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A new exponential generalization of the Hardy–Hilbert integral inequality

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Abstract. The Hardy–Hilbert integral inequality has been the subject of extensive study in recent decades. In this article, we present a two-parameter exponential generalization of this inequality. The associated constant factor involves the upper incomplete gamma function. Interestingly, it can also be expressed in terms of the classical error function. Through precise analysis, we prove the optimality of this constant. We then derive several integral inequalities of various types, some involving primitive functions, while others incorporating auxiliary functions.

1. Introduction. We begin by recalling the classical Hardy–Hilbert integral inequality, originally established by G. H. Hardy in [HLP67]. Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,\infty) \to [0,\infty)$ be two measurable functions such that

$$\int_0^\infty f^p(x)\,dx < \infty, \qquad \int_0^\infty g^q(y)\,dy < \infty.$$

Then

$$\iint_0^\infty \frac{1}{x+y} f(x)g(y)\,dx\,dy \le \frac{\pi}{\sin(\pi/p)} \left[\int_0^\infty f^p(x)\,dx\right]^{1/p} \left[\int_0^\infty g^q(y)\,dy\right]^{1/q},$$

where $\iint_0^\infty$ denotes $\int_0^\infty \int_0^\infty$.

The constant factor $\pi/\sin(\pi/p)$ is optimal in the sense that replacing it with a smaller constant would invalidate the inequality for certain admissible functions f and g. The Hardy–Hilbert integral inequality has numerous applications in mathematical analysis, particularly in the theory of function spaces and integral operators and in harmonic analysis. See [HLP67, MPK91]. Over the past few decades, significant progress has been

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made in generalizing or modifying it in various ways. This includes refining the upper bound, incorporating weight functions, modifying the integrand structure, and establishing entirely new classes of related inequalities. These developments have expanded the scope and application of integral inequalities in both theoretical and practical contexts. A selection of relevant contributions is [BBPV17, Ch24a, Ch24b, Ch25, GY17, GR07, HC23, HH23, HWY22, KPPV12, Ku99, KL08, Pa98, RY14, RY15, Xu07, Ya96, Ya98, Ya04, Ya09, Ya17, Ya21, YR19].

For the purposes of this article, we highlight two important results.

Result 1. The inequality below is a well-known generalization of the Hardy–Hilbert integral inequality involving one parameter. Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, and $f, g : [0,\infty) \to [0,\infty)$ be such that

$$\int_0^\infty x^{p/2-1} f^p(x) dx < \infty, \quad \int_0^\infty y^{q/2-1} g^q(y)\, dy < \infty.$$

Then

$$\iint_0^\infty \frac{1}{\alpha x + y} f(x) g(y)\, dx\, dy \le \frac{\pi}{\sqrt{\alpha}} \Big[\int_0^\infty x^{p/2-1} f^p(x)\, dx\Big]^{1/p} \Big[\int_0^\infty y^{q/2-1} g^q(y)\, dy\Big]^{1/q}. \tag{1.1}$$

Setting $\alpha = 1$, we get the following known variant of the Hardy–Hilbert integral inequality:

$$\iint_0^\infty \frac{1}{x + y} f(x) g(y)\, dx\, dy \le \pi \Big[\int_0^\infty x^{p/2-1} f^p(x)\, dx\Big]^{1/p} \Big[\int_0^\infty y^{q/2-1} g^q(y)\, dy\Big]^{1/q}.$$

Result 2. The inequality below is a famous exponential modification of the Hardy–Hilbert integral inequality established in [Ya11] for $p = 2$. Let $f, g : [0,\infty) \to [0,\infty)$ be such that

$$\int_0^\infty f^2(x) dx < \infty, \quad \int_0^\infty g^2(y)\, dy < \infty.$$

Then

$$\iint_0^\infty \exp(-xy) f(x) g(y)\, dx\, dy \le \sqrt{\pi} \sqrt{\int_0^\infty f^2(x)\, dx} \sqrt{\int_0^\infty g^2(y)\, dy}.$$

The interest of this inequality lies in its elegant use of the exponential term $\exp(-xy)$, which naturally arises in various contexts such as Laplace transforms, heat kernel estimates, and problems in harmonic analysis. This result has inspired numerous exponential generalizations and modifications of the Hardy–Hilbert-type integral inequality [HL15, Liu15, Liu14, Liu25, LL13a, LL13b, LS13, LS14].

In this article, we investigate an exponential generalization of the Hardy–Hilbert integral inequality, centered on the following double integral:

$$\iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g(y)\, dx\, dy,$$

where $\alpha > 0$ and $\beta \geq 0$ are two adjustable parameters. When $\alpha = 1$ and $\beta = 0$, it clearly reduces to the double integral of the classical Hardy–Hilbert integral inequality. Although this generalization has an intuitive expression, it appears to have received little or no attention in the literature. Therefore, our main objective is to determine a sharp constant factor, say ξ, such that

$$\iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g(y)\, dx\, dy$$
$$\leq \xi \left[\int_0^\infty x^{p/2-1} f^p(x)\, dx\right]^{1/p} \left[\int_0^\infty y^{q/2-1} g^q(y)\, dy\right]^{1/q}.$$

Our candidate for this constant will be expressed using the upper incomplete gamma function, that is,

$$\Gamma(a, b) = \int_b^\infty x^{a-1} e^{-x}\, dx,$$

with $a, b > 0$. This can also be expressed in terms of the classical error function, which will be described in more detail later. Based on this result, we will derive several complementary integral inequalities of different forms, including variants involving primitives and variants incorporating auxiliary functions. This work supplements past research on exponential-type generalizations of the Hardy–Hilbert integral inequality, as discussed in [HL15, Liu15, Liu14, Liu25, LL13a, LL13b, LS13, LS14].

The remainder of this article is organized as follows. In Section 2 we present our main integral inequality and examine the optimality of the constant factor. Section 3 contains the complementary results. A conclusion is given in Section 4, with some perspectives for future research.

2. Main contributions

2.1. A key theorem. The main exponential generalization of the Hardy–Hilbert integral inequality is presented in the theorem below. The proof relies primarily on a suitable decomposition of the integrand, the Hölder integral inequality, changes of variables and several integral results presented in the form of lemmas.

THEOREM 2.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta \geq 0$, and $f, g : [0,\infty) \to [0,\infty)$ be two functions such that*

$$\int_0^\infty x^{p/2-1} f^p(x)\,dx < \infty, \quad \int_0^\infty y^{q/2-1} g^q(y)\,dy < \infty.$$

Then

$$\iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g(y)\,dx\,dy$$
$$\leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_0^\infty x^{p/2-1} f^p(x)\,dx\right]^{1/p} \left[\int_0^\infty y^{q/2-1} g^q(y)\,dy\right]^{1/q}.$$

Proof. Making a judicious decomposition of the integrand via the identity $1/p + 1/q = 1$, and using the Hölder integral inequality, we obtain

$$\text{(2.1)} \quad \iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g(y)\,dx\,dy$$
$$= \iint_0^\infty \left\{ x^{1/(2q)} y^{-1/(2p)} \frac{1}{(\alpha x + y)^{1/p}} \exp\left[-\frac{\beta}{p}\left(\frac{x}{y}\right)\right] f(x) \right\}$$
$$\times \left\{ x^{-1/(2q)} y^{1/(2p)} \frac{1}{(\alpha x + y)^{1/q}} \exp\left[-\frac{\beta}{q}\left(\frac{x}{y}\right)\right] g(y) \right\} dx\,dy$$
$$\leq \mathcal{A}^{1/p} \mathcal{B}^{1/q},$$

where

$$\mathcal{A} = \iint_0^\infty x^{p/(2q)} \frac{1}{\sqrt{y}(\alpha x + y)} \exp\left(-\beta \frac{x}{y}\right) f^p(x)\,dx\,dy,$$
$$\mathcal{B} = \iint_0^\infty y^{q/(2p)} \frac{1}{\sqrt{x}(\alpha x + y)} \exp\left(-\beta \frac{x}{y}\right) g^q(y)\,dx\,dy.$$

In order to examine $\mathcal{A}$ and $\mathcal{B}$, we present a key integral formula, extracted from [GR07].

LEMMA 2.2 ([GR07], a particular case of Entry 3.471.13). *Let $\gamma > 0$ and $\theta \geq 0$. Then*

$$\int_0^\infty \frac{1}{\sqrt{x}(x + \gamma)} \exp\left(-\theta \frac{1}{x}\right) dx = \sqrt{\frac{\pi}{\gamma}} \exp\left(\frac{\theta}{\gamma}\right) \Gamma\left(\frac{1}{2}, \frac{\theta}{\gamma}\right).$$

For $\mathcal{A}$, using the Fubini–Tonelli theorem, performing the change of variables $u = y/x$, applying Lemma 2.2 with $\gamma = \alpha$ and $\theta = \beta$, and using the identity $p/(2q) = (p-1)/2$, we obtain

$$\text{(2.2)}\qquad \mathcal{A} = \int_0^\infty x^{p/(2q)} f^p(x) \left[\int_0^\infty \frac{1}{\sqrt{y}(\alpha x + y)} \exp\left(-\beta \frac{x}{y}\right) dy \right] dx$$

$$= \int_0^\infty x^{p/(2q)-1/2} f^p(x) \left[\int_0^\infty \frac{1}{\sqrt{y/x}(y/x + \alpha)} \exp\left(-\beta \frac{x}{y}\right) \frac{1}{x}\, dy \right] dx$$

$$= \int_0^\infty x^{p/(2q)-1/2} f^p(x) \left[\int_0^\infty \frac{1}{\sqrt{u}(u + \alpha)} \exp\left(-\beta \frac{1}{u}\right) du \right] dx$$

$$= \int_0^\infty x^{p/(2q)-1/2} f^p(x) \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) dx$$

$$= \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^\infty x^{p/2-1} f^p(x)\, dx.$$

For $\mathcal{B}$, we need the lemma below.

LEMMA 2.3. *Let $\gamma > 0$ and $\theta \geq 0$. Then*

$$\int_0^\infty \frac{1}{\sqrt{x}(\gamma x + 1)} \exp(-\theta x)\, dx = \sqrt{\frac{\pi}{\gamma}} \exp\left(\frac{\theta}{\gamma}\right) \Gamma\left(\frac{1}{2}, \frac{\theta}{\gamma}\right).$$

Proof. Performing the change of variables $u = 1/x$ and using Lemma 2.2, we get

$$\int_0^\infty \frac{1}{\sqrt{x}(\gamma x + 1)} \exp(-\theta x)\, dx$$

$$= \int_\infty^0 \frac{1}{\sqrt{(1/u)}[\gamma(1/u) + 1]} \exp\left(-\theta \frac{1}{u}\right) \left(-\frac{1}{u^2}\, du\right)$$

$$= \int_0^\infty \frac{1}{\sqrt{u}(u + \gamma)} \exp\left(-\theta \frac{1}{u}\right) du = \sqrt{\frac{\pi}{\gamma}} \exp\left(\frac{\theta}{\gamma}\right) \Gamma\left(\frac{1}{2}, \frac{\theta}{\gamma}\right). \qquad \blacksquare$$

Using the Fubini–Tonelli theorem, performing the change of variables $v = x/y$, applying Lemma 2.3 with $\gamma = \alpha$ and $\theta = \beta$, and using the identity $q/(2p) = (q-1)/2$, we obtain

$$\text{(2.3)}\qquad \mathcal{B} = \int_0^\infty y^{q/(2p)} g^q(y) \left[\int_0^\infty \frac{1}{\sqrt{x}(\alpha x + y)} \exp\left(-\beta \frac{x}{y}\right) dx \right] dy$$

$$= \int_0^\infty y^{q/(2p)-1/2} g^q(y) \left[\int_0^\infty \frac{1}{\sqrt{(x/y)}(\alpha x/y + 1)} \exp\left(-\beta \frac{x}{y}\right) \frac{1}{y}\, dx \right] dy$$

$$= \int_0^{\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{\infty} \frac{1}{\sqrt{v}(\alpha v+1)} \exp(-\beta v)\, dv \right] dy$$
$$= \int_0^{\infty} y^{q/(2p)-1/2} g^q(y) \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) dy$$
$$= \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^{\infty} y^{q/2-1} g^q(y)\, dy.$$

Combining (2.1)–(2.3), and simplifying via $1/p + 1/q = 1$, we get

$$\iint_0^{\infty} \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g(y)\, dx\, dy$$
$$\leq \left[\sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^{\infty} x^{p/2-1} f^p(x)\, dx \right]^{1/p}$$
$$\times \left[\sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^{\infty} y^{q/2-1} g^q(y)\, dy \right]^{1/q}$$
$$= \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_0^{\infty} x^{p/2-1} f^p(x)\, dx \right]^{1/p} \left[\int_0^{\infty} y^{q/2-1} g^q(y)\, dy \right]^{1/q}.$$

This ends the proof of Theorem 2.1. ■

Setting $\beta = 0$ and noting that $\Gamma(1/2, 0) = \Gamma(1/2) = \sqrt{\pi}$, we obtain equation (1.1). Theorem 2.1 can thus be viewed as a generalization of this variant of the Hardy–Hilbert integral inequality.

Furthermore, performing the change of variables $x = z^2$ and using the Gauss integral $\int_0^{\infty} e^{-z^2}\, dz = (1/2)\sqrt{\pi}$, we obtain

$$\Gamma\left(\frac{1}{2}, b\right) = \int_b^{\infty} x^{-1/2} e^{-x} dx = 2 \int_{\sqrt{b}}^{\infty} e^{-z^2}\, dz = 2\left[\int_0^{\infty} e^{-z^2}\, dz - \int_0^{\sqrt{b}} e^{-z^2}\, dz \right]$$
$$= 2\left[\frac{1}{2}\sqrt{\pi} - \int_0^{\sqrt{b}} e^{-z^2}\, dz \right] = \sqrt{\pi}\{1 - \operatorname{erf}[\sqrt{b}]\},$$

where $\operatorname{erf}(c)$ denotes the error function defined by

$$\operatorname{erf}(c) = \frac{2}{\sqrt{\pi}} \int_0^{c} e^{-x^2}\, dx.$$

This well-known special function quantifies the probability of a value in a normal distribution falling within a certain range of the mean. It is widely used in statistics and probability theory, as well as in solving differential equations involving heat and diffusion.

By expressing the constant factor in terms of the error function, the inequality in Theorem 2.1 takes the more explicit form

$$\iint_0^\infty \frac{1}{\alpha x+y}\exp\left(-\beta\frac{x}{y}\right)f(x)g(y)\,dx\,dy$$
$$\le \frac{\pi}{\sqrt{\alpha}}\exp\left(\frac{\beta}{\alpha}\right)\left\{1-\operatorname{erf}\left[\sqrt{\frac{\beta}{\alpha}}\right]\right\}\left[\int_0^\infty x^{p/2-1}f^p(x)\,dx\right]^{1/p}$$
$$\times\left[\int_0^\infty y^{q/2-1}g^q(y)\,dy\right]^{1/q}.$$

In this formulation, the appearance of the constant π agrees naturally with the structure seen in equation (1.1). This also provides a clearer insight into the relationship between the parameters and the special functions involved.

2.2. Optimality of the constant. The proposition below discusses the optimality of the constant factor exhibited in Theorem 2.1.

PROPOSITION 2.4. *In Theorem* 2.1*, the constant factor*

$$\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{2},\frac{\beta}{\alpha}\right)$$

is optimal. This means that replacing this constant with a smaller one would invalidate the inequality in Theorem 2.1 *for certain admissible functions f and g.*

Proof. Suppose that the constant

$$\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{2},\frac{\beta}{\alpha}\right)$$

is not optimal. We will demonstrate that this leads to a contradiction. If this assumption were true, then there would exist a constant

$$\vartheta\in\left(0,\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{2},\frac{\beta}{\alpha}\right)\right)$$

satisfying, for any $f,g:[0,\infty)\to[0,\infty)$,

$$\iint_0^\infty \frac{1}{\alpha x+y}\exp\left(-\beta\frac{x}{y}\right)f(x)g(y)\,dx\,dy \tag{2.4}$$
$$\le\vartheta\left[\int_0^\infty x^{p/2-1}f^p(x)\,dx\right]^{1/p}\left[\int_0^\infty y^{q/2-1}g^q(y)\,dy\right]^{1/q}.$$

To reach a contradiction, we construct appropriate test functions depending on a parameter $\epsilon>0$, and analyze the behavior of both sides of (2.4) as

$\epsilon \to 0^+$. Specifically, for any $\epsilon > 0$, we set

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-1/2-\epsilon/p} & \text{if } x \in [1,\infty), \end{cases}$$

$$g_\epsilon(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-1/2-\epsilon/q} & \text{if } y \in [1,\infty). \end{cases}$$

Some classical integral developments give

$$\int_0^\infty x^{p/2-1} f_\epsilon^p(x)\,dx = \int_1^\infty x^{p/2-1}(x^{-1/2-\epsilon/p})^p\,dx = \int_1^\infty x^{-\epsilon-1}\,dx = 1/\epsilon,$$

and similarly

$$\int_0^\infty y^{q/2-1} g_\epsilon^q(y)\,dy = 1/\epsilon.$$

This, with the identity $1/p + 1/q = 1$ and (2.4), gives

$$\begin{aligned} \vartheta &= \vartheta\epsilon \times \frac{1}{\epsilon^{1/p}} \times \frac{1}{\epsilon^{1/q}} \\ &= \epsilon\left\{\vartheta\left[\int_0^\infty x^{p/2-1} f_\epsilon^p(x)\,dx\right]^{1/p}\left[\int_0^\infty y^{q/2-1} g_\epsilon^q(y)\,dy\right]^{1/q}\right\} \\ &\geq \epsilon \iint_0^\infty \frac{1}{\alpha x + y}\exp\left(-\beta\frac{x}{y}\right) f_\epsilon(x) g_\epsilon(y)\,dx\,dy. \end{aligned} \tag{2.5}$$

Now, let us focus on the double integral. Performing the change of variables $x = uy$, applying the Fubini–Tonelli theorem and the identity $1/p+1/q = 1$, we obtain

$$\begin{aligned} &\iint_0^\infty \frac{1}{\alpha x + y}\exp\left(-\beta\frac{x}{y}\right) f_\epsilon(x) g_\epsilon(y)\,dx\,dy \\ &= \int_1^\infty \int_1^\infty \frac{1}{\alpha x + y}\exp\left(-\beta\frac{x}{y}\right) x^{-1/2-\epsilon/p} y^{-1/2-\epsilon/q}\,dx\,dy \\ &= \int_1^\infty \left[\int_1^\infty \frac{1}{\alpha x + y}\exp\left(-\beta\frac{x}{y}\right) x^{-1/2-\epsilon/p}\,dx\right] y^{-1/2-\epsilon/q}\,dy \\ &= \int_1^\infty \left[\int_{1/y}^\infty \frac{1}{\alpha u y + y}\exp\left(-\beta\frac{uy}{y}\right)(uy)^{-1/2-\epsilon/p} y\,du\right] y^{-1/2-\epsilon/q}\,dy \\ &= \int_1^\infty \left[\int_{1/y}^\infty \frac{1}{\sqrt{u}(\alpha u + 1)}\exp(-\beta u) u^{-\epsilon/p}\,du\right] y^{-(1+\epsilon)}\,dy. \end{aligned} \tag{2.6}$$

Now, let us decompose this double integral. Applying the Chasles integral relation, the Fubini–Tonelli theorem and the identity $1/p+1/q=1$, we get

$$\begin{aligned}
(2.7)\quad & \int_1^{\infty}\left[\int_{1/y}^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right]y^{-(1+\epsilon)}\,dy \\
&= \int_1^{\infty}\left[\int_{1/y}^{1}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right]y^{-(1+\epsilon)}\,dy \\
&\quad + \int_1^{\infty}\left[\int_{1}^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right]y^{-(1+\epsilon)}\,dy \\
&= \int_0^{1}\left[\int_{1/u}^{\infty}y^{-(1+\epsilon)}dy\right]\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du \\
&\quad + \left[\int_{1}^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right]\left[\int_1^{\infty}y^{-(1+\epsilon)}\,dy\right] \\
&= \int_0^{1}\left(\frac{1}{\epsilon}u^{\epsilon}\right)\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du \\
&\quad + \frac{1}{\epsilon}\left[\int_{1}^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right] \\
&= \frac{1}{\epsilon}\left[\int_0^{1}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{\epsilon/q}\,du\right. \\
&\quad \left.+ \int_1^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du\right].
\end{aligned}$$

Combining (2.5)–(2.7), we find that

$$\begin{aligned}
\vartheta \geq & \int_0^{1}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{\epsilon/q}\,du \\
& + \int_1^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du.
\end{aligned}$$

Now, let us work with the inferior limit at $\epsilon\to 0^+$, denoted by $\underline{\lim}$. It follows from the Fatou integral lemma, $\underline{\lim}\, u^{\epsilon/q}=1$ for $u\in(0,1)$, $\underline{\lim}\, u^{-\epsilon/p}=1$ for $u\in[1,\infty)$, the Chasles integral relation and Lemma 2.3 with $\gamma=\alpha$ and $\theta=\beta$ that

$$\begin{aligned}
\vartheta \geq & \underline{\lim}\int_0^{1}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{\epsilon/q}\,du \\
& + \underline{\lim}\int_1^{\infty}\frac{1}{\sqrt{u}(\alpha u+1)}\exp(-\beta u)u^{-\epsilon/p}\,du
\end{aligned}$$

$$\geq \int_0^1 \frac{1}{\sqrt{u}(\alpha u+1)} \exp(-\beta u)[\underline{\lim}\, u^{\epsilon/q}]\, du + \int_1^\infty \frac{1}{\sqrt{u}(\alpha u+1)} \exp(-\beta u)[\underline{\lim}\, u^{-\epsilon/p}]\, du$$

$$= \int_0^1 \frac{1}{\sqrt{u}(\alpha u+1)} \exp(-\beta u)\, du + \int_1^\infty \frac{1}{\sqrt{u}(\alpha u+1)} \exp(-\beta u)\, du$$

$$= \int_0^\infty \frac{1}{\sqrt{u}(\alpha u+1)} \exp(-\beta u)\, du = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right).$$

Consequently,

$$\vartheta \notin \left(0, \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right)\right).$$

This contradiction ends the proof of Proposition 2.4. ■

We complete the analysis with some additional integral inequality results presented in the section below.

3. Complementary results. Three complementary results based on Theorem 2.1 are now proposed. These are a primitive variant, a one-function variant, and a multiple-function variant.

3.1. Primitive variant. The result below offers a primitive variant of Theorem 2.1. The proof uses the Hardy integral inequality, explaining the expression of the derived constant factor.

PROPOSITION 3.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta \geq 0$, $f, g : [0,\infty) \to [0,\infty)$ be such that*

$$\int_0^\infty f^p(x)\, dx < \infty, \quad \int_0^\infty g^q(y)\, dy < \infty$$

and $F, G : [0,\infty) \to [0,\infty)$ be the primitives defined by

$$F(x) = \int_0^x f(t)\, dt, \quad G(y) = \int_0^y g(t)\, dt.$$

Then

$$\iint_0^\infty (xy)^{-3/2} x^{1/p} y^{1/q} \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) F(x) G(y)\, dx\, dy \leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \frac{p^2}{p-1} \left[\int_0^\infty f^p(x)\, dx\right]^{1/p} \left[\int_0^\infty g^q(y)\, dy\right]^{1/q}.$$

This inequality is sharp.

Proof. The double integral can be expressed as

$$\iint_0^\infty (xy)^{-3/2}x^{1/p}y^{1/q}\frac{1}{\alpha x+y}\exp\Big(-\beta\frac{x}{y}\Big)F(x)G(y)\,dx\,dy = \iint_0^\infty \frac{1}{\alpha x+y}\exp\Big(-\beta\frac{x}{y}\Big)f_*(x)g_*(y)\,dx\,dy, \tag{3.1}$$

where

$$f_*(x)=\frac{1}{x^{3/2-1/p}}F(x),\quad g_*(y)=\frac{1}{y^{3/2-1/q}}G(y).$$

Applying Theorem 2.1 to f_* and g_*, we get

$$\iint_0^\infty \frac{1}{\alpha x+y}\exp\Big(-\beta\frac{x}{y}\Big)f_*(x)g_*(y)\,dx\,dy \le \sqrt{\frac{\pi}{\alpha}}\exp\Big(\frac{\beta}{\alpha}\Big)\Gamma\Big(\frac{1}{2},\frac{\beta}{\alpha}\Big)\Big[\int_0^\infty x^{p/2-1}f_*^p(x)\,dx\Big]^{1/p}\Big[\int_0^\infty y^{q/2-1}g_*^q(y)\,dy\Big]^{1/q}. \tag{3.2}$$

Now, let us majorize the two integrals of this upper bound. Using the definitions of f_* and g_* and the Hardy integral inequality, we obtain

$$\int_0^\infty x^{p/2-1}f_*^p(x)\,dx = \int_0^\infty x^{p/2-1}\Big[\frac{1}{x^{3/2-1/p}}F(x)\Big]^p dx = \int_0^\infty \frac{1}{x^p}F^p(x)\,dx \le \Big(\frac{p}{p-1}\Big)^p\int_0^\infty f^p(x)\,dx. \tag{3.3}$$

In a similar way, using the identity $1/p+1/q=1$, we get

$$\int_0^\infty y^{q/2-1}g_*^q(y)\,dy = \int_0^\infty y^{q/2-1}\Big[\frac{1}{y^{3/2-1/q}}G(y)\Big]^q dy = \int_0^\infty \frac{1}{y^q}G^q(y)\,dy \le \Big(\frac{q}{q-1}\Big)^q\int_0^\infty g^q(y)\,dy = p^q\int_0^\infty g^q(y)\,dy. \tag{3.4}$$

Combining (3.1)–(3.4), and using $1/p+1/q=1$, we establish that

$$\iint_0^\infty (xy)^{-3/2}x^{1/p}y^{1/q}\frac{1}{\alpha x+y}\exp\Big(-\beta\frac{x}{y}\Big)F(x)G(y)\,dx\,dy$$
$$\le \sqrt{\frac{\pi}{\alpha}}\exp\Big(\frac{\beta}{\alpha}\Big)\Gamma\Big(\frac{1}{2},\frac{\beta}{\alpha}\Big)\Big[\Big(\frac{p}{p-1}\Big)^p\int_0^\infty f^p(x)\,dx\Big]^{1/p}\Big[p^q\int_0^\infty g^q(y)\,dy\Big]^{1/q}$$
$$= \sqrt{\frac{\pi}{\alpha}}\exp\Big(\frac{\beta}{\alpha}\Big)\Gamma\Big(\frac{1}{2},\frac{\beta}{\alpha}\Big)\frac{p^2}{p-1}\Big[\int_0^\infty f^p(x)\,dx\Big]^{1/p}\Big[\int_0^\infty g^q(y)\,dy\Big]^{1/q}.$$

The sharpness of the upper bound is demonstrated by considering the functions

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-1/2-\epsilon/p} & \text{if } x \in [1,\infty), \end{cases} \qquad g_\epsilon(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-1/2-\epsilon/q} & \text{if } y \in [1,\infty), \end{cases}$$

and carrying out a similar limiting argument to the proof of Proposition 2.4. We leave the details to the reader. ∎

We emphasize the unweighted norms of f and g. We also note the modification of the constant factor, which is now defined as follows:

$$\sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \frac{p^2}{p-1}.$$

The extra term $p^2/(p-1)$ is a consequence of the application of the Hardy integral inequality.

3.2. One-function variant. In the result below, a one-function integral inequality is derived from Theorem 2.1.

PROPOSITION 3.2. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta \geq 0$, *and* $f : [0,\infty) \to [0,\infty)$ *be such that*

$$\int_0^\infty x^{p/2-1} f^p(x)\, dx < \infty.$$

Then

$$\int_0^\infty y^{-(q/2-1)(p-1)} \left[\int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^p dy \\ \leq \frac{\pi^{p/2}}{\alpha^{p/2}} \exp\left(\frac{\beta p}{\alpha}\right) \Gamma^p\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^\infty x^{p/2-1} f^p(x)\, dx.$$

This inequality is sharp.

Proof. We set

$$\mathcal{C} = \int_0^\infty y^{-(q/2-1)(p-1)} \left[\int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^p dy.$$

Then we can decompose $\mathcal{C}$ as

$$\mathcal{C} = \int_0^\infty \left[\int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right] \\ \times \left[y^{-(q/2-1)} \int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{p-1} dy \\ = \iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g_\circ(y)\, dx\, dy, \tag{3.5}$$

where

$$g_\circ(y) = \left[y^{-(q/2-1)} \int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{p-1}.$$

Applying Theorem 2.1 to to the functions f and $g_\circ$, we find that

$$\begin{aligned}(3.6)\qquad & \iint_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x) g_\circ(y)\, dx\, dy \\ & \leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_0^\infty x^{p/2-1} f^p(x)\, dx\right]^{1/p} \left[\int_0^\infty y^{q/2-1} g_\circ^q(y)\, dy\right]^{1/q}.\end{aligned}$$

Now, let us determine the last integral in this upper bound. Using the definition of $g_\circ$ and the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}(3.7)\qquad & \int_0^\infty y^{q/2-1} g_\circ^q(y)\, dy \\ & = \int_0^\infty y^{q/2-1} \left[y^{-(q/2-1)} \int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{q(p-1)} dy \\ & = \int_0^\infty y^{q/2-1} \left[y^{-(q/2-1)} \int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{p} dy \\ & = \int_0^\infty y^{-(q/2-1)(p-1)} \left[\int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{p} dy = \mathcal{C}.\end{aligned}$$

Combining (3.5)–(3.7) gives

$$\mathcal{C} \leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_0^\infty x^{p/2-1} f^p(x)\, dx\right]^{1/p} \mathcal{C}^{1/q}.$$

By the identity $1 - 1/q = 1/p$, we obtain

$$\mathcal{C}^{1/p} \leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_0^\infty x^{p/2-1} f^p(x)\, dx\right]^{1/p}.$$

This implies that

$$\begin{aligned}& \int_0^\infty y^{-(q/2-1)(p-1)} \left[\int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx\right]^{p} dy \\ & \leq \left\{\sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right)\right\}^p \int_0^\infty x^{p/2-1} f^p(x)\, dx \\ & = \frac{\pi^{p/2}}{\alpha^{p/2}} \exp\left(\frac{\beta p}{\alpha}\right) \Gamma^p\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \int_0^\infty x^{p/2-1} f^p(x)\, dx.\end{aligned}$$

To see that the upper bound is sharp, we consider the function

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-1/2-\epsilon/p} & \text{if } x \in [1,\infty), \end{cases}$$

and let $\epsilon \to 0^+$, arguing as in the proof of Proposition 2.4. ■

Based on this result, we can establish the continuity of the following integral operator under the indicated weighted L_p norms:

$$\mathfrak{I}(f)(y) = \int_0^\infty \frac{1}{\alpha x + y} \exp\left(-\beta \frac{x}{y}\right) f(x)\, dx.$$

The presence of the adjustable parameters α and β adds a certain dimension to the applicability of this operator. We can consider $\mathfrak{I}(f)$ as a generalized smoothing operator that interpolates between different types of decay and singularity behaviors depending on the chosen parameters. This makes it particularly relevant to the study of weighted inequalities, kernel methods and integral equations, in which controlling regularity and decay is essential.

3.3. Multiple-function variant. A multiple-function variant of Theorem 2.1 is proposed below.

PROPOSITION 3.3. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$ *with* $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{\infty\}$ *with* $d > c$, $f : [a,b] \to [0,\infty)$ *and* $g : [c,d] \to [0,\infty)$ *be any two functions, and* $h : [a,b] \to [0,\infty)$ *and* $k : [c,d] \to [0,\infty)$ *be two differentiable strictly increasing functions such that* $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = \infty$, $\lim_{t\to c} k(t) = 0$, $\lim_{t\to d} k(t) = \infty$, *and*

$$\int_a^b f^p(x) \frac{h^{p/2-1}(x)}{[h'(x)]^{p-1}}\, dx < \infty, \quad \int_c^d g^q(y) \frac{k^{q/2-1}(y)}{[k'(y)]^{q-1}}\, dy < \infty.$$

Then

$$\int_c^d \int_a^b \frac{1}{\alpha h(x) + k(y)} \exp\left[-\beta \frac{h(x)}{k(y)}\right] f(x) g(y)\, dx\, dy$$
$$\leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2}, \frac{\beta}{\alpha}\right) \left[\int_a^b f^p(x) \frac{h^{p/2-1}(x)}{[h'(x)]^{p-1}}\, dx\right]^{1/p}$$
$$\times \left[\int_c^d g^q(y) \frac{k^{q/2-1}(y)}{[k'(y)]^{q-1}}\, dy\right]^{1/q}.$$

This inequality is sharp.

Proof. Performing the change of variables $s = h(x)$ and $t = k(y)$, which implies that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = \{1/[h'(h^{-1}(s))]\}\,ds$ and $dy = \{1/[k'(k^{-1}(t))]\}\,dt$, we get

$$\begin{aligned}(3.8)\qquad &\int_c^d\int_a^b \frac{1}{\alpha h(x)+k(y)}\exp\left[-\beta\frac{h(x)}{k(y)}\right]f(x)g(y)\,dx\,dy\\ &=\int_0^\infty\int_0^\infty \frac{1}{\alpha s+t}\exp\left(-\beta\frac{s}{t}\right)f(h^{-1}(s))g(k^{-1}(t))\frac{1}{h'(h^{-1}(s))}\,\frac{1}{k'(k^{-1}(t))}\,ds\,dt\\ &=\int_0^\infty\int_0^\infty \frac{1}{\alpha s+t}\exp\left(-\beta\frac{s}{t}\right)f_\dagger(s)g_\dagger(t)\,ds\,dt,\end{aligned}$$

where

$$f_\dagger(s)=f(h^{-1}(s))\frac{1}{h'(h^{-1}(s))},\qquad g_\dagger(t)=g(k^{-1}(t))\frac{1}{k'(k^{-1}(t))}.$$

Applying Theorem 2.1 to $f_\dagger$ and $g_\dagger$, we obtain

$$\begin{aligned}(3.9)\qquad &\int_0^\infty\int_0^\infty \frac{1}{\alpha s+t}\exp\left(-\beta\frac{s}{t}\right)f_\dagger(s)g_\dagger(t)\,ds\,dt\\ &\le\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{2},\frac{\beta}{\alpha}\right)\left[\int_0^\infty s^{p/2-1}f_\dagger^p(s)\,ds\right]^{1/p}\left[\int_0^\infty t^{q/2-1}g_\dagger^q(t)\,dt\right]^{1/q}.\end{aligned}$$

Let us now determine the expression of the two integrals in the upper bound. Using the definition of $f_\dagger$ and performing the change of variables $s = h(u)$, we get

$$\begin{aligned}(3.10)\qquad &\int_0^\infty s^{p/2-1}f_\dagger^p(s)\,ds=\int_0^\infty s^{p/2-1}\left[f(h^{-1}(s))\frac{1}{h'(h^{-1}(s))}\right]^p ds\\ &=\int_a^b h^{p/2-1}(u)f^p(u)\frac{1}{[h'(u)]^p}h'(u)\,du=\int_a^b f^p(u)\frac{h^{p/2-1}(u)}{[h'(u)]^{p-1}}\,du.\end{aligned}$$

In a similar way, using the definition of $g_\dagger$ and performing the change of variables $t = k(v)$, we have

$$\begin{aligned}(3.11)\qquad &\int_0^\infty t^{q/2-1}g_\dagger^q(t)\,dt=\int_0^\infty t^{q/2-1}\left[g(k^{-1}(t))\frac{1}{k'(k^{-1}(t))}\right]^q dt\\ &=\int_c^d k^{q/2-1}(v)g^q(v)\frac{1}{[k'(v)]^q}k'(v)\,dv=\int_c^d g^q(v)\frac{k^{q/2-1}(v)}{[k'(v)]^{q-1}}\,dv.\end{aligned}$$

Combining (3.8)–(3.10) and (3.11), and standardizing the notation, we get

$$\int_c^d\int_a^b \frac{1}{\alpha h(x)+k(y)} \exp\left[-\beta\frac{h(x)}{k(y)}\right] f(x)g(y)\,dx\,dy$$

$$\leq \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta}{\alpha}\right) \Gamma\left(\frac{1}{2},\frac{\beta}{\alpha}\right) \left[\int_a^b f^p(x)\frac{h^{p/2-1}(x)}{[h'(x)]^{p-1}}\,dx\right]^{1/p}$$

$$\times \left[\int_c^d g^q(y)\frac{k^{q/2-1}(y)}{[k'(y)]^{q-1}}\,dy\right]^{1/q}.$$

To see the sharpness, simply note that the above change of variables, expressed in terms of h and k, transforms the inequality into the upper bound of Theorem 2.1. ■

The variety of possible choices for the functions h and k makes this result applicable to many mathematical scenarios involving integral inequalities of this form. Therefore, it is particularly useful in harmonic analysis, weighted norm inequalities and the theory of integral operators.

4. Conclusion. Several perspectives for future research can be explored. One promising direction is to extend our results to weighted integral inequalities and multidimensional domains. Furthermore, their applications to partial differential equations, particularly in diffusion theory, warrant further investigation. Finally, the numerical analysis of the constants and their asymptotic behavior could provide valuable insights in both theoretical and applied contexts.

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