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Elliptic integrals and transcendence

by

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Abstract. Since the 2022 work of Huber and Wüstholz one knows practically all there is to know about the transcendence and linear independence of path integrals of algebraic functions in a single variable. Here we focus on elliptic integrals and give simple necessary and sufficient conditions for the transcendence. Our conditions use the classical concept of elementary differential and are particularly suitable as an effective decision procedure, even for families of integrals. We supply some lemniscatic examples, also for genus 0 and 2.

1. Introduction. Let X be an absolutely irreducible algebraic curve, usually smooth and projective but not always, defined over the field $\overline{\mathbf{Q}}$ of all algebraic numbers (embedded in the complex field $\mathbf{C}$). Let ω be an algebraic differential form on X also defined over $\overline{\mathbf{Q}}$. And let Γ be a path in $X(\mathbf{C})$, not containing any poles of ω, whose endpoints are defined over $\overline{\mathbf{Q}}$. Then the integral $\int_\Gamma \omega$ exists, and its value can easily be a classical constant in mathematics.

There is evidence that the problem of the transcendence of such integrals arose already in 1691. With X as the (projective completion of) the circle $x^2 + y^2 = 1$ in real $\mathbf{R}^2$, ω coming from $y\mathrm{d}x$ and Γ the straight line joining $(0,1)$ to $(1,0)$ we get the area

$$\int_\Gamma \omega = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4},$$

proved to be transcendental by Lindemann in 1882. But Leibniz with the (singular) lemniscate

$$y^2 = x^2(1-x^2) \tag{1.1}$$

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and similar ω, Γ observed that

$$\int_\Gamma \omega = \int_0^1 x\sqrt{1-x^2}\,dx = \frac{1}{3}.$$

This is known as the Huygens lemniscate. For more on this story see the book of Baker and Wüstholz [1, pp. 127, 128] and Wüstholz [18, p. 35].

Another lemniscate

$$x^2 - y^2 = (x^2+y^2)^2 \tag{1.2}$$

due to Bernoulli was studied later by Fagnano. Its length amounts to the integral

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{\sqrt{1+6t^2+t^4}} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} \tag{1.3}$$

proved to be transcendental by Siegel. We shall investigate its area below.

Since then there has been a great deal of work on this problem for general X; see for example the book of Huber and Wüstholz [5, pp. xi–xiii] for a short account. For our purposes it will suffice to mention first that Baker's Theorems of 1966-67 led soon afterwards in 1971 to a decision procedure by van der Poorten [12] when the geometric genus of X is 0 (as with the two lemniscates above). The second crucial step came with Wüstholz's Analytic Subgroup Theorem in [17] from 1989. This allowed arbitrary genus, at least when the path Γ is closed. In that case $\int_\Gamma \omega$ is algebraic if and only if it is zero. See also Wüstholz [16, p. 482] and Baker and Wüstholz [1, p. 126].

After Wüstholz [19] the most recent advance was the book [5] in 2022; now Γ need not be closed. A noteworthy result is that $\int_\Gamma \omega$ is algebraic if and only if $\omega = \mathrm{d}f + \omega'$ for some f in $\overline{\mathbf{Q}}(X)$ and some ω' with $\int_\Gamma \omega' = 0$. See [5, Theorem 1.1 (p. 2) or Theorem 13.9 (p. 125)]. There is also much discussion of when $\int_\Gamma \omega' = 0$ can occur; see in particular [5, Chapter 14].

Our modest contribution here is first to give a streamlined version of van der Poorten's result for genus 0 and then develop this to give a close analogue for genus 1. That will provide a lot more information about ω' above. Both new results make use of topological winding numbers in the integers $\mathbf{Z}$ and so are particularly suited to deciding effectively whether $\int_\Gamma \omega$ is transcendental or algebraic. We shall illustrate this by several examples (even in genus 2). Important is the concept of *elementary* differential as in a recent paper [10] of Zannier and the author.

We now state our two results. Following the fundamental works of Liouville (apparently unrelated to his famous construction of transcendental numbers), we call a differential *elementary* if it has a representation

$$\omega = \mathrm{d}f_0 + \gamma_1\frac{\mathrm{d}f_1}{f_1} + \cdots + \gamma_s\frac{\mathrm{d}f_s}{f_s} \tag{1.4}$$

with f_0 in $\overline{\mathbf{Q}}(X)$, non-zero $f_1, \ldots, f_s$ in $\overline{\mathbf{Q}}(X)$, and $\gamma_1, \ldots, \gamma_s$ in $\overline{\mathbf{Q}}$.

A standard linear algebra argument (see Section 2) shows that we may assume that $\gamma_1, \ldots, \gamma_s$ are linearly independent over the rationals $\mathbf{Q}$. In that case we shall call (1.4) *reduced*. It follows easily (see also Section 2) that the poles of the resulting $\mathrm{d}f_0, \mathrm{d}f_1/f_1, \ldots, \mathrm{d}f_s/f_s$ lie among the poles of ω.

For non-zero f in $\overline{\mathbf{Q}}(X)$ and a path Γ in $X(\mathbf{C})$ from A in $X(\overline{\mathbf{Q}})$ to B in $X(\overline{\mathbf{Q}})$ containing no poles or zeroes of f and such that $f(A) = f(B)$, we define

$$n_\Gamma(f) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{\mathrm{d}f}{f}.$$

The integral is some logarithm of $f(B)/f(A) = 1$ and so $n_\Gamma(f)$ lies in $\mathbf{Z}$.

In genus 0 it is easy to show that all differentials are elementary, for example by writing

$$\overline{\mathbf{Q}}(X) = \overline{\mathbf{Q}}(x); \tag{1.5}$$

see also Section 3.

THEOREM 0. *In genus* 0, *the integral* $I = \int_\Gamma \omega$ *is algebraic if and only if*

$$f_1(A) = f_1(B), \ldots, f_s(A) = f_s(B), \quad n_\Gamma(f_1) = 0, \ldots, n_\Gamma(f_s) = 0$$

for a reduced representation, in which case $I = f_0(B) - f_0(A)$.

In genus 1 not all differentials are elementary. Now a convenient analogue of (1.5) is

$$\overline{\mathbf{Q}}(X) = \overline{\mathbf{Q}}(x, y) = \overline{\mathbf{Q}}(E)$$

for a Weierstrass elliptic curve E defined by

$$y^2 = 4x^3 - g_2x - g_3$$

with g_2, g_3 in $\overline{\mathbf{Q}}$.

It is easy to see that $\mathrm{d}x/y$, with no poles, is not elementary. For us a more important class of examples consists of the

$$\theta(P) = \frac{1}{2}\frac{y+\eta}{x-\xi}\frac{\mathrm{d}x}{y}$$

defined for every $O \neq P = (\xi, \eta)$ on E, with poles only at P and the point at infinity O. This is hardly ever elementary.

For Γ in $E(\mathbf{C})$ which is closed, homologically trivial, and not passing through P and O we define

$$n_\Gamma(P) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \theta(P).$$

This too is in $\mathbf{Z}$, because the residue of $\theta(P)$ at P is $+1$ and at O is -1, so when we shrink Γ to a point we change the right-hand side of the above by an integer.

Theorem 1. *In genus* 1, *the integral* $I = \int_\Gamma \omega$ *is algebraic if and only if either*

(a) ω *is elementary with*

$$f_1(A) = f_1(B),\ \ldots,\ f_s(A) = f_s(B), \qquad n_\Gamma(f_1) = 0,\ \ldots,\ n_\Gamma(f_s) = 0$$

for a reduced representation, in which case $I = f_0(B) - f_0(A)$, *or*

(b) Γ *is closed, homologically trivial, and*

$$\sum_{P \neq O} n_\Gamma(P)\,\mathrm{res}(\omega, P) = 0 \tag{1.6}$$

for the residues $\mathrm{res}(\omega, P)$, *in which case* $I = 0$.

In particular, if $\int_\Gamma \omega$ in genus 1 is algebraic, then either Γ is closed or ω is elementary. In some vague probabilistic sense the first is unlikely, and the second is also unlikely. For example we can fix a finite set $\mathcal{S}$ in $E(\mathbf{C})$ with cardinality N. The $\mathbf{C}$-vector space of all differentials with no poles outside $\mathcal{S}$ modulo the exact ones is well-known to have dimension $N+1$ (for a hint see (5.1) below). But it is not difficult to show that if one restricts to elementary, the corresponding dimension drops to something at most $N-1$ (and usually quite smaller). Thus the elementary ones are "of codimension at least 2".

One sees also expressions similar to (1.6) in [5, pp. 138, 140].

An easy consequence is the following analogue of Wüstholz's earlier result on closed paths, but now for non-closed paths Γ and with differentials ω of the third kind (that is, the only poles are simple). If $I = \int_\Gamma \omega$ is algebraic then Theorem 1 implies (1.4); and now f_0 must be constant, otherwise it would have a pole so $\mathrm{d}f_0$ would have a higher order pole which could not then be cancelled by $\mathrm{d}f_1/f_1, \ldots, \mathrm{d}f_s/f_s$ in (1.4). So by (a) we have $I = 0$ and therefore the following result.

Corollary. *In genus* 1, *when the only poles of* ω *are simple and* Γ *is non-closed,* $\int_\Gamma \omega$ *is algebraic if and only if it is zero.*

In fact, this corollary remains valid if we allow as well a single double pole. On the other hand, $\int_0^1 \mathrm{d}x = 1$ and the elliptic $\mathrm{d}x$ has a triple pole at the origin. Or $\int_0^1 \mathrm{d}x/(x+1)^2 = 1/2$ where there are usually two double poles.

It should be reasonably clear that these results supply effective decision procedures. For example, it is relatively easy to see if a given ω is elementary or not, in fact, a matter of determining torsion (this caused great trouble in [10], but that was because we were working with parametrized families of differentials). Also the integers $n_\Gamma(f), n_\Gamma(P)$ can be computed, and the residues in (1.6) are in $\overline{\mathbf{Q}}$. We will see a bit more of this in Sections 11–13.

Our proof of Theorem 0 is straightforward enough following van der Poorten [12]. But Theorem 1 has to rely on the Analytic Subgroup Theorem (however not the motivic version of [5, Theorem 9.7, p. 85]). To a large extent we can follow [5]; but a crucial part is played by the interaction between differentials of the third kind like $\theta(P)$ and "differentials of the logarithmic kind" $\mathrm{d}f/f$. That accounts for an extra power of multiplicative $\mathbf{G}_{\mathrm{m}}$ in the group varieties under consideration. We also have to make sure that the extra $\mathrm{d}f/f$ coming in do not happen to acquire poles on Γ, which seems a little delicate.

Here is how the present paper is arranged.

After a few preliminaries in Section 2 we prove Theorem 0 in Section 3.

Then Section 4 contains preparations for the proof of Theorem 1, leading in Section 5 to various reductions of the differential ω. In Section 6 we perform (along well-established lines) the actual integration, and this allows one more reduction, but only of $I = \int_\Gamma \omega$.

Section 7 starts some preparations for "doing transcendence", which is carried out in Section 8 when Γ is not closed. When Γ is closed, it suffices to apply the results of Wüstholz [15] from 1984, and this takes up Section 9. In Section 10 we indicate a connection with a nice side result in [5].

We finish with some new examples, motivated mainly by area and length: first for genus 0 in Section 11, then for genus 1 in Section 12; and then in Section 13 we make some comments regarding genus 2.

We reported on our results in [8].

2. Some preliminaries. We reduce (1.4) as follows. If $s \geq 2$ we can annihilate a relation

$$m_s\gamma_s = m_1\gamma_1 + \cdots + m_{s-1}\gamma_{s-1}$$

with $m_s \neq 0, m_1, \ldots, m_{s-1}$ in $\mathbf{Z}$ simply by writing

$$m_s\left(\gamma_1\frac{\mathrm{d}f_1}{f_1} + \cdots + \gamma_{s-1}\frac{\mathrm{d}f_{s-1}}{f_{s-1}} + \gamma_s\frac{\mathrm{d}f_s}{f_s}\right) = \gamma_1\frac{\mathrm{d}\tilde{f}_1}{\tilde{f}_1} + \cdots + \gamma_{s-1}\frac{\mathrm{d}\tilde{f}_{s-1}}{\tilde{f}_{s-1}}$$

with

$$\tilde{f}_1 = f_1^{m_s}f_s^{m_1}, \ldots, \tilde{f}_{s-1} = f_{s-1}^{m_s}f_s^{m_{s-1}}.$$

By repeating this process we end up with $\mathbf{Q}$-independent coefficients as desired.

Now any pole of $\mathrm{d}f_0$ is at least double and so cannot be cancelled in (1.4) by $\mathrm{d}f_1/f_1, \ldots, \mathrm{d}f_s/f_s$ all of whose poles are simple. So it is a pole of ω. Also let P be a pole of some $\mathrm{d}f_j/f_j$. Then $\mathrm{res}(\omega, P) = \sum_{j=1}^{s} \gamma_j \,\mathrm{ord}_P\, f_j$ is a linear combination of $\gamma_1, \ldots, \gamma_s$ with rational coefficients not all zero, thus cannot itself be zero. So P is a pole of ω. This verifies a claim made in Section 1.

In [10, Section 11] we considered "shortest" representations, but a reduced representation is not necessarily shortest; for example $\mathrm{d}f_1/f_1 + \sqrt{2}\mathrm{d}f_2/f_2$

with $f_1 = f_2$. Even shortest representations are not unique. This situation could be improved as follows (but we will not need to do so).

Let $f_1, \dots, f_s$ be as in a reduced representation, and consider the group F generated over $\overline{\mathbf{Q}}^*$ by them. The "relative division group" F_ω, consisting of all f in $\overline{\mathbf{Q}}(E)^*$ for which there is a positive integer n with f^n in F, is in fact independent of the reduced representation. Its $\mathbf{Z}$-rank modulo $\overline{\mathbf{Q}}^*$ is the length of a shortest representation, which is also the dimension of the $\mathbf{Q}$-vector space generated by all the residues of ω.

For example in Theorems 0 and 1(a) the conditions on $f_1, \dots, f_s$ can be replaced by $f(A) = f(B), n_\Gamma(f) = 0$ for all f in F_ω.

We shall require a "reduced" version of Baker's Theorems. Specifically, if $\gamma_1, \dots, \gamma_s$ are algebraic numbers linearly independent over $\mathbf{Q}$, and $\ell_1, \dots, \ell_s$ are logarithms of non-zero algebraic numbers such that $\gamma_1\ell_1 + \dots + \gamma_s\ell_s$ is itself algebraic, then $\ell_1 = \dots = \ell_s = 0$. This follows easily by the same sort of linear algebra (and induction) from the original result that if $1, \ell_1, \dots, \ell_s$ are $\overline{\mathbf{Q}}$-linearly dependent, then $\ell_1, \dots, \ell_s$ are $\mathbf{Q}$-linearly dependent.

Perhaps paradoxically, we shall not need any analogue for elliptic logarithms.

3. Genus 0. Now we assume (1.5). So $\omega = R(x)\mathrm{d}x$ for $R(x)$ in $\overline{\mathbf{Q}}(x)$. To see (1.4) we write $\xi_1, \dots, \xi_s$ for the (finite) poles of R, and $\gamma_1, \dots, \gamma_s$ for their residues (in $\overline{\mathbf{Q}}$). Then partial fractions show easily that

$$R - \left(\frac{\gamma_1}{x - \xi_1} + \dots + \frac{\gamma_s}{x - \xi_s}\right) = \frac{\mathrm{d}S}{\mathrm{d}x}$$

for $S = S(x)$ also in $\overline{\mathbf{Q}}(x)$. This gives (1.4) with $f_0 = S$ and $f_1 = x - \xi_1, \dots, f_s = x - \xi_s$. Here $\gamma_1, \dots, \gamma_s$ might not be $\mathbf{Q}$-independent but we can achieve this as in Section 2.

Now

$$I = \int_\Gamma \omega = f_0(B) - f_0(A) + \sum_{j=1}^{s} \gamma_j \int_\Gamma \frac{\mathrm{d}f_j}{f_j}$$

and as above each integral on the far right is $\log_\Gamma(f_j(B)/f_j(A))$ for some choice of the logarithm depending on Γ.

Finally, suppose I is algebraic. Then Baker's Theorem and the independence of $\gamma_1, \dots, \gamma_s$ imply that these logarithms are all zero and $I = f_0(B) - f_0(A)$. So the other $f_j(B)/f_j(A)$ are 1 and these logarithms are just $2\pi\sqrt{-1}n_\Gamma(f_j)$. This proves the hard part of Theorem 0; and the easier part is trivial.

4. Genus 1 – preparations. Let us define $\theta(O)$ as the zero differential. Then the following observation will enable us to deal with the various $\theta(P)$ that will arise.

LEMMA 4.1. *The map θ defines a homomorphism from $E(\overline{\mathbf{Q}})$ to the group of all differentials modulo the subgroup Ω_0 generated by all $\mathrm{d}f/f$ and all algebraic multiples of $\mathrm{d}x/y$.*

Proof. It suffices to show that

$$\theta(P+Q)-\theta(P)-\theta(Q)+\frac{\mathrm{d}f}{f}=a\frac{\mathrm{d}x}{y}$$

for all P,Q in $E(\overline{\mathbf{Q}})$, where $f=f_{PQ}, a=a_{PQ}$. If O,P,Q and $R=P+Q$ are all distinct, there is f with divisor $(f)=P+Q-R-O$. Now one checks that the left-hand side of the above has no poles at all. So it must have the form of the right-hand side. The same proof works for any P,Q (and of course one can replace $\overline{\mathbf{Q}}$ throughout by $\mathbf{C}$). ■

We need two consequences of this.

LEMMA 4.2. *For $r\geq 2$ let $P_1,\ldots,P_r$ in $E(\overline{\mathbf{Q}})$ be such that some non-zero integer multiple of P_r is an integer linear combination of $P_1,\ldots,P_{r-1}$. Then there are $\mu_1,\ldots,\mu_{r-1},\mu$ in $\mathbf{Q}$ and α in $\overline{\mathbf{Q}}$ such that*

$$\theta(P_r)=\mu_1\theta(P_1)+\cdots+\mu_{r-1}\theta(P_{r-1})+\mu\frac{\mathrm{d}f}{f}+\alpha\frac{\mathrm{d}x}{y} \tag{4.1}$$

for non-zero f in $\overline{\mathbf{Q}}(E)$ with no poles or zeroes except possibly at $O,P_1,\ldots,P_r$.

Proof. We have

$$m_rP_r=m_1P_1+\cdots+m_{r-1}P_{r-1}$$

for integers $m_r\neq 0,m_1,\ldots,m_{r-1}$. Then modulo Ω_0 we deduce

$$m_r\theta(P_r)\equiv\theta(m_rP_r)\equiv m_1\theta(P_1)+\cdots+m_{r-1}\theta(P_{r-1}).$$

Dividing by m_r we get (4.1) for some f (and $\mu=1/m_r$). Now taking residue divisors (see for example [10, p. 256]) gives

$$(f)=\mathrm{Res}\left(\frac{\mathrm{d}f}{f}\right)=m_rP_r-m_rO-m_1P_1+m_1O-\cdots-m_{r-1}P_{r-1}+m_{r-1}O,$$

confirming the remark about poles and zeroes. ■

LEMMA 4.3. *Let $\mathcal{S}$ be any finite subset of $E(\overline{\mathbf{Q}})$. For $r\geq 2$ let $P_1,\ldots,P_r$ in $E(\overline{\mathbf{Q}})$ be $\mathbf{Z}$-independent, and let $\delta_1,\ldots,\delta_r$ in $\overline{\mathbf{Q}}$ be such that δ_r is a rational linear combination of $\delta_1,\ldots,\delta_{r-1}$. Then there are $n\neq 0$ in $\mathbf{Z}$, α in $\overline{\mathbf{Q}}$, and $Q_1,\ldots,Q_{r-1}$ in $E(\overline{\mathbf{Q}})$ also $\mathbf{Z}$-independent and not in $\mathcal{S}$, such that $\delta_1\theta(P_1)+\cdots+\delta_r\theta(P_r)$ is*

$$\frac{\delta_1}{n}\theta(Q_1)+\cdots+\frac{\delta_{r-1}}{n}\theta(Q_{r-1})+\frac{\delta_1}{n}\frac{\mathrm{d}f_1}{f_1}+\cdots+\frac{\delta_{r-1}}{n}\frac{\mathrm{d}f_{r-1}}{f_{r-1}}+\alpha\frac{\mathrm{d}x}{y} \tag{4.2}$$

for non-zero $f_1,\ldots,f_{r-1}$ *in* $\overline{\mathbf{Q}}(E)$ *with no poles or zeroes except possibly at*

$$O, P_1,\ldots,P_r, Q_1,\ldots,Q_{r-1}.$$

Proof. We have

$$n_r\delta_r = n_1\delta_1 + \cdots + n_{r-1}\delta_{r-1} \tag{4.3}$$

for integers $n_r \neq 0, n_1,\ldots,n_{r-1}$. Much as in the proof of Lemma 4.2 we get (4.2) for $n = n_r$ and

$$Q_1 = n_rP_1 + n_1P_r,\ \ldots\ , Q_{r-1} = n_rP_{r-1} + n_{r-1}P_r$$

with

$$n_r\theta(P_1) + n_1\theta(P_r) \equiv \theta(Q_1),\ \ldots\ , n_r\theta(P_{r-1}) + n_{r-1}\theta(P_r) \equiv \theta(Q_{r-1})$$

modulo Ω_0. Thus the divisors $(f_1),\ldots,(f_{r-1})$ are

$$n_rP_1 - n_rO + n_1P_r - n_1O - Q_1 + O,\ \ldots,$$
$$n_rP_{r-1} - n_rO + n_{r-1}P_r - n_{r-1}O - Q_{r-1} + O.$$

It is easy to see that $Q_1,\ldots,Q_{r-1}$ are **Z**-independent. As for avoiding $\mathcal{S}$, we can use a daft trick of multiplying (4.3) by any integer k. Then $Q_1,\ldots,Q_{r-1}$ become $kQ_1,\ldots,kQ_{r-1}$, and we can choose k so that none of those lie in $\mathcal{S}$. This completes the proof. ∎

5. Genus 1 – reducing the differential. The standard elliptic substitute for (1.4) consists in writing our differential form ω as

$$\omega = \mathrm{d}f_0 + \delta_1\theta(P_1) + \cdots + \delta_r\theta(P_r) + \alpha\frac{\mathrm{d}x}{y} + \beta\frac{x\mathrm{d}x}{y}. \tag{5.1}$$

We may take f_0 in $\overline{\mathbf{Q}}(E)$, non-zero $P_1,\ldots,P_r$ in $E(\overline{\mathbf{Q}})$, and $\delta_1 \neq 0,\ldots,\delta_r \neq 0$ in $\overline{\mathbf{Q}}$ as well as α,β. Of course $r = 0$ is allowed.

We wish to simplify this as we did for (1.4) – but the simplification will not be so simple.

If $P_1,\ldots,P_r$ are **Z**-independent we do nothing at this stage. Otherwise we start working towards that situation. If $P_1,\ldots,P_r$ are **Z**-dependent then we can suppose we are in the situation of Lemma 4.2. Via (4.1) this enables us to reduce $P_1,\ldots,P_r$ to $P_1,\ldots,P_{r-1}$ at the expense of an algebraic multiple of some $\mathrm{d}f/f$, this f having no poles or zeroes except possibly at $O, P_1,\ldots,P_r$.

If $P_1,\ldots,P_{r-1}$ are **Z**-dependent then we can repeat the procedure to get a second $\mathrm{d}f/f$ with similar properties.

We carry on until we reach **Z**-independence, say of $P_1,\ldots,P_{\tilde r}$ for some $\tilde r \leq r$ (also $\tilde r$ can be zero). A similar procedure was used in [15, p. 166]. We write this out as

$$\omega = \mathrm{d}f_0 + \tilde{\delta}_1\theta(P_1) + \cdots + \tilde{\delta}_{\tilde{r}}\theta(P_{\tilde{r}}) + \gamma_1\frac{\mathrm{d}f_1}{f_1} + \cdots + \gamma_s\frac{\mathrm{d}f_s}{f_s} + \tilde{\alpha}\frac{\mathrm{d}x}{y} + \beta\frac{x\mathrm{d}x}{y}; \tag{5.2}$$

note that β has not changed but α probably has (and s can be zero).

Now we can start to apply Lemma 4.3, with a set $\mathcal{S}$ that will be chosen later. If $\tilde{\delta}_1, \ldots, \tilde{\delta}_{\tilde{r}}$ are $\mathbf{Q}$-independent we again do nothing. Otherwise, if $\tilde{\delta}_1, \ldots, \tilde{\delta}_{\tilde{r}}$ are $\mathbf{Q}$-dependent, then using (4.2) we can replace $P_1, \ldots, P_{\tilde{r}}$ by $Q_1, \ldots, Q_{\tilde{r}-1}$ at the expense of yet more $\mathrm{d}f/f$. This time the f have no poles or zeroes except possibly at

$$O, P_1, \ldots, P_{\tilde{r}}, Q_1, \ldots, Q_{\tilde{r}-1};$$

the $Q_1, \ldots, Q_{\tilde{r}-1}$ remain independent and do not lie in $\mathcal{S}$.

If the coefficients of $\theta(Q_1), \ldots, \theta(Q_{\tilde{r}-1})$ are $\mathbf{Q}$-dependent, we can repeat the procedure to get $R_1, \ldots, R_{\tilde{r}-2}$ and f with no poles or zeroes outside

$$O, P_1, \ldots, P_{\tilde{r}}, Q_1, \ldots, Q_{\tilde{r}-1}, R_1, \ldots, R_{\tilde{r}-2},$$

with $R_1, \ldots, R_{\tilde{r}-2}$ independent and not in $\mathcal{S}$.

We carry on until we reach $\mathbf{Q}$-independence (analogous to what we achieved in (1.4) above). The outcome, expressed in notation for which I sincerely apologize (one should think of the dot being tildetilde), is

$$\omega = \mathrm{d}f_0 + \dot{\delta}_1\theta(\tilde{P}_1) + \ldots + \dot{\delta}_{\dot{r}}\theta(\tilde{P}_{\dot{r}}) + \tilde{\gamma}_1\frac{\mathrm{d}\tilde{f}_1}{\tilde{f}_1} + \cdots + \tilde{\gamma}_{\tilde{s}}\frac{\mathrm{d}\tilde{f}_{\tilde{s}}}{\tilde{f}_{\tilde{s}}} + \dot{\alpha}\frac{\mathrm{d}x}{y} + \beta\frac{x\mathrm{d}x}{y} \tag{5.3}$$

where now $\tilde{P}_1, \ldots, \tilde{P}_{\dot{r}}$ are $\mathbf{Z}$-independent and are either just $P_1, \ldots, P_{\tilde{r}}$ or not in $\mathcal{S}$, and $\dot{\delta}_1, \ldots, \dot{\delta}_{\dot{r}}$ are $\mathbf{Q}$-independent (and $\dot{r}, \tilde{s}$ can be zero – and α has changed yet again). Now $\tilde{f}_1, \ldots, \tilde{f}_{\tilde{s}}$ have no poles or zeroes except possibly at $O, P_1, \ldots, P_r$ and some other points not in $\mathcal{S}$.

We are not quite done with our reductions and one more will follow after we have carried out some integration.

6. Genus 1 – integrating the differential. We now integrate (5.3) along the path Γ from A to B. We take $\mathcal{S}$ to consist of $A, B, B-A$. We make two temporary assumptions:

(s1) the residue of ω at O is non-zero.
(s2) $B - A$ is not among $P_1, \ldots, P_{\tilde{r}}$ in (5.2).

It follows from (s1) and (5.1) that $O, P_1, \ldots, P_r$ are really poles of ω, so cannot lie on Γ, and in particular are not A, B.

For $\mathrm{d}f_0$ in (5.3) we observe that

$$\int_\Gamma \mathrm{d}f_0 = f_0(B) - f_0(A)$$

because any non-zero pole of f_0 gives rise to a (repeated) pole of ω and so cannot lie on Γ, and O itself is not on Γ.

For $\mathrm{d}x/y$ in (5.3) we make the usual Weierstrass substitution $x = \wp(z)$, $y = \wp'(z)$ to see that

$$\int_\Gamma \frac{\mathrm{d}x}{y} = u_B - u_A.$$

Here u_A is picked arbitrarily with $A = (\wp(u_A), \wp'(u_A))$ (recall $A \neq O$); and then u_B with $B = (\wp(u_B), \wp'(u_B))$ is determined by continuation along Γ.

Similarly

$$\int_\Gamma \frac{x\,\mathrm{d}x}{y} = -(\zeta(u_B) - \zeta(u_A))$$

because of the Weierstrass $\zeta'(z) = -\wp(z)$.

For a typical $\theta(P)$ in (5.3) the indefinite integral is any logarithm (exponential not elliptic) of $\varphi(z)$, where

$$\varphi(z) = \frac{\sigma(z-u)}{\sigma(z)\sigma(u)} e^{\zeta(u)z} \tag{6.1}$$

and $P = (\wp(u), \wp'(u))$ (recall $P \neq O$) for the Weierstrass function with $\sigma'(z)/\sigma(z) = \zeta(z)$. All this is classical; in our context see [5, pp. 178–182]. In fact, φ itself is actually independent of the choice of u, but we do not need this (see for example [7, p. 257]).

Here $\varphi(u_A)$ is defined and non-zero, first because $A \neq O$ and second because $u_A - u$ in the period lattice would imply $A = P$. The latter is impossible if $P = P_1, \ldots, P_{\tilde{r}}$, but also if P is not in $\mathcal{S}$. Similarly for $\varphi(u_B)$. Thus

$$\int_\Gamma \theta(P) = \log_\Gamma \frac{\varphi(u_B)}{\varphi(u_A)} \tag{6.2}$$

for some choice of the logarithm (now depending on Γ).

As for the $\mathrm{d}f/f$ in (5.3) we note that $f(A)$ is defined and non-zero because $A \neq O, P_1, \ldots, P_r$ and A is in $\mathcal{S}$. Similarly $f(B)$; thus

$$\int_\Gamma \frac{\mathrm{d}f}{f} = \log_\Gamma \frac{f(B)}{f(A)}$$

like (6.2).

Collecting these up, we find for $I = \int_\Gamma \omega$ in (5.3) that

$$I = f_0(B) - f_0(A) + \sum_{i=1}^{\dot r} \dot\delta_i \log_\Gamma \frac{\varphi_i(u_B)}{\varphi_i(u_A)} + \sum_{j=1}^{\tilde s} \tilde\gamma_j \log_\Gamma \frac{\tilde f_j(B)}{\tilde f_j(A)} + \dot\alpha(u_B - u_A) - \beta(\zeta(u_B) - \zeta(u_A)) \tag{6.3}$$

where $\varphi_1, \dots, \varphi_{\dot r}$ correspond to $\tilde P_1, \dots, \tilde P_{\dot r}$ in (6.1).

We next express this entirely in terms of

$$u = u_B - u_A$$

and certain algebraic numbers. That will be done by using suitable addition formulae (of course those lie behind the group law on the group varieties below). But we need the extra temporary assumption

(s3) $B \neq A$.

This corresponds of course to Γ non-closed. It follows that this u is not in the period lattice.

To start with, we recall the well-known formula

$$\zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2}\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

It follows easily that

$$\zeta(u_B) - \zeta(u_A) - \zeta(u)$$

is an algebraic number.

We also have the less well-known formula

$$\varphi(z_1+z_2) = \frac{1}{2}\varphi(z_1)\varphi(z_2)\frac{\wp(z_1)\wp'(z_2) - \wp'(z_1)\wp(z_2) + \xi(\wp'(z_1) - \wp'(z_2)) + \eta(\wp(z_1) - \wp(z_2))}{(\wp(z_1) - \wp(z_2))(\wp(z_1) - \xi)(\wp(z_2) - \xi)}$$

for φ and $P = (\xi, \eta)$ as in (6.1); see for example [7, p. 256] after changing the sign of v there. In a similar way it follows that each

$$\frac{\varphi_i(u_B)}{\varphi_i(u_A)\varphi_i(u)}$$

is an algebraic number; here we have to note that $\varphi_i(u)$ is also defined and non-zero. For u is neither a period nor congruent to u_i with $\tilde P_i = (\wp(u_i), \wp'(u_i))$. The latter is because $B - A = \tilde P_i$ is impossible when $\tilde P_i$ is among $P_1, \dots, P_{\tilde r}$ by assumption (s2) above, and impossible when $\tilde P_i$ is not in $\mathcal{S}$, which contains $B - A$ by definition.

Now (6.3) simplifies to

$$I = \sum_{i=1}^{\dot r} \dot\delta_i \log_\Gamma(\varepsilon_i \varphi_i(u)) + \sum_{j=1}^{\tilde s} \tilde\gamma_j \log_\Gamma \lambda_j + \dot\alpha u - \beta\zeta(u) + \kappa \tag{6.4}$$

for algebraic κ and non-zero algebraic $\varepsilon_1, \dots, \varepsilon_{\dot r}, \lambda_1, \dots, \lambda_{\tilde s}$.

We are all set up for our final reduction. If $\dot{\delta}_1, \ldots, \dot{\delta}_{\dot{r}}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{s}}$ are $\mathbf{Q}$-dependent, then by the independence of $\dot{\delta}_1, \ldots, \dot{\delta}_{\dot{r}}$ we can assume that $\tilde{\gamma}_{\tilde{s}}$ is a $\mathbf{Q}$-linear combination of $\dot{\delta}_1, \ldots, \dot{\delta}_{\dot{r}}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{s}-1}$. Now a simple calculation as in the proofs of Lemmas 4.2 and 4.3 shows that $\tilde{s}$ may be reduced to $\tilde{s}-1$ in (6.4) at the expense of changing $\varepsilon_1, \ldots, \varepsilon_{\dot{r}}$ and $\lambda_1, \ldots, \lambda_{\tilde{s}-1}$.

As usual we may repeat this operation and (paying the usual attention to dots and tildes) we end up with

$$I = \sum_{i=1}^{\dot{r}} \dot{\delta}_i \log_\Gamma(\tilde{\varepsilon}_i \varphi_i(u)) + \sum_{j=1}^{\dot{s}} \dot{\gamma}_j \log_\Gamma \tilde{\lambda}_j + \dot{\alpha} u - \beta\zeta(u) + \kappa \tag{6.5}$$

with $\dot{\delta}_1, \ldots, \dot{\delta}_{\dot{r}}, \dot{\gamma}_1, \ldots, \dot{\gamma}_{\dot{s}}$ $\mathbf{Q}$-independent (and not only $\dot{r}$ but also $\dot{s}$ can be zero).

7. Genus 1 – preparing for transcendence. We shall of course use the Analytic Subgroup Theorem, which involves a commutative group variety G defined over $\overline{\mathbf{Q}}$. It has an exponential map $\exp = \exp_G$ from its complex tangent space $T(G)$ to $G(\mathbf{C})$. Suppose we have a complex subspace Z of $T(G)$ such that $\exp(Z)$ contains a non-zero point of $G(\overline{\mathbf{Q}})$. Then the conclusion is that there is a non-zero connected group subvariety H of G such that Z contains $T(H)$ (see for example [5, p. 48]). We may obviously assume that H is minimal in that there is no H' strictly smaller than H with this property.

Our choice of G here depends on β in (6.5) (which has remained unchanged under all our various reductions).

We start with the case $\beta \neq 0$. We take

$$G = G_1 \times_E \cdots \times_E G_{\dot{r}} \times_E E_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}}^{\dot{s}} \tag{7.1}$$

where G_i is the extension of E by multiplicative $\mathbf{G}_{\mathrm{m}}$ associated with $\tilde{P}_i$ in (5.3), E_{a} is the non-split extension of E by additive $\mathbf{G}_{\mathrm{a}}$, and the subscripts E denote fibre product. Without $\mathbf{G}_{\mathrm{m}}^{\dot{s}}$ this was used in [15, p. 167]. The dimension of G is $d = \dot{r} + \dot{s} + 2$, and so $T(G)$ is isomorphic to

$$\mathbf{C}^d = \mathbf{C} \times \mathbf{C}^{\dot{r}} \times \mathbf{C} \times \mathbf{C}^{\dot{s}}$$

parametrized by $\mathbf{z} = (z, w_1, \ldots, w_{\dot{r}}, t, x_1, \ldots, x_{\dot{s}})$, and exp involves the functions

$$\wp(z), \varphi_1(z)e^{w_1}, \ldots, \varphi_{\dot{r}}(z)e^{w_{\dot{r}}}, \beta\zeta(z) + t, e^{x_1}, \ldots, e^{x_{\dot{s}}}$$

(needing good eyesight).

For well-known technical reasons one should throw in $\wp'(z)$ and more elaborate stuff, and also take cubes everywhere (see [15, pp. 167–168] and the work of Faltings and Wüstholz [3] especially), but we shall ignore this precision. It will be convenient to include extra coefficients as in (6.5) and so we use

$$\exp_0(\mathbf{z}) = (\wp(z), \tilde{\varepsilon}_1\varphi_1(z)e^{w_1}, \ldots, \tilde{\varepsilon}_{\dot{r}}\varphi_{\dot{r}}(z)e^{w_{\dot{r}}}, \beta\zeta(z) + t, e^{x_1}, \ldots, e^{x_{\dot{s}}}).$$

LEMMA 7.1. *Every non-zero connected group subvariety of G contains a group subvariety $H_0 = \exp(V_0)$, where the line V_0 in $\mathbf{C}^d$ is defined either by*

$$z = w_1 = \cdots = w_{\dot{r}} = x_1 = \cdots = x_{\dot{s}} = 0 \tag{7.2}$$

parametrized by t, or by

$$z = 0,\ w_1 = \mu_1 y, \ldots, w_{\dot{r}} = \mu_{\dot{r}} y,\ t = 0,\ x_1 = \nu_1 y, \ldots, x_{\dot{s}} = \nu_{\dot{s}} y \tag{7.3}$$

parametrized by y, where $\mu_1, \ldots, \mu_{\dot{r}}, \nu_1, \ldots, \nu_{\dot{s}}$ are in $\mathbf{Q}$ not all zero.

Proof. We have $G = G' \times G''$ with

$$G' = G_1 \times_E \cdots \times_E G_{\dot{r}} \times_E E_{\mathrm{a}}, \qquad G'' = \mathbf{G}_{\mathrm{m}}^{\dot{s}}.$$

Here G' is an extension of E by its linear part $G_l = \mathbf{G}_{\mathrm{m}}^{\dot{r}} \times \mathbf{G}_{\mathrm{a}}$. As $\tilde{P}_1, \ldots, \tilde{P}_{\dot{r}}$ are $\mathbf{Z}$-independent and E_{a} is non-split, the only connected group subvarieties of G' either are G' itself or are contained in G_l (see Bertrand [2, Corollaire, p. 10] with $F = \mathbf{Q}$ and $t = \dot{r}$, $l = 1$). To find the group subvarieties H of $G' \times G''$ we use the ideas of Goursat–Kolchin–Ribet. Write H', H'' for the projections of H to G', G'' respectively. If K' is the set of h' in H' such that $(h', 0)$ lies in H, and similarly for K'' in H'', then it is known that $H'/K', H''/K''$ are isomorphic (see for example [9, p. 262]).

If $H' = G'$ then $K' = G'$ too, because otherwise the neutral connected component of K' would be in G_l, and there would be a non-zero homomorphism from H'/K' to E, impossible as H'/K' is isomorphic to the linear H''/K''. So $K'' = H''$ too, and now $H = G' \times H''$. This contains $G' \times 0$ which contains $G_l \times 0$ and we can use V_0 defined by (7.2) above (note $\exp(V_0) = \mathbf{G}_{\mathrm{a}}$).

If $H' \neq G'$ then as we observed H' is in G_l so H is in $G_l \times \mathbf{G}_{\mathrm{m}}^{\dot{s}} = \mathbf{G}_{\mathrm{m}}^{\dot{r}} \times \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}}^{\dot{s}}$ of which the group subvarieties are well-known. We can take again (7.2) if H projects onto all of $\mathbf{G}_{\mathrm{a}}$, or otherwise as a one-dimensional group subvariety of $\mathbf{G}_{\mathrm{m}}^{\dot{r}} \times \mathbf{G}_{\mathrm{m}}^{\dot{s}}$ as in (7.3). ∎

8. Genus 1 – doing transcendence. We start by showing that $I = \int_\Gamma \omega$ is transcendental if $\beta \neq 0$ in (6.5) (still under the assumptions (s1)–(s3) of Section 6). As usual we assume I is algebraic and reach a contradiction.

We look at the point

$$\Pi = (u, -\log_\Gamma(\tilde{\varepsilon}_1\varphi_1(u)), \ldots, -\log_\Gamma(\tilde{\varepsilon}_{\dot{r}}\varphi_{\dot{r}}(u)), t_0, \log_\Gamma \tilde{\lambda}_1, \ldots, \log_\Gamma \tilde{\lambda}_{\dot{s}})$$

in $T(G)$, where

$$t_0 = -\sum_{i=1}^{\dot{r}} \dot{\delta}_i \log_\Gamma(\tilde{\varepsilon}_i\varphi_i(u)) - \sum_{j=1}^{\dot{s}} \dot{\gamma}_j \log_\Gamma \tilde{\lambda}_j - \dot{\alpha} u.$$

Then

$$\exp_0(\Pi) = (\wp(u), 1, \ldots, 1, \kappa - I, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\dot{s}})$$

is defined over $\overline{\mathbf{Q}}$. It is not zero because u is not a lattice point. So $\exp(\Pi)$ lies in $G(\overline{\mathbf{Q}})$ and is non-zero.

Also Π lies in the subspace Z defined by

$$\dot{\alpha} z - (\dot{\delta}_1 w_1 + \cdots + \dot{\delta}_{\dot{r}} w_{\dot{r}}) + t + (\dot{\gamma}_1 x_1 + \cdots + \dot{\gamma}_{\dot{s}} x_{\dot{s}}) = 0.$$

So by the Analytic Subgroup Theorem, Z contains some $T(H_0)$ and therefore some V_0 as in Lemma 7.1. Clearly (7.2) is impossible. If $\dot{r} = \dot{s} = 0$ then (7.3) has zero dimension so this is also impossible. But if $\dot{r}, \dot{s}$ are not both zero then (7.3) leads to

$$-(\mu_1 \dot{\delta}_1 + \cdots + \mu_{\dot{r}} \dot{\delta}_{\dot{r}}) + (\nu_1 \dot{\gamma}_1 + \ldots + \nu_{\dot{s}} \dot{\gamma}_{\dot{s}}) = 0;$$

and this contradicts the $\mathbf{Q}$-independence attained at the end of Section 6.

All this was for $\beta \neq 0$. If however $\beta = 0$ in (6.5) then $\zeta(u)$ disappears and so we can dispense with E_{a} in (7.1). If $\dot{\alpha} \neq 0$ we can simply work with

$$G_1 \times_E \cdots \times_E G_{\dot{r}} \times \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}}^{\dot{s}} \tag{8.1}$$

instead, at least if $\dot{r}, \dot{s}$ are not both zero. And if $\dot{r} = \dot{s} = 0$ then we should use just $E \times \mathbf{G}_{\mathrm{a}}$. We leave the verifications to the reader; again we get the transcendence of I.

Finally, what if $\beta = \dot{\alpha} = 0$ in (6.5)? We use (8.1), at least if $\dot{r}, \dot{s}$ are not both zero. But if $\dot{r} = \dot{s} = 0$ then we revert back to (5.3), which shows (note that $\tilde{s}$ there is probably larger than $\dot{s} = 0$) at last that ω is elementary.

The outcome is that we have very nearly proved the part of Theorem 1 corresponding to (a). Namely if I is algebraic, and (s1)–(s3) of Section 6 are satisfied, then ω must be elementary and the rest of this part of Theorem 1 follows, just as in Section 3.

So what about these (s1)–(s3)?

If (s2) fails, then $B - A$ lies in the set $\{P_1, \ldots, P_{\tilde{r}}\}$. Now we do a very daft trick of reversing the direction of Γ. If (s2) still fails, then the above holds for $A - B$. So $B - A$ lies in the set $\{-P_1, \ldots, -P_{\tilde{r}}\}$. However, these two sets are disjoint because $P_1, \ldots, P_{\tilde{r}}$ are $\mathbf{Z}$-independent.

If (s1) fails then the simple poles of ω are just $P_1, \ldots, P_r$ in the original (5.1). We may now apply a translation $\tau(P) = P + P_r$ to get a new differential $\tau^*\omega$ whose simple poles are $P_1 - P_r, \ldots, P_{r-1} - P_r, O$. The analogue of (5.1) then has $r-1$ in place of r so we can use an inductive argument. Note that

$$\int_{\Gamma} \omega = \int_{\tau^{-1}\Gamma} \tau^*\omega, \quad \mathrm{d}(\tau^* f) = \tau^*(\mathrm{d}f),$$

leading to $n_\Gamma(f) = n_{\tau^{-1}\Gamma}(\tau^* f)$, so the conditions in Theorem 1(a) are translation invariant.

Finally, if (s3) fails then Γ is closed, and now part (b) of Theorem 1 can be deduced from the results of [15]. This we carry out in the next section.

9. Genus 1 – doing a bit more. We now assume that Γ is closed, and it will suffice to start with (5.3). Then the integration is much easier. Thus Γ corresponds to a period p of the lattice in the sense that

$$\int_\Gamma \frac{\mathrm{d}x}{y} = p, \qquad \int_\Gamma \frac{x\,\mathrm{d}x}{y} = -q$$

for the corresponding quasi-period $q = \zeta(z+p) - \zeta(z)$. And then

$$\int_\Gamma \theta(P) = t(u,p) + 2\pi\sqrt{-1}n$$

for

$$t(u,p) = p\zeta(u) - qu$$

and some integer n, probably some sort of winding number but depending on the choice of u in $P = (\wp(u), \wp'(u))$.

Now (5.3) gives

$$I = \sum_{i=1}^{\dot r} \dot\delta_i(t(u_i,p) + 2\pi\sqrt{-1}n_i) + 2\pi\sqrt{-1}\sum_{j=1}^{\tilde s} \tilde\gamma_j n_\Gamma(\tilde f_j) + \dot\alpha p - \beta q$$

with $u_1, \ldots, u_{\dot r}$ corresponding to $\tilde P_1, \ldots, \tilde P_{\dot r}$.

We start with the case $p \neq 0$. We write

$$I = \sum_{i=1}^{\dot r} \dot\delta_i t(u_i,p) + 2\pi\sqrt{-1}\gamma + \dot\alpha p - \beta q$$

with γ in $\overline{\mathbf{Q}}$. Let p^* be a period $\mathbf{Q}$-independent of p with $p^*/2$ not a period. Then

$$t\left(\frac{p^*}{2}, p\right) = \frac{pq^* - p^*q}{2} = 2\pi\sqrt{-1}\delta$$

for $\delta \neq 0$ in $\mathbf{Q}$ and q^* the quasi-period corresponding to p^*. So I is a linear combination of

$$p, q, t(u_1,p), \ldots, t(u_{\dot r},p), t(p^*/2,p) \tag{9.1}$$

with algebraic coefficients. Note that here $u_1, \ldots, u_{\dot r}, p^*/2$ are $\mathbf{Q}$-linearly independent modulo $\mathbf{Q}p$, because a relation $l_1u_1 + \cdots + l_{\dot r}u_{\dot r} + l^*(p^*/2) = lp$ with integer coefficients would imply first $l_1 = \cdots = l_{\dot r} = 0$ by the independence of $\tilde P_1, \ldots, \tilde P_{\dot r}$, and then $l^* = l = 0$. So Satz 2 of [15, p. 165] implies that the numbers in (9.1) together with 1 are linearly independent over $\overline{\mathbf{Q}}$. Thus I in $\overline{\mathbf{Q}}$ implies at once

$$I = \dot\delta_1 = \cdots = \dot\delta_{\dot r} = \gamma = \dot\alpha = \beta = 0.$$

Hence by (5.3),

$$\omega = \mathrm{d}f_0 + \tilde{\gamma}_1 \frac{\mathrm{d}\tilde{f}_1}{\tilde{f}_1} + \cdots + \tilde{\gamma}_{\tilde{s}} \frac{\mathrm{d}\tilde{f}_{\tilde{s}}}{\tilde{f}_{\tilde{s}}},$$

and we are back to elementary. We should now assume as usual that $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{s}}$ are $\mathbf{Q}$-independent, and then

$$0 = I = 2\pi\sqrt{-1} \sum_{j=1}^{\tilde{s}} \tilde{\gamma}_j n_\Gamma(\tilde{f}_j)$$

leads back to (a) of Theorem 1.

At last what if $p = 0$ (and so $q = 0$)? This means that the closed path Γ is homologically trivial. We go back to (5.1), which now yields simply

$$I = 2\pi\sqrt{-1}(\delta_1 n_\Gamma(P_1) + \cdots + \delta_r n_\Gamma(P_r)).$$

So we see that I is algebraic if and only if the inner sum is zero, which is exactly the sum in Theorem 1(b).

This completes the proof of the hard part of Theorem 1; and as with Theorem 0, the easier part is trivial.

10. Yet more transcendence. Huber and Wüstholz noted a nice consequence of their studies [5]. Namely, their Theorem 18.6 (p. 185) states that if u is such that $(\wp(u), \wp'(u))$ is non-torsion in $E(\overline{\mathbf{Q}})$, then

$$\chi(u) = 2\log\sigma(u) - u\zeta(u)$$

is transcendental for any choice of the logarithm.

Our arguments above provide a proof of something stronger: in fact, $\chi(u)$ is not a linear combination with algebraic coefficients of $1, u, \zeta(u)$, and logarithms of algebraic numbers. For suppose on the contrary

$$\chi(u) = -\kappa - \alpha u + \beta\zeta(u) - \gamma_1 \log\lambda_1 - \cdots - \gamma_n \log\lambda_n$$

with algebraic $\kappa, \alpha, \beta, \gamma_1, \ldots, \gamma_n, \lambda_1 \neq 0, \ldots, \lambda_n \neq 0$. We can assume that $\gamma_1, \ldots, \gamma_n$ are $\mathbf{Q}$-linearly independent by the familiar trickery. Assume for the moment that even $1, \gamma_1, \ldots, \gamma_n$ are $\mathbf{Q}$-linearly independent. Defining $u_1 = -u$ and φ_1 as in (6.1) with u_1 instead of u, we have

$$\varphi_1(u) = -\frac{\sigma(2u)}{\sigma(u)^2} e^{-u\zeta(u)} = \varepsilon_1^{-1}\sigma(u)^2 e^{-u\zeta(u)}$$

for the algebraic $\varepsilon_1 = 1/\wp'(u)$. Thus we get a relation

$$0 = \log(\varepsilon_1\varphi_1(u)) + \sum_{j=1}^{n} \gamma_j \log\lambda_j + \alpha u - \beta\zeta(u) + \kappa \tag{10.1}$$

as in (6.5). As $\dot{r} = 1$ the arguments of Sections 7 and 8 give a flat contradiction.

And should $1, \gamma_1, \ldots, \gamma_n$ be $\mathbf{Q}$-linearly dependent, for example $\gamma_n - \nu$ being a $\mathbf{Q}$-linear combination of $\gamma_1, \ldots, \gamma_{n-1}$ for some ν in $\mathbf{Q}$, then with a final flourish the sum in (10.1) becomes

$$\gamma_1 \log \tilde{\lambda}_1 + \cdots + \gamma_{n-1} \log \tilde{\lambda}_{n-1} + \log(\lambda_n^\nu);$$

and the last logarithm can be thrown into the first term of (10.1); note that now $1, \gamma_1, \ldots, \gamma_{n-1}$ must be $\mathbf{Q}$-independent.

11. Genus 0 – an example. By solving for y in the Bernoulli lemniscate (1.2) we see that a quarter of the area is given by

$$I = \int_0^1 y\,dx = \int_0^1 \sqrt{-\frac{1}{2} - x^2 + \frac{\sqrt{1+8x^2}}{2}}\,\mathrm{d}x$$

(with both square roots non-negative). When asked for an explicit evaluation, even of the indefinite integral, curiously Maple 2024, although not too bad at algebraic integration, here simply refuses to answer. It is much better at numerical integration, when it answers with .2500000000. Here the genus is indeed 0. By examining poles and residues we checked that

$$y\mathrm{d}x = \mathrm{d}f_0, \quad f_0 = -\frac{xy^3}{x^2 - y^2}$$

(thus providing the information refused by Maple). Our path goes from $(x, y) = (0, 0)$ to $(x, y) = (1, 0)$. Unfortunately, we have a singularity at $(0, 0)$, very visible from the real graph. To resolve this we need something like (1.5). Siegel [14, pp. 1–4] gives various parametrizations, but they amount to only strict inclusions. We found

$$x = \frac{1 - t^4}{1 + 6t^2 + t^4}, \quad y = \frac{2t(1 - t^2)}{1 + 6t^2 + t^4} \tag{11.1}$$

(compare (1.3) above) with

$$t = \frac{y}{x + x^2 + y^2} = \frac{x - x^2 - y^2}{y};$$

under which f_0 becomes

$$-\frac{8t^3(1 + t^2)}{(1 + 6t^2 + t^4)^2}.$$

We now go from $t = 1$ to $t = 0$, and we get indeed $I = 1/4$.

After these somewhat non-trivial calculations, a tip from Daniel Bernoulli made us ask Wikipedia instead. There we saw parametrizations slightly simpler than (11.1). We also saw the polar equation $r^2 = \cos 2\theta$, from which the area is now obvious and almost certainly known to Jacob Bernoulli.

12. Genus 1 – examples. We already noted (1.3) for the length of the Bernoulli lemniscate. This amounts to a closed Γ, and so is transcendental by earlier results. We may even compute the length from the origin $(0,0)$ to the highest point $(\sqrt{6}/4, \sqrt{2}/4)$. This leads to

$$\int\limits_{\sqrt{3}-\sqrt{2}}^{1} \frac{\mathrm{d}t}{\sqrt{1+6t^2+t^4}}.$$

We go to Weierstrass form via $t = \sqrt{x-2}$ to get $y^2 = 4(x^3 - 11x + 14)$, of course not with the same x, y, and we get

$$\int\limits_{7-2\sqrt{6}}^{3} \frac{\mathrm{d}x}{y}$$

(with $y > 0$) still classically transcendental. Incidentally, the elliptic curve above has complex multiplication by the order $\mathbf{Z} + 2\sqrt{-1}\mathbf{Z}$, whereas Siegel's parametrizations lead to the more familiar $y^2 = 4(x^3 - x)$ (isogenous) and the maximal order $\mathbf{Z} + \sqrt{-1}\mathbf{Z}$.

A non-classical example, chosen at random, is

$$I = \int\limits_0^1 \frac{\mathrm{d}x}{(1+x)\sqrt{1+x^3}}$$

which Maple can do algebraically but using five long lines and engineering-type elliptic functions. Here the transcendence follows quickly from Theorem 1. Part (b) is ruled out because $I > 0$; and the differential on $y^2 = 4(x^3+1)$ cannot be elementary, because its only pole, at the (torsion) point $(x,y) = (-1,0)$, is double, which would then have to come from $\mathrm{d}f_0$ with f_0 having its only pole there and simple; impossible.

The example

$$I = \int\limits_0^1 \frac{\mathrm{d}x}{(2+x)\sqrt{1+x^3}}$$

needs ten Maple lines, and now there are simple poles at $P = (-2, \sqrt{-7})$ and $-P$. So the differential is of the third kind, and I algebraic would imply by our Corollary that $I = 0$. Alternatively classical arguments (or see [10, Section 14]) show that P would have to be torsion if the differential were elementary. It is well-known that this can be decided effectively (see for example [6] for much more). Here we can reduce $y^2 = 4(x^3+1)$ to Legendre form $y_1^2 = x_1(x_1-1)(x_1-t)$ with $x = (-t+2)x_1 - 1$ for $t = e^{\pi i/3}$ and the point P_1 with abscissa $-(t+1)/3$. By [4, (5.7), p. 467] we have

$$|h(2P_1) - \hat{h}(2P_1)| \le \tfrac{1}{3}\log 58.$$

As $2P_1$ has abscissa $\xi = -(t+1)/21$ we find $h(2P_1) = h(\xi) = \log\sqrt{147}$ exceeding this, so the above Néron–Tate height cannot be zero and so P_1, P cannot be torsion.

Similarly we can treat families and show that

$$I = \int_0^1 \frac{x}{x^2 - \tau x + \tau^2} \frac{\mathrm{d}x}{\sqrt{1+x^3}}$$

is transcendental for all real algebraic $\tau > 1$, as then the differential is of the third kind and $I > 0$. Here the situation could become a bit delicate; already in [10, p. 300] we mentioned this differential (with x replaced by $-x$) and showed that it actually becomes elementary for infinitely many τ. And in [11] we show that this happens when $-\tau$ is the abscissa of a torsion point of order at least 3. In fact, there are infinitely many such real $\tau > 1$; for example, there is a non-zero real period and multiplying by $\sqrt{-3}$ we get a purely imaginary period p and then $\tau = -\wp(p/n)$ is real and asymptotically $-n^2/p^2$, so $\tau > 1$ for all large integers n.

Incidentally, just for $\tau = 2$ the above I could not be calculated in explicit terms by Maple in less than an hour (if indeed at all).

13. Genus 2 – counterexamples and examples. At present we have no "Theorem 2". In fact, Bertrand remarked that it could not hold for all X of genus 2. As an actual counterexample we have

$$\int_{-1}^{+1} \frac{x\,\mathrm{d}x}{\sqrt{1+x^6}} = 0$$

with a non-closed path. Here $\omega = x\mathrm{d}x/y$ on $y^2 = x^6 + 1$ is not elementary as would be required in the analogue of Theorem 1(a). We can see this because $\omega = \psi^*\omega_1$ for $\omega_1 = \mathrm{d}x_1/2y_1$ on $y_1^2 = x_1^3 + 1$ and $\psi(x,y) = (x^2, y) = (x_1, y_1)$, so ω_1 is of the first kind and ω too, thus not elementary. This argument shows that the jacobian of X is not simple, and one might hope this to be the only obstruction. And indeed (see the acknowledgements below), Sertöz has communicated a "Theorem g" for all genus $g \geq 2$ when the jacobian is simple.

At first sight the similar integral

$$\int_{-1}^{+1} \frac{x\,\mathrm{d}x}{\sqrt{1+x^4}} = 0$$

now in genus 1 appears to contradict Theorem 1; but here the differential is $(1/2)\mathrm{d}f_0/f_0$ for $f_0 = x^2 + \sqrt{x^4+1}$.

Still using maps like ψ above we can construct amusing transcendence examples in genus 2. Consider the length of the Huygens lemniscate (1.1).

The total length is probably transcendental by the older work of Wüstholz. So again we consider the length from $(0,0)$ to the highest point, now $(\sqrt{2}/2, 1/2)$. We find this to be

$$\int_0^{\sqrt{2}/2} \frac{4x^4 - 5x^2 + 2}{\sqrt{-4x^6 + 9x^4 - 7x^2 + 2}}\,\mathrm{d}x, \tag{13.1}$$

and we can use the above argument to reduce this to an elliptic $\int_\Gamma \omega$ for

$$\omega = \frac{x^2 - x + 16}{(x+7)^2}\,\frac{\mathrm{d}x}{y}, \quad y^2 = 4(x^3 + 15x + 16)$$

and non-closed Γ. Now the only poles of ω are double, at $P = (-7, 12\sqrt{-3})$ and $-P$. Thus if ω were elementary, the $\mathrm{d}f_0$ would have to involve f_0 whose only poles are simple at these points. The only such f_0 are $c/(x+7)$ for $c \neq 0$. We can indeed find a unique c such that $\omega - \mathrm{d}f_0$ has at worst a single pole at P; but then we find that the double pole at $-P$ persists. So ω is not elementary and the length (13.1) is transcendental.

Implicit in the above are the isogeny formulae due to Legendre (see [10, p. 267]). We could also use those due to Hermite (see [10, p. 270]) to construct transcendence examples on say X of genus 2 defined by $y^2 = (x^2 - a)(8x^3 - 6ax - b)$.

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