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REVISITING THE LAVRENTIEV PHENOMENON IN ONE DIMENSION

BY

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Abstract. We clarify and extend insights from Lavrentiev's seminal paper. We examine the original theorem dealing with the absence of the Lavrentiev phenomenon, a cornerstone issue in the calculus of variations. We point out some inconsistencies in the original proof by providing a counterexample and supply the result with a new, concise, and complete reasoning. In the appendix, we also provide additional details to supplement the original proof.

1. Introduction. Let us consider the variational functional

$$\mathcal{F}(u) = \int_a^b f(x, u(x), u'(x))\, dx,$$

where $f\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and u belongs to some class $\mathcal{E}$ (to be specified). The calculus of variations focuses on the study of such functionals, in particular on the problem of finding a function that minimizes $\mathcal{F}$ within a given class $\mathcal{E}$ of functions, that is,

$$\inf_{u\in\mathcal{E}} \mathcal{F}(u).$$

This involves investigating the properties of $\mathcal{F}$ and of admissible functions $u \in \mathcal{E}$, with the goal of determining whether a minimizer exists and, if so, characterizing such minimizers. Note that the infimum is not always attained within a given class of functions. A thorough discussion of such situations, with illustrative examples, can be found in the work of Buttazzo, Hildebrandt, and Giaquinta [11]. In these instances, the infimum cannot be reached by restricting to regular functions alone, which highlights the necessity of considering broader classes of functions to obtain a reachable minimizer.

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We would want to calculate the infimum by restricting to regular functions. This raises an important question: under what conditions is such a restriction sufficient? Clearly, the following inequality always holds:

$$\inf_{\text{all}\, u} \mathcal{F}(u) \leq \inf_{\text{regular}\, u} \mathcal{F}(u).$$

However, there are situations in which there exists a constant $c \in \mathbb{R}$ such that:

$$\inf_{\text{all}\, u} \mathcal{F}(u) < c < \inf_{\text{regular}\, u} \mathcal{F}(u).$$

Such a situation is called the *Lavrentiev phenomenon* after the prominent contribution by Mikhail Lavrentiev ([1]) in [19]. Therein the first example of the occurrence of the phenomenon is provided together with the conditions necessary for its absence. The phenomenon was later popularized by Manià's example [23], the proof of which was later simplified (see e.g. [21]). A new example of a functional for which the Lavrentiev phenomenon occurs was presented by Cerf and Mariconda [13], under the assumption that only one endpoint value of u is prescribed. Other examples can be found in [10]. In the context of the one-dimensional Lavrentiev phenomenon, the topic was thoroughly examined in the book by Buttazzo, Hildebrandt, and Giaquinta [11]. Mariconda [24] explored its emergence in Sobolev spaces, identifying conditions under which it can be avoided, particularly through boundary considerations, while Gómez [22] extended the analysis to capillarity-type problems and solution regularity in bounded variation spaces. Furthermore, Buttazzo and Mizel [12] showed that the Lavrentiev phenomenon is local in nature.

The exclusion of the Lavrentiev phenomenon is significant in the regularity theory of minimizers to variational functionals. In the case of non-autonomous functionals with nonstandard growth conditions, the presence of the Lavrentiev phenomenon can obstruct the regularity of minimizers and their gradients. Recent work has made advances in this area, with studies on double-phase integrals, (p, q)-growth conditions, and higher integrability of minima, as well as comprehensive reviews of these complex variational problems [2, 4, 14, 15, 16, 25, 28, 29]. Notably, absence of the Lavrentiev phenomenon was established without relying on growth conditions or convexity in [8, 9]. New studies identify conditions preventing the Lavrentiev gap in both scalar and vectorial cases in the anisotropic setting, under certain balance conditions [5, 6, 7, 18]. Results concerning the absence of the Lavrentiev phenomenon are crucial to the stability and convergence of numerical methods. This phenomenon can lead to failures in traditional nu-

([1]) The most widely used Latin spelling, Lavrentiev, does not match the spelling used in the author's original French publication, where his name appears as "M. Lavrientieff". His name has appeared in a variety of other forms in the literature, including Lavrientieff, Lavrientief, Lavrent'ev, Lavrentjev, Lavrientiev, and Lavrentyev.

merical techniques, such as finite element methods, which may struggle to detect singular minimizers. This issue was studied in the one-dimensional setting by Ball and Knowles [3], who examined Manià's example to illustrate the phenomenon's impact on numerical methods. Similar challenges in higher dimensions are addressed in [1, 20, 26, 27], which propose more sophisticated numerical methods for simulating the rapid expansion of voids and other complex deformations under tensile stress.

In this paper, we revisit Lavrentiev's seminal 1927 work [19]. Specifically, we examine his original example, including a review of the flaws in the original proof and the presentation of a new proof (Example 2.2). Then, we focus on the proof of the absence of the Lavrentiev phenomenon as a consequence of his Approximation Lemma. We review Lavrentiev's original proof from his seminal paper, refining the details and presenting a simple counterexample to demonstrate that certain aspects are not immediately obvious. Subsequently, in Section 4, we present a new concise and complete proof of Lavrentiev's Approximation Lemma. In the Appendix A we correct Lavrentiev's original proof. Specifically, we explain how to construct the sets required for his argument, as the details provided in his paper are misleading.

Notation. We define $AC_*[a,b]$ to be the set of absolutely continuous functions on the interval $[a,b]$ that satisfy the boundary conditions $x(a)=A$ and $x(b)=B$:

$$AC_*[a,b] = \{x \in AC([a,b])\colon x(a)=A,\ x(b)=B\}.$$

We define $\mathrm{Lip}_*([a,b])$ and $C^k_*([a,b])$ for $k=1,\dots,\infty$ to consist of the functions in the respective spaces that also satisfy these boundary conditions.

A set is called *perfect* if it is closed and contains no isolated points.

2. The Lavrentiev example. In his original paper [19], Mikhail Lavrentiev presented an implicit example of a functional for which the Lavrentiev phenomenon occurs, and a complicated justification of this fact. He postulated the existence of a smooth function $f\colon \mathbb{R}\to\mathbb{R}$ satisfying certain assumptions, for which a gap occurs. His functional took the following form:

$$\mathcal{F}(u) := \int_0^1 e^{-\frac{2}{(u(x)-\sqrt{x})^2}} f(u'(x))\,dx \quad \text{with} \quad u(0)=0 \text{ and } u(1)=1. \tag{2.1}$$

We will outline the original proof from [19], show some of its flaws, and present a new elementary reasoning instead.

Lavrentiev required the function f to have a particular set of properties. To introduce these conditions, we first define an auxiliary function $p\colon [0,\infty)\to\mathbb{R}$ as follows:

$$p(x) := 7x + 4\sqrt{3}x. \tag{2.2}$$

This function arises from the following geometric construction. For each $x_0 \in \mathbb{R}$, we consider the tangent line to the parabola $y(x) = \frac{1}{4}\sqrt{x}$ at the point $\left(x_0, \frac{1}{4}\sqrt{x_0}\right)$. The value $p(x_0)$ represents the abscissa (i.e., the x-coordinate) of the second intersection point of this tangent line with the parabola $y = \frac{1}{2}\sqrt{x}$, moving from left to right (see Figure 1).

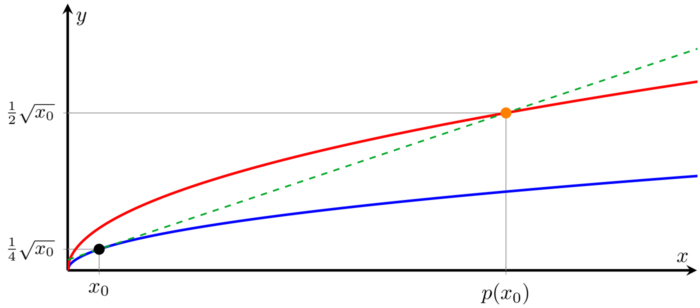

Fig. 1. In blue: The plot of $y = \frac{1}{4}\sqrt{x}$. In red: The plot of $y = \frac{1}{2}\sqrt{x}$. In green: The tangent line to the graph of $y = \frac{1}{4}\sqrt{x}$ at x_0.

Now we are ready to present the properties outlined by Lavrentiev in his paper. Some of them have been reformulated for better readability. First, Lavrentiev required that f is increasing and convex with a minimum at 0 that is greater than 1:

$$\text{(I)} \qquad \frac{d^2 f}{dx^2} > 0 \quad \text{and} \quad \min_{x\in\mathbb{R}} f(x) = f(0) \geq 1.$$

The second property yields a lower bound for the growth of f, outside of a neighborhood of 0:

$$\text{(II)} \qquad \inf_{\substack{1/2 \leq a < 1 \\ p_0 \in (a,1]}} \left((p_0 - a) f\left[\frac{\frac{1}{4}\sqrt{\frac{1}{2}}}{p_0 - a} \right] \right) > e^{8\sqrt{2}}.$$

Additionally, Lavrentiev specified that the growth of f should satisfy the following condition across the entire domain:

$$\text{(III)} \qquad e^{-\frac{8}{\sqrt{x}}}(p(x) - x) f\left[\frac{1}{4} \left(\frac{\sqrt{p(x)} - \sqrt{x}}{p(x) - x} \right) \right] > 1 \qquad (0 < x \leq 1).$$

The fourth condition ensures that f is bounded from below:

$$\text{(IV)} \qquad \frac{e^{-8/\sqrt{x}}}{4} \sqrt{x} f\left(\frac{1}{8\sqrt{x}} \right) \cdot \frac{1}{\sqrt{1 + \left(\frac{1}{8\sqrt{x}}\right)^2}} > 1 \qquad (0 < x < 1).$$

Finally, Lavrentiev's additional monotonicity condition was

$$\text{(V)} \qquad \frac{d}{dt}\left(f(t)\cdot\frac{1}{\sqrt{1+t^2}}\right) > 0 \qquad (t>0).$$

He did not provide a complete proof for the existence of a function satisfying the above properties, offering only brief remarks. Before presenting the original reasoning, we will need the following lemma, which follows from Jensen's inequality.

LEMMA 2.1. *The minimum of*

$$\mathcal{F}(u) := \int_a^b f(u'(x))\,dx$$

for $u \in \mathrm{Lip}_*([a,b])$ *and any convex function* $f\colon \mathbb{R}\to\mathbb{R}$ *is attained when* $u(x)$ *is a straight line. Therefore,*

$$\mathcal{F}(u) \geq (b-a)f\left(\frac{u(b)-u(a)}{b-a}\right).$$

Let us now examine the original proof, which involves assumptions (I)–(V).

Sketch of Lavrentiev's original proof.

STEP 1. Define φ as the infimal value of the functional $\mathcal{F}$ from (2.1) on the interval from a, the abscissa of the first point of intersection of the graph of u with the graph of $y(x)=\sqrt{x}/4$, to 1. Namely ([2]),

$$\varphi(a) := \inf\left\{\int_a^1 e^{-\frac{2}{(u(x)-\sqrt{x})^2}} f(u'(x))\,dx\colon \right.$$
$$\left. u\in C^1([a,1]),\ u(a)=\tfrac14\sqrt{a},\ u(1)=1\right\}.$$

Note that the integral above is taken over the interval $[a,1]$ instead of $[0,1]$. Consequently, it is sufficient to prove that $\varphi(a)>1$ for all $a\in(0,1)$.

STEP 2. Define a sequence $\{x_n\}_{n=1}^{\infty}$ recursively by setting $x_1=1/2$, and for $n\geq 1$, let x_{n+1} satisfy the equation $p(x_{n+1})=x_n$, where the function p is defined as in (2.2). Notice that $x_n\to 0$ as $n\to\infty$. Therefore, it is sufficient to prove that $\varphi(a)>1$ for all $a\geq x_n$ for all $n\in\mathbb{N}$. This can be accomplished by induction.

STEP 3. *Base case.* Conditions (I) and (II), combined with Lemma 2.1, imply that $\varphi(a)>1$ for all $a>x_1=1/2$.

([2]) The notation has been modified from the original: $a:=x_0$, $b:=x'$, $u(x):=y(x)$, and $\varphi(a):=\varphi(a,\frac14\sqrt{a})$.

STEP 4. *Induction step.* Assume that $\varphi(a) > 1$ for all $a \geq x_n$. To show that $\varphi(a) > 1$ for $a \geq x_{n+1}$, consider the following three cases (see Figure 2), where $y_1 = u(p(a))$:

CASE 1: $y_1 \leq \frac{1}{4}\sqrt{p(a)}$. This case is straightforward because $x_n < p(a)$, allowing us to use our induction assumption. Indeed, in such a case there always will be a point $\bar{a} \geq x_n$ at which the graphs of u and $x \mapsto \frac{1}{4}\sqrt{x}$ intersect.

CASE 2: $\frac{1}{4}\sqrt{p(a)} \leq y_1 \leq \frac{1}{2}\sqrt{p(a)}$, and $u(\xi) < \frac{1}{2}\sqrt{\xi}$ for $a \leq \xi \leq p(a)$. This is the case where $u(x)$ is separated from $\sqrt{x}$ over a long interval. It is slightly more involved than the first case, as it requires the use of conditions (I) and (III).

CASE 3: $y_1 > \frac{1}{2}\sqrt{b}$ for $a < b < p(a)$, meaning the curve $u = u(x)$ intersects the parabola $u = \frac{1}{2}\sqrt{x}$ at least once. This case is the most complicated and contains some flaws. The author attempted to use condition (IV) and (V). He claimed that

$$\frac{\frac{1}{2}\sqrt{b} - \frac{1}{4}\sqrt{a}}{b - a} \geq \frac{1}{8\sqrt{a}},$$

which can be written as

$$14ab \geq b^2 + a^2.$$

Notice that this cannot hold in general. As an example, when $b = 10a$, we have $10a < 13.92a \approx p(a)$ and the above fails to hold.

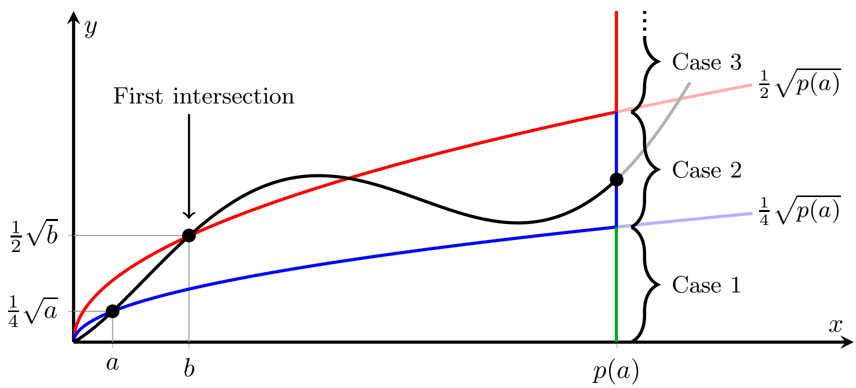

Fig. 2. In blue: The plot of $y = \frac{1}{4}\sqrt{x}$. In red: The plot of $y = \frac{1}{2}\sqrt{x}$. In green: The example of $y(\cdot)$, with the first intersection point marked. The vertical line at $p(x_0)$ indicates cases: for functions that do not intersect $y = \frac{1}{2}\sqrt{x}$, we focus on the intersection point relative to the position of $y = \frac{1}{4}\sqrt{p(x_0)}$.

We now present a new, alternative proof for this example.

Example 2.2 (Lavrentiev, 1926). Consider the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u) := \int_0^1 e^{-\frac{2}{(u(x)-\sqrt{x})^2}} f(u'(x))\, dx \quad \text{with} \quad u(0) = 0 \text{ and } u(1) = 1. \tag{2.3}$$

Then there exists a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that the Lavrentiev phenomenon occurs between $AC_*([0,1])$ and $C^1_*([0,1])$, namely

$$\inf_{u\in AC_*([0,1])} \mathcal{F}(u) = 0 \quad \text{and} \quad \inf_{u\in C^1_*([0,1])} \mathcal{F}(u) > 0.$$

Proof. Consider the sequence of functions

$$u_n(x) := \sqrt{x} + \frac{x(x-1)}{n} \qquad \text{for } n \in \mathbb{N}.$$

We have

$$\begin{aligned}\inf_{u\in AC_*([0,1])} \mathcal{F}(u) = \lim_{n\to\infty} \mathcal{F}(u_n) &\le \lim_{n\to\infty} \int_{1/4}^{3/4} \left|e^{-\frac{2n^2}{x^2(1-x)^2}} f(u_n'(x))\right| dx \\ &\le \lim_{n\to\infty} C \int_{1/4}^{3/4} e^{-n^2}\, dx = 0,\end{aligned}$$

indicating that the infimum over absolutely continuous functions is indeed 0. We demonstrate that, for the functional $\mathcal{F}$ with the choice of

$$f(x) := e^{8\cdot 256x^2}, \tag{2.4}$$

we have

$$\inf_{u\in C^1_*([0,1])} \mathcal{F}(u) > e.$$

Assume $u \in \mathrm{Lip}_*([0,1])$. Its derivative $u'(x)$ is bounded by a Lipschitz constant L for all $x \in [0,1]$. Since the derivative of $x \mapsto \sqrt{x}$ becomes unbounded as $x \to 0$, we conclude that there exist $a, b \in (0,1)$ such that

$$\sqrt{x}/4 < u(x) < \sqrt{x}/2 \quad \text{for all } x \in (a,b), \tag{2.5}$$

and equality holds on the boundary, i.e.

$$u(a) = \sqrt{a}/4 \quad \text{and} \quad u(b) = \sqrt{b}/2.$$

Thus, for all $x \in [a,b]$,

$$e^{-\frac{2}{(u(x)-\sqrt{x})^2}} \ge e^{-\frac{2}{(\sqrt{x}/2-\sqrt{x})^2}} = e^{-8/x}.$$

We divide the reasoning into two cases.

Case 1: $b < 2a$. Young's inequality can be written as

$$\beta^2\varepsilon \ge 2\alpha\beta - \alpha^2/\varepsilon,$$

so by substituting $\varepsilon := e^{-8/x}$, $\alpha := 1$, and $\beta^2 := f(u'(x))$, we obtain

$$f(u'(x))e^{-8/x} \geq 2\sqrt{f(u'(x))} - e^{8/x}.$$

Integrating this inequality over $[a, b]$, we get

$$\int_a^b f(u'(x))e^{-8/x}\,dx \geq \int_a^b \left(2\sqrt{f(u'(x))} - e^{8/x}\right) dx. \tag{2.6}$$

From the assumption of Case 1, that is, $0 < a < b < 2a$, and since $\sqrt{2}+\frac{1}{2} < 2$, we have

$$\begin{aligned}\int_a^b u'(x)\,dx &= (b-a)\frac{\frac{1}{2}\sqrt{b} - \frac{1}{4}\sqrt{a}}{b-a} \\ &= (b-a)\frac{b - \frac{1}{4}a}{2(b-a)(\sqrt{b} + \frac{1}{2}\sqrt{a})} \\ &\geq (b-a)\frac{\frac{3}{16}\sqrt{a}}{b-a} = \frac{3}{16}\sqrt{a}.\end{aligned} \tag{2.7}$$

For convenience, we define

$$g(x) := \sqrt{f(x)} = e^{4\cdot 256x^2}, \tag{2.8}$$

which is an increasing and convex function. We now apply (2.6) along with Jensen's inequality and the estimates from (2.7). This yields the following chain of inequalities:

$$\begin{aligned}\int_a^b f(u'(x))e^{-8/x}\,dx &\overset{(2.6)}{\geq} \int_a^b (2\sqrt{f(u'(x))} - e^{8/x})\,dx \\ &\geq (b-a)\left(\frac{1}{b-a}\int_a^b 2\sqrt{f(u'(x))}\,dx - e^{8/a}\right) \\ &\overset{(2.8)}{=} (b-a)\left(\frac{1}{b-a}\int_a^b 2g(u'(x))\,dx - e^{8/a}\right) \\ &\overset{\text{Jensen}}{\geq} (b-a)\left(2g\left(\frac{1}{b-a}\int_a^b u'(x)\,dx\right) - e^{8/a}\right) \\ &\overset{(2.7)}{\geq} (b-a)\left(2g\left(\frac{\frac{3}{16}\sqrt{a}}{b-a}\right) - e^{8/a}\right).\end{aligned}$$

By the definition of g in (2.8), we can estimate the integral (2.3) under the assumptions of Case 1, i.e., $0 < a < b < 2a$:

$$\begin{aligned}\int_a^b f(u'(x))e^{-8/x}\,dx &\geq (b-a)(2e^{\frac{8a}{(b-a)^2}} - e^{8/a}) \\ &\geq (b-a)(e^{\frac{8a}{(b-a)^2}} + e^{8/a} - e^{8/a}) \\ &\geq (b-a)e^{\frac{1}{b-a}} \geq e.\end{aligned}$$

The last inequality follows from fact that $x \mapsto xe^{1/x}$ is decreasing for $x \in (0,1]$. This concludes the proof in the first case.

Case 2: $b \geq 2a$. Keeping in mind (2.5) we estimate

$$\begin{aligned}\int_a^b f(u'(x))e^{-8/x}\,dx &\geq e^{-8/a}\int_a^{2a} f(u'(x))\,dx \\ &\overset{\text{Lemma 2.1}}{\geq} (2a-a)e^{-\frac{8}{a}} f\left(\frac{u(2a)-\frac{1}{4}\sqrt{a}}{2a-a}\right) \\ &\geq ae^{-8/a} f\left(\frac{\sqrt{2}-1}{4\sqrt{a}}\right) = ae^{\frac{8\cdot 16(3-2\sqrt{2})-8}{a}} \geq e.\end{aligned}$$

The last inequality holds since $a \geq e^{-1/a}$ for $a > 0$. This concludes the proof in the second case. By combining both cases, we establish that the Lavrentiev phenomenon occurs between $AC_*([0,1])$ and $C^1_*([0,1])$ for the functional (2.3), with f defined as in (2.4). ■

3. Absence of the Lavrentiev phenomenon in one-dimensional problems. In 1927 Lavrentiev published an article in which he formulated conditions sufficient for the absence of a gap between absolutely continuous functions and C^1. Firstly, we will present his Approximation Lemma and theorems resulting from it. Then, we will discuss the original proof of the lemma and provide a simplified proof.

Lemma 3.1 (Approximation Lemma, [19]). *Let $f\colon [0,1]\times\mathbb{R}\times\mathbb{R} \to [0,\infty)$ be a continuous function of variables (x,y,ξ) such that $\left|\frac{\partial f}{\partial y}\right| < M$. Let $u \in AC_*([0,1])$ be such that the function $x \mapsto f(x,u(x),u'(x))$ is integrable. Then, for every $\varepsilon > 0$, there exists a function $\phi \in C^\infty([0,1])$ satisfying the following conditions:*

(L1) $\phi(0) = u(0)$ *and* $\phi(1) = u(1)$,
(L2) $|u(x) - \phi(x)| < \varepsilon$ *for all* $x \in [0,1]$,
(L3) $|\int_0^1 f(x,u(x),u'(x))\,dx - \int_0^1 f(x,\phi(x),\phi'(x))\,dx| < \varepsilon$.

Thanks to Lemma 3.1, we can exclude Lavrentiev's phenomenon for functions that have bounded derivatives with respect to the second variable.

Thus, we have

$$\inf_{AC_*([0,1])} \mathcal{F} = \inf_{C^1_*([0,1])} \mathcal{F}.$$

This is the content of Theorem 5.2 below.

In his 1927 paper, to prove Lemma 3.1, Lavrentiev constructed a piecewise linear function that satisfied conditions (L1)–(L3) from the lemma. While the original formulation of Lavrentiev's Approximation Lemma is accurate, there is a step in his proof that is not adequately demonstrated. Let us examine this issue.

3.1. Sketch of the original proof. The original proof of Lavrentiev's Approximation Lemma from [19] revolves around constructing a piecewise constant function that approximates u in a suitable manner. Here's a sketch of the original proof:

1. It suffices to prove that there exists a piecewise linear function satisfying the same conditions as ϕ.

2. Let $\varepsilon > 0$ be given and use Luzin's Theorem to choose P_ε such that

(a) u' is continuous on P_ε,
(b) the measure of P_ε approaches 1 as $\varepsilon \to 0$,
(c) P_ε is perfect – it is closed and contains no isolated points.

3. Take a finite partition $\mathcal{A} = \{A_1, \dots, A_n\}$ of P_ε such that

$$\forall_{i\in\{1,\dots,n\}}\ \exists_{c_i\in\mathbb{R}}\ \forall_{x\in A_i} \quad |u'(x) - c_i| < \varepsilon.$$

4. Expand $\mathcal{A}$ to a family $\mathcal{B} = \{B_1, \dots, B_n\}$ of disjoint intervals such that for all $i \in \{1, \dots, n\}$, we have $A_i \subseteq B_i$. Take the complement $B_i \setminus A_i$, divide it into intervals, and denote them by $\{C_{i,k}\}_{k=1}^{\infty}$.

5. Consider the piecewise constant function

$$\psi(x) := \begin{cases} c_i & \text{if } x \in A_i \text{ or } x \in C_{i,k} \text{ for } k > k_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sum_{k=k_i}^{\infty} \lambda(C_{i,k})$ is sufficiently small.

6. Define the piecewise linear function

$$\bar{u}(x) := \int_0^x \psi(t)\,dt + u(0)$$

and show that it satisfies conditions (L2) and (L3).

7. Fix the boundary condition: to do this, choose a subset E of P_ε of measure $1/2$ and construct the function accordingly, similarly to Steps 5 and 6: set

$$\bar{\psi}(x) := \begin{cases} \psi(x) + 2(\bar{u}(1) - u(1)) & \text{for } x \in E, \\ \psi(x) & \text{otherwise,} \end{cases}$$

and finally define

$$\bar{u}(x) := \int_0^x \bar{\psi}(t)\,dt + u(0).$$

Verify that this function satisfies conditions (L1)–(L3).

3.2. Simple counterexample. In the original proof, it is not clarified why we can choose a family from Step 4. Firstly, let us consider a simple example showing it is not obvious that such sets exist.

LEMMA 3.2. *For every* $\varepsilon \in (0, 1/4)$, *there exists a set* P_ε *and* $u \in AC_*([0,1])$ *which has the following properties:*

(1) P_ε *is perfect (closed and without isolated points),*
(2) $\lambda(P_\varepsilon) > 1 - \varepsilon$,
(3) u' *is continuous on* P_ε,
(4) *it is not possible to partition* P_ε *into a finite number of intervals.*

Proof. The idea is to take P_ε that contains infinitely many connected components. Take any $\varepsilon \in (0, 1/4)$. We define $u'(x)$ as follows:

$$u'(x) := \begin{cases} -1 & \text{if } x \in \left[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}\right],\ n \in \mathbb{N} \cup \{0\}, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the set $P_\varepsilon := \bigcup_{n=1}^{\infty} \left[\frac{1+\varepsilon}{2^n}, \frac{1-\varepsilon}{2^{n-1}}\right] \cup \{0\}$. This set is perfect, and its measure is $1 - 3\varepsilon$. Additionally, u' is continuous on P_ε.

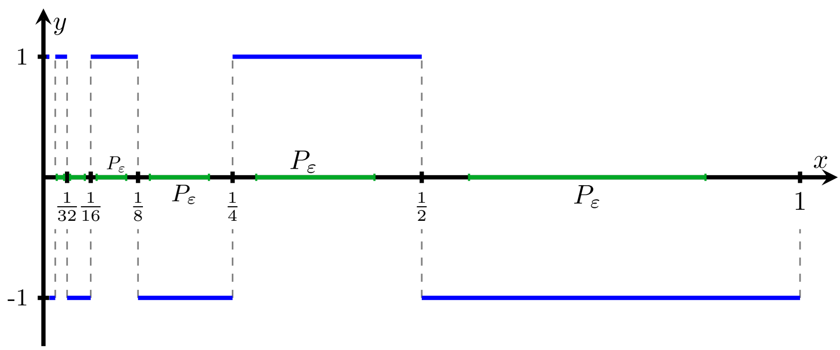

Fig. 3. In blue: The graph of u'. In green: The set P_ε on the x-axis.

For any partition $\mathcal{B} = \{B_1, \ldots, B_N\}$ of $[0,1]$ into intervals, there exists a set B_i containing $x, y \in P_\varepsilon$ such that $u'(x) = 1$ and $u'(y) = -1$. That contradicts the assumption that we can expand the division of P_ε into a finite number of pairwise disjoint intervals. ∎

In this example, P_ε is specifically chosen to demonstrate that Lavrentiev's argument is not correct. This raises the question of whether there exists a function for which there is no partition that can be extended to a finite number of closed, nonintersecting intervals. This problem is discussed in detail in the Appendix A, where we present a method of constructing such intervals.

4. A simplified proof of Lavrentiev's Approximation Lemma. Here, we present a new, concise proof of Lemma 3.1. The idea of our proof is as follows. Since our function u is assumed to be absolutely continuous (hence bounded), we can further assume without loss of generality that $0 \le u \le 1$. We will construct an approximate sequence in three steps: first, we approximate the function by Lipschitz functions; next, we smooth the approximation using the standard mollifier; finally, we correct the end-point values. At each step, we ensure that the new function is sufficiently close to the previous one. Ultimately, we verify that the final function closely approximates u using the triangle inequality.

Step 1: *Approximation by Lipschitz functions* (*without keeping end-point values*). Let us begin the construction from the derivative

$$v_k(x) := \begin{cases} u'(x) & \text{if } |u'(x)| < k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

It is clear that $\{v_k\}_{n=1}^{\infty}$ converges a.e. to u'. Now, let

$$u_k(x) := \int_0^x v_k(t)\,dt.$$

Next, let us check that $u_k \to u$ uniformly on $[0,1]$, that is,

$$\forall_{\varepsilon>0}\ \exists_k\ \forall_x \quad |u_k(x) - u(x)| < \varepsilon. \tag{4.2}$$

From the definition of an $AC([0,1])$ function, we get

$$\begin{aligned} |u_k(x) - u(x)| &= \Big| \int_{[0,x]\cap\{|u'(t)|>k\}} u'(t)\,dt \Big| \\ &\le \int_{\{|u'(t)|>k\}} |u'(t)|\,dt \xrightarrow{k\to\infty} 0. \end{aligned}$$

Thus, condition (L2) is satisfied. Now, let us check (L3). We can infer that

$$\Big| \int_0^1 f(x, u(x), u'(x))\,dx - \int_0^1 f(x, u_k(x), u_k'(x))\,dx \Big| \le \varepsilon \tag{4.3}$$

using the Dominated Convergence Theorem. Given (4.1), from the definition of u_k we see that $u_k'(x) = 0$ or $u_k'(x) = u'(x)$, so that

$$\begin{aligned} |f(x, u_k(x), u_k'(x))| &\le |f(x, u_k(x), u'(x))| + |f(x, u_k(x), 0)| \\ &=: G_k(x) + H_k(x). \end{aligned}$$

The first term can be bounded as

$$G_k(x) \leq |f(x,u(x),u'(x))| + M|u(x)-u_k(x)|,$$

because we have assumed that $\left|\frac{\partial f}{\partial y}\right| < M$. This term is integrable since $f(x,u(x),u'(x))$ is integrable by our assumptions, and $u(x)-u_k(x)$ is small by uniform convergence.

For $H_k(x)$, notice that the sequence $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded (since it is a uniformly convergent sequence of bounded real-valued functions), so we can approximate it by the supremum of f on a compact set. Thus, by the Dominated Convergence Theorem, (4.3) is arbitrarily small for k large enough.

STEP 2: *Approximation by smooth functions* (*without keeping end-point values*). Let us apply mollification to u_k. Define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right),$$

where $\eta \in C_c^\infty(\mathbb{R})$ is the standard mollifier. Then the mollification of u_k is defined by

$$u_{k,n} := u_k * \eta_{1/n},$$

where $*$ denotes convolution. To define this properly, we extend u_k to a neighborhood of $[0,1]$ – by setting $u_k(x) = u_k(0)$ on the left, and $u_k(x) = u_k(1)$ on the right. It follows that $u_{k,n} \in C^\infty(\mathbb{R})$ and that $\{u_{k,n}\}$ uniformly converges to u_k as $n \to \infty$ (see [17, Appendix C.4]). Additionally, since u_k is Lipschitz with constant k (see (4.1)), we obtain

$$|u_{k,n}(x) - u_k(x)| \leq \int_{-1}^{1} |(u_k(x-t/n) - u_k(x))|\eta(t)\,dt \leq \int_{-1}^{1} \left|k\frac{t}{n}\right|\eta(t)\,dt \leq \frac{k}{n}.$$

We will check conditions (L2) and (L3) between $u_{k,n}$ and u_k. In the last step, we will check the closeness of approximation and the energy between u and the constructed function. We can see that condition (L2) is satisfied:

$$\forall_{\varepsilon>0}\ \forall_k\ \exists_{n_k}\ \forall_{n\geq n_k}\ \forall_x \quad |u_{k,n}(x) - u_k(x)| \leq k/n < \varepsilon. \tag{4.4}$$

Condition (L3) is slightly more involved; we need to use the Dominated Convergence Theorem. Firstly, observe that each $u_{k,n}$ is a Lipschitz function with constant k:

$$\begin{aligned}|u_{k,n}(x_1) - u_{k,n}(x_2)| &= \left|\int_{-1}^{1} (u_k(x_1-y) - u_k(x_2-y))\eta_{1/n}(y)\,dy\right| \\ &\leq \int_{-1}^{1} |u_k(x_1-y) - u_k(x_2-y)|\eta_{1/n}(y)\,dy \leq k|x_1-x_2|.\end{aligned}$$

Therefore a suitable bound is given by

$$|f(x,u_{k,n}(x),u'_{k,n}(x))| \le \sup_{(x,y,z)\in[0,1]\times[-c_k,c_k]\times[k,k]} |f(x,y,z)|,$$

for some c_k, as $u_{k,n}$ is smooth and $[0,1]$ is closed. Then

$$\int_0^1 f(x,u_{k,n}(x),u'_{k,n}(x))\,dx \xrightarrow{n\to\infty} \int_0^1 f(x,u_k(x),u'_k(x))\,dx, \tag{4.5}$$

so the difference becomes arbitrarily small.

STEP 3: *Correcting boundary values.* Without loss of generality we can assume that $u(0)=0$ and $u(1)=1$. If this is not the case, we can adjust the function by adding $u(0)+x\big(u(1)-u(0)\big)$ to the definition of $\phi_{n,k}$ below, and the calculations remain valid under this adjustment. Let us take

$$w_{k,n}(x) := u'_{k,n}(x) + (u_{k,n}(0)+1-u_{k,n}(1)),$$

and define our final function

$$\phi_{k,n}(x) := \int_0^x w_{k,n}(t)\,dt. \tag{4.6}$$

This allows us to fix the boundary values: $\phi_{k,n}(0)=0$ and $\phi_{k,n}(1)=1$. Finally, we need to check the remaining two conditions.

Condition (L2)*:* Since u_k differs from u by no more than ε, we obtain

$$\begin{aligned} |\phi_{k,n}(x)-u_{k,n}(x)| &= \Big|\int_0^x w_{k,n}\,dt - \int_0^x u'_{k,n}(t)\,dt - u_{k,n}(0)\Big| \\ &= \Big|\int_0^x (u_{k,n}(0)+1-u_{k,n}(1))\,dt - u_{k,n}(0)\Big| \\ &= |(x-1)u_{k,n}(0) + x(1-u_{k,n}(1))| \\ &\le |u_{k,n}(0)| + |1-u_{k,n}(1)| \le 4\varepsilon. \end{aligned} \tag{4.7}$$

Condition (L3)*:* Firstly, we will break the relevant expression into smaller parts:

$$\begin{aligned} &\Big|\int_0^1 f(x,u_{k,n}(x),u'_{k,n}(x))\,dx - \int_0^1 f(x,\phi_{k,n}(x),\phi'_{k,n}(x))\,dx\Big| \\ &\quad\le \Big|\int_0^1 f(x,u_{k,n}(x),u'_{k,n}(x))\,dx - \int_0^1 f(x,u_{k,n}(x),\phi'_{k,n}(x))\,dx\Big| \\ &\quad\quad + \Big|\int_0^1 f(x,u_{k,n}(x),\phi'_{k,n}(x))\,dx - \int_0^1 f(x,\phi_{k,n}(x),\phi'_{k,n}(x))\,dx\Big| \\ &\quad =: I_1 + I_2. \end{aligned} \tag{4.8}$$

Let us begin by analyzing I_1. Using continuity and estimates similar to those in (4.7), we observe that

$$\forall_{\delta>0}\ \exists_{k_0}\ \forall_{k>k_0}\ \exists_{n_k}\ \forall_{n\geq n_k}\quad |u_{k,n}(0)+1-u_{k,n}(1)|\leq\delta\quad \text{for all } x\in[0,1].$$

Next, consider the mapping $\psi\colon x\mapsto(x,u_{k,n}(x),u'_{k,n}(x))$. Since $u_{k,n}$ is Lipschitz continuous with constant k,

$$\psi([0,1])\subset[0,1]\times[-C,C]\times[-k,k].$$

As f is continuous, it is uniformly continuous on $\psi([0,1])\subset[0,1]\times[-C,C]\times[-k,k]$. Denoting the modulus of continuity by ω, we get

$$\begin{aligned}\big|f(x,u_{k,n}(x),u'_{k,n}(x))-f\big(x,u_{k,n}(x),u'_{k,n}(x)+u_{k,n}(0)+1-u_{k,n}(1)\big)\big|\\ \leq\omega|u_{k,n}(0)+1-u_{k,n}(1)|\leq\omega\,\delta.\end{aligned}$$

Thus,

$$I_1\leq\omega\delta,$$

which is arbitrarily small. It remains to show that I_2 is negligible. Indeed,

$$\begin{aligned}I_2&=\Big|\int_0^1 f(x,u_{k,n}(x),\phi'_{k,n}(x))\,dx-\int_0^1 f(x,\phi_{k,n}(x),\phi'_{k,n}(x))\,dx\Big|\\ &\leq\Big|\sup_{[0,1]\times\mathbb{R}^2}\frac{\partial f}{\partial y}(x,y,z)(u_{k,n}-\phi_{k,n})\Big|\leq M\big\|u_{k,n}-\phi_{k,n}\big\|_\infty\leq M\varepsilon.\end{aligned}$$

Final verification. Let us show that the function $\phi_{k,n}$ from (4.6) satisfies all conditions of Lemma 3.1. Note that it agrees with u on the boundary, so we only need to check the last two conditions.

Let us verify condition (L2) for u. From (4.2), (4.4), and (4.7), we obtain

$$\begin{aligned}&\forall_{\varepsilon>0}\ \exists_{k_0}\ \forall_{k>k_0}\ \exists_{n_k}\ \forall_{n\geq n_k}\ \forall_x\\ &|\phi_{k,n}(x)-u(x)|\leq|\phi_{k,n}(x)-u_{k,n}(x)|+|u_{k,n}(x)-u_k(x)|+|u_k(x)-u(x)|\leq3\varepsilon.\end{aligned}$$

Condition (L3)*:* From (4.3), (4.5), and (4.8), we get

$$\begin{aligned}&\forall_{\varepsilon>0}\ \exists_{k_0}\ \forall_{k>k_0}\ \exists_{n_k}\ \forall_{n\geq n_k}\\ &\Big|\int_0^1 f(x,\phi_k(x),\phi'_k(x))\,dx-\int_0^1 f(x,u(x),u'(x))\,dx\Big|\\ &\qquad\leq\Big|\int_0^1 f(x,\phi_k(x),\phi'_k(x))\,dx-\int_0^1 f(x,u_{k,n}(x),u'_{k,n}(x))\,dx\Big|\\ &\qquad\quad+\Big|\int_0^1 f(x,u_{k,n}(x),u'_{k,n}(x))\,dx-\int_0^1 f(x,u_k(x),u'_k(x))\,dx\Big|\\ &\qquad\quad+\Big|\int_0^1 f(x,u_k(x),u'_k(x))\,dx-\int_0^1 f(x,u(x),u'(x))\,dx\Big|\leq3\varepsilon.\end{aligned}$$

This proves the lemma. ∎

From the proof of Lemma 3.1, it turns out that the conclusion can be strengthened to additionally yield convergence a.e. of the derivatives.

REMARK 4.1. Let f and u be as in Lemma 3.1. Then for every $\varepsilon > 0$, there exists $\phi \in C^\infty([0,1])$ such that conditions (L1)–(L3) are satisfied and moreover

(L4) $\int_0^1 |u'(x) - \phi'(x)|\,dx < \varepsilon$.

Proof. Fix $\varepsilon > 0$ and assume that x is such that $u'(x)$ is well-defined. It is enough to prove that $\{\phi_{k,n}\}$ from the proof above (see (4.6)) satisfies

$$\int_0^1 |u'(x) - \phi'_{k,n}(x)|\,dx < \varepsilon \tag{4.9}$$

for all k, n large enough. Let us observe that

$$|u'(x) - \phi'_{k,n}(x)| \le |u'(x) - u'_k(x)| + |u'_k(x) - u'_{k,n}(x)| + |u'_{k,n}(x) - \phi'_{k,n}(x)|,$$

where $\{u_k\}$ is defined in Step 1, $\{u_{k,n}\}$ is defined in Step 1, whereas k, n are sufficiently large as described in the proof of Lemma 3.1. By the very definition of u_k (cf. (4.2)), we obtain

$$\int_0^1 |u'(x) - u'_k(x)|\,dx \le \int_{\{|u'(t)|>k\}} |u'(t)|\,dt \xrightarrow{k\to\infty} 0.$$

The convergence of $u_{k,n}$ defined in Step 1 is verified as follows:

$$\begin{aligned} |u'_k(x) - u'_{k,n}(x)| &= \left| u'_k(x) - \frac{d}{dx} \int_{-1/n}^{1/n} u_k(x-y)\eta_{1/n}(y)\,dy \right| \\ &= \left| u'_k(x) - \int_{-1/n}^{1/n} u'_k(x-y)\eta_{1/n}(y)\,dy \right| \\ &\le \left| \int_{-1/n}^{1/n} (u'_k(x) - u'_k(x-y))\eta_{1/n}(y)\,dy \right| \\ &\le \int_{-1/n}^{1/n} |ky|\eta_{1/n}(y)\,dy < \frac{2k}{n}. \end{aligned}$$

We can change the order of differentiation and integration by invoking the Dominated Convergence Theorem, since u_k is a Lipschitz function. In Step 1, by the definition of $\phi_{k,n}$, we can estimate, for $x \in [0,1]$,

$$\left|\phi'_{k,n}(x) - u'_{k,n}(x)\right| = |u_{k,n}(0) + 1 - u_{k,n}(1)| \le |u_{k,n}(0) + 1 - u_{k,n}(1)|,$$

where the right-hand side is arbitrarily small (cf. (4.7)). Inserting the above observations to (4.9) and integrating, we conclude the proof. ∎

5. Consequences of the Approximation Lemma. From Lemma 3.1, we can derive two important theorems:

THEOREM 5.1 (Lavrentiev 1927, [19]). *Suppose f is as in Lemma 3.1. Then the infimum (resp. supremum) of the integral $\int_0^1 f(x,u(x),u'(x))\,dx$ in (resp. $AC_*([0,1])$) is equal to the lower (resp. upper) bound of the same integral in $C_*^\infty([0,1])$.*

Proof. Since $C_*^\infty([0,1])$ is a subset of $AC_*([0,1])$, we immediately see that

$$\inf_{u\in AC_*([0,1])}\int_0^1 f(x,u(x),u'(x))\,dx \le \inf_{u\in C_*^\infty([0,1])}\int_0^1 f(x,u(x),u'(x))\,dx.$$

It suffices to prove the reverse inequality. Let $\{u_n\}\subset AC_*([0,1])$ be a minimizing sequence. Then for every $\varepsilon>0$, we can find n_ε such that for all $n>n_\varepsilon$,

$$\int_0^1 f(x,u_n(x),u_n'(x))\,dx \le \inf_{u\in AC_*([0,1])}\int_0^1 f(x,u(x),u'(x))\,dx+\varepsilon.$$

We fix an arbitrary $\varepsilon>0$ and choose $n>n_\varepsilon$. By Lemma 3.1, we obtain $\varphi_n\in C_*^\infty([0,1])$ such that

$$\Big|\int_0^1 f(x,u_n(x),u_n'(x))\,dx-\int_0^1 f(x,\varphi_n(x),\varphi_n'(x))\,dx\Big|<\varepsilon.$$

Combining the inequalities, for all $n>n_\varepsilon$, we have

$$\begin{aligned}\int_0^1 f(x,\varphi_n(x),\varphi_n'(x))\,dx &\le \int_0^1 f(x,u_n(x),u_n'(x))\,dx+\varepsilon\\ &\le \inf_{u\in AC_*([0,1])}\int_0^1 f(x,u(x),u'(x))\,dx+2\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ concludes the proof. ∎

THEOREM 5.2 (Lavrentiev 1927, [19]). *If there exists a map in the class $C_*^1([0,1])$ that minimizes the integral $\int_0^1 f(x,u(x),u'(x))\,dx$ (where f satisfies the conditions of Theorem 3.1), then there also exists a minimum for this integral in the class $AC_*([0,1])$, and this minimum is attained by the same map.*

Proof. Since $C_*^\infty([0,1]) \subset C_*^1([0,1]) \subset AC_*([0,1])$, and Theorem 5.1 establishes that the lower bounds in $C_*^\infty([0,1])$ and $AC_*([0,1])$ are the same, it follows that the function above is also a minimizer of the integral in the class $AC_*([0,1])$. ∎

Appendix A. Supplementing Lavrentiev's proof. As discussed in Section 3.2, here we focus on the issue of constructing a partition into a finite number of nonintersecting intervals for any absolutely continuous function, as required in item 4 of Section 3.1. To address this, we provide a methodical construction of such intervals, emphasizing the critical role played by measure regularity. We shall use the following lemma.

LEMMA A.1. *In a separable metric space (X, d), given a nonatomic measure μ, the set of isolated points has measure zero.*

As a consequence of Lemma A.1, we have the following fact.

LEMMA A.2. *In a separable metric space (X, d), given a nonatomic measure μ, consider a measurable set A. If μ is inner regular, then*

$$\mu(A) = \sup\{\mu(F) \colon F \subseteq A,\ F \text{ is perfect}\}.$$

The following lemma provides the foundation for our construction.

LEMMA A.3. *Let P_1 and P_2 be two disjoint perfect subsets of $[0,1]$. Then there exists a finite collection $\mathcal{B} = \{B_1, \ldots, B_n\}$ of disjoint intervals such that*

(1) $\bigcup_{i=1}^n B_i = [0,1]$,
(2) *for each interval B_i, either $P_1 \cap B_i = \emptyset$ or $P_2 \cap B_i = \emptyset$,*
(3) *each interval B_i for $i \in \{1, \ldots, n-1\}$ is of the form $[a, b)$, and B_n is closed.*

Proof. Define

$$y_x := \begin{cases} \inf\{y > x \colon y \in P_2\} & \text{if } x \in P_1, \\ \inf\{y > x \colon y \in P_1\} & \text{if } x \in P_2, \end{cases}$$

and if $0 \notin P_1 \cup P_2$, set

$$y_0 := \inf\{y > 0 \colon y \in P_1 \cup P_2\}.$$

For each $x \in (P_1 \cup P_2) \cup \{0\}$ set

$$I_x := \begin{cases} [x, y_x) & \text{if } y_x < \infty, \\ [x, 1] & \text{otherwise.} \end{cases}$$

This yields a family $\{I_x\}$ of intervals, each intersecting at most one of the sets P_1 or P_2. Next, remove from this collection any interval strictly contained in another and denote the resulting disjoint collection by $\mathcal{I}$. These intervals are not degenerate, since P_1 and P_2 are disjoint closed sets.

We now prove that $\mathcal{I}$ is finite. Suppose, for contradiction, that there are infinitely many intervals in $\mathcal{I}$ with left end-points in P_1 (the same argument works symmetrically for P_2). Then we can choose a strictly monotone sequence $\{a_i\}_{i=1}^{\infty} \subset P_1$ of such left end-points converging to some $a \in P_1$, by the closedness of P_1. By construction, between a_i and a_{i+1} there must exist a point $b_i \in P_2$. For instance, if $\{a_i\}$ is increasing, one may take $b_i = y_{a_i}$; if it is decreasing, then $b_i = y_{a_{i+1}}$. Hence, we obtain a sequence $\{b_i\}_{i=1}^{\infty} \subset P_2$ with $b_i \in (a_i, a_{i+1})$ (or $b_i \in (a_{i+1}, a_i)$ if the sequence is decreasing), and thus $\lim_{i\to\infty} b_i = a \in P_1$. This contradicts the assumption that $P_1 \cap P_2 = \emptyset$.

Therefore, the collection $\mathcal{I}$ must be finite, completing the proof. ∎

To construct the required intervals, instead of searching for the sets A_i mentioned in item 3 of the sketch of the original proof, we can begin directly by identifying the family $\mathcal{B}$ described in item 4. Then, the sets A_i are simply the intersections of P_ε and B_i.

Lemma A.4. *For every $\varepsilon > 0$, there exists a family $\mathcal{B} = \{B_1, \dots, B_n\}$ of disjoint intervals and a set $P_\varepsilon \subseteq [0,1]$ such that*

(1) $\lambda(P_\varepsilon) > 1 - \varepsilon$ *and P_ε is perfect,*
(2) $\bigcup B_i = [0,1]$,
(3) *for each i we have* $\operatorname{diam}(u'(B_i \cap P_\varepsilon)) \le \varepsilon$, *where* diam *denotes diameter.*

Proof. Fix $\varepsilon > 0$. Let n be so large that $1/n < \varepsilon$. For each $k \in \mathbb{Z}$, define

$$C_k := (u')^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right).$$

Those sets are measurable and pairwise disjoint. Since $u' \in L^1([0,1])$, there exists $M \in \mathbb{N}$ such that

$$\lambda\Big(\bigcup_{k<-M} C_k \cup \bigcup_{k>M} C_k\Big) \le \frac{\varepsilon}{4}.$$

We notice that

$$\lambda\Big(\bigcup_{k=-M}^{M} C_k\Big) \ge 1 - \frac{\varepsilon}{4}.$$

From the inner regularity of the Lebesgue measure, for each $k \in [-M, M] \cap \mathbb{Z}$ we can find compact sets $P_{\varepsilon,k}$ such that

$$\lambda(C_k \setminus P_{\varepsilon,k}) \le \frac{\varepsilon}{4M}.$$

Without loss of generality, we can assume that $P_{\varepsilon,k}$ are perfect sets as per Lemma A.2. Let

$$P_\varepsilon := \bigcup_{k=-M}^{M} P_{\varepsilon,k}.$$

Then P_ε is closed and

$$\lambda(P_\varepsilon) > 1 - \frac{(2M+1)\varepsilon}{4M} - \frac{\varepsilon}{4}. \tag{A.1}$$

Define

$$F_m := \bigcup_{k=-M}^{m} P_{\varepsilon,k}, \quad G_m := \bigcup_{k=m+1}^{M} P_{\varepsilon,k}, \quad \text{for } m \in [-M, M-1] \cap \mathbb{Z}.$$

Applying Lemma A.3 to the pairs (G_m, F_m), we obtain finite families $\mathcal{B}'_m$ of intervals. Let $\mathcal{B}' = \bigcup_{m=-M}^{M-1} \mathcal{B}'_m$.

To construct $\mathcal{B}$, take the end-points of all intervals in $\mathcal{B}'$ and sort them to get a sequence $\{a_n\}_{n=1}^{N}$. Set $a_0 = 0$, and

$$\mathcal{B} = \big\{[a_i, a_{i+1})\colon i \in \{0, 1, \dots, N\}\big\} \cup \{[a_N, 1]\}\,.$$

In (A.1), we ensured that condition (1) is satisfied. The construction also guarantees that the union of these intervals covers $[0,1]$.

Now, let us check (3). Consider any interval $B \in \mathcal{B}$. If B does not intersect P_ε, there is nothing to prove. If B intersects P_ε, it must intersect $P_{\varepsilon,k}$ for some k. Suppose it also intersects $P_{\varepsilon,j}$ for some $j > k$. Then $\mathcal{B}'_k$ is a partition of $[0,1]$ such that every interval from this family intersects exactly one of the sets $P_{\varepsilon,k}$ or $P_{\varepsilon,j}$. Since the interval B is contained in some interval from $\mathcal{B}'_k$, this yields a contradiction. Therefore, $\operatorname{diam}(u'(B_i \cap P_\varepsilon)) = \operatorname{diam}(u'(B_i \cap P_{\varepsilon,k})) \le \varepsilon$, as $P_{\varepsilon,k} \subset C_k$. ∎

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