

Orbit misbehavior, isotropy discontinuity, and large isotypic components

by

ALEXANDRU CHIRVASITU

Abstract. Let $\mathbb{G}$ be a compact Hausdorff group acting on a compact Hausdorff space X , α an irreducible $\mathbb{G}$ -representation, and $C(X)$ the C^* -algebra of complex-valued continuous functions on X . We prove that the isotypic component $C(X)_\alpha$ is finitely generated as a module over the invariant subalgebra $C(X/\mathbb{G}) \subseteq C(X)$ precisely when the map sending $x \in X$ to the dimension of the space of vectors in α invariant under the isotropy group $\mathbb{G}_x$ is locally constant. This (a) specializes back to an observation of De Commer–Yamashita equating the finite generation of all $C(X)_\alpha$ with the Vietoris continuity of $x \mapsto \mathbb{G}_x$, and (b) recovers and extends Watatani’s examples of infinite-index expectations resulting from non-free finite-group actions.

We also show that the action of a compact group $\mathbb{G}$ on the maximal equivariant compactification on the disjoint union of its Lie-group quotients has tubes about all orbits precisely when $\mathbb{G}$ is Lie. This is the converse (via a canonical construction) of the well-known fact that actions of compact Lie groups on Tikhonov spaces admit tubes.

Introduction. For every *representation* [24, §2] of a compact (always Hausdorff in the sequel) group $\mathbb{G}$ on a Banach space F and every irreducible (hence finite-dimensional [24, Corollary 5.8]) representation $\alpha \in \text{Irr}(\mathbb{G})$ we have the corresponding *spectral projection* (the P_{e_α} of [17, Definition 4.21])

$$F \xrightarrow{\pi_\alpha} F_\alpha := \{v \in F \mid \text{span } \mathbb{G}v \cong \alpha^n \text{ for some } n \in \mathbb{Z}_{>0}\} \leq F$$

onto the (α) -*isotypic component*. This all applies in particular to the $\mathbb{G}$ -action $s \triangleright f := f(s^{-1} \triangleright \bullet)$ on the C^* -algebra $F := C(X)$ of complex-valued continuous functions on X for a compact Hausdorff left $\mathbb{G}$ -space $\mathbb{G} \times X \xrightarrow{\triangleright} X$.

This paper was initially motivated by the cursory [7, Remark 2 preceding Proposition 3.4], to the effect that the $C(X)_\alpha$ are *all* finitely generated (f.g.

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for short) over the invariant subalgebra $C(X)_{\text{TRIV}} = C(X)^{\mathbb{G}} \cong C(X/\mathbb{G})$ of $C(X)$ precisely when

$$(0.1) \quad X \ni x \mapsto \mathbb{G}_x := \text{isotropy subgroup } \{s \in \mathbb{G} \mid s \triangleright x = x\} \leq \mathbb{G}$$

is continuous for the *Vietoris topology* [19, §III.4.2] on the space of closed subsets (or subgroups) of $\mathbb{G}$. This means that, regarded as a map to $2^{\mathbb{G}}$ (the power-set of $\mathbb{G}$), (0.1) is both

- *upper semicontinuous* (sometimes *hemicontinuous* [1, Definition 17.2]) in the sense that

$$\forall x \in X, \forall \text{neighborhood } U \supseteq \mathbb{G}_x, \\ \{x' \in X \mid \mathbb{G}_{x'} \subseteq U\} \subseteq X \text{ is a neighborhood of } x$$

(this much is automatic for every action [17, Proposition 10.6]);

- and *lower semicontinuous*:

$$\forall x \in X, \forall \text{open } U \text{ with } \mathbb{G}_x \cap U \neq \emptyset, \\ \{x' \in X \mid \mathbb{G}_{x'} \cap U \neq \emptyset\} \subseteq X \text{ is a neighborhood of } x.$$

That motivating observation can be retrieved (Corollary 1.6) as a consequence of a “robust” version thereof (Theorem 1.5), characterizing the α for which finite generation holds in terms of isotropy behavior:

THEOREM A. *Let $\mathbb{G}$ be a compact group acting on a compact Hausdorff space X . The isotypic component $C(X)_{\alpha}$, $\alpha \in \text{Irr}(\mathbb{G})$ is finitely generated as a $C(X)^{\mathbb{G}}$ -module if and only if the function*

$$X \ni x \mapsto \dim \alpha^{\mathbb{G}_x} \in \mathbb{Z}_{\geq 0}, \\ \alpha^{\mathbb{H}} := \mathbb{H}\text{-invariant vectors in the carrier space of } \alpha,$$

is locally constant. ■

Having thus accorded some attention to irregularities of the isotropy-group map (0.1), Section 2 switches focus to the adjacent issue of action misbehavior for non-Lie $\mathbb{G}$. By way of a brief reminder, recall that an *open* or *closed tube* [4, §II.4] about an orbit $\mathbb{G}x$ is a $\mathbb{G}$ -equivariant homeomorphism of

$$\mathbb{G} \times_{\mathbb{G}_x} Y := \mathbb{G} \times Y / ((ts, y) = (t, sy), \forall t \in \mathbb{G}, s \in \mathbb{G}_x, y \in Y)$$

onto an open or, respectively, closed neighborhood of $\mathbb{G}x$ for a $\mathbb{G}_x$ -space Y (then referred to as a *slice* [4, Definition 4.1]). Unqualified tubes are typically understood to be open.

It is well known ([4, Theorem II.5.4], [27, Theorem I.5.7], etc.) that all orbits of compact-Lie-group actions on *Tikhonov* (or $T_{3\frac{1}{2}}$) [32, Definition 14.8] spaces admit tubes, and (the proof of) Theorem 2.1 inverts that implication canonically:

THEOREM B. *A compact group $\mathbb{G}$ is Lie if and only if its action on the universal equivariant compactification [8, §2.8] of*

$$\coprod_{\mathbb{K}} (\text{Lie-group quotients } \mathbb{G}/\mathbb{K})$$

admits tubes around all of its orbits. ■

An earlier, more involved and less direct attempt at proving Theorem A resorted to an analysis of an arbitrary compact-group action based on restricting attention to tubes. It quickly became apparent that the Lie/non-Lie dichotomy would have to be central to any such attempt: tubes are significantly better behaved when $\mathbb{G}$ is Lie, which in turn forces any such route to Theorem A into a reduction of the general problem to Lie-group approximations of arbitrary compact $\mathbb{G}$. Section 2 (summarized here in the form of Theorem B) is an offshoot of those earlier considerations.

1. Orbits, isotropy jumps, and (failure of) finite generation.

Topological (mostly compact) groups are assumed to be Hausdorff, and $\mathbb{G}$ -spaces X are at least *Tikhonov* (or $T_{3\frac{1}{2}}$) [32, Definition 14.8], and usually themselves compact Hausdorff. A few basic observations will be implicit in much that follows:

- For a compact Hausdorff $\mathbb{G}$ -space with compact $\mathbb{G}$ the restrictions

$$(1.1) \quad C(X)_\alpha \xrightarrow{\text{restriction}} C(Z)_\alpha, \quad \text{closed } \mathbb{G}\text{-invariant } Z \subseteq X,$$

are onto by the *Tietze–Gleason theorem* [4, Theorem I.2.3].

- Whence the non-zero $C(X)_\alpha$ are precisely those with $\alpha \leq C(\mathbb{G}/\mathbb{G}_x)$ for some isotropy group $\mathbb{G}_x$. Equivalently [24, pp. 82–83], α must have non-zero vectors invariant under some $\mathbb{G}_x$.

The following example is a paraphrase (and extension) of sorts of [7, p. 711, 2nd paragraph], where $\mathbb{G} := \mathbb{S}^1$. See also [30, Proposition 2.8.2] for a similar discussion concerning non-free finite-group actions with “most” isotropy groups trivial.

EXAMPLE 1.1. Extend the (left, say) translation self-action of a compact group $\mathbb{G}$ in the obvious fashion to the *cone* [15, Chapter 0, pp. 8–9]

$$X := \mathcal{C}\mathbb{G} := (\mathbb{G} \times I)/(\mathbb{G} \times \{0\}), \quad I := [0, 1].$$

I claim that the only isotypic component $C(X)_\alpha$ that is finitely generated as a module over $C(X)_{\text{TRIV}}$ is $C(X)_{\text{TRIV}}$ itself.

Note first that

$$C(X)_{\text{TRIV}} = C(X)^{\mathbb{G}} \cong C(X/\mathbb{G}) \cong C(I),$$

consisting of continuous functions defined arbitrarily on any single interval

$$I \cong \{s\} \times I \subset X, \quad \text{for fixed } s \in \mathbb{G},$$

and then extended uniquely to X by equivariance. Now fix $\text{TRIV} \neq \alpha \in \text{Irr}(\mathbb{G})$ and observe:

- On the one hand, all elements of $C(X)_\alpha$ must vanish at the tip p_0 of the cone $X = \mathcal{C}\mathbb{G}$: regarding an $f \in C(X)_\alpha$ as a function on $\mathbb{G} \times I$, its restriction to the slice $\mathbb{G} \times \{0\}$ collapsed in the cone is both constant and an element of $C(\mathbb{G})_\alpha$ (so zero).
- On the other hand, every $0 \leq f \in C(X)^\mathbb{G}$ vanishing at p_0 can be recovered as

$$f = \pi(g^*g), \quad g \in C(X)_\alpha,$$

$$\text{for the usual expectation } C(X) \xrightarrow{\pi := \pi_{\text{TRIV}}} C(X)^\mathbb{G}.$$

It follows that the ideal

$$(1.2) \quad \text{span} \{ \pi(g^*g') \mid g, g' \in C(X)_\alpha \} \trianglelefteq C(X)^\mathbb{G} \cong C(I)$$

is precisely the ideal

$$(1.3) \quad M_0 := \{f \in C(I) \mid f(0) = 0\} \trianglelefteq C(I) \quad \text{of [12, Theorem 4.6].}$$

If $C(X)_\alpha$ were finitely generated over $C(I)$, so too would be (1.3). On the other hand, [12, Problem 14C] shows that (the image of) M_0 is not finitely generated even after passing to the quotient ring $C(I)/O_0$ by the ideal

$$O_0 := \{f \in C(I) \text{ vanishing in a neighborhood of } 0\} \trianglelefteq C(I)$$

of [12, Problem 4I].

REMARKS 1.2. (1) It is also clear from Example 1.1 that isotypic components can easily be *topologically* finitely generated over $C(X)^\mathbb{G}$ without being algebraically so, in the sense of containing a dense finitely-generated module. Indeed, all of them are (topologically f.g.): they are easily seen to be of the form

$$C(\mathcal{C}\mathbb{G})_\alpha \cong C_0((0, 1]) \otimes C(\mathbb{G})_\alpha \quad (\text{ordinary algebraic tensor product})$$

(with C_0 denoting, as usual [31, General notation], the continuous functions on a locally compact space vanishing at infinity), and hence [13, pre Theorem 1.1] each is respectively topologically generated by

$$\dim_{\mathbb{C}} C(\mathbb{G})_\alpha \xrightarrow{\text{[24, pre Corollary 5.7]}} (\dim \alpha)^2$$

elements.

(2) On the other hand, it is also perfectly possible for isotypic components $C(X)_\alpha$ not to be even topologically f.g. over $C(X/\mathbb{G})$: Example 1.3.

EXAMPLE 1.3. Construct the action $\mathbb{G} \times X \rightarrow X$, for instance, as follows:

- Set $X_0 := \mathbb{G} \times Y$ with the left-hand-factor translation $\mathbb{G}$ -action for any locally compact (Hausdorff) Y that is not σ -compact (i.e. [26, §I.3, p.19] a union of countably many compact subspaces); examples include any uncountable discrete set or

$$\begin{aligned} Y &:= [0, \Omega) = \{\text{set of countable ordinals}\} \\ &= \{\text{set of ordinals} < \Omega\} \\ &= \Omega := \text{first uncountable ordinal} \end{aligned}$$

endowed with the *order topology* of [26, §II.39] (the space also denoted by $[0, \Omega)$ in [26, §II.42]).

- Extend that action in the only way possible to the *one-point compactification* [26, §II.35] $X := X_0 \sqcup \{*\}$ of X_0 .

The quotient space $X/\mathbb{G}$ is the one-point compactification $\bar{Y} = Y \sqcup \{*\}$ of Y , and *no* $C(X)_\alpha$ is topologically finitely generated for non-trivial α . One can argue as in Example 1.1: note that every $\pi(g^*g') \in C(X/\mathbb{G}) \cong C(\bar{Y})$ with $g, g' \in C(X)_{\alpha \neq \text{TRIV}}$ vanishes along some non-singleton closed subset $F \supset * \in \bar{Y}$ because the assumed non- σ -compactness ensures that $\{*\}$ is not a G_δ set (i.e. [26, §I.1, p. 4] a countable intersection of open sets). Indeed: $\{*\}$'s being G_δ is by the very definition of the one-point-compactification topology equivalent to its complement $Y = \bar{Y} \setminus \{*\}$ being a countable union of closed (hence compact) subsets of Y . The singleton $\{*\}$ being non- G_δ , finitely many such $\pi(g^*g')$ cannot generate the entire ideal defined on the left-hand side of (1.2).

The phenomenon underlying Example 1.1 (and [30, Proposition 2.8.2 and Example 2.8.4]) is of ampler scope, and the pertinent proof techniques allow for the generalization of Corollary 1.8, in turn following from Theorem 1.5. The latter is very much in line with [7, Remark after Theorem 3.3], noting the ill behavior resulting, for an action of a compact group $\mathbb{G}$ on a compact space X , from the Vietoris-discontinuity of (0.1).

It will be convenient to have short-hand vocabulary for referring to finite generation of isotypic components.

DEFINITION 1.4. Let $\mathbb{G}$ be a compact group acting on a compact Hausdorff space X . A representation $\alpha \in \text{Irr}(\mathbb{G})$ is *tame* (for the given action) if $C(X)_\alpha$ is finitely $C(X)^\mathbb{G}$ -generated.

Variations of the phrase, such as *tameness over X* , also appear and should be self-explanatory. We also refer to X or the action as α -*taming*.

For $\mathbb{G}$ -representations α and closed subgroups $\mathbb{H} \leq \mathbb{G}$ we write

$$\alpha^\mathbb{H} := \{\mathbb{H}\text{-invariant vectors in } \alpha\}.$$

THEOREM 1.5. *A representation $\alpha \in \text{Irr}(\mathbb{G})$ is tame for an action of a compact group $\mathbb{G}$ on compact Hausdorff X if and only if*

$$(1.4) \quad X \ni x \mapsto \dim \alpha^{\mathbb{G}_x} \in \mathbb{Z}_{\geq 0}$$

is continuous (equivalently, locally constant).

Proof. The continuity of (1.4) is equivalent to that of

$$(1.5) \quad X/\mathbb{G} \ni \mathbb{G}x \mapsto \dim C(\mathbb{G}x)_\alpha = \dim C(\mathbb{G}/\mathbb{G}_x)_\alpha \quad \underline{\underline{[24, \text{pp. 82–83}]}} \dim \alpha \cdot \dim \alpha^{\mathbb{G}_x},$$

and we switch between the two freely.

($\Rightarrow$) $C(X)_\alpha$ is a *Hilbert module* [20, §1, p. 4] over $C(X/\mathbb{G})$ with the inner product

$$\langle f | g \rangle := \pi_{\text{TRIV}}(f^*g) \in C(X/\mathbb{G}), \quad f, g \in C(X).$$

Its assumed finite generation then entails its projectivity [31, Corollary 15.4.8], whence the conclusion: for any commutative ring A and projective finitely-generated A -module M the *rank* function [29, §1]

$$\{\text{prime ideals of } A\} =: \text{Spec } A \ni \mathfrak{p} \mapsto \dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} M_{\mathfrak{p}} \in \mathbb{Z}_{\geq 0}$$

(with $\bullet_{\mathfrak{p}}$ denoting *localization* [2, §3]) is locally constant [29, Proposition 1.4] even in the (coarser, in our case of $A := C(X/\mathbb{G})$ and $M := C(X)_\alpha$) *Zariski* topology [2, p. 12].

($\Leftarrow$) Conversely, the discussion following [13, Theorem 2.2] explains how $C(X)_\alpha$ can be recovered as the space of continuous *sections* (*cross-sections* in [11, Definitions II.13.1]) of a *Hilbert bundle* [11, Definition II.13.4] $F = F_\alpha$ over $X/\mathbb{G}$. The fiber $F_{\mathbb{G}x}$ over $\mathbb{G}x \in X/\mathbb{G}$ is easily seen to be precisely $C(\mathbb{G}x)_\alpha$:

- the functions on X annihilated, in the $C(X/\mathbb{G})$ -module action on $C(X)$, by the annihilator of $C(\mathbb{G}x)$ are tautologically precisely the elements of $C(\mathbb{G}x)$;
- now simply apply the functor $(-)_\alpha$ to each $C(\mathbb{G}x)$.

The local constancy of the map (1.5) thus entails its *local triviality* [9, Proposition 2.3]. Yet another application of [25, Theorem 1.6.3] then finishes the proof (that the space $C(X)_\alpha$ of sections of F is projective finitely generated over $C(X/\mathbb{G})$). ■

In particular, this recovers the aforementioned [7, Remark 2, p. 725]:

COROLLARY 1.6. *Given an action of a compact group $\mathbb{G}$ on a compact Hausdorff space X , the representations $\alpha \in \text{Irr}(\mathbb{G})$ are all tame precisely when the map $x \mapsto \mathbb{G}_x$ of (0.1) is continuous for the Vietoris topology on closed subgroups of $\mathbb{G}$.*

Proof. The backward implication ($\Leftarrow$) follows from Theorem 1.5 and the continuity of

$$\mathbb{H} \begin{array}{c} \nearrow \alpha^{\mathbb{H}} \\ \searrow \dim \alpha^{\mathbb{H}} \end{array}$$

with $\mathbb{H}$ ranging over the Vietoris-topologized space of closed subgroups of $\mathbb{G}$ and $\alpha^{\mathbb{H}}$ regarded as a subspace of the carrier space of V , and hence an element of the Grassmannian $\text{Gr}(V)$.

For the converse ($\Rightarrow$), suppose (0.1) is *not* continuous. Because *upper* semicontinuity is automatic [17, Proposition 10.6], discontinuity at x (and the compactness [10, Theorem 1 and remark (IV) following it] of the Vietoris topology) then means there is some convergent *net* [32, Definition 11.2]

$$x_\lambda \xrightarrow{\lambda} x \quad \text{with} \quad \mathbb{G}_{x_\lambda} \xrightarrow[\lambda]{\text{Vietoris}} \mathbb{L} \not\leq \mathbb{G}_x.$$

There is [23, p. 306] a non-trivial irreducible representation $\beta \in \text{Irr}(\mathbb{G}_x)$ with non-zero $\beta^{\mathbb{L}}$, and also

$$\alpha \in \text{Irr}(\mathbb{G}) \quad \text{with} \quad \beta \leq \text{restriction} \downarrow_{\mathbb{G}_x}^{\mathbb{G}} \alpha$$

(by *Frobenius reciprocity* [24, Theorem 8.9]: *induce* β up to $\mathbb{G}$ [24, §8, p. 84] and pick an appropriate irreducible summand therein). But then

$$\dim \alpha^{\mathbb{G}_{x_\lambda}} \xrightarrow{\lambda} \dim \alpha^{\mathbb{L}} > \dim \alpha^{\mathbb{G}_x},$$

contradicting tameness. (0.1) will thus indeed be continuous if all irreducible representations are tame. ■

REMARK 1.7. Free actions, naturally, fall into the class delineated by Corollary 1.6, as all $\mathbb{G}_x$ are then trivial. So too do homogeneous actions $\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H}$ for closed $\mathbb{H} \leq \mathbb{G}$, the isotropy of gH being the conjugate $g\mathbb{H}g^{-1}$.

[30, Proposition 2.8.2] can be rephrased as saying that for finite $\mathbb{G}$ acting on compact X with

$$(1.6) \quad X_{(1)} := \{x \in X \mid \mathbb{G}_x = \{1\}\} \subseteq X$$

dense and proper, $\alpha \in \text{Irr}(\mathbb{G})$ cannot *all* be tame. The following consequence of Theorem 1.5 generalizes and sharpens that result.

COROLLARY 1.8. *Let X be a compact Hausdorff $\mathbb{G}$ -space for compact $\mathbb{G}$, $X_{(1)}$ defined as in (1.6), and $x \in X_{(1)}$. If $\alpha \in \text{Irr}(\mathbb{G})$ restricts non-trivially to $\mathbb{G}_x$, then it is not tame. ■*

2. Tube existence as a Lie-marking property. Given the noted [17, Proposition 10.6] upper semicontinuity of (0.1), the triviality of $\mathbb{G}_{x_0}$ means that the $\mathbb{G}_x$ of nearby x are contained in arbitrarily small neighborhoods of $1 \in \mathbb{G}$, hence by [17, Corollary 2.40] they are trivial if $\mathbb{G}$ is Lie (more

generally, if $\mathbb{G}$ is Lie then $\mathbb{G}_x$ is conjugate to a subgroup of $\mathbb{G}_{x_0}$ for $x \sim x_0$ [4, Corollary II.5.5]). In short, for compact $\mathbb{G}$ acting on compact X ,

$$(2.1) \quad \mathbb{G} \text{ is Lie} \implies \{x \in X \mid \mathbb{G}_x = \{1\}\} \subseteq X \text{ is open.}$$

Nothing like this holds without the Lie constraint. In fact, for non-Lie $\mathbb{G}$ the negation of (2.1) holds in the strongest possible form, per Theorem 2.1.

THEOREM 2.1. *For a compact group $\mathbb{G}$ the following conditions are equivalent:*

- (a) $\mathbb{G}$ is Lie.
- (b) Every orbit in every $T_{3\frac{1}{2}}$ (or just compact Hausdorff) $\mathbb{G}$ -space admits a tube.
- (c) For any compact Hausdorff X acted upon by $\mathbb{G}$ and any closed subgroup $\mathbb{H} \leq \mathbb{G}$, the set

$$X_{(\leq \mathbb{H})} := \{x \in X \mid \mathbb{G}_x \text{ conjugate to a subgroup of } \mathbb{H}\} \subseteq X$$

is open.

- (d) For any compact Hausdorff X acted upon by $\mathbb{G}$, the set

$$X_{(1)} := \{x \in X \mid \mathbb{G}_x = \{1\}\} \subseteq X$$

is open.

Proof. The downward implications are easily handled: (a) $\implies$ (b) is [4, Theorem II.5.4], (b) $\implies$ (c) follows as in the proof of [4, Corollary II.5.5], and (c) $\implies$ (d) is obvious. The interesting implication is thus (d) $\implies$ (a).

$\mathbb{G}$ acts on each

$$\text{Lie quotient } \mathbb{G}/\mathbb{K}, \quad \{1\} \neq \mathbb{K} \trianglelefteq \mathbb{G},$$

hence also on their disjoint union and on

$$X := \text{universal } \mathbb{G}\text{-equivariant compactification [8, §2.8] of } \coprod_{\mathbb{K}} \mathbb{G}/\mathbb{K} =: Y.$$

Observe that the elements of Y all have non-trivial isotropy groups (since we are ranging over *non-trivial* $\mathbb{K} \trianglelefteq \mathbb{G}$) and, X being the closure of Y , the complement $X \setminus Y$ has empty interior. It remains to argue that the complement contains elements with trivial isotropy provided $\mathbb{G}$ is not Lie.

Recall next [17, Corollary 2.43] that

$$\mathbb{G} \text{ not Lie} \implies \mathbb{G} \cong \varprojlim_{\mathbb{K}} \mathbb{G}/\mathbb{K}.$$

Ordering the neighborhoods $U \ni 1 \in \mathbb{G}$ by reverse inclusion, we can construct a net

$$U \mapsto 1_U := 1 \in \mathbb{G}/\mathbb{K}_U \subset Y \quad \text{for some } \{1\} \neq \mathbb{K}_U \trianglelefteq \mathbb{G} \text{ contained in } U.$$

The *cluster points* [32, Definition 11.3] of (1_U) in X (which do exist [32, Theorem 17.4] by compactness) all belong to $X \setminus Y$ under the present assumption that $\mathbb{G}$ is not Lie: sufficiently small neighborhoods $U \ni 1$ will fail to contain any given $\{1\} \neq \mathbb{K} \supseteq \mathbb{G}$ with Lie quotient $\mathbb{G}/\mathbb{K}$, so no subnet can converge to a point in the component $\mathbb{G}/\mathbb{K}$ of Y . I claim that every such cluster point $p \in X \setminus Y$ has trivial isotropy.

To see this, fix $1 \neq s \in \mathbb{G}$. For a sufficiently small neighborhood $U_0 \ni 1 \in \mathbb{G}$ the image $s \in \mathbb{G}/\mathbb{K}_{U_0}$ is non-trivial, so we can define a bounded continuous function

$$(2.2) \quad Y = \coprod_{\mathbb{K}} \mathbb{G}/\mathbb{K} \xrightarrow{f} [0, 1]$$

by restricting some

$$\mathbb{G}/\mathbb{K}_{U_0} \xrightarrow{f_0} [0, 1], \quad f(1) = 0, f(s) = 1,$$

along the surjection $\mathbb{G}/\mathbb{K} \rightarrow \mathbb{G}/\mathbb{K}_{U_0}$ whenever $\mathbb{K} \leq \mathbb{K}_{U_0}$, and extending by zero to the other $\mathbb{G}/\mathbb{K}$. The resulting function (2.2)

- satisfies the uniform continuity requirements of [8, Lemma 2.2] and hence by [8, Propositions 2.6, 2.7 and 2.9] extends continuously to all of X ;
- vanishes at p because it vanishes along the net $(1_U)_{U \supseteq U_0}$ clustering at p ;
- takes the value 1 at $s \triangleright p$ because it does so along the net

$$(s \triangleright 1_U)_{U \supseteq U_0} = (\text{image of } s \text{ in } \mathbb{G}/\mathbb{K}_U)_{U \supseteq U_0}.$$

This means that indeed $s \triangleright p \neq p$ (for *arbitrary* $1 \neq s \in \mathbb{G}$), and we are done. ■

In roughly the same spirit of action misbehavior for non-Lie groups, recall ([4, Theorem II.5.8], [17, Theorem 10.34]) that for compact Lie $\mathbb{G}$, any action on a locally compact Hausdorff X with *constant orbit type* $\mathbb{H} \leq \mathbb{G}$ (i.e. with all stabilizers conjugate to $\mathbb{H}$) results in a *fiber bundle*

$$X \twoheadrightarrow X/\mathbb{G} \quad \text{with fiber } \mathbb{G}/\mathbb{H}$$

in the sense of [4, §II.1], whose source in particular assumes local triviality. In particular, a free action gives a *principal* $\mathbb{G}$ -*bundle* with total space X , the term once again subsuming local triviality (in concert also with [28, §14.1], say). This is again not the case in general, for $\mathbb{G}$ non-Lie: [33, §III] gives an example with compact abelian $\mathbb{G}$, and a large class of examples can be produced by means of either Example 2.2 or Lemma 2.4.

EXAMPLE 2.2. Take $\mathbb{G} = \prod_i \mathbb{G}_i$, a product of an arbitrary infinite family of compact groups $\mathbb{G}_i \neq \{1\}$. Each $\mathbb{G}_i$ operates freely on the n th *truncation* $X_i := E_{n_i} \mathbb{G}_i$ of its *classifying space* [18, §7.2.7] $E\mathbb{G}$, so the product $\mathbb{G}$ acts in the obvious fashion, freely again, on the product $X := \prod_i X_i$ (compact, because the X_i are).

Suppose the set $\{n_i\}_i$ of truncation parameters is unbounded. By [18, §7.2.5] a local trivialization of the action would yield a $\mathbb{G}$ -equivariant map $X \rightarrow E_n \mathbb{G}$ for some n , i.e. the action would have *finite $\mathbb{G}$ -index* [21, Definition 6.2.3] (as argued in [6, Theorem 3.4] for $\mathbb{G} = \mathbb{Z}/2$ for instance). This, then, would give $\mathbb{G}_i$ -equivariant maps

$$E_{n_i} \mathbb{G}_i \cong X_i \cong X_i \times (p_j)_{j \neq i} \hookrightarrow X \rightarrow E_n \mathbb{G} \rightarrow E_n \mathbb{G}_i,$$

for fixed points $p_j \in X_j$, contradicting the Borsuk–Ulam-type [5, Theorem 1.1] for $n_i > n$ (or [21, Theorem 6.2.5] for *finite $\mathbb{G}_i$*).

DEFINITION 2.3. For a compact group $\mathbb{G}$, the *canonical embedding* is the morphism

$$\mathbb{G} \xrightarrow{\iota = \iota_{\mathbb{G}} := (\alpha)_{\alpha}} \mathbb{U} := \prod_{\alpha \in \text{Irr}(\mathbb{G})} U(\dim \alpha), \quad U(n) := n \times n \text{ unitary group},$$

obtained by collating all individual $\alpha \in \text{Irr}(\mathbb{G})$ regarded as maps $\mathbb{G} \xrightarrow{\alpha} U(\dim \alpha)$.

It is indeed an *embedding*, since all (locally) compact groups have “enough” irreducible representations [16, Theorem 22.12].

LEMMA 2.4. *A profinite group $\mathbb{G}$ is finite if and only if its canonical embedding $\mathbb{G} \xrightarrow{\iota_{\mathbb{G}}} \mathbb{U}$ makes $\mathbb{U}$ the total space of a principal $\mathbb{G}$ -bundle.*

Proof. The direction $\Rightarrow$ is of course [17, Theorem 10.34] valid for arbitrary Lie groups, while the converse $\Leftarrow$ is what the argument in the paragraph following [22, Theorem 1] (ostensibly concerned with infinite profinite subgroups of infinite-dimensional tori $(\mathbb{S}^1)^S$ for sets S) in fact proves: the product $\mathbb{U}$ in Definition 2.3 is connected and locally connected, so it cannot have open subsets homeomorphic to

$$V \times \mathbb{G}, \quad V \subseteq \mathbb{U}/\mathbb{G} \text{ open and } \mathbb{G} \text{ infinite profinite,}$$

(because such sets in turn have no connected open subsets). ■

The construction in the proof of Theorem 2.1 provides a natural free $\mathbb{G}$ -space to work with; it will turn out to be trivial (i.e. of the form $\mathbb{G} \times Y$ with left-hand-factor action), but this is perhaps not immediately obvious in the non-Lie case. In any event, we take a detour to examine the matter.

For compact $\mathbb{G}$ we write

- $\beta_{\mathbb{G}} X$ for the above-mentioned [8, §2.8] (universal) $\mathbb{G}$ -equivariant compactification of a Tikhonov $\mathbb{G}$ -space (so that plain $\beta X := \beta_{\{1\}} X$ is the usual *Stone–Čech compactification* of X [32, Definition 19.4]);
- $\gamma_{\mathbb{G}} X := \beta_{\mathbb{G}} X \setminus X$, which we term the $\mathbb{G}$ -equivariant *corona* of the $\mathbb{G}$ -space X (following the analogous non-equivariant terminology in [14, Introduction, p. 125], say).

DEFINITION 2.5. Let $\mathbb{G}$ be a compact group.

- (1) A closed normal $\mathbb{K} \trianglelefteq \mathbb{G}$ is *co-Lie* if the quotient $\mathbb{G}/\mathbb{K}$ is Lie.
- (2) The *Lie $\mathbb{G}$ -space* (or *Lie (action) space of $\mathbb{G}$*) is

$$\Lambda(\mathbb{G}) := \coprod_{\text{co-Lie } \mathbb{K} \trianglelefteq \mathbb{G}} \mathbb{G}/\mathbb{K},$$

equipped with the obvious $\mathbb{G}$ -action on each $\mathbb{G}/\mathbb{H}$.

- (3) The *compactified Lie $\mathbb{G}$ -space* is the universal $\mathbb{G}$ -equivariant compactification $\beta_{\mathbb{G}}\Lambda(\mathbb{G})$.

PROPOSITION 2.6. *Let $\mathbb{G}$ be a compact group. The subspace*

$$\beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)} := \{x \in \beta_{\mathbb{G}}\Lambda(\mathbb{G}) \mid \mathbb{G}_x = \{1\}\} \subseteq \beta_{\mathbb{G}}\Lambda(\mathbb{G})$$

is closed and trivial as a $\mathbb{G}$ -space. Specifically, writing

$$(\mathcal{CL}(\mathbb{G}), \preceq) := (\text{co-Lie closed normal } \mathbb{K} \trianglelefteq \mathbb{G}, \text{ reverse inclusion}),$$

we have a $\mathbb{G}$ -equivariant isomorphism $\beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)} \cong \mathbb{G} \times Y$ for

$$(2.3) \quad Y := \{\text{cluster points of the net } (\mathbb{K})_{\mathbb{K}} \subset \mathcal{CL}(\mathbb{G})\} \subseteq \beta \mathcal{CL}(\mathbb{G}).$$

Proof. There are two qualitatively distinct cases to consider.

$\mathbb{G}$ is Lie. The poset $(\mathcal{CL}(\mathbb{G}), \preceq)$ (ordered, recall, by *reverse inclusion* $\preceq := \supseteq$) has a largest element $\{1\} \leq \mathbb{G}$. The only cluster point of the net $(\mathbb{K})_{\mathbb{K} \in \mathcal{CL}(\mathbb{G})}$ in the sense of [32, Definition 11.3] is then $\{1\}$ itself, and the statement is interpreted as the claim that

$$\beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)} \cong \mathbb{G} \quad \mathbb{G}\text{-equivariantly.}$$

More precisely, I claim that in this case $\beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)}$ is exactly the component $\mathbb{G} \cong \mathbb{G}/\{1\}$ of $\Lambda(\mathbb{G})$. Since of course no points in any other $\mathbb{G}/\mathbb{K} \subseteq \Lambda(\mathbb{G})$, $\mathbb{K} \neq \{1\}$ have trivial isotropy, an equivalent formulation is that (for Lie $\mathbb{G}$)

$$\beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)} \cap (\mathbb{G}\text{-equivariant corona } \gamma_{\mathbb{G}}\Lambda(\mathbb{G})) = \emptyset.$$

To verify this, consider a convergent net

$$\Lambda(\mathbb{G}) \supseteq \mathbb{G}/\mathbb{K}_{\alpha} \ni p_{\alpha} \xrightarrow[\alpha \nearrow]{} p \in \gamma_{\mathbb{G}}\Lambda(\mathbb{G}) \quad \text{for } \mathbb{K}_{\alpha} \in \mathcal{CL}(\mathbb{G}).$$

The right-hand ambient space in the following display being compact [10, Theorem 1 and remark (IV) following it], the net

$$(\mathbb{K}_{\alpha})_{\alpha} \subset \{\text{closed normal } \mathbb{K} \leq \mathbb{G}\} \text{ with the Vietoris topology}$$

has a cluster point $\mathbb{K} \trianglelefteq \mathbb{G}$. There is an origin neighborhood $U \ni 1 \in \mathbb{G}$ containing no non-trivial subgroups ($\mathbb{G}$ has *no small subgroups* [17, Corollary 2.40], being Lie). Because the p_{α} converge to a point *outside* $\Lambda(\mathbb{G})$,

$$\mathbb{K}_{\alpha} \cap (\mathbb{G} \setminus U) \neq \emptyset \text{ for large } \alpha \implies \mathbb{K} \cap (\mathbb{G} \setminus U) \neq \emptyset \implies \mathbb{K} \neq \{1\}.$$

But then $\{1\} \neq \mathbb{K} \leq \mathbb{G}_p$, and we are done.

$\mathbb{G}$ is not Lie. This time around the co-Lie $\mathbb{K} \trianglelefteq \mathbb{G}$ are all non-trivial, the points in $\mathbb{G}/\mathbb{K} \subset \Lambda(\mathbb{G})$ all have $\mathbb{K}$ -isotropy, so trivial-isotropy points, if any, *must* belong to the corona $\gamma_{\mathbb{G}}\Lambda(\mathbb{G})$.

First identify the space Y of (2.3) with the subspace of $\gamma_{\mathbb{G}}\Lambda(\mathbb{G})$ consisting of cluster points of the net $(1_{\mathbb{G}/\mathbb{K}})_{\mathbb{K} \in \mathcal{CL}(\mathbb{G})}$. We have already argued in the proof of Theorem 2.1 that $Y \subseteq \Lambda(\mathbb{G})_{(1)}$. We now have an embedding

$$(2.4) \quad \mathbb{G} \times Y \ni (s, y) \xrightarrow[\cong]{} sy \in \left(\coprod_{s \in \mathbb{G}} \text{cluster points of } (s\mathbb{K})_{\mathbb{K}} \right) \subseteq \beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)},$$

and it will suffice to argue that that embedding is onto. Consider, then, a convergent net

$$\mathbb{G}/\mathbb{K}_{\lambda} \ni s_{\lambda}\mathbb{K}_{\lambda} \xrightarrow[\lambda]{} x \in \beta_{\mathbb{G}}\Lambda(\mathbb{G})_{(1)}.$$

Because x has trivial isotropy, the net $(\mathbb{K}_{\lambda})_{\lambda}$ must (Vietoris-)converge to $\{1\}$. Next, convergence in $\beta_{\mathbb{G}}\Lambda(\mathbb{G})$ also entails (as in the proof of Theorem 2.1, via [8, Lemma 2.2]) the convergence of (s_{λ}) to some $s \in \mathbb{G}$. But then x belongs to the image of $\{s\} \times Y$ through (2.4), completing the proof. ■

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Although the aforementioned [7, Remark 2, p. 725] does not cite a source for the claim recovered below as Corollary 1.6, in the late stages of preparing a draft of the present paper it was brought to my attention that there is work [3] (albeit not in circulation at the time of this writing) proving that result and presumably overlapping the material below in a few ways. I am very grateful to K. De Commer for informing me of this.

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Alexandru Chirvasitu
Department of Mathematics
University at Buffalo
Buffalo, NY 14260-2900, USA
E-mail: achirvas@buffalo.edu