of Orlicz spaces endowed with s-norms (Theorems 4.3, 4.4 and 4.6 respectively). We also provide a diagrammatic summary of the results obtained. Finally, in Section 5, we derive some results on best approximation and an extremal problem in approximation theory as well as norm differentiability by applying the results of Section 4.

We extend and synthesize the results on local convexities obtained for the Orlicz and Luxemburg norms. But also, using the s-norm, we present new results for p-Amemiya norms such as equi-continuity and a relation between norm and measure in addition to local convexities.

2. Preliminaries. A function $\Phi : \mathbb{R} \to [0,\infty]$ is an *Orlicz function* if $\Phi(0) = 0$, Φ is not identically equal to zero, Φ is even and convex on the interval $(-b_\Phi, b_\Phi)$, and Φ is left continuous at b_Φ, where $b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}$. From these properties it follows that an Orlicz function Φ is continuous on $(-b_\Phi, b_\Phi)$, increasing on $[0, b_\Phi)$, and satisfies $\lim_{u\to\infty} \Phi(u) = \infty$. Let also $a_\Phi = \sup\{u \geq 0 : \Phi(u) = 0\}$. Then $a_\Phi = 0$ means that Φ vanishes only at 0, while $b_\Phi = \infty$ means that Φ takes only finite values.

For an Orlicz function Φ, we define its *complementary function* Ψ by the formula

$$\Psi(v) = \sup_{u\geq 0} \{u|v| - \Phi(u)\}.$$

It is well-known that the complementary function is also an Orlicz function [19]. Let p_+ and p_- denote the right and left derivatives of an Orlicz function Φ respectively, and let q_+ denote the right derivative of the complementary function Ψ, with the conventions that $\lim_{u\to\infty} p_+(u) = p_+(\infty)$ and $p_+(u) = \infty$ for all $u \geq b_\Phi$. We know that the pair (Φ, Ψ) satisfies Young's inequality, that is,

$$xy \leq \Phi(x) + \Psi(y) \quad (x, y \in \mathbb{R}),$$

where equality holds when $y = p_+(x)$ or $x = q_+(y)$ for $x, y \in \mathbb{R}$ [19].

We say that an Orlicz function Φ satisfies the Δ_2 *condition* and write $\Phi \in \Delta_2$ if there exists a constant $K > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for all $u \geq 0$ (respectively for all $u \geq u_0$ and with $\Phi(u_0) < \infty$) provided the measure space is infinite (respectively finite).

We say that an Orlicz function Φ satisfies the ∇_2 *condition* and write $\Phi \in \nabla_2$ if there exists a constant $l > 0$ such that $\Phi(u) \leq \frac{1}{2l}\Phi(lu)$ for all $u \geq 0$ (respectively for all $u \geq u_0$ with $\Phi(u_0) < \infty$) provided the measure space is infinite (respectively finite). It is known that $\Phi \in \nabla_2$ is equivalent to $\Psi \in \Delta_2$.

Throughout the paper, we will assume that (X, Σ, μ) is a measure space with a σ-finite, non-atomic and complete measure μ and denote by $L^0(X, \Sigma, \mu)$ the space of all μ-equivalence classes of real-valued and Σ-measurable functions defined on X. For $1 \leq p < \infty$, we will also denote by $L^p(X, \Sigma, \mu)$ or just by L^p the space of Lebesgue p-integrable functions with respect to (X, Σ, μ). In addition, we use the conventions $0 \cdot \infty = 0$, $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

For a given Orlicz function Φ, we define on $L^0(X, \Sigma, \mu)$ a convex functional I_Φ by

$$I_\Phi(f) = \int_X \Phi(f(t))\, d\mu.$$

The *Orlicz space* $L^\Phi(X, \Sigma, \mu)$ generated by an Orlicz function Φ is a linear space of measurable functions [17] defined by

$$L^\Phi(X, \Sigma, \mu) = \{f \in L^0(X, \Sigma, \mu) : I_\Phi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

We denote $L^\Phi(X, \Sigma, \mu)$ briefly by L^Φ. Also, the subspace E^Φ of L^Φ is defined by

$$E^\Phi = \{f \in L^0(X, \Sigma, \mu) : I_\Phi(\lambda f) < \infty \text{ for all } \lambda > 0\}.$$

It is well-known that $E^\Phi = L^\Phi$ if and only if $\Phi \in \Delta_2$ [19].

The Orlicz space L^Φ is usually equipped with the *Orlicz norm* [17]

$$\|f\|_\Phi^o = \sup\Big\{\int_X |f(t)g(t)|\, d\mu : g \in L^\Psi,\, I_\Psi(g) \leq 1\Big\},$$

where Ψ is the complementary function to Φ, or with the equivalent *Luxem-*

burg norm

$$\|f\|_{\Phi} = \inf\left\{\lambda > 0 : I_{\Phi}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Further, for all $1 \le p \le \infty$ the *p-Amemiya norm* is defined on L^{Φ} by

$$\|f\|_{\Phi,p} = \begin{cases} \inf_{k>0} k^{-1}(1 + I_{\Phi}(kf)^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \inf_{k>0} k^{-1} \max\{1, I_{\Phi}(kf)\} & \text{if } p = \infty. \end{cases}$$

The family of p-Amemiya norms includes the Orlicz and the Luxemburg norms [10, 14].

The notion of the s-norm was introduced by M. Wisła and the following definitions can be found in [25].

Definition 2.1. A function $s : [0, \infty] \to [1, \infty]$ is called an *outer function* if it is convex and satisfies the inequality

$$\max\{u, 1\} \le s(u) \le u + 1$$

for all $u \ge 0$.

Let us note that an outer function s is continuous and increasing on $[0, \infty)$. Evidently $s(0) = 1$ and we set $s(\infty) = \infty$.

Since it is convex, an outer function s has both right and left derivatives. Let s'_+ be the right derivative of s so that $s'_+ : [0, \infty) \to [0, 1]$ is an increasing function. Let ${s'_+}^{-1} : [0, 1] \to [0, \infty]$ be a general inverse of s'_+ as defined in [25, p. 11]. Then ${s'_+}^{-1}$ is also an increasing function.

Let us give some examples of families of outer functions.

Example 2.2.

(i) For $1 \le p \le \infty$,

$$s_p(u) = \begin{cases} (1 + u^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1, u\} & \text{if } p = \infty. \end{cases} \tag{2.1}$$

(ii) For $0 \le c \le 1$,

$$s_c(u) = \max\{1, u + c\}. \tag{2.2}$$

(iii) For $0 \le \alpha \le 1$,

$$s_{\alpha}(u) = \alpha \max\{1, u\} + (1 - \alpha)(1 + u). \tag{2.3}$$

Definition 2.3. Let s be an outer function and Φ be an Orlicz function. Then the *s-norm* of $f \in L^{\Phi}$ is defined by

$$\|f\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_{\Phi}(kf)).$$

We denote the Orlicz space equipped with the s-norm by L^{Φ}_s and its closed unit ball (respectively the unit sphere) by $B(L^{\Phi}_s)$ (respectively $S(L^{\Phi}_s)$).

Observe that each of the families given in Example 2.2 generates both the Orlicz norm and the Luxemburg norm. In (2.1), if we take $s = s_1$ then $\|f\|_{\Phi,s} = \|f\|^o_{\Phi}$; if $s = s_\infty$, then $\|f\|_{\Phi,s} = \|f\|_{\Phi}$; if $s = s_p$ for $1 < p < \infty$ then $\|f\|_{\Phi,s} = \|f\|_{\Phi,p}$; see [10, 14]. Similarly, in (2.2), $c = 0$ gives the Luxemburg norm and $c = 1$ the Orlicz norm. Finally, in (2.3), $\alpha = 0$ gives rise to the Orlicz norm and $\alpha = 1$ to the Luxemburg norm.

It is known that the s-norm $\|\cdot\|_{\Phi,s}$ is equivalent to the Luxemburg norm $\|\cdot\|_{\Phi}$ with $\|f\|_{\Phi} \le \|f\|_{\Phi,s} \le 2\|f\|_{\Phi}$ for any $f \in L^{\Phi}_s$.

Definition 2.4. Two outer functions s, σ are called *conjugate to each other in the Hölder sense* if

$$u + v \le s(u)\sigma(v)$$

for all $u, v \ge 0$.

Note that for any outer function s the function s^* defined by $s^*(v) = \sup_{u\ge 0} \frac{u+v}{s(u)}$ is the minimal outer function conjugate to s in the Hölder sense.

Definition 2.5. Let s be an outer function. For all $0 \le v \le 1$, we define

$$w(v) = \int_0^v {s'_+}^{-1}(t)\, dt. \tag{2.4}$$

It is clear that w is a non-negative, increasing and continuous function on $[0, 1]$.

Definition 2.6. Let s be an outer function. For all $0 \le u < \infty$ and $0 \le v \le \infty$, set

$$\beta_s(u, v) = 1 - w(s'_+(u)) - v s'_+(u).$$

Denote also $\beta_s(kf) = \beta_s(I_\Phi(kf), I_\Psi(p_+(k|f|))$ for all $f \in L^{\Phi}_s$.

Note that the function $k \mapsto \beta_s(kf)$ is decreasing on $[0, \infty)$.

Definition 2.7. Let s be an outer function and Φ be an Orlicz function. For $f \in L^{\Phi} \setminus \{0\}$ and $0 < k < \infty$, we define the following functions:

$$k^* : L^{\Phi}_s \to [0, \infty), \qquad k^*(f) = \inf\{k > 0 : \beta_s(kf) \le 0\},$$
$$k^{**} : L^{\Phi}_s \to (0, \infty], \qquad k^{**}(f) = \sup\{k > 0 : \beta_s(kf) \ge 0\}.$$

It is easy to see that $0 < k^*(f) \le k^{**}(f) \le \infty$. Let us also define

$$K(f) := \{0 < k < \infty : k^*(f) \le k \le k^{**}(f)\}.$$

Obviously, $K(f) \ne \emptyset$ if and only if $k^*(f) < \infty$. If $k^*(f) < \infty$ for any $f \in L^{\Phi}_s$, then the s-norm is called k^*-*finite*; if $k^{**}(f) < \infty$ for any $f \in L^{\Phi}_s$, then the s-norm is called k^{**}-*finite*. Further, if $k^*(f) = k^{**}(f) < \infty$ for any $f \in L^{\Phi}_s$, then the s-norm is called k-*unique*.

3. Convergence and equi-continuity of s-norms on unit balls. In this section, by using s-norms, we extend some results that have been obtained for Luxemburg and Orlicz norms [6], but have not yet been considered for the p-Amemiya norm. More specifically, we investigate a relation between convergence in measure and in s-norm. Then we formulate a sufficient condition for equi-continuity of a sequence under the s-norm. We shall use these results later.

LEMMA 3.1. *Let Φ be a strictly convex Orlicz function and s an outer function such that the s-norm is k^*-finite. Suppose $f_n, g_n \in B(L_s^\Phi)$, $k_n \in K(f_n)$, $l_n \in K(g_n)$ and $b := \sup_n \max\{k_n, l_n\} < \infty$. Then $\|f_n + g_n\|_{\Phi,s} \to 2$ implies $k_n f_n - l_n g_n \xrightarrow{\mu} 0$.*

Proof. Assume that $\|f_n + g_n\|_{\Phi,s} \to 2$ and $k_n f_n - l_n g_n$ does not converge to 0 in measure. Thus there exists $\varepsilon_0 > 0$ such that $E_n = \{t \in X : |k_n f_n(t) - l_n g_n(t)| \geq \varepsilon_0\}$ is of positive measure for $n \in \mathbb{N}$. Let $\mu(E_n) > 2\gamma > 0$ and $m_0 = \Phi^{-1}(2/\gamma)$.

Let $A_n = \{t \in X : |f_n(t)| > m_0\}$ and $B_n = \{t \in X : |g_n(t)| > m_0\}$. Since $\|f_n\|_{\Phi,s} \geq \|f_n\|_\Phi$, we have $\|f_n\|_\Phi \leq 1$, leading to

$$1 \geq I_\Phi(f_n) \geq I_\Phi(f_n \chi_{A_n}) = \int_{A_n} \Phi(f_n(t))d\mu(t) \geq \int_{A_n} \Phi(m_0)d\mu(t) = \frac{2}{\gamma}\mu(A_n).$$

Therefore $\mu(A_n) \leq \gamma/2$ and similarly $\mu(B_n) \leq \gamma/2$.

Let $X_n = E_n \setminus (A_n \cup B_n)$. Then

$$\mu(X_n) \geq \mu(E_n) - \mu(A_n) - \mu(B_n) > 2\gamma - \gamma/2 - \gamma/2 = \gamma.$$

In view of [6, Proposition 1.4], there exists $\delta > 0$ such that

$$\Phi(\alpha k_n f_n(t) + (1-\alpha)l_n g_n(t)) \leq (1-\delta)(\alpha\Phi(k_n f_n(t)) + (1-\alpha)\Phi(l_n g_n(t))$$

for any $\alpha \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right] \subset (0,1)$ and all $u, v \in \mathbb{R}$ with $|u|, |v| \leq bm_0$ and $|u - v| \geq \varepsilon_0$.

Moreover, since $k_n \in K(f_n)$ and $l_n \in K(g_n)$ we have

$$1 \geq \|f_n\|_{\Phi,s} = \frac{1}{k_n} s(I_\Phi(k_n f_n)) \geq \frac{1}{k_n}.$$

Hence $k_n \geq 1$ and similarly $l_n \geq 1$ for $n \in \mathbb{N}$, so $\frac{k_n}{k_n+l_n}, \frac{l_n}{k_n+l_n} \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right]$.

It follows that for any $t \in X_n$,

$$\Phi\left(\frac{k_n l_n}{k_n+l_n}(f_n(t)+g_n(t))\right) \leq (1-\delta)\left(\frac{l_n}{k_n+l_n}\Phi(k_n f_n(t))+\frac{k_n}{k_n+l_n}\Phi(l_n g_n(t))\right).$$

Thus using the definitions of the outer function s and the Orlicz function Φ, we obtain

$$\begin{aligned}
\|f_n+g_n\|_{\Phi,s} &\le \frac{l_n+k_n}{l_nk_n}\, s\bigg(I_\Phi\bigg(\frac{l_nk_n}{l_n+k_n}(f_n(t)+g_n(t))\bigg)\bigg)\\
&\le 2-\delta\bigg(\frac{1}{k_n}\int\limits_{X_n}\Phi(k_nf_n(t))\,d\mu+\frac{1}{l_n}\int\limits_{X_n}\Phi(l_ng_n(t))\,d\mu\bigg)\\
&\le 2-\frac{2\delta}{b}\int\limits_{X_n}\Phi\bigg(\frac{k_nf_n(t)-l_ng_n(t)}{2}\bigg)\,d\mu\\
&< 2-\frac{2\delta}{b}\Phi\bigg(\frac{\varepsilon_0}{2}\bigg)\gamma<2.
\end{aligned}$$

However, this contradicts $\|f_n+g_n\|_{\Phi,s}\to 2$. Therefore, $k_nf_n-l_ng_n\xrightarrow{\mu}0$ under the assumptions of the lemma. ∎

LEMMA 3.2. *Assume that $\Phi\in\Delta_2\cap\nabla_2$, the corresponding s-norm is k^*-finite, $f_n,g_n\in B(L_s^\Phi)$ and $\|f_n+g_n\|_{\Phi,s}\to 2$. Then for any $\varepsilon>0$ there exist $n'\in\mathbb{N}$ and $\delta>0$ such that for any $n\ge n'$ and $E\in\Sigma$, $\|g_n\chi_E\|_{\Phi,s}<\delta$ implies $\|f_n\chi_E\|_{\Phi,s}<\varepsilon$.*

Proof. Let $\varepsilon>0$ and $E\in\Sigma$. Since $\Phi\in\Delta_2$, convergence in the s-norm is equivalent to that in I_Φ [25]. Thus there exists $\alpha_\varepsilon>0$ such that $I_\Phi(f)<\alpha_\varepsilon$ implies $\|f\|_{\Phi,s}<\varepsilon$ for $f\in L_s^\Phi$.

Let $k_n\in K(f_n)$ and $l_n\in K(g_n\chi_E)$. Hence $k_n,l_n\ge 1$ since $f_n,g_n\chi_E\in B(L_s^\Phi)$. Moreover, $\Phi\in\nabla_2$ leads to $b:=\sup\max\{k_n,l_n\}<\infty$ by [2, Proposition 4.6].

Now let $l_n>1$ for $n\in\mathbb{N}$. Since (l_n) is bounded and $I_\Phi(f_n\chi_E)\le I_\Phi(f_n)\le 1$, we take $\gamma>0$ such that $I_\Phi(f_n\chi_E)/\gamma\le 1-1/l_n$ for $n\in\mathbb{N}$. Without loss of generality, let $\delta\le\frac{\alpha_\varepsilon}{2\gamma}$ and suppose that $\|g_n\chi_E\|_{\Phi,s}<\delta$.

Since $u\le s(u)\le 1+u$ and $\|f_n+g_n\|_{\Phi,s}\to 2$, there exists $n'\in\mathbb{N}$ such that for any $n\ge n'$,

$$\begin{aligned}
\frac{\alpha_\varepsilon}{2\gamma}>2-\|f_n+g_n\|_{\Phi,s}\ge 2-\|f_n\|_{\Phi,s}-\|g_n\|_{\Phi,s}&\\
&\ge 2-\frac{1}{k_n}s(I_\Phi(k_nf_n))-\frac{1}{l_n}s(I_\Phi(l_ng_n))\\
&\ge 1-\frac{1}{l_n}s(I_\Phi(l_ng_n))\\
&\ge 1-\frac{1}{l_n}-\frac{1}{l_n}I_\Phi(l_ng_n)\\
&\ge \frac{I_\Phi(l_n\chi_E)}{\gamma}-\frac{I_\Phi(l_ng_n\chi_E)}{l_n}.
\end{aligned}$$

Since $\|g_n\chi_E\|_{\Phi,s} < \delta$, we have

$$\frac{I_\Phi(l_n g_n \chi_E)}{l_n} \le \frac{1}{l_n} s(I_\Phi(l_n g_n \chi_E)) < \delta.$$

It follows that

$$\frac{\alpha_\varepsilon}{2\gamma} > 2 - \|f_n + g_n\|_{\Phi,s} \ge \frac{I_\Phi(f_n\chi_E)}{\gamma} - \delta \ge \frac{I_\Phi(f_n\chi_E)}{\gamma} - \frac{\alpha_\varepsilon}{2\gamma},$$

and so $I_\Phi(f_n\chi_E) < \alpha_\varepsilon$.

Now let $l_n = 1$ for $n \in \mathbb{N}$. Hence $1 \ge \|g_n\chi_E\|_{\Phi,s} = s(I_\Phi(g_n\chi_E)) \ge 1$, so $\|f_n\|_{\Phi,s} \le \|g_n\|_{\Phi,s} = 1$. Therefore,

$$\begin{aligned}\alpha_\varepsilon > 2 - \|f_n + g_n\|_{\Phi,s} \ge 2 - \|f_n\|_{\Phi,s} - \|g_n\|_{\Phi,s} &\ge 2 - 2\|g_n\chi_E\|_{\Phi,s} \\ &\ge 2I_\Phi(f_n\chi_E) - 2\delta.\end{aligned}$$

Assuming $\delta \le \alpha_\varepsilon/2$ yields $I_\Phi(f_n\chi_E) < \alpha_\varepsilon$. In both cases, we have $I_\Phi(f_n\chi_E) < \alpha_\varepsilon$, and thus $\|f_n\chi_E\|_{\Phi,s} < \varepsilon$ holds for any $n \ge n'$. ∎

A sequence (f_n) is said to be *norm equi-continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$ then $\|f_n\chi_E\| < \varepsilon$.

The next theorem provides a sufficient condition for equi-continuity of a sequence with respect to s-norm.

THEOREM 3.3. *Assume $\Phi \in \Delta_2 \cap \nabla_2$ and $\|f_n + f\|_{\Phi,s} \to 2$ for $f, f_n \in B(L_s^\Phi)$. Then (f_n) is norm equi-continuous.*

Proof. Let $\varepsilon > 0$ be given. By Lemma 3.2, there exist $\delta > 0$ and $n' \in \mathbb{N}$ such that for all $n \ge n'$ and $E \in \Sigma$ if $\|f\chi_E\|_{\Phi,s} < \delta$, then $\|f_n\chi_E\|_{\Phi,s} < \varepsilon$.

Since μ is σ-finite, we can take an increasing sequence $(X_n) \subset \Sigma$ with $\mu(X_n) < \infty$ such that $\bigcup X_n = X$. Due to the Beppo Levi theorem, $I_\Phi(f\chi_{X_n}) \to I_\Phi(f)$, so $I_\Phi(f\chi_{X\setminus X_n}) \to 0$. Moreover, since $\Phi \in \Delta_2$, we have $\|f\chi_{X\setminus X_n}\|_{\Phi,s} \to 0$. Accordingly, there exists $n'' \in \mathbb{N}$ such that $\|f\chi_{X\setminus X_n}\|_{\Phi,s} < \delta$ for all $n \ge n''$. Hence, $\|f_n\chi_{X\setminus X_n}\|_{\Phi,s} < \varepsilon$ for any $n \ge n_0 := \max\{n', n''\}$.

Further, since $\Phi \in \Delta_2$, we have $E_s^\Phi = L_s^\Phi$ which implies that $f\chi_{X_n}$ is norm absolutely continuous. Take a measurable subset $S \subset X_n$ for $n \in \mathbb{N}$. Thus there exists $\alpha > 0$ such that if $\mu(S) < \alpha$, then $\|f\chi_S\|_{\Phi,s} < \varepsilon$ and it follows from Lemma 3.2 that $\|f_n\chi_S\|_{\Phi,s} < \varepsilon$ for any $n \ge n_0$; this shows that (f_n) is norm equi-continuous. ∎

The following lemma reveals a relation between convergence in measure and convergence in s-norm.

THEOREM 3.4. *Assume $\Phi \in \Delta_2 \cap \nabla_2$, $\|f_n + f\|_{\Phi,s} \to 2$ for $f, f_n \in B(L_s^\Phi)$, and $f_n \xrightarrow{\mu} f$. Then $\|f_n - f\|_{\Phi,s} \to 0$.*

Proof. Take an increasing sequence $(X_n) \subset \Sigma$ such that $\mu(X_n) < \infty$ and $\bigcup X_n = X$. Let $\varepsilon > 0$ be given. By Lemma 3.2, there exist $\delta > 0$ and $n' \in \mathbb{N}$ such that for any $n \ge n'$ and $E \in \Sigma$, if $\|f\chi_E\|_{\Phi,s} < \delta$, then

$\|f_n\chi_E\|_{\Phi,s} < \varepsilon/5$. Assume that $\delta \leq \varepsilon/5$. Also, since $\Phi \in \Delta_2$, there exists $n \in \mathbb{N}$ such that $\|f\chi_{X\setminus X_n}\|_{\Phi,s} < \delta$. Then $\|f_n\chi_{X\setminus X_n}\|_{\Phi,s} < \varepsilon$ for any $n \geq n'$.

From Theorem 3.3, (f_n) is norm equi-continuous, so there exist $n_1 \in \mathbb{N}$ and $\alpha > 0$ such that for any $E \in \Sigma$, if $\mu(E) < \alpha$, then $\|f_n\chi_E\|_{\Phi,s} < \varepsilon/5$ for any $n \geq n_1$.

Moreover, $f\chi_E$ has absolutely continuous norm since $\Phi \in \Delta_2$, so there exists $\gamma > 0$ such that for any measurable $E \subset X_n$, if $\mu(E) < \gamma$, then $\|f\chi_E\|_{\Phi,s} < \delta$.

On the other hand, we have pointwise convergence of $f_n(t) \to f(t)$ for μ-a.e. $t \in X$ since $f_n \xrightarrow{\mu} f$. Due to the Egorov theorem, there exists a measurable $A \subset X_n$ with $\mu(A) < \min\{\alpha, \gamma\}$ such that $f_n \to f$ uniformly on $X_n \setminus A$. Then for some $a < b_\Phi$, there exists $n_2 \in \mathbb{N}$ such that $|f_n(t) - f(t)| < a$ for any $t \in X_n \setminus A$.

Using the Lebesgue dominated convergence theorem, we deduce that $I_\Phi((f_n - f)\chi_{X_n\setminus A}) \to 0$, equivalently $\|(f_n - f)\chi_{X_n\setminus A}\|_{\Phi,s} \to 0$ since $\Phi \in \Delta_2$. This implies there exists $n_0 \geq \max\{n_1, n_2\}$ such that $\|(f_n - f)\chi_{X_n\setminus A}\|_{\Phi,s} < \varepsilon/5$ for any $n \geq n_0$. Consequently, for any $n \geq n_0$, we have

$$\begin{aligned}\|f_n - f\|_{\Phi,s} \leq \|f_n\chi_{X\setminus X_n}\|_{\Phi,s} + \|f\chi_{X\setminus X_n}\|_{\Phi,s} + \|(f_n - f)\chi_{X_n\setminus A}\|_{\Phi,s} \\ + \|f_n\chi_A\|_{\Phi,s} + \|f\chi_A\|_{\Phi,s} < \varepsilon,\end{aligned}$$

which shows $\|f_n - f\|_{\Phi,s} \to 0$. ■

COROLLARY 3.5.

(i) *If $\Phi \in \Delta_2 \cap \nabla_2$ and $\|f_n + f\|_{\Phi,p} \to 2$ for $f, f_n \in B(L_p^\Phi)$ with $1 \leq p \leq \infty$, then (f_n) is norm equi-continuous.*

(ii) *If $\Phi \in \Delta_2 \cap \nabla_2$, $\|f_n + f\|_{\Phi,p} \to 2$ for $f, f_n \in B(L_p^\Phi)$ with $1 \leq p \leq \infty$, and $f_n \xrightarrow{\mu} f$, then $\|f_n - f\|_{\Phi,p} \to 0$.*

4. Main results. This section is devoted to presenting criteria for some local convexities in Orlicz spaces endowed with the s-norm. In Theorems 4.3, 4.4 and 4.6, we characterize local uniform convexity, weak local uniform convexity and compact local uniform convexity of the Orlicz space L_s^Φ respectively. Moreover, we deduce criteria for local convexities of the Orlicz space endowed with the p-Amemiya norm, which is denoted by L_p^Φ, in Corollary 4.7. Furthermore, we briefly illustrate the interrelations between some convexity notions in the Orlicz space L_s^Φ.

Let us recall the definitions of the convexity notions to be used.

DEFINITION 4.1. A function $f \in S(L_s^\Phi)$ is called an *extreme point* of $B(L_s^\Phi)$ if for any $g, h \in B(L_s^\Phi)$, $2f = g + h$ implies $g = h$. The set of all extreme points of $B(L_s^\Phi)$ is denoted by $\operatorname{Ext} B(L_s^\Phi)$. The Orlicz space L_s^Φ is called *strictly convex* or *rotund* (abbreviated as SC) if $\operatorname{Ext} B(L_s^\Phi) = S(L_s^\Phi)$.

Definition 4.2.

(i) If for any $f \in S(L_s^\Phi)$, the convergence $\|f_n + f\|_{\Phi,s} \to 2$ for $(f_n) \subset S(L_s^\Phi)$ implies $\|f_n - f\|_{\Phi,s} \to 0$, i.e., $f_n \to f$, then the Orlicz space L_s^Φ is called *locally uniformly convex* (LUC).

(ii) If for any $f \in S(L_s^\Phi)$, the convergence $\|f_n + f\|_{\Phi,s} \to 2$ for $(f_n) \subset S(L_s^\Phi)$ implies $f_n \xrightarrow{w} f$, i.e., (f_n) converges to f weakly, then the Orlicz space L_s^Φ is called *weakly locally uniformly convex* (WLUC).

(iii) If for any $f \in S(L_s^\Phi)$, the convergence $\|f_n + f\|_{\Phi,s} \to 2$ for $(f_n) \subset S(L_s^\Phi)$ implies that the sequence (f_n) is relatively compact, then the Orlicz space L_s^Φ is called *compactly locally uniformly convex* (CLUC).

Theorem 4.3. *The Orlicz space L_s^Φ is locally uniformly convex if and only if*

(i) *the s-norm is k-unique,*

(ii) *Φ is strictly convex,*

(iii) $\Phi \in \Delta_2 \cap \nabla_2$.

Proof. Necessity. Assume the Orlicz space L_s^Φ is locally uniformly convex. Then L_s^Φ is strictly convex, so the s-norm is k-unique and Φ is strictly convex by [2, Theorems 5.2 and 5.3].

Now we show that $\Phi \in \Delta_2$ is a necessary condition for locally uniform convexity of the Orlicz space L_s^Φ. Assume $\Phi \notin \Delta_2$. Then $E_s^\Phi \neq L_s^\Phi$ and we can take $f \in L_s^\Phi \setminus E_s^\Phi$.

Since μ is σ-finite, we can take an increasing sequence of measurable sets of finite measures such that $\bigcup X_n = X$. Define

$$f_n(t) = \begin{cases} f(t)\chi_{X_n}(t) & \text{if } |f(t)| \leq n, \\ 0 & \text{if } |f(t)| > n. \end{cases}$$

This yields $(f_n) \subset E_s^\Phi$. By the semi-Fatou property of the s-norm [25], we have $\|f_n\|_{\Phi,s} \to \|f\|_{\Phi,s}$. Therefore,

$$\left\| \frac{f_n}{\|f_n\|_{\Phi,s}} + \frac{f}{\|f\|_{\Phi,s}} \right\|_{\Phi,s} \geq \left(\frac{1}{\|f_n\|_{\Phi,s}} + \frac{1}{\|f\|_{\Phi,s}} \right) \|f_n\|_{\Phi,s} \to 2.$$

This leads to

$$\left\| \frac{f_n}{\|f_n\|_{\Phi,s}} + \frac{f}{\|f\|_{\Phi,s}} \right\|_{\Phi,s} \to 2.$$

On the other hand, by the Hahn–Banach theorem, there is a functional $F : L_s^\Phi \to \mathbb{R}$ such that $F(E_s^\Phi) = \{0\}$ and $F(f) \neq 0$, and so

$$F\left(\frac{f_n}{\|f_n\|_{\Phi,s}} - \frac{f}{\|f\|_{\Phi,s}} \right) = -\frac{F(f)}{\|f\|_{\Phi,s}} \neq 0 \quad \text{for any } n \in \mathbb{N}.$$

It follows that $f_n/\|f_n\|_{\Phi,s}$ does not converge to $f/\|f\|_{\Phi,s}$ weakly. Evidently, $f_n/\|f_n\|_{\Phi,s} \nrightarrow f/\|f\|_{\Phi,s}$. We conclude that if the Orlicz space L_s^{Φ} is locally uniformly convex, then $\Phi \in \Delta_2$.

Now we show that if L_s^{Φ} is locally uniformly convex, then $\Phi \in \nabla_2$. Assume that, on the contrary, $\Phi \notin \nabla_2$, equivalently $\Psi \notin \Delta_2$. Then there exists a sequence (v_n) such that

$$I_\Psi(v_n) \to \infty, \quad \operatorname{sgn}(v_n(t)) = \operatorname{sgn}(f(t)) \text{ for any } t \in \operatorname{supp} f(t),$$

and

$$\Psi\left(\frac{v_n}{1-1/n}\right) > 2n\,\Psi(v_n)$$

for $n \in \mathbb{N}$.

On the other hand, since $\Phi \in \Delta_2$, we have $E_s^{\Phi} = L_s^{\Phi}$. Thus, $(L_s^{\Phi})^* = L_{s^*}^{\Psi}$, where $(L_s^{\Phi})^*$ denotes the Köthe dual of L_s^{Φ} and s^* is the minimal Hölder conjugate of s [25]. If $f \in S(L_s^{\Phi})$, we can find $g \in S(L_{s^*}^{\Psi})$ such that $\int_X g(t)f(t)\,d\mu = \langle g, f\rangle = \|f\|_{\Phi,s} = 1$.

Since μ is non-atomic and σ-finite, we can find a sequence $(E_n) \subset \Sigma$ such that $\Psi(v_n)\mu(E_n) = 1/n$ and $E_n \subset \{t \in X : |f(t)| \le a\}$ for some $a > 0$. Note also that $\mu(E_n) \to 0$ since $\Psi(v_n) \to \infty$. We have $I_\Psi(v_n\chi_{E_n}) = 1/n$ and so

$$I_\Psi\left(\frac{v_n\chi_{E_n}}{1-1/n}\right) \ge 2n\Psi(v_n)\mu(E_n) = 2$$

for $n \in \mathbb{N}$. This shows that

$$1 \ge \|v_n\chi_{E_n}\|_\Psi = \inf\left\{\lambda > 0 : I_\Psi\left(\frac{v_n\chi_{E_n}}{\lambda}\right) \le 1\right\} \ge 1 - \frac{1}{n}.$$

Hence, we find that $\|v_n\chi_{E_n}\|_{\Psi,s^*} \ge 1 - 1/n$ by using $\|v_n\chi_{E_n}\|_\Psi \le \|v_n\chi_{E_n}\|_{\Psi,s^*}$. Moreover,

$$1 + \frac{1}{n} = 1 + I_\Psi(v_n\chi_{E_n}) \ge \|v_n\chi_{E_n}\|_\Psi^o \ge \|v_n\chi_{E_n}\|_{\Psi,s^*}.$$

This leads to

$$1 + \frac{1}{n} \ge \|v_n\chi_{E_n}\|_{\Psi,s^*} \ge 1 - \frac{1}{n},$$

and so $v_n\chi_{E_n} \in E_{s^*}^{\Psi}$. Moreover, since $(E_{s^*}^{\Psi})^* = L_s^{\Phi}$, there exists a sequence $(f_n) \subset S(L_s^{\Phi})$ such that

$$\int_X f_n(t)v_n\chi_{E_n}(t)\,d\mu = \langle f_n, v_n\chi_{E_n}\rangle = \|v_n\chi_{E_n}\|_{\Psi,s^*} \ge 1 - \frac{1}{n}.$$

Without loss of generality, we assume that $f_n = f_n\chi_{E_n}$. Now let

$$h_n(t) = \frac{1}{1+1/n}(g\chi_{X\setminus E_n}(t) + v_n\chi_{E_n}(t)),$$

and so $h_n \in B(L_{s^*}^{\Psi})$.

Using the Hölder inequality and absolute continuity of the s-norm [25], we obtain

$$\begin{aligned}\|f_n+f\|_{\Phi,s} &\geq \langle h_n, f_n+f\rangle = \int\limits_X f_n(t)h_n(t)\,d\mu + \int\limits_X f(t)h_n(t)\,d\mu \\ &\geq \frac{1}{1+1/n}\Bigg(0+\int\limits_{E_n} f_n(t)v_n\chi_{E_n}(t)\,d\mu + \int\limits_X f(t)g(t)\,d\mu \\ &\qquad - \int\limits_{E_n} f(t)g(t)\,d\mu - \Bigg|\int\limits_{E_n} f(t)v_n\chi_{E_n}(t)\,d\mu\Bigg|\Bigg) \\ &\geq \frac{1}{1+1/n}\left(1-\frac{1}{n}+1-\|a\chi_{E_n}\|_{\Phi,s}-\|a\chi_{E_n}\|^o_{\Phi}\right)\to 2.\end{aligned}$$

Since $\|f_n+f\|_{\Phi,s}\leq 2$, we obtain $\|f_n+f\|_{\Phi,s}\to 2$.

Now, take $u\in E^{\Psi}_{s^*}$ with $\int_X f(t)u(t)\,d\mu>0$. Then by the Hölder inequality,

$$\begin{aligned}\int\limits_X (f(t)-f_n(t))u(t)\,d\mu &\geq \int\limits_X f(t)u(t)\,d\mu - \|f_n\|_{\Phi,s}\|u\chi_{E_n}\|_{\Psi,s^*} \\ &\to \int\limits_X f(t)u(t)\,d\mu > 0,\end{aligned}$$

which yields $f_n \not\rightharpoonup f$. However, this contradicts the assumption that the Orlicz space L^{Φ}_s is locally uniformly convex. We conclude that if L^{Φ}_s is locally uniformly convex, then $\Phi\in\nabla_2$.

Sufficiency. Let $f_n, f\in S(L^{\Phi}_s)$ with $\|f_n+f\|_{\Phi,s}\to 2$.

By (i), the s-norm is k^*-finite so we can take $k\in K(f)$, $k_n\in K(f_n)$ for all $n\in\mathbb{N}$. Since μ is σ-finite, we can find an increasing sequence $(X_n)\subset\Sigma$ such that $X=\bigcup X_n$, and $\mu(X_n)<\infty$.

On the other hand, $b:=\sup\{k_n,k\}<\infty$ since $\Phi\in\nabla_2$ and $\|f_n+f\|_{\Phi,s}\to 2$. Then we have $k_nf_n\chi_{X_n}\xrightarrow{\mu} kf\chi_{X_n}$ in view of Lemma 3.1. Further, by Lemma 3.2, for any $\varepsilon>0$, there are $n'\in\mathbb{N}$ and $\delta>0$ such that

$$\|f\chi_{X\setminus X_n}\|_{\Phi,s}<\delta \quad \text{implies} \quad \|f_n\chi_{X\setminus X_n}\|_{\Phi,s}<\frac{\varepsilon}{3b}. \tag{4.1}$$

Let $\delta<\frac{\varepsilon}{3b}$. By Theorem 3.3, the sequence (f_n) is norm equi-continuous. It follows that the sequence (k_nf_n) is also norm equi-continuous since $\|k_nf_n\chi_E\|_{\Phi,s}=k_n\|f_n\chi_E\|_{\Phi,s}\leq b\|f_n\chi_E\|_{\Phi,s}$ for any $E\in\Sigma$. Also, we have $\mu(X_n)<\infty$ and so by using the analogous method in Theorem 3.4, we obtain $\|(k_nf_n-kf)\chi_{X_n}\|_{\Phi,s}\to 0$ when $n\to\infty$, i.e., there exists $n''\in\mathbb{N}$ such that

$$\|(k_nf_n-kf)\chi_{X_n}\|_{\Phi,s}<\frac{\varepsilon}{3} \quad \text{for any } n\geq n''.$$

Let $N = \max\{n', n''\}$. Using $f_n, f \in S(L_s^{\Phi})$, we obtain

$$\begin{aligned}|k_n - k| &= \big|\|k_n f_n\|_{\Phi,s} - \|kf\|_{\Phi,s}\big| \le \|k_n f_n - kf\|_{\Phi,s} \\ &\le \|(k_n f_n - kf)\chi_{X_n}\|_{\Phi,s} + \|k_n f_n \chi_{X\setminus X_n}\|_{\Phi,s} + \|kf\chi_{X\setminus X_n}\|_{\Phi,s} \\ &\le \|(k_n f_n - kf)\chi_{X_n}\|_{\Phi,s} + b(\|f_n\chi_{X\setminus X_n}\|_{\Phi,s} + \|f\chi_{X\setminus X_n}\|_{\Phi,s}) < \varepsilon\end{aligned}$$

for any $n \ge N$. This shows that $k_n - k \to 0$ as $n \to \infty$. Therefore, $k(f_n - f)\chi_{X_n} \xrightarrow{\mu} 0$, equivalently, $(f_n - f)\chi_{X_n} \xrightarrow{\mu} 0$. It follows from Theorem 3.4 that $\|(f_n - f)\chi_{X_n}\|_{\Phi,s} \to 0$. Moreover, $\|(f_n - f)\chi_{X\setminus X_n}\|_{\Phi,s} \to 0$ in virtue of (4.1). Therefore, using the inequality

$$\|f_n - f\|_{\Phi,s} \le \|(f_n - f)\chi_{X\setminus X_n}\|_{\Phi,s} + \|(f_n - f)\chi_{X_n}\|_{\Phi,s},$$

we obtain $\|f_n - f\|_{\Phi,s} \to 0$.

Consequently, $f_n \to f$ whenever $\|f_n + f\|_{\Phi,s} \to 2$ for $f_n, f \in S(L_s^{\Phi})$, and so the Orlicz space L_s^{Φ} is locally uniformly convex if (i)–(iii) hold. ∎

THEOREM 4.4. *The Orlicz space L_s^{Φ} is weakly locally uniformly convex if and only if*

(i) *s-norm is k-unique,*
(ii) *Φ is strictly convex,*
(iii) $\Phi \in \Delta_2 \cap \nabla_2$.

Proof. Since sufficiency is straightforward from the definitions of LUC and WLUC spaces, we only prove the necessity part. Assume that the Orlicz space L_s^{Φ} is weakly locally uniformly convex and one of the conditions (i), (ii) or (iii) does not hold. Failure to satisfy (i) or (ii) implies that L_s^{Φ} is not strictly convex, so it cannot be weakly locally uniformly convex. Suppose that (iii) does not hold.

If $\Phi \notin \Delta_2$, as established in the proof of Theorem 4.3, L_s^{Φ} is not WLUC.

If $\Phi \notin \nabla_2$, then following the proof of Theorem 4.3, there exist $f_n, f \in S(L_s^{\Phi})$ with $\|f_n + f\|_{\Phi,s} \to 2$ and

$$\lim_{n\to\infty} \int_X (f(t) - f_n(t))\, u(t)\, d\mu > 0$$

for some $u \in E_{s^*}^{\Psi}$. This shows that f_n does not converge to f in the weak* sense, so f_n does not converge to f weakly. Therefore, if L_s^{Φ} is WLUC, then $\Phi \in \nabla_2$. ∎

The following remark emphasizes that local uniform convexity and weak local uniform convexity are not always equivalent.

REMARK 4.5. From Theorems 4.3 and 4.4, it follows that for the Orlicz space L_s^{Φ} local uniform convexity and weak local uniform convexity are equivalent. The key point for this equivalence is the relation between convergence in measure and convergence in s-norm established in Theorem 3.4.

On the other hand, these two notions are not necessarily equivalent in every Banach space. For instance, the space c_0 endowed with

$$|||x||| = \|x\|_\infty + \Big(\sum_{n=1}^{\infty} 2^{-n}|x_n|^2\Big)^{1/2}$$

is WLUC but not LUC [12].

THEOREM 4.6. *The Orlicz space L_s^Φ is compactly locally uniformly convex if and only if*

(i) *the s-norm is k^*-finite,*
(ii) *Φ is strictly convex,*
(iii) *$\Phi \in \Delta_2 \cap \nabla_2$.*

Proof. Necessity. Assume that L_s^Φ is compactly locally uniformly convex. Then it satisfies the Kadec–Klee property [18]. This implies that Φ is strictly convex and $\Phi \in \Delta_2$ [11].

Assume that (i) does not hold, i.e., the s-norm is not k^*-finite. Hence Φ admits an oblique asymptote, that is,

$$\lim_{u\to\infty} p_+(u) = \lim_{u\to\infty} \frac{\Phi(u)}{u} = c < \infty$$

by [25, Theorem 7.3]. Using the definition of the outer function s, we obtain

$$c\|f\|_1 = \lim_{k\to\infty} \frac{I_\Phi(kf)}{k} \le \|f\|_{\Phi,s} \le \lim_{k\to\infty} \frac{1}{k} + \frac{I_\Phi(kf)}{k} = c\|f\|_1.$$

Thus, L_s^Φ includes a linearly isometric copy of L^1. Since L^1 is not compactly locally uniformly convex, this contradicts our assumption [5].

Now assume that $\Phi \notin \nabla_2$ and $f \in S(L_s^\Phi)$. Let $g \in S(L_{s^*}^\Psi)$ be such that $\int_X g(t)f(t)\,d\mu = \langle g, f\rangle = \|f\|_{\Phi,s} = 1$. Take the sequence (v_n) from Theorem 4.3 and take $(A_n) \subset \Sigma_n$ such that $\Psi(v_n)\mu(X \setminus A_n) = 1/n$ and $X \setminus A_n \subset \{t \in X : |f(t)| \le a\}$ for some $a > 0$. We have $\mu(X \setminus A_n) \to 0$ as above.

Similarly,

$$1 + \frac{1}{n} \ge \|v_n\chi_{X\setminus A_n}\|_{\Psi,s^*} \ge 1 - \frac{1}{n}$$

for $n \in \mathbb{N}$, so $v_n\chi_{X\setminus A_n} \in E_{s^*}^\Psi$. For $n_1 \in \mathbb{N}$, we have $\|v_{n_1}\chi_{X\setminus A_{n_1}}\|_{\Psi,s^*} \ge 1 - 1/n_1$. Since the measure μ is non-atomic, we can find a measurable set A_{n_2} such that $\|v_{n_1}\chi_{A_{n_2}\setminus A_{n_1}}\|_{\Psi,s^*} = 1 - 1/n_1$. Continuing, we construct a sequence (A_{n_k}) of measurable sets such that

$$\|v_{n_k}\chi_{A_{n_{k+1}}\setminus A_{n_k}}\|_{\Psi,s^*} = 1 - \frac{1}{n_k} = \frac{n_k - 1}{n_k}$$

and a sequence $(f_k) \subset S(L_s^{\Phi})$ such that

$$\int_X f_k(t) v_{n_k} \chi_{A_{n_{k+1}} \setminus A_{n_k}}(t)\, d\mu = \langle f_k, v_{n_k} \chi_{A_{n_{k+1}} \setminus A_{n_k}} \rangle = \| v_{n_k} \chi_{A_{n_{k+1}} \setminus A_{n_k}} \|_{\Psi, s^*}$$

for $k \in \mathbb{N}$. We also have $\operatorname{supp}(f_k) = \operatorname{supp}(v_{n_k} \chi_{A_{n_{k+1}} \setminus A_{n_k}})$. Letting

$$h_k(t) = \frac{1}{1 + \frac{1}{n_k}} (g \chi_{X \setminus (A_{n_{k+1}} \setminus A_{n_k})}(t) + v_{n_k} \chi_{A_{n_{k+1}} \setminus A_{n_k}}(t)),$$

we obtain

$$\lim_{k \to \infty} \| f_k + f \|_{\Phi, s} \geq \lim_{k \to \infty} \langle h_k, f_k + f \rangle \geq 2$$

much as in the proof of Theorem 4.3. It follows that $\lim_{k \to \infty} \| f_k + f \|_{\Phi, s} = 2$. However, for any $k \neq m$ we have $\| f_k - f_m \|_{\Phi, s} \geq \| f_k \|_{\Phi, s} = 1$ since the supports of f_k and f_m are disjoint. This shows that (f_k) has no convergent subsequence, i.e., (f_k) is not relatively compact. As a result, if L_s^{Φ} is CLUC, then $\Phi \in \nabla_2$.

Sufficiency. Suppose (i)–(iii) hold. Let $f \in S(L_s^{\Phi})$ and $\| f_n + f \|_{\Phi, s} \to 2$ for $f_n \in S(L_s^{\Phi})$. The s-norm is k^*-finite by (i). Hence there are $k \in K(f)$ and $k_n \in K(f_n)$ for any $n \in \mathbb{N}$. Therefore $f_n \to f$ as in the proof of Theorem 4.3. The sequence (f_n) is relatively compact since it is convergent and this shows that the Orlicz space L_s^{Φ} is compactly locally uniformly convex. ■

We summarize the relevant connections between the above-mentioned convexity notions of the Orlicz space L_s^{Φ} in the following diagram.

$$\begin{tikzcd}
\text{LUC} \arrow[r, shift left] \arrow[d, shift left] & \text{WLUC} \arrow[l, shift left] \arrow[dr, shift right] & \\
\text{CLUC} \arrow[u, shift left, "k\text{-uniqueness}"] & & \text{SC} \arrow[ul, shift right, "\begin{array}{l}\Phi \in \nabla_2 \text{ if } s \in \mathcal{S}_+ \\ \Phi \in \nabla_2 \cap \Delta_2 \text{ if } s \in \mathcal{S}_0\end{array}"']
\end{tikzcd}$$

In addition to the diagram above, we know that L_s^{Φ} is SC and CLUC if and only if it is LUC.

In the following corollary, we provide criteria for the local convexities of Orlicz function spaces endowed with the p-Amemiya norm. Note also that the criteria for Luxemburg and Orlicz norms have already been given in [7, 15].

COROLLARY 4.7.

(a) *The Orlicz space* $(L^{\Phi}, \| \cdot \|_{\Phi, p})$ *for* $1 < p < \infty$ *is LUC (equivalently WLUC) if and only if*

(i) Φ *is strictly convex,*
(ii) $\Phi \in \Delta_2 \cap \nabla_2$.

(b) *The Orlicz space* $(L^{\Phi}, \|\cdot\|_{\Phi,p})$ *for* $1 < p < \infty$ *is CLUC if and only if*

(i) Φ *is strictly convex,*

(ii) $\Phi \in \Delta_2 \cap \nabla_2$.

5. Some consequences of local convexities in Orlicz spaces. In this section, we provide several results concerning aspects of approximation theory in the Orlicz spaces L_s^{Φ} and Fréchet differentiability of the s-norms via the results obtained in Section 4.

Let us recall some basic concepts from approximation theory.

DEFINITION 5.1.

(i) Let $C \subseteq L_s^{\Phi}$ and $f \in L_s^{\Phi}$. If there exists $h \in C$ such that

$$\|f-h\|_{\Phi,s} = \inf\{\|f-g\|_{\Phi,s} : g \in C\} = \mathrm{dist}(f, C)$$

then h is called a *best approximation* of f in C.

(ii) Let $P_C(f)$ denote the set of best approximations to f in C, i.e.,

$$P_C(f) = \{g \in C : \|f-g\|_{\Phi,s} = \mathrm{dist}(f, C)\}.$$

Then the set C is called *proximinal* if $P_C(f) \neq \emptyset$ for any $f \in L_s^{\Phi}$.

(iii) The operator $P_C : L_s^{\Phi} \to \mathcal{P}(C)$ which maps $f \mapsto P_C(f)$ is called *the metric projection*, where $\mathcal{P}(C)$ denotes the power set of C.

(iv) The set C is said to be *approximatively compact* if for each $f \in L_s^{\Phi}$ every sequence (g_n) in C satisfying $\lim_{n\to\infty} \|f-g_n\| = \mathrm{dist}(f, C)$ has a convergent subsequence.

Now we give some results on best approximation in Orlicz spaces endowed with s-norms through local convexities. By combining Theorems 4.3 and 4.6 with the results from [20, 21], we derive the following conclusions.

THEOREM 5.2. *Suppose the s-norm is k^*-finite. If Φ is strictly convex and $\Phi \in \Delta_2 \cap \nabla_2$, then every closed convex subset of L_s^{Φ} is approximatively compact. Hence any closed convex subset $C \subseteq L_s^{\Phi}$ is proximinal.*

THEOREM 5.3. *Let $Q(C) = \{f \in L_s^{\Phi} : \mathrm{card}(P_C(f)) \leq 1\}$ for $C \subseteq L_s^{\Phi}$. If the s-norm is k-unique, and Φ is strictly convex and $\Phi \in \Delta_2 \cap \nabla_2$, then the set $Q(C)$ is of second Baire category for any $C \subseteq L_s^{\Phi}$. Therefore, for any proximinal subset C, the set $P_C(f)$ is a singleton.*

The following corollary provides a sufficient condition for the continuity of the metric projection on the Orlicz space L_s^{Φ}.

COROLLARY 5.4. *If the s-norm is k-unique, and Φ is strictly convex and $\Phi \in \Delta_2 \cap \nabla_2$, then the metric projection P_C is continuous for any closed convex $C \subseteq L_s^{\Phi}$.*

Recall that for any $c > 0$, if $\Phi(u) = c|u|^2$ for all $u \in \mathbb{R}$, then the Orlicz space L_s^Φ is a Hilbert space with the inner product $\langle f, g\rangle_{\Phi,s} = \frac{1}{\zeta^2}\int_X |fg|\,d\mu$, where $\zeta = \inf_{u>0}\frac{1}{u}s(cu^2)$ [3]. It is well-known that in a closed convex subset of a Hilbert space, there is a unique best approximation. Therefore, we obtain the following.

Corollary 5.5. *If $\Phi(u) = c|u|^2$ for all $u \in \mathbb{R}$, and $\langle\cdot,\cdot\rangle_{\Phi,s}$ is as above, then any closed convex subset $C \subseteq L_s^\Phi$ is proximinal and $P_C(f)$ is a singleton for $f \in C$. In addition, the metric projection P_C is continuous.*

In the following theorem, we give a result regarding the category of the set of farthest points which is a contrast to best approximation, by using Theorem 4.3 and Asplund's result in [1].

Theorem 5.6. *Let $E \subseteq L_s^\Phi$ be a closed bounded subset. If the s-norm is k-unique, and Φ is strictly convex and $\Phi \in \Delta_2 \cap \nabla_2$, then the set of all points which have farthest points in E, that is, $F(E) = \{f \in L_s^\Phi :$ there exists $h \in E$ such that $\|f - h\|_{\Phi,s} \geq \|f - g\|_{\Phi,s}$ for all $g \in E\}$ is residual, i.e., has a complement of the first Baire category.*

Definition 5.7. Let Φ be a Orlicz function. Then the norm $\|\cdot\|_{\Phi,s}$ is said to be *Fréchet* or *strongly differentiable* at a point $f \in L_s^\Phi$ if the limit

$$\lim_{t\to 0}\frac{\|f + tg\|_{\Phi,s} - \|f\|_{\Phi,s}}{t} \tag{5.1}$$

exists uniformly in $g \in S(L_s^\Phi)$. If the norm $\|\cdot\|_{\Phi,s}$ is Fréchet differentiable at every point of $S(L_s^\Phi)$, then the norm $\|\cdot\|_{\Phi,s}$ is *Fréchet differentiable* on $S(L_s^\Phi)$.

By using Theorem 4.3 and [16, Theorem 2.3], we give a sufficient condition for Fréchet differentiability of the s-norm below.

Theorem 5.8. *Let s be an outer function with Hölder conjugate s^* and (Φ, Ψ) be a complementary pair of Orlicz functions with $\Phi, \Psi \in \Delta_2$. If the s^*-norm is k-unique and Ψ is strictly convex, then $\|\cdot\|_{\Phi,s}$ is Fréchet differentiable.*

Funding. This study was partially funded by the Scientific and Technological Research Council of Turkey (TÜBİTAK), project number 123F368.

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Esra Başar
Institute of Graduate Studies in Sciences
İstanbul University
İstanbul, Türkiye
and
Department of Mathematics
Faculty of Arts and Sciences
Yeditepe University
İstanbul, Türkiye
E-mail: esra.basar@yeditepe.edu.tr

Serap Öztop
Department of Mathematics
Faculty of Science
İstanbul University
İstanbul, Türkiye
E-mail: oztops@istanbul.edu.tr