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Unique solvability for a Schrödinger equation with Robin boundary condition on Lipschitz domains

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Abstract. Let $1 < p \leq 2$. Using the layer potential method, we establish the unique solvability in L^p-spaces of the Schrödinger equation $-\Delta u + \mathbb{V}u = 0$ on a Lipschitz domain subject to a Robin boundary condition. The potential $\mathbb{V}$ belongs to the reverse Hölder class. The case of C^1-domains is also discussed.

1. Introduction.

In this paper, we examine the Schrödinger equation with Robin boundary condition

$$\begin{cases} -\Delta u + \mathbb{V}u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + bu = g \in L^p(\partial\Omega), \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases} \tag{1.1}$$

The *underlying structure* is as follows:

- $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with connected boundary whose unit outward normal vector is ν.
- $1 < p \leq 2$.
- The potential $0 \leq \mathbb{V}$ is in RH_σ with $n < \sigma \leq \infty$. See Definition 1.1 below.
- The Robin coefficient $b : \partial\Omega \to [0, \infty)$ satisfies $b \neq 0$ and

$$b \in \begin{cases} L^{n-1}(\partial\Omega) & \text{if } 1 < p \leq 2 \text{ and } n \geq 4, \\ & \text{or } 1 < p < 2 \text{ and } n = 3, \\ L^s(\partial\Omega) & \text{for some } s > 2 \text{ if } p = 2 \text{ and } n = 3. \end{cases} \tag{1.2}$$

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- $(\nabla u)^*$ denotes the non-tangential maximal function of ∇u, in the sense that
$$(\nabla u)^*(Q) := \sup\{|\nabla u(X)| : X \in \gamma(Q)\}$$
for each $Q \in \partial\Omega$, where
$$\gamma(Q) := \{X \in \Omega : |X - Q| < 2\,\mathrm{dist}(X, \partial\Omega)\}$$
is the interior non-tangential region.
- All boundary relations herein are interpreted as holding non-tagentially almost everywhere with respect to the surface measure on $\partial\Omega$. In particular,
$$\frac{\partial u}{\partial \nu} + bu = g \quad \text{ on } \partial\Omega$$
means
$$\lim_{\substack{X\to Q\\ X\in\gamma(Q)}} [\langle \nabla u(X), \nu(Q)\rangle + b(Q)u(X)] = g(Q)$$
for a.e. $Q \in \partial\Omega$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$.

The reverse Hölder class is defined as follows.

Definition 1.1. Let $1 < q \le \infty$. We say that $\mathbb{V} \in RH_q$ if $\mathbb{V} \ge 0$ is locally integrable and there exists a constant $C > 0$ such that the reverse Hölder inequality
$$\left(\frac{1}{|B|}\int_B \mathbb{V}^q\,dX\right)^{1/q} \le \frac{C}{|B|}\int_B \mathbb{V}\,dX, \qquad 1 < q < \infty,$$
and
$$\|\mathbb{V}\|_{L^\infty(B)} \le \frac{C}{|B|}\int_B \mathbb{V}\,dX, \qquad q = \infty,$$
hold for every ball $B \subset \mathbb{R}^n$.

Note that the reverse Hölder class contains non-negative polynomials. In fact, given a non-negative polynomial P one has
$$\max_{x\in B} P(x) \le \frac{C}{|B|}\int_B P(x)\,dX,$$
where C is a constant depending on the degree of P and the spatial dimension n only. This implies $P \in RH_q$ for all $q \in (1,\infty]$. In particular, the Schrödinger operator $-\Delta + \mathbb{V}$ encapsulates well the Hermite operator $-\Delta + |x|^2$ which plays a significant role in harmonic oscillators. At the same time, this class also has a close tie with Muckenhoupt weights, which makes it highly appropriate for doing analysis in harmonic analysis and partial differential equations. Furthermore, the reverse Hölder class has a remarkable property that if $\mathbb{V} \in RH_q$ for some $q \in (1,\infty)$ then $\mathbb{V} \in RH_{q+\epsilon}$ for some $\epsilon > 0$ depending on n and the constant C in Definition 1.1. Since

$RH_s \subset RH_q$ for all $s > q$, the aforementioned property is often referred to as the *self-improving property* of the reverse Hölder class. For more information, one may consult [S95, Introduction], [SY10, Lemma 2.1], [T93], and the references therein.

REMARK 1.2. Condition (1.2) is adopted from [LS05], which is believed to be optimal for the unique solvability of (1.1) when $\mathbb{V} = 0$. See [LS05, p. 93] for a further discussion. Our main results imply that (1.2) remains suitable for the unique solvability of (1.1) in the presence of $\mathbb{V} \neq 0$.

Having said that, the focus is to investigate the existence and uniqueness of a solution to (1.1). To do so, we employ the layer potential method. This is a powerful tool for solving problems related to Laplace's equations with boundary conditions. The geometries of various domains are also taken into consideration. In this direction, a partial account is as follows:

- [FJR78] showed the existence of unique solutions to the Dirichlet and Neumann problems with boundary data in L^p-spaces on C^1 domains, where $1 < p < \infty$.
- For the Dirichlet problem on a Lipschitz domain, [D79] established the unique solvability with boundary datum in L^p-spaces and $2 \leq p \leq \infty$. Whereas, [V84] asserted that the same result remains valid for $1 < p \leq 2$ if the boundary datum and also its first derivatives belong to L^p-spaces.
- Optimal results for the unique solvability of the Neumann problem on a Lipschitz domain with boundary datum in L^p-spaces were then derived in [DK87] for $1 < p < 2 + \epsilon$, where $\epsilon > 0$ is a constant depending on the domain. The authors also discussed the end-point results for the Hardy spaces H^1. The extension to the Hardy spaces H^p with $1 - 1/n < p < 1$ can be found in [B95].
- The Robin problem on a Lipschitz domain was considered in [LS05]. Therein, the unique solvability was proved for $1 < p < 2 + \epsilon$, where $\epsilon > 0$ is a constant depending on the domain. If the domain is C^1, the exponent range can be extended to $1 < p < \infty$.

Recently it has been observed that the layer potential method can be applied to the Schrödinger equation of the form

$$-\Delta u + \mathbb{V} u = 0 \quad \text{in } \Omega,$$

where $\mathbb{V} \geq 0$ belongs to the reverse Hölder class. The following papers are of special interest to us:

- [S94] obtained a unique solution to the Neumann equation for $1 < p \leq 2$ and $\mathbb{V} \in RH_\infty$.
- [TW04] extended [S94] to $\mathbb{V} \in RH_n$. The Neumann problem in Hardy spaces H^p was also examined. Later on, the Dirichlet and Neumann prob-

lems in Lebesgue and Hardy spaces were studied in [T12]. The results are in the spirit of [V84, B95, S94].

- [Y18] established characterizations for the unique solvability of the Schrödinger equation with either the Dirichlet or the Neumann boundary condition:

$$\begin{cases} -\Delta u + \mathbb{V}u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f \in L^p(\partial\Omega;\omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p(\partial\Omega;\omega) \end{cases}$$

or

$$\begin{cases} -\Delta u + \mathbb{V}u = 0 & \text{in } \Omega, \\ u = f \in L^p_{1,\mathbb{V}}(\partial\Omega;\omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p(\partial\Omega;\omega), \end{cases}$$

where Ω is a bounded Lipschitz domain, $\mathbb{V} \in RH_n$, $p > 2$ and the weight ω belongs to the class $A_q(\partial\Omega)$ of Muckenhoupt weights for some $q \geq 1$. The space $L^p_{1,\mathbb{V}}(\partial\Omega;\omega)$ is a suitable subspace of $L^p_1(\partial\Omega;\omega)$ which takes into account the potential $\mathbb{V}$. The boundary values are understood in the non-tangential sense. As an application, the author proved the unique solvability of the Dirichlet problem for any $p \in (1,\infty)$ in bounded (semi-)convex domains.

Back to our consideration, (1.1) can be thought of as a Schrödinger version of the equation in [LS05]. Taking into account the latest development in [TW04] and [T12], we wish to establish the unique solvability of the Robin problem for $\mathbb{V} \in RH_n$. Due to the self-improving property of the reverse Hölder class, we may further assume that $\mathbb{V} \in RH_\sigma$ with $\sigma > n$ as stated above.

Hereafter, by a *solution* u to the equation

$$-\Delta u + \mathbb{V}u = 0 \quad \text{in } \Omega, \tag{1.3}$$

we mean u is a locally weak solution, in the sense that $u \in L^2(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ is such that

$$\int_\Omega (\nabla u \cdot \nabla\varphi + \mathbb{V}u\varphi)\, dX = 0$$

for all $\varphi \in C_c^\infty(\Omega)$. Consequently, a solution u to (1.1) is a locally weak solution which also satisfies

$$\frac{\partial u}{\partial \nu} + bu = g \in L^p(\partial\Omega) \quad \text{and} \quad (\nabla u)^* \in L^p(\partial\Omega).$$

A remark is immediate.

REMARK 1.3. Technically speaking, (1.3) requires $u \in L^2(\Omega, \mathbb{V}dX)$, in the sense that

$$\int_{\Omega} \mathbb{V}|u|^2 \, dX < \infty.$$

However, since $\mathbb{V} \in RH_\sigma$ with $\sigma > n$, one has $L^2(\Omega, \mathbb{V}dX) \hookrightarrow L^2(\Omega)$.

With the aforementioned fact in mind, our main result can be stated as shown in Theorem 6.1. The proof of this result is split into two cases: $p = 2$ and $1 < p < 2$. The essential background is collected in Section 2. Then we prove the case $p = 2$ in Section 3. The case $1 < p < 2$ is more involved and requires the idea of Robin functions. We pursue this idea in Section 4. Then we prove the case $1 < p < 2$ in Section 5. Section 6 concludes our investigation with the main theorem and further remarks.

Notation. Throughout the paper the following notation is used without mentioning. Set $\mathbb{N} = \{0, 1, \ldots\}$. The constants C and c are positive, independent of the main parameters, and their values may change from line to line. For any two functions f and g, we write $f \lesssim g$ and $f \sim g$ to mean $f \le Cg$ and $cg \le f \le Cg$ respectively.

Throughout assumptions. The underlying structure is always imposed in the whole paper. The parameters and indices therein are also fixed.

2. Background. In this section, we define auxiliary functions and list some of their basic properties. We also discuss the fundamental solution of the Schrödinger operator associated with our main equation (1.1).

2.1. Auxiliary functions. For all $X \in \mathbb{R}^n$, define

$$m(X, \mathbb{V}) := \inf \left\{ \frac{1}{r} > 0 : \frac{1}{r^{n-2}} \int_{B(X,r)} \mathbb{V}(Y) \, dY \le 1 \right\}, \tag{2.1}$$

which is called the *auxiliary function*. This function originated from the work [S95] and has since played a significant role in the study of Schrödinger operators with reverse Hölder potentials.

Note that $\mathbb{V} \in RH_\sigma$ guarantees $0 < m(X, \mathbb{V}) < \infty$ for every $X \in \mathbb{R}^n$. Moreover,

$$\frac{1}{r^{n-2}} \int_{B(X,r)} \mathbb{V}(Y) \, dY \sim 1 \iff r \sim \frac{1}{m(X, \mathbb{V})}$$

for every $X \in \mathbb{R}^n$. In fact, more is true.

PROPOSITION 2.1 ([S95, Lemma 1.4]). *There exist constants $C > 0$ and $k_0 > 0$ such that for all $X, Y \in \mathbb{R}^n$, one has the following properties:*

(i) $m(X, \mathbb{V}) \sim m(Y, \mathbb{V})$ *if* $|X - Y| \le \dfrac{C}{m(X, \mathbb{V})}$.

(ii) $m(Y,\mathbb{V}) \le C(1+|X-Y|\,m(X,\mathbb{V}))^{k_0} m(X,\mathbb{V})$.

(iii) $m(Y,\mathbb{V}) \ge \dfrac{Cm(X,\mathbb{V})}{(1+|X-Y|\,m(X,\mathbb{V}))^{k_0/(k_0+1)}}$.

(iv) $$C^{-1}(1+|X-Y|\,m(X,\mathbb{V}))^{1/(k_0+1)} \le 1+|X-Y|\,m(X,\mathbb{V}) \le C(1+|X-Y|\,m(Y,\mathbb{V}))^{k_0+1}.$$

(v) *If $Rm(X,\mathbb{V}) \ge 1$, then*

$$\frac{1}{R^{n-2}} \int_{B(X,R)} \mathbb{V}\,dy \le C(Rm(X,\mathbb{V}))^{k_0}.$$

LEMMA 2.2. *Let $u \in C_c^1(\mathbb{R}^n)$. Then*

$$\int_\Omega |u(X)|^2 m(X,\mathbb{V})^2\,dX \le C\Big\{\int_\Omega |\nabla u(X)|^2\,dX + \int_\Omega |u(X)|^2\mathbb{V}(X)\,dX\Big\}.$$

Proof. This is essentially [S94, Lemma 1.11], in which $V \in RH_\infty$ is assumed. Here we note that the new assumption $V \in RH_\sigma$ causes no harm to the arguments which are used to obtain [S94, Lemma 1.11]. See also [S95, Lemma 1.9]. ∎

2.2. A uniform estimate. We recall the space $L_1^q(\partial\Omega)$ with $1 < q < \infty$ defined in [V84, Definition 1.7]. Loosely speaking, $L_1^q(\partial\Omega)$ consists of functions in $L^q(\partial\Omega)$ whose first derivatives are also in $L^q(\partial\Omega)$. In the sequel, $d\sigma$ denotes the surface measure on $\partial\Omega$.

The next estimate is known as the Friedrichs inequality.

LEMMA 2.3 ([LS05, Theorem 6.1]). *Let Ω be a bounded, connected Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$. Let $E \subset \partial\Omega$ satisfy $\sigma(E) > 0$. Then there exists a constant $C = C(n,\Omega,E) > 0$ such that*

$$\int_\Omega |u|^2\,dX \le C\Big(\int_\Omega |\nabla u|^2\,dX + \int_E |u|^2\,d\sigma\Big)$$

for all $u \in L_1^2(\Omega)$.

In what follows, we write

$$D(X,R) := B(X,R) \cap \Omega$$

for each ball $B(X,R) \subset \mathbb{R}^n$. Since $\partial\Omega$ is connected, we may also let $R_0 > 0$ be sufficiently small so that $B(Q,R_0)\cap\Omega$ is a region above a Lipschitz graph after a possible rotation of the coordinate system for all $Q \in \partial\Omega$, according to [PV95, Section 5]. One has the following uniform estimate.

LEMMA 2.4. *Let $X_0 \in \overline{\Omega}$ and $0 < R < R_0$. Let u be a solution to*

$$\begin{cases} -\Delta u + \mathbb{V}u = 0 & \text{in } D(X_0, 2R), \\ \dfrac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D(X_0, 2R) \cap \partial\Omega. \end{cases}$$

Then for each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$\sup_{X \in D(X_0,R)} |u(X)| \le \frac{C_k}{(1 + Rm(X_0, \mathbb{V}))^k} \left(\frac{1}{R^n} \int_{D(X_0,2R)} |u|^2 \, dY \right)^{1/2}. \tag{2.2}$$

Proof. We divide the proof into two steps.

STEP 1. We show that

$$\sup_{X \in D(X_0,R/2)} |u(X)| \le C \Big(\fint_{D(X_0,R)} |u|^2 \, dX \Big)^{1/2}.$$

To this end, fix $0 < R < R_0$. There are two cases, either $D(X_0, 2R) \neq \emptyset$ or $D(X_0, 2R) = \emptyset$. The treatments for these two cases are essentially the same, except for the extra appearance of the boundary term in the former. So we focus on the former only. In this case, we may assume without loss of generality that $X_0 \in \partial\Omega$. For each $M > 0$ and $X \in D(X_0, 2R)$, define

$$v_M(X) := \begin{cases} u(X) & \text{if } -M < u(X) < M, \\ M \operatorname{sign}(u) & \text{otherwise,} \end{cases}$$

whence

$$\nabla v_M(X) = \begin{cases} \nabla u(X) & \text{if } -M < u(X) < M, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varphi \in C_c^\infty(B(X_0, R))$ and $D = D(X_0, R)$. Then

$$\begin{aligned} \int_D \nabla u \cdot \nabla(v_M \varphi^2) \, dX &= \int_{\partial D} \frac{\partial u}{\partial \nu} v_M \varphi^2 d\sigma - \int_D v_M \varphi^2 \Delta u \, dX \\ &= - \int_{\partial\Omega \cap B(X_0,R)} b u v_M \varphi^2 d\sigma - \int_D v_M \varphi^2 \mathbb{V} u \, dX \le 0 \end{aligned}$$

since $b|_{\partial\Omega} \ge 0$ and $\mathbb{V} \ge 0$ by hypothesis. It follows that

$$\int_D |\nabla v_M|^2 \varphi^2 \, dX + 2 \int_D (\nabla v_M \cdot \nabla\varphi) \varphi v_M \, dX \le 0.$$

This implies

$$\int_D |\nabla(v_M \varphi)|^2 \, dX \le \int_D (v_M |\nabla\varphi|)^2 \, dX. \tag{2.3}$$

In turn, we obtain

$$\Big(\int_D |v_M\varphi|^{\frac{2n}{n-2}}\,dX\Big)^{\frac{n-2}{n}} \le C\Big(\int_D |v_M\varphi|^2\,dX + \int_D |\nabla(v_M\varphi)|^2\,dX\Big) \tag{2.4}$$

by the Sobolev embedding theorem $W^{1,2}(D) \hookrightarrow L^{\frac{2n}{n-2}}(D)$.

Using the Friedrichs inequality in Lemma 2.3, we arrive at

$$\Big(\int_D |v_M\varphi|^{\frac{2n}{n-2}}\,dX\Big)^{\frac{n-2}{n}} \le C\int_D (v_M|\nabla\varphi|)^2\,dX. \tag{2.5}$$

Since φ is arbitrary, in (2.5) we may choose $\varphi \in C_0^\infty(B(X_0,s))$ such that

$$0 \le \varphi \le 1, \quad \varphi|_{B(X_0,t)} = 1, \quad \text{and} \quad |\nabla\varphi| \le \frac{C}{s-t},$$

where $0 < t < s < R$. This gives

$$\Big(\int_{D(X_0,t)} |v_M|^{\frac{2n}{n-2}}\,dX\Big)^{\frac{n-2}{2n}} \le \frac{C}{s-t}\Big(\int_{D(X_0,s)} v_M^2\,dX\Big)^{1/2}.$$

By setting

$$\gamma := \frac{n}{n-2} > 1 \quad \text{and} \quad \varPhi(q,t) := \Big(\int_{D(X_0,t)} |v_M|^q\,dX\Big)^{1/q} \tag{2.6}$$

for each $q > 1$, the last estimate takes the form

$$\varPhi(2\gamma,t) \le \frac{C}{s-t}\varPhi(2,s). \tag{2.7}$$

By iterating (2.7) $m \in \mathbb{N}$ times with

$$t = \frac{R}{2} + \frac{R}{2^{m+1}} \quad \text{and} \quad s = \frac{R}{2} + \frac{R}{2^m},$$

we conclude that

$$\varPhi\bigg(2\gamma^{m+1}, \frac{R}{2} + \frac{R}{2^{m+1}}\bigg) \le \bigg(\frac{C}{R}\bigg)^{\sum_{j=0}^m 1/\gamma^3} (2\gamma)^{\sum_{j=0}^m j/\gamma^3}\varPhi(2,R). \tag{2.8}$$

Note that

$$\sum_{j\in\mathbb{N}} \frac{1}{\gamma^3} = \frac{\gamma}{\gamma-1} = \frac{n}{2}.$$

Hence we may let $m \to \infty$ in (2.8) to conclude that

$$\sup_{D(X_0,R/2)} |v_M| \le C\Big(\fint_{D(X_0,R)} v_M^2\,dX\Big)^{1/2}. \tag{2.9}$$

Lastly, we let $M \to \infty$ in (2.9) to obtain

$$\sup_{D(X_0,R/2)} |u| \le C\Big(\fint_{D(X_0,R)} |u|^2\,dX\Big)^{1/2}$$

as required.

STEP 2. We show that (2.2) holds.

To this end, we may assume that $R > \frac{2}{m(X_0,\mathbb{V})}$. By Caccioppoli's inequality,

$$\int_{D(X_0,R)} |\nabla u|^2 \, dX + \int_{D(X_0,R)} |u|^2 \mathbb{V} \, dX \le \frac{C}{R^2} \int_{D(X_0,3R/2)} |u|^2 \, dX.$$

Let $\eta \in C_c^\infty(B(X_0,R))$ be such that

$$\eta|_{B(X_0,R/2)} = 1 \quad \text{and} \quad |\nabla \eta| \le \frac{C}{R}.$$

Applying Lemma 2.2 to the function $u\eta$, together with Caccioppoli's inequality above, we obtain

$$\begin{aligned}
&\int_{D(X_0,R/2)} m(X,\mathbb{V})^2 |u|^2 \, dX \\
&\qquad \le C \int_{D(X_0,R)} |\nabla u|^2 \, dX + C \int_{D(X_0,R)} |u|^2 \mathbb{V} \, dX + \frac{C}{R^2} \int_{D(X_0,3R/2)} |u|^2 \, dX \\
&\qquad \le \frac{C}{R^2} \int_{D(X_0,3R/2)} |u|^2.
\end{aligned}$$

Since

$$m(X,\mathbb{V}) \ge \frac{C m(X_0,\mathbb{V})^{\frac{1}{k_0+1}}}{(R/2)^{\frac{k_0}{k_0+1}}}$$

for all $X \in D(x_0, R/2)$ by Proposition 2.1(iii), it follows that

$$\int_{D(X_0,R/2)} |u|^2 \, dX \le \frac{C}{(Rm(X_0,\mathbb{V}))^{\frac{2}{k_0+1}}} \int_{D(X_0,3R/2)} |u|^2 \, dX.$$

Iterating this last estimate shows that

$$\int_{D(X_0,R)} |u|^2 \, dX \le \frac{C_k}{(Rm(X_0,\mathbb{V}))^k} \int_{D(X_0,3R/2)} |u|^2 \, dX$$

for all integer $k > 0$. In view of Step 1, we further derive

$$\sup_{D(X_0,R)} |u| \le \frac{C_k}{(1 + Rm(X_0,\mathbb{V}))^k} \left(\frac{1}{R^n} \int_{D(X_0,2R)} |u|^2 \, dX \right)^{1/2}.$$

The proof is complete. ∎

2.3. Fundamental solutions. Let $\Gamma(X,Y)$ denote the fundamental solution for the Schrödinger operator $-\Delta + \mathbb{V}$ in $\mathbb{R}^n$. Clearly,

$$\Gamma(X,Y) = \Gamma(Y,X)$$

for each $X, Y \in \mathbb{R}^n$. Since $\mathbb{V} \geq 0$, it is well-known that

$$0 \leq \Gamma(X,Y) \leq \Gamma^0(X,Y),$$

where $\Gamma^0(X,Y)$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^n$.

The study of fundamental solutions is of independent interest. Interested readers are directed to [GW82, CW88, S99, BDT21, CK10, D22] and references therein for more information. Some essential properties of the fundamental solution are listed next.

Proposition 2.5 ([S94, Theorem 1.14]). *We have*

(i) $|\Gamma(X,Y)| \leq \dfrac{C_k}{\{1+|X-Y|\,m(X,\mathbb{V})\}^k} \cdot \dfrac{1}{|X-Y|^{n-2}}$,

(ii) $|\nabla_X \Gamma(X,Y)| \leq \dfrac{C_k}{(1+|X-Y|\,m(X,\mathbb{V}))^k} \cdot \dfrac{1}{|X-Y|^{n-1}}$

for all $X, Y \in \mathbb{R}^n$, where $k \geq 1$ is an arbitrary integer.

Lemma 2.6 ([TW04, Lemma 2.5]). *Assume $\mathbb{V} \in RH_n$. Then there exists a constant $C > 0$ such that*

$$|\nabla_X \Gamma(X,Y) - \nabla_X \Gamma_0(X,Y)| \leq \frac{Cm(X,\mathbb{V})}{|X-Y|^{n-2}}$$

for all $X, Y \in \mathbb{R}^n$ satisfying $|X-Y| \leq 2/m(X,\mathbb{V})$.

2.4. Layer potentials. Let $1 < q < \infty$. For each $f \in L^q(\partial\Omega)$ and $X \in \mathbb{R}^n \setminus \partial\Omega$, define

$$\mathcal{S}(f)(X) := \int_{\partial\Omega} \Gamma(X,Q) f(Q)\, dQ.$$

which is called the *single layer potential.* It is known that $\mathcal{S}(f)$ solves the Schrödinger equation

$$-\Delta u + \mathbb{V} u = 0 \quad \text{in } \mathbb{R}^n \setminus \partial\Omega.$$

Our main task herein is to incorporate the Robin boundary condition in the layer potential to achieve a solution to (1.1).

Lemma 2.7 ([TW04, Lemma 2.6]). *Let $1 < q < \infty$ and $f \in L^q(\partial\Omega)$. Let $u = \mathcal{S}(f)$. Then*

$$\|(\nabla u)^*\|_{L^q(\partial\Omega)} \leq C\|f\|_{L^q(\partial\Omega)}$$

and

$$\frac{\partial u}{\partial X_i}(P) = \tfrac{1}{2}\nu^i(P) f(P) + \text{p.v.} \int_{\partial\Omega} \nabla_P \Gamma(P,Q) f(Q)\, dQ$$

for a.e. $P \in \partial\Omega$.

In view of Lemma 2.7, the Robin boundary term can be written as

$$\frac{\partial}{\partial \nu}\mathcal{S}(f) + b\mathcal{S}(f) = \big(\tfrac{1}{2}I + K\big)(f) + b\mathcal{S}(f)$$

where

$$K(f)(P) = \text{p.v.} \int_{\partial\Omega} \big(\nabla_P \Gamma(P,Q) \cdot \nu(P)\big) f(Q)\, d\sigma(Q)$$

for a.e. $P \in \partial\Omega$. The solvability of (1.1) is then converted to the invertibility of the operator

$$\mathcal{T} := \tfrac{1}{2} I + K + b\mathcal{S} \tag{2.10}$$

in appropriate Lebesgue spaces. We make use of the Fredhohm alternative theorem to examine this.

We indicate the special case when $\mathbb{V} = 0$ by writing K_0 in place of K. As such, $\frac{1}{2}I + K_0$ is the boundary operator associated with the Laplacian $-\Delta$.

Proposition 2.8 ([V84] or [DK87]). *There exists $\epsilon = \epsilon(\Omega) > 0$ such that the operator*

$$\tfrac{1}{2} I + K_0 : L^q(\partial\Omega) \to L^q(\partial\Omega)$$

is a Fredholm operator with index zero for all $1 < q < 2 + \epsilon$.

Proposition 2.9. *There exists $\epsilon = \epsilon(\Omega) > 0$ such that the operator*

$$\tfrac{1}{2} I + K : L^q(\partial\Omega) \to L^q(\partial\Omega)$$

is a Fredholm operator with index zero for all $1 < q < 2 + \epsilon$.

Proof. Define

$$K_1 := K - K_0.$$

Lemma 2.6 implies that the operator K_1 is compact. Then we can deduce that $\frac{1}{2}I + K$ is a Fredholm operator with index zero. ∎

Next we show that the operator $b\mathcal{S}$ is compact. The proof makes use of the boundedness of $\mathcal{S}$ which is stated next.

Proposition 2.10. *The operator*

$$\mathcal{S} : L^q(\partial\Omega) \to \text{Range}(\mathcal{S}) \subset L^q_1(\partial\Omega)$$

is bounded for all $1 < q < \infty$. Furthermore, $\mathcal{S}$ is invertible for all $1 < q \le 2$.

Proof. The boundedness is a consequence of Proposition 2.5 and Lemma 2.7. The invertibility follows from [T12, Corollary 5.6]. ∎

Remark 2.11. The range of $\mathcal{S}$ in Proposition 2.10 can be described explicitly. The idea is to take the potential $\mathbb{V}$ into account and define a new function space $L^q_{1,\mathbb{V}}(\partial\Omega) \subset L^q_1(\partial\Omega)$ which is a modification of $L^q_1(\partial\Omega)$. Then one can show that

$$\text{Range}(\mathcal{S}) = L^q_{1,\mathbb{V}}(\Omega).$$

We do not pursue the details here, since Proposition 2.10 is sufficient for our purpose. More details are available in [T12, Definition 1.2, Lemma 4.1 and Corollary 5.6].

Now we are ready to state the compactness of $b\mathcal{S}$.

PROPOSITION 2.12. *Suppose $b \in L^s(\partial\Omega)$ for some $s \in [n-1,\infty]$. Then the operator $b\mathcal{S} : L^q(\partial\Omega) \to L^q(\partial\Omega)$ is compact for all $q \in [1,s)$. If $s > n-1$ then the compactness also holds for $q = s$.*

Proof. [LS05, proof of Lemma 2.3] applies verbatim to our setting. Therein, the essential ingredient is [LS05, Theorem 2.1] which is now replaced by Proposition 2.10. ∎

Combining Propositions 2.9 and 2.12, we may conclude the following.

COROLLARY 2.13. *The operator $\mathcal{T} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator with index zero.*

Recall that if a Fredholm operator with index zero is one-to-one, then it is invertible. Concerning the operator $\mathcal{T}$, Corollary 2.13 yields the following result.

LEMMA 2.14. *Suppose that a solution to* (1.1) *is unique if it exists. Then $\mathcal{T} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is one-to-one. In particular, $\mathcal{T} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is invertible.*

Proof. [LS05, proof of Lemma 2.5] applies verbatim to our setting. ∎

3. The case $p = 2$. In this section, we will prove the unique solvability of (1.1) for $p = 2$.

PROPOSITION 3.1. *Let $p = 2$. Then $u = \mathcal{S}(\mathcal{T}^{-1}(g))$ uniquely solves* (1.1). *Moreover,*

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

Proof. The "Moreover" part follows from Lemma 2.7 once we show that $u = \mathcal{S}(\mathcal{T}^{-1}(g))$ is the unique solution to (1.1) for $p = 2$. The rest of the proof is divided into two parts: existence and uniqueness.

Uniqueness. Let u_1 and u_2 be two solutions to (1.1) for $p = 2$. Then $v = u_1 - u_2$ is a solution to (1.1) with $p = 2$ and $g = 0$. Note that $(\nabla v)^* \in L^2(\partial\Omega)$ implies that $(v)^* \in L^2(\partial\Omega)$. By approximating Ω by a sequence of smooth domains from inside as was done in [V84, Theorem 1.12], we may use integration by parts to obtain

$$\begin{aligned}\int_\Omega |\nabla v|^2\,dX &= -\int_\Omega v\Delta u\,dX + \int_{\partial\Omega} v\frac{\partial v}{\partial\nu}\,d\sigma \\ &= -\int_\Omega \mathbb{V}|v|^2\,dX - \int_{\partial\Omega} b|v|^2\,d\sigma \le 0\end{aligned}$$

as $b \ge 0$ and $\mathbb{V} \ge 0$. Hence v is constant in Ω. The Robin boundary condition now reduces to $bv = 0$ on $\partial\Omega$. But $b \ne 0$ on $\partial\Omega$ by hypothesis. This means $v = 0$ in Ω.

Existence. With the above uniqueness in mind, we refer to Lemma 2.14 to obtain a solution to (1.1) for $p = 2$ which is of the form $\mathcal{S}(\mathcal{T}^{-1}(g))$. ■

4. Robin functions. We will construct Robin functions and their estimates. Following [V84, Theorem 1.12] and [LS05, p. 100], we define the sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of smooth domains which approximates Ω uniformly and non-tangentially as follows:

(a) There are homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$ such that $\Lambda_j(Q) \in \gamma(Q)$ for all $Q \in \partial\Omega$ and

$$\lim_{j\to\infty}\Big(\sup_{Q\in\partial\Omega} |Q - \Lambda_j(Q)|\Big) = 0.$$

(b) There are positive functions h_j on $\partial\Omega$ such that h_j and h_j^{-1} are uniformly bounded, and

$$\int_E h_j\, d\sigma = \int_{\Lambda_j(E)} d\sigma_j$$

for any Borel set $E \subset \partial\Omega$. Moreover,

$$\lim_{j\to\infty} h_j = 1 \quad \text{pointwise in } \partial\Omega.$$

(c) Let ν_j be the outer unit normal to $\partial\Omega_j$. Then ν_j satisfies

$$\lim_{j\to\infty} \nu_j(\Lambda_j(Q)) = \nu(Q) \quad \text{pointwise in } \partial\Omega.$$

For each $j \in \mathbb{N}$, define

$$b_j(Q) = (\eta_j b)(\Lambda_j^{-1}(Q)),$$

where η_j is the characteristic function of the set $\{P \in \partial\Omega : b(P) \le j\}$. Then $b_j \in L^\infty(\partial\Omega)$ and by virtue of Proposition 2.5 we also have

$$\frac{\partial\Gamma^Y}{\partial\nu_j} + b_j\Gamma^Y \in L^2(\partial\Omega_j)$$

for all $Y \in \Omega_j$, where $\Gamma^Y(\cdot) := \Gamma(\cdot, Y)$.

In turn, we infer from Proposition 3.1 that for each $Y \in \Omega_j$ there exists a unique function $W_j^Y \in L^2(\Omega_j) \cap W^{1,2}_{\mathrm{loc}}(\Omega_j)$ satisfying,

$$\begin{cases} -\Delta W_j^Y + \mathbb{V} W_j^Y = 0 \quad \text{in } \Omega_j, \\ \dfrac{\partial W_j^Y}{\partial\nu_j} + b_j W_j^Y = \dfrac{\partial\Gamma^Y}{\partial\nu_j} + b_j\Gamma^Y \quad \text{on } \partial\Omega_j, \\ (\nabla W_j^Y)^* \in L^2(\partial\Omega_j). \end{cases}$$

The *Robin function* is defined by

$$R_j(X,Y) := R_j^Y(X) := \Gamma^Y(X) - W_j^Y(X)$$

for all $X, Y \in \Omega_j$ such that $X \ne Y$.

Next we prove some essential properties of the Robin function to be used in the sequel.

PROPOSITION 4.1. *Let $X, Y \in \Omega_j$ and $X \neq Y$. Then $R_j(X,Y) = R_j(Y,X)$.*

Proof. Since $\Gamma(X,Y) = \Gamma(Y,X)$, it suffices to show that

$$W_j^Y(X) = W_j^X(Y).$$

Using Green's second identity, we obtain

$$\begin{aligned}
&\int_{\partial\Omega_j} W_j^Y(Q) \frac{\partial}{\partial\nu_j} \Gamma^X(Q)\, d\sigma(Q) \\
&\qquad = \int_{\partial\Omega_j} \Gamma^X(Q) \frac{\partial}{\partial\nu_j} W_j^Y(Q)\, d\sigma(Q) \\
&\qquad\quad + \int_{\Omega_j} [W_j^Y(Q) \Delta\Gamma^X(Q) - \Gamma^X(Q) \Delta W_j^Y(Q)]\, dQ \\
&\qquad = \int_{\partial\Omega_j} \Gamma^X(Q) \frac{\partial}{\partial\nu_j} W_j^Y(Q)\, d\sigma(Q) - W_j^Y(X).
\end{aligned}$$

Hence,

$$-W_j^Y(X) = \int_{\partial\Omega_j} \left[W_j^Y(Q) \frac{\partial}{\partial\nu_j} \Gamma^X(Q) - \Gamma^X(Q) \frac{\partial}{\partial\nu_j} W_j^Y(Q) \right] d\sigma(Q).$$

Similarly,

$$-W_j^X(Y) = \int_{\partial\Omega_j} \left[W_j^X(Q) \frac{\partial}{\partial\nu_j} \Gamma^Y(Q) - \Gamma^Y(Q) \frac{\partial}{\partial\nu_j} W_j^X(Q) \right] d\sigma(Q).$$

Therefore,

$$\begin{aligned}
&W_j^X(Y) - W_j^Y(X) \\
&\qquad = \int_{\partial\Omega_j} \left[W_j^X(Q) \frac{\partial}{\partial\nu_j} W_j^Y(Q) - W_j^Y(Q) \frac{\partial}{\partial\nu_j} W_j^X(Q) \right] d\sigma(Q) \\
&\qquad\quad + \int_{\partial\Omega_j} \left[\Gamma^Y(Q) \frac{\partial}{\partial\nu_j} \Gamma^X(Q) - \Gamma^X(Q) \frac{\partial}{\partial\nu_j} \Gamma^Y(Q) \right] d\sigma(Q) \\
&\qquad = 0,
\end{aligned}$$

where the second step follows from the divergence theorem. ■

LEMMA 4.2. *Let $j \in \mathbb{N}$ and $f \in C_c^\infty(\Omega_j)$. Set*

$$F_j(X) = \int_{\Omega_j} R_j(X,Y) f(Y)\, dY. \tag{4.1}$$

Then $F_j \in W^{1,2}(\Omega_j)$ is a weak solution to

$$\begin{cases} -\Delta F_j + \mathbb{V} F_j = f & \text{in } \Omega_j, \\ \dfrac{\partial F_j}{\partial \nu_j} + b_j F_j = 0 & \text{on } \partial\Omega_j. \end{cases} \tag{4.2}$$

Moreover, there exists a constant $C = C(n, \Omega, b) > 0$ such that

$$\Big(\int_{\Omega_j} |F_j|^{\frac{2n}{n-2}} \, dX \Big)^{\frac{n-2}{2n}} \le C \Big(\int_{\Omega_j} |f|^{\frac{2n}{n+2}} \, dX \Big)^{\frac{n+2}{2n}}.$$

Proof. In view of Proposition 3.1, $\{\|(\nabla W_j^Y)^*\|_{L^2(\Omega_j)}\}_{Y \in \operatorname{supp} f}$ is uniformly bounded. As a consequence, $F_j \in W^{1,2}(\Omega_j)$. From the definition of R_j and F_j, we infer that F_j is a weak solution to (4.2).

Next, we use F_j as a test function in (4.2) to obtain

$$\int_{\Omega_j} |\nabla F_j|^2 \, dX + \int_{\Omega_j} \mathbb{V} |F_j|^2 + \int_{\partial\Omega_j} b_j |F_j|^2 \, d\sigma = \int_{\Omega_j} f F_j \, dX. \tag{4.3}$$

Regarding the right-hand side, Hölder's inequality implies

$$\int_{\Omega_j} f F_j \, dX \le \|f\|_{L^{\frac{2n}{n+2}}(\Omega_j)} \|F_j\|_{L^{\frac{2n}{n-2}}(\Omega_j)}.$$

Regarding the left-hand side, we first choose a constant $\delta = \delta(b) > 0$ such that

$$E_\delta := \{Q \in \partial\Omega : \delta \le b(Q) \le 1/\delta\}$$

and set $E_{j,\delta} := \Lambda_j(E_\delta)$. Then Lemma 2.3 and the fact that $\mathbb{V} \ge 0$ give

$$\begin{aligned} \int_{\Omega_j} |F_j|^2 \, dX &\le C(n, \Omega, b) \Big(\int_{\Omega_j} |\nabla F_j|^2 \, dX + \int_{E_{j,\delta}} b_j |F_j|^2 \, dX \Big) \\ &\le C(n, \Omega, b) \Big(\int_{\Omega_j} |\nabla F_j|^2 \, dX + \int_{\Omega_j} \mathbb{V} \, |F_j|^2 \, dX + \int_{\Omega_j} b_j |F_j|^2 \, dX \Big) \end{aligned}$$

for all $j \ge 1/\delta$. This in combination with the Sobolev embedding theorem shows that

$$\begin{aligned} &\|F_j\|^2_{L^{\frac{2n}{n-2}}(\Omega_j)} \\ &\le C(n, \Omega, b) \Big(\int_{\Omega_j} |\nabla F_j|^2 \, dX + \int_{\Omega_j} \mathbb{V} \, |F_j|^2 \, dX + \int_{\Omega_j} b_j |F_j|^2 \, dX \Big). \end{aligned}$$

Now we put the estimates together to arrive at the claim. ■

In what follows, we write

$$D_j(X, r) := B(X, r) \cap \Omega_j$$

for each $X \in \Omega_j$, $r > 0$ and $j \in \mathbb{N}$.

LEMMA 4.3. *For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that*

$$|R_j(X,Y)| \le \frac{C_k}{(1+|X-Y|\,m(X,\mathbb{V}))^k} \cdot \frac{1}{|X-Y|^{n-2}}$$

for all $X, Y \in \Omega_j$ and $j \in \mathbb{N}$.

Proof. Let $j, k \in \mathbb{N}$. Let $X_0, Y_0 \in \Omega_j$ be such that $|X_0 - Y_0| > 0$. We choose $\kappa \in \mathbb{N}$ be such that $\operatorname{diam}(\Omega) < 2^\kappa R_0$, where R_0 is as in Lemma 2.4. Define

$$\rho_0 := \min\{1/2^\kappa, 1/8\}|X_0 - Y_0|.$$

Let $f \in C_c^\infty(D(Y_0, 2\rho_0))$. Let F_j be defined by (4.1). Then Lemma 4.2 asserts that

$$\Big(\int_{\Omega_j} |F_j|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}} \le C(n,\Omega,b)\Big(\int_{D(Y_0,2\rho_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{2n}}. \tag{4.4}$$

We consider two cases.

CASE 1. Suppose

$$|X_0 - Y_0| \le \frac{1}{m(X_0,\mathbb{V})}.$$

Observe that $-\Delta F_j + \mathbb{V}F_j = 0$ in $D(X_0, 2\rho_0)$. Therefore,

$$\begin{aligned}
|F_j(X_0)| &\le \sup_{X \in B(X_0,\rho_0)} F_j(X) \\
&\le C\Big(\frac{1}{\rho_0^n}\int_{D_j(X_0,2\rho_0)} |F_j|^2\,dY\Big)^{1/2} \\
&\le \frac{C}{\rho_0^{\frac{n}{2}}}\|D_j(X_0,2\rho_0)\|^{1/n}\Big(\int_{D_j(X_0,2\rho_0)} |F_j|^{\frac{2n}{n-2}}\,dY\Big)^{\frac{n-2}{2n}} \\
&\le \frac{C}{\rho_0^{\frac{n-2}{2}}}\Big(\int_{\Omega_j} |F_j|^{\frac{2n}{n-2}}\,dY\Big)^{\frac{n-2}{2n}} \\
&\le \frac{C}{\rho_0^{\frac{n-2}{2}}}\Big(\int_{D(Y_0,2\rho_0)} |f|^{\frac{2n}{n+2}}\,dY\Big)^{\frac{n+2}{2n}},
\end{aligned}$$

where we have used Lemma 2.4 in the second step and (4.4) in the last step. By duality, this implies that

$$\Big(\int_{D(Y_0,2\rho_0)} |R_j(X_0,Y)|^{\frac{2n}{n-2}}\,dY\Big)^{\frac{n-2}{2n}} \le C/\rho_0^{\frac{n-2}{2}},$$

whence

$$|R_j(X_0,Y_0)| \le C\biggl(\frac{1}{\rho_0^n}\int\limits_{D(Y_0,2\rho_0)} |R_j(X_0,Y)|^{\frac{2n}{n-2}}\,dY\biggr)^{\frac{n-2}{2n}} \le C/\rho_0^{n-2}$$

by Lemma 2.4.

Case 2. Suppose

$$|X_0-Y_0| \sim \frac{2^\ell}{m(X_0,\mathbb{V})} \quad \text{for some } \ell\in\mathbb{N}.$$

According to Lemma 2.2 and (4.3),

$$\begin{aligned}
&\int\limits_{\Omega_j} |F_j|^2\, m(X,\mathbb{V})^2\,dX\\
&\qquad\le C\Bigl(\int\limits_{\Omega_j}|\nabla F_j|^2\,dX+\int\limits_{\Omega_j}\mathbb{V}|F_j|^2\,dX\Bigr)\\
&\qquad\le C\Bigl(\int\limits_{\Omega_j}|\nabla F_j|^2\,dX+\int\limits_{\Omega_j}\mathbb{V}|F_j|^2\,dX+\int\limits_{\partial\Omega_j} b_j|F_j|^2\,dX\Bigr)^{1/2}\\
&\qquad= C\int\limits_{\Omega_j} fF_j\,dX = C\int\limits_{D_j(Y_0,2\rho_0)} fF_j\,dX.
\end{aligned}$$

This implies

$$\begin{aligned}
\Bigl(\int\limits_{D_j(X_0,2\rho_0)}|F_j|^2 m(X,\mathbb{V})^2\,dX\Bigr)^{1/2}&\Bigl(\int\limits_{D_j(Y_0,2\rho_0)}|F_j|^2 m(X,\mathbb{V})^2\,dX\Bigr)^{1/2}\\
&\le C\int\limits_{D_j(Y_0,2\rho_0)} fF_j\,dX.
\end{aligned}$$

But

$$\frac{Cm(X_0,\mathbb{V})}{(2^\ell)^{\frac{k_0}{k_0+1}}} \le m(X,\mathbb{V}) \quad \text{for all } X\in D_j(X_0,2\rho_0)\cup D_j(Y_0,2\rho_0)$$

by Lemma 2.1(iii). It follows that

$$\begin{aligned}
\frac{m(X_0,\mathbb{V})^2}{(2^\ell)^{\frac{2k_0}{k_0+1}}}&\Bigl(\int\limits_{D_j(X_0,2\rho_0)}|F_j|^2\,dX\Bigr)^{1/2}\Bigl(\int\limits_{D_j(Y_0,2\rho_0)}|F_j|^2\,dX\Bigr)^{1/2}\\
&\le C\Bigl(\int\limits_{D_j(Y_0,2\rho_0)}|F_j|^2\,dX\Bigr)^{1/2}\Bigl(\int\limits_{D_j(Y_0,2\rho_0)}|f|^2\,dX\Bigr)^{1/2},
\end{aligned}$$

or equivalently

$$\frac{m(X_0,\mathbb{V})^2}{(2^\ell)^{\frac{2k_0}{k_0+1}}}\Big(\int_{D_j(X_0,2\rho_0)}|F_j|^2\,dX\Big)^{1/2}\le C\Big(\int_{D_j(Y_0,2\rho_0)}|f|^2\,dX\Big)^{1/2}.$$

Again, using the fact that

$$-\Delta F_j+\mathbb{V}F_j=0\quad\text{ in } D(X_0,2\rho_0)$$

and then applying Lemma 2.4, we obtain

$$\begin{aligned}|F_j(X_0)|&\le\frac{C}{\rho_0^{n/2}}\Big(\int_{D_j(X_0,2\rho_0)}|F_j|^2\,dX\Big)^{1/2}\\&\le\frac{C(2^\ell)^{\frac{2k_0}{k_0+1}}}{\rho_0^{n/2}m(X_0,\mathbb{V})^2}\Big(\int_{D_j(Y_0,2\rho_0)}|f|^2\,dX\Big)^{1/2}.\end{aligned}$$

By duality,

$$\Big(\int_{D(Y_0,2\rho_0)}|R_j(X_0,Y)|^2\,dY\Big)^{1/2}\le\frac{C(2^\ell)^{\frac{2k_0}{k_0+1}}}{\rho_0^{n/2}m(X_0,\mathbb{V})^2}.$$

Applying Lemma 2.4 once more to $R_j^{Y_0}$ shows that

$$\begin{aligned}|R_j(X_0,Y_0)|&\le\frac{C_k}{(1+2^\ell)^k}\bigg(\frac{1}{\rho_0^n}\int_{D(Y_0,2\rho_0)}|R_j(X_0,Y)|^2dY\bigg)^{1/2}\\&\le\frac{C_k}{(1+2^\ell)^{k-\frac{2k_0}{k_0+1}}}\cdot\frac{1}{\rho_0^n m(X_0,\mathbb{V})^2}\\&\le\frac{C_k}{(1+2^\ell)^{k+2-2k_0/(k_0+1)}}\cdot\frac{1}{\rho_0^{n-2}}\\&\le\frac{C_k}{(1+|X_0-Y_0|m(X_0,\mathbb{V}))^k}\cdot\frac{1}{|X_0-Y_0|^{n-2}},\end{aligned}$$

where we have used $m(X_0,\mathbb{V})\sim 2^\ell/\rho_0$ in the third step. This finishes our proof. ■

5. The case $1<p<2$**.** In this section, we will prove the unique solvability of (1.1) for $1<p<2$. We begin with a representation formula using the Robin function constructed in Section 4. Also recall the sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of smooth domains from Section 4.

LEMMA 5.1. *Let $j\in\mathbb{N}$. Let u be a solution to*

$$-\Delta u+\mathbb{V}u=0\quad \textit{in } \Omega.$$

Then

$$u(X) = \int_{\partial\Omega_j} R_j(X,Q)\left(\frac{\partial u}{\partial\nu_j}(Q) + b_j(Q)u(Q)\right) d\sigma(Q) \tag{5.1}$$

for all $X \in \Omega_j$*.*

Proof. Recall the definition of the Robin function

$$R_j(X,Y) = \Gamma^Y(X) - W_j^Y(X) \quad \text{for each } X, Y \in \Omega_j.$$

Hence we may split the right-hand side of (5.1) into $I_1 - I_2$, where

$$I_1 := \int_{\partial\Omega_j} \Gamma^X(Q)\left(\frac{\partial u}{\partial\nu_j}(Q) + b_j(Q)u(Q)\right) d\sigma(Q),$$

$$I_2 := \int_{\partial\Omega_j} W_j^X(Q)\left(\frac{\partial u}{\partial\nu_j}(Q) + b_j(Q)u(Q)\right) d\sigma(Q).$$

Observe that $-\Delta u + \mathbb{V}u = 0$ in Ω_j and $u \in W^{1,2}(\Omega_j)$. We claim that $u \in W^{2,2}(\Omega_j)$. Indeed, by writing

$$-\Delta u = -\mathbb{V}u \quad \text{in } \Omega_{j+1}$$

and noting $\mathbb{V}u \in L^2(\Omega_{j+1})$, we infer that $u \in W^{2,2}(\Omega_j)$ by [GT82, Theorem 8.8].

Now we may use Green's second identity to obtain

$$\begin{aligned} I_1 &= \int_{\partial\Omega_j} \Gamma^X(Q)\frac{\partial}{\partial\nu_j}u(Q)\,d\sigma(Q) + \int_{\partial\Omega_j} \Gamma^X(Q)b_j(Q)u(Q)\,d\sigma(Q) \\ &= \int_{\partial\Omega_j} \frac{\partial}{\partial\nu_j}\Gamma^X(Q)u(Q)\,d\sigma(Q) + \int_{\Omega_j} [\Gamma^X(Q)\Delta u - u\Delta\Gamma^X(Q)]\,dQ \\ &\quad + \int_{\partial\Omega_j} \Gamma^X(Q)b_j(Q)u(Q)\,d\sigma(Q) \\ &= \int_{\partial\Omega_j} \left(\frac{\partial}{\partial\nu_j}\Gamma^X(Q) + b_j(Q)\Gamma^X(Q)\right)u(Q)\,d\sigma(Q) \\ &\quad + \int_{\Omega_j} [\Gamma^X(Q)\mathbb{V}u - u\mathbb{V}\Gamma^X(Q) + \delta_X u]\,dQ \\ &= \int_{\partial\Omega_j} \left(\frac{\partial}{\partial\nu_j}\Gamma^X(Q) + b_j(Q)\Gamma^X(Q)\right)u(Q)\,d\sigma(Q) + u(X), \end{aligned}$$

where δ_X is Dirac's function concentrated at X.

Similarly,

$$
\begin{aligned}
I_2 &= \int\limits_{\partial\Omega_j} \left(\frac{\partial W_j^X}{\partial \nu_j}(Q) + b_j(Q) W_j^X(Q) \right) u(Q)\, d\sigma(Q) \\
&= \int\limits_{\partial\Omega_j} \left(\frac{\partial}{\partial \nu_j} \Gamma^X(Q) + b_j(Q) \Gamma^X(Q) \right) u(Q)\, d\sigma(Q).
\end{aligned}
$$

Therefore, $u(X) = I_1 - I_2$. ∎

Theorem 4.3 and Lemma 5.1 together establish the uniqueness property for the Robin problem in the case $1 < p < 2$.

Lemma 5.2. *Let $1 < p < 2$. Let u be a solution to*

$$
\begin{cases}
-\Delta u + \mathbb{V} u = 0 & \text{in } \Omega, \\
\dfrac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial\Omega, \\
(\nabla u)^* \in L^p(\partial\Omega).
\end{cases}
$$

Then $u = 0$ in Ω.

Proof. Fix $X \in \Omega$. There exists an integer $j_0 = j_0(X) > 0$ such that $X \in \Omega_j$ for all $j \geq j_0$. Using Lemma 5.1 and the homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, we may write

$$
\begin{aligned}
u(X) &= \int\limits_{\partial\Omega} R_j^X(\Lambda_j(Q)) [\langle \nabla u(\Lambda_j(Q)), \nu_j(\Lambda_j(Q)) \rangle + b_j(\Lambda_j(Q)) u(\Lambda_j(Q))] \\
&\qquad \times h_j(Q)\, d\sigma(Q) \\
&=: \int\limits_{\partial\Omega} R_j^X(\Lambda_j(Q)) H_j(Q)\, d\sigma(Q)
\end{aligned}
$$

for all $j \geq j_0$, where h_j is given in part (b) of the definition of $\{\Omega_j\}$. It then follows from Theorem 4.3 that

$$
|u(X)| \leq C_X \int\limits_{\partial\Omega} |H_j(Q)|\, d\sigma(Q). \tag{5.2}
$$

Recall that

- $b_j(\Lambda_j(Q)) = (\eta_j b)(Q)$, where η_j is the characteristic function of the set $\{P \in \partial\Omega : b(P) \leq j\}$,
- $\Lambda_j(Q) \in \gamma(Q)$,
- $\nu_j(\Lambda_j(Q)) \to \nu(Q)$ and $h_j(Q) \to 1$ a.e. on $\partial\Omega$ as $j \to \infty$.

These imply

$$
\lim_{j\to\infty} H_j = \frac{\partial u}{\partial \nu} + bu = 0 \quad \text{pointwise in } \partial\Omega.
$$

Furthermore,

$$\sup_{j\geq j_0} |H_j(Q)| \leq |(\nabla u)^*(Q) + b(Q)(u)^*(Q)| \sup_{j\geq j_0} \|h_j\|_{L^\infty(\partial\Omega)}$$

and

$$(\nabla u)^* + b(u)^* \in L^1(\partial\Omega).$$

In view of the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty} \int_{\Omega} |H_j(Q)|\, d\sigma(Q) = 0.$$

Combining with (5.2) implies that $u(X) = 0$. Since X is arbitrary, we conclude that $u = 0$ in Ω as required. ∎

The following theorem is the existence part of Theorem 6.1.

Proposition 5.3. *Let* $1 < p < 2$. *Then* $u = \mathcal{S}(\mathcal{T}^{-1}(g))$ *uniquely solves* (1.1). *Moreover,*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}.$$

Proof. The uniqueness is proved in Lemma 5.2. By Lemma 2.14, the uniqueness implies that the operator $\mathcal{T}$ is invertible on $L^p(\partial\Omega)$. The existence and the representation $u = \mathcal{S}(\mathcal{T}^{-1}(g))$ now follow. The “Moreover” part is contained in Lemma 2.7. ∎

6. Main results. Combining Propositions 3.1 and 5.3, we arrive at our main results in this paper.

Theorem 6.1. *Assume the underlying structure. Then* $u = \mathcal{S}(\mathcal{T}^{-1}(f))$ *uniquely solves* (1.1). *Moreover,*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}.$$

If Ω is C^1-smooth, the Robin problem (1.1) is uniquely solvable for all $1 < p < \infty$. The precise statement is as follows. For comparison, see [LS05, Theorem 5.1].

Theorem 6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* C^1 *domain with* $n \geq 3$. *Let* $1 < p < \infty$ *and* $0 \leq \mathbb{V} \in RH_\sigma$ *with* $n < \sigma \leq \infty$. *Suppose* $0 \leq b \in L^s(\partial\Omega)$ *satisfies* $b \neq 0$, *where*

$$s = \begin{cases} n-1 & \text{if } 1 < p < n-1, \\ n-1+\epsilon,\ \epsilon > 0 & \text{if } p = n-1, \\ p & \text{if } p > n-1. \end{cases} \tag{6.1}$$

Then $u = \mathcal{S}(\mathcal{T}^{-1}(f))$ *uniquely solves* (1.1). *Moreover,*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}.$$

Proof. Uniqueness. This follows from Lemma 5.2 and the simple fact that $L^w(\partial\Omega) \hookleftarrow L^t(\partial\Omega)$ for all $1 < t < 2 \le w$.

Existence. Since Ω is a C^1 domain, Proposition 2.8 can be improved. In particular, the operator

$$\tfrac{1}{2}I + K_0 : L^q(\partial\Omega) \to L^q(\partial\Omega)$$

is compact for all $1 < q < \infty$. In turn, this improves Proposition 2.9, which implies that the operator

$$\tfrac{1}{2}I + K : L^q(\partial\Omega) \to L^q(\partial\Omega)$$

is compact for all $1 < q < \infty$. In addition, (6.1) ensures that the operator $b\mathcal{S} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact in view of Proposition 2.12. At this point, we may conclude that the operator $\mathcal{T} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact, where $\mathcal{T}$ is defined by (2.10). To finish, we refer to Lemma 2.14 to obtain a solution to (1.1) which is of the form $\mathcal{S}(\mathcal{T}^{-1}(g))$.

Lastly, the "Moreover" part is contained in Lemma 2.7. ∎

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