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A new problem from zero-sum theory

by

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Abstract. Let p be a prime number. We denote by s_p^* the smallest integer l such that, out of any given l integers coprime with p, one can select p integers such that the sum of the p integers is a multiple of p, but not a multiple of p^2. It is conjectured that $\mathsf{s}_p^* = 2p+1$ for any prime number $p \geq 3$. We give a non-trivial upper bound $\mathsf{s}_p^* \leq 3p-2$.

1. Introduction. Let p be a prime. This paper deals with the following combinatorics constant related to p, which arises in the study of certain problems related to the cross number in the zero-sum theory (see [5, Section 3]).

Definition 1.1. We denote by s_p^* the smallest integer l such that, out of any given l integers coprime with p, one can select p integers such that the sum of the p integers is a multiple of p, but not a multiple of p^2.

Throughout this paper, let G be a finite additive abelian group with identity element 0. We denote by C_n a cyclic group of order n. By the structure theorem of finite abelian groups, every non-trivial finite abelian group G can be decomposed into a direct sum of cyclic groups

$$C_{n_1} \oplus \cdots \oplus C_{n_r},$$

where the integers $n_1, \ldots, n_r$ satisfy $1 < n_1 \mid \cdots \mid n_r$. Here r and n_r are called the *rank* and the *exponent* of G, respectively. The exponent of G is denoted by $\exp(G)$, and the order of an element $g \in G$ is written as $\mathrm{ord}(g)$. Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G. We denote by $\sigma(S)$ the sum $\sum_{i=1}^{l} g_i \in G$, and by $\mathsf{k}(S)$ the cross number $\sum_{i=1}^{l} \mathrm{ord}(g_i)^{-1} \in \mathbb{Q}_{\geq 0}$.

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Let $G = C_{p^m}$ for some prime p and integer $m \geq 2$, and let $H = pG = C_{p^{m-1}}$ be the subgroup of G. Then s_p^* is the smallest integer l such that, every sequence S over $G \setminus H$ with $|S| = l$ has a subsequence T with $\sigma(T) \in H$, $|T| = p$ and $\mathsf{k}(T) = \mathsf{k}(\sigma(T))$.

Let $\mathsf{s}(G)$ be the smallest integer l such that every sequence of length l has a zero-sum subsequence of length $\exp(G)$. The Erdős–Ginzburg–Ziv Theorem (see [4]) states that $\mathsf{s}(C_n) = 2n - 1$ and the invariant $\mathsf{s}(G)$ is the so-called Erdős–Ginzburg–Ziv constant. It is not easy to determine the exact values of $\mathsf{s}(G)$ for non-cyclic groups G. The result that $\mathsf{s}(C_p \oplus C_p) = 4p - 3$ was conjectured by Kemnitz in 1983, and was confirmed by Christian Reiher in 2007 (see [8]).

From $\mathsf{s}(C_p \oplus C_p) = 4p-3$, one can deduce that $\mathsf{s}_p^* \leq 4p-3$ for any prime p. Let $4p - 3$ integers coprime to p be given. Without loss of generality, write them as $a_1+b_1p, \ldots, a_{4p-3}+b_{4p-3}p$, where $a_i \in [1, p-1]$ for each $i \in [1, 4p-3]$. By Reiher's result, there exists a subset $I \subseteq [1, 4p-3]$ with $|I| = p$ such that p divides both $\sum_{i \in I} a_i$ and $\sum_{i \in I} b_i$. Since $p \leq \sum_{i \in I} a_i \leq p(p-1) < p^2$, it is impossible that $p^2 \mid \sum_{i \in I}(a_i + b_i p)$. Therefore, these p integers $a_i + b_i p$ with $i \in I$ have the desired properties.

Consider the set of $2p$ integers, for odd prime p, consisting of $p - 1$ elements equal to 1, $p-1$ elements equal to -1, the elements $1-p$ and $p-1$. Then the sum of p elements from this set is a multiple of p only if those elements are all $1^{p-1}(1-p)$, or all $(-1)^{p-1}(p-1)$. In both cases, the sum of the p elements is divisible by p^2. From this we obtain the lower bound $\mathsf{s}_p^* \geq 2p + 1$. Furthermore, the following conjecture was first posed in [5].

Conjecture 1.2. [5, Conjecture 3.2] For any odd prime number p, $\mathsf{s}_p^* = 2p + 1$.

Clearly, $\mathsf{s}_2^* = 3$ and the proof of it is very easy. Conjecture 1.2 has been verified for $p = 3, 5$ and 7. But to verify Conjecture 1.2 for $p = 11$ is beyond the power of our computer. In this paper, we give a non-trivial upper bound of s_p^* as follows.

Theorem 1.3. *For any prime number p, $\mathsf{s}_p^* \leq 3p - 2$.*

Its proof will be given in Section 3. The final section is devoted to some remarks.

2. Notation and preliminaries. We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. For convenience, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ for any two real numbers a, b. For a real number x, let $\lfloor x \rfloor$ be the largest integer no greater than x.

We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We can write a sequence $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

with $\mathsf{v}_g(S) \in \mathbb{N}_0$ for all $g \in G$, where $\mathsf{v}_g(S)$ is called the *multiplicity* of g in S. We say that S contains the element g of G if $\mathsf{v}_g(S) > 0$. A sequence S' is called a *subsequence* of S, denoted by $S' \mid S$, if $\mathsf{v}_g(S') \le \mathsf{v}_g(S)$ for all $g \in G$, and SS'^{-1} denotes the subsequence obtained from S by deleting S'. The unit element $\emptyset \in \mathcal{F}(G)$ is called the *empty* sequence.

For a sequence

$$S = g_1 \cdots g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G),$$

we define the following notions:

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$, the length of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$, the sum of S,
- $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subseteq G$, the support of S,
- S is a *squarefree sequence* if $\mathsf{v}_g(S) \le 1$ for all $g \in G_0$.
- S is a *zero-sum sequence* if $\sigma(S) = 0 \in G$,
- $b + S = (b + g_1) \cdots (b + g_l)$ where $b \in G$,
- $\Sigma_r(S) = \{\sigma(T) : T \mid S, |T| = r\}$ where $r \in [1, l]$.

Every map of abelian groups $\varphi : G \to H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \to \mathcal{F}(H)$, where $\varphi(S) = \varphi(g_1) \cdots \varphi(g_l)$. If φ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence over H if and only if $\sigma(S) \in \ker(\varphi)$. Obviously, $\varphi(\sigma(S)) = \sigma(\varphi(S))$.

The following well-known result, called the Cauchy–Davenport Theorem, plays an important role in the development of zero-sum theory.

Lemma 2.1 (Cauchy–Davenport [2]). *Let p be a prime number, and let $A_1, \ldots, A_k$ be nonempty subsets of C_p. Then*

$$|A_1 + \cdots + A_k| \ge \min\{p, |A_1| + \cdots + |A_k| - k + 1\}.$$

A direct application of Lemma 2.1 in zero-sum theory is the following.

Lemma 2.2. *Let $S \in \mathcal{F}(C_p)$ and $k \in [1, p]$. If $|S| \ge p + k - 1$ and $\mathsf{h}(S) \le k$, then $\Sigma_k(S) = C_p$.*

Proof. Without loss of generality, let $|S| = p + k - 1$ and

$$S = a_1 \cdots a_{p+k-1}.$$

Moreover, we can require that, if $a_i = a_j$ with $i \le j$, then $a_l = a_i$ for every $l \in [i, j]$. It follows from $\mathsf{h}(S) \le k$ that $a_i \ne a_{i+k}$ for every $i \in [1, p-1]$.

For every $j \in [1,k]$, let

$$A_j = \prod_{l=0}^{\lfloor \frac{p+k-1-j}{k} \rfloor} a_{j+lk}.$$

Then $S = \prod_{j=1}^{k} A_j$ and every A_j is a squarefree subsequence. Regarding every A_j as a set and applying Lemma 2.1, we obtain

$$\Big|\sum_{j=1}^{k} A_j\Big| \geq \min\Big\{p, \sum_{j=1}^{k} |A_j| - k + 1\Big\} = p.$$

Noticing that $S \in \mathcal{F}(C_p)$, we have

$$C_p = \sum_{j=1}^{k} A_j \subseteq \Sigma_k(S) \subseteq C_p.$$

Lemma 2.2 is proved. ∎

There are many generalizations of Lemma 2.1. One of them is the following lemma stated by Chowla in 1935 and recently as an exercise in [6, Exercise 6.6] by Grynkiewicz. It is an immediate consequence of Kneser's Theorem. For more generalizations, we refer to [3].

Lemma 2.3 ([1]). *Let G be an abelian group and $A, B \subseteq G$ be non-empty sets with $A + B \neq G$. If there is $b \in B$ such that* $\mathrm{ord}(g - b) = |G|$ *for all $g \in B \setminus \{b\}$, then*

$$|A + B| \geq |A| + |B| - 1.$$

3. A non-trivial upper bound of s_p^*. In this section, we focus on the group $C_{p^2} \cong (\mathbb{Z}/p^2\mathbb{Z}, +)$ and give a proof of Theorem 1.3. Throughout this section, let $C_{p^2}^{\times} = C_{p^2} \setminus pC_{p^2}$ be the set of all generators in the cyclic group C_{p^2}. The sequence S will always be over $C_{p^2}^{\times}$.

First, we introduce a lemma from [5] to simplify the structure of the sequence that we need to deal with in the proof of Theorem 1.3.

Lemma 3.1 ([5, Lemma 3.4]). *Let p be an odd prime number and let $S \in \mathcal{F}(C_{p^2}^{\times})$ be a sequence of length $|S| = 2p + 1$. If there are two elements a, b in S satisfying $a - b \in pC_{p^2} \setminus \{0\}$, then there is a subsequence T of S such that $|T| = p$ and $\sigma(T) \in pC_{p^2} \setminus \{0\}$.*

Next, we give two corollaries of Lemma 2.3 which are necessary in our proof of Theorem 1.3. The first one follows by applying Lemma 2.3 inductively.

Lemma 3.2. *Let $B_1, \ldots, B_k$ with $k \geq 2$ be non-empty subsets of an abelian group C_{p^2}. For every $i \in [2, k]$, suppose that there exists some $b_i \in B_i$*

such that $\operatorname{ord}(g - b_i) = p^2$ *for all* $g \in B_i \setminus \{b_i\}$. *Then*

$$\Big|\sum_{i=1}^{k} B_i\Big| \ge \min\Big\{p^2, \sum_{i=1}^{k} |B_i| - k + 1\Big\}.$$

LEMMA 3.3. *Let* $S \in \mathcal{F}(C_{p^2}^{\times})$ *and* $k \in [1, p-1]$. *Then at least one of the following statements holds:*

(1) *There are two elements* a, b *in* S *such that* $a - b \in pC_{p^2} \setminus \{0\}$.
(2) *If* $|S| \ge p + k$ *and* $\mathsf{h}(S) \le k$, *then there are two subsequences* T_1, T_2 *of* S *such that* $|T_1| = |T_2| = k$ *and* $\sigma(T_1) - \sigma(T_2) \in pC_{p^2} \setminus \{0\}$.

Proof. Assume that (1) does not hold. That is, if a, b are different elements in S, then we have $a - b \in C_{p^2}^{\times}$, or equivalently, $\operatorname{ord}(a - b) = p^2$. We are going to prove that statement (2) must then hold.

Without loss of generality, let $|S| = p + k$ and

$$S = a_1 \cdots a_{p+k}.$$

Moreover, we can require that, if $a_i = a_j$ with $i \le j$, then $a_l = a_i$ for every $l \in [i, j]$. It follows from $\mathsf{h}(S) \le k$ that $a_i \ne a_{i+k}$ for every $i \in [1, p]$.

For every $j \in [1, k]$, set

$$B_j = \prod_{l=0}^{\lfloor \frac{p+k-j}{k} \rfloor} a_{j+lk}.$$

Then $S = \prod_{j=1}^{k} B_j$ and every B_j is a squarefree subsequence of length $|B_j| \ge 2$. By assumption, for every $j \in [1, k]$, there is some $b_j \in B_j$ such that $\operatorname{ord}(g - b_j) = p^2$ for all $g \in B_j \setminus \{b_j\}$. Regarding every B_j as a set and applying Lemma 3.2, we obtain

$$\Big|\sum_{j=1}^{k} B_j\Big| \ge \min\Big\{p^2, \sum_{j=1}^{k} |B_j| - k + 1\Big\} = p + 1.$$

Since $\sum_{j=1}^{k} B_j \subseteq C_{p^2}$, there are two different sums $u_1 + \cdots + u_k$ and $v_1 + \cdots + v_k$ in $\sum_{j=1}^{k} B_j$ such that

$$(u_1 + \cdots + u_k) - (v_1 + \cdots + v_k) \in pC_{p^2} \setminus \{0\}.$$

Let $T_1 = \prod_{i=1}^{k} u_i$ and $T_2 = \prod_{i=1}^{k} v_i$. Then T_1, T_2 are two subsequences of S such that $|T_1| = |T_2| = k$ and $\sigma(T_1) - \sigma(T_2) \in pC_{p^2} \setminus \{0\}$. ■

In the rest of this section, let $\varphi : C_{p^2} \to C_{p^2}/pC_{p^2}$ be the canonical homomorphism. Then for any sequence $S \in \mathcal{F}(C_{p^2})$, we have $\varphi(S) \in \mathcal{F}(C_{p^2}/pC_{p^2}) \cong \mathcal{F}(C_p)$. Now, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.1, we only need to deal with the case

$$S = g_1^{h_1} \cdots g_s^{h_s},$$

where $1 \le h_s \le \cdots \le h_1 \le p-1$, $4 \le s \le p-1$, $|S| = \sum_{i=1}^{s} h_i = 3p-2$ and $pg_i \neq pg_j$ for $i \neq j \in [1, s]$. We will prove that S has two subsequences of length p such that their respective sums are both in pC_{p^2} but different in C_{p^2}.

For $j \in [1, h_1]$, let

$$A_j = \prod_{\substack{i \in [1,s] \\ h_i \ge j}} g_i.$$

Then $S = \prod_{j=1}^{h_1} A_j$ and every A_j is a squarefree subsequence. Furthermore,

$$\emptyset \neq A_{h_1} \subseteq \cdots \subseteq A_1 = \operatorname{supp}(S)$$

as sets and $|A_{h_i}| \ge i$ for $i \in [1, s]$. Hence $2 \le |A_{h_2}| \le \cdots \le |A_1| \le p-1$.

Note that

$$\sum_{j=1}^{1} (|A_j| - 1) + 1 < \cdots < \sum_{j=1}^{h_2} (|A_j| - 1) + 1,$$

where $|A_1| = s \le p-1$ and

$$\sum_{j=1}^{h_2} (|A_j| - 1) + 1 = \sum_{i=2}^{s} h_i + 1 = |S| - h_1 + 1 \ge 2p$$

since $h_1 \le p-1$, $|S| = 3p-2$ and $\prod_{j=1}^{h_2} (g_1^{-1} A_j) = g_2^{h_2} \cdots g_s^{h_s}$. There exists an integer $k \in [2, h_2]$ such that

$$\sum_{j=1}^{k-1} (|A_j| - 1) + 1 < p + 1 \le \sum_{j=1}^{k} (|A_j| - 1) + 1.$$

Clearly, $k < h_1 < p-1$ since $\sum_{i=1}^{h_1 - 1} (|A_i| - 1) + 1 \ge |S| - |A_{h_1}| - h_1 + 2 \ge p+2$.

Next, we divide the proof into the following two cases.

Case 1: $p + 1 = \sum_{j=1}^{k} (|A_j| - 1) + 1$. Let $W = \prod_{j=1}^{k} A_j$. Then $|W| = \sum_{j=1}^{k} |A_j| = p + k > 2k$ and $\mathsf{h}(W) = k$. By Lemma 3.3, there exist two subsequences W_1, W_2 of W such that $|W_1| = |W_2| = k$ and

$$\sigma(W_1) - \sigma(W_2) \in pC_{p^2} \setminus \{0\}. \tag{3.1}$$

Set $W_3 = \prod_{g \in C_{p^2}} g^{\max\{\mathsf{v}_g(W_1), \mathsf{v}_g(W_2)\}}$. Then $W_3 \neq W$ since $|W| > 2k$, so there exists an element x in $W_3^{-1} W$.

Consider the subsequence

$$X = x \cdot \prod_{j=k+1}^{h_1} A_j$$

of S. We have $|\varphi(X)| = |X| = |S| - |W| + 1 = 2p - k - 1$ and $\mathsf{h}(\varphi(X)) = \mathsf{h}(X) \le h_1 - k + 1 \le p - k$. By Lemma 2.2, $\sum_{p-k}(\varphi(X)) = C_{p^2}/pC_{p^2}$ and so there is $X' \mid X$ such that $|X'| = p - k$ and

$$\sigma(\varphi(X')) = -\sigma(\varphi(W_1)) \in C_{p^2}/pC_{p^2},$$

Hence $\sigma(\varphi(X'W_1)) = 0 + pC_{p^2}$ in C_{p^2}/pC_{p^2}, or equivalently,

$$\sigma(X'W_1) \in \ker(\varphi) = pC_{p^2}. \tag{3.2}$$

Now, we get two subsequences $X'W_1, X'W_2$ of S satisfying $|X'W_1| = p = |X'W_2|$. It follows from (3.1) and (3.2) that

$$\sigma(X'W_2) = \sigma(X'W_1) - \sigma(W_1) + \sigma(W_2) \in pC_{p^2},$$
$$\sigma(X'W_1) - \sigma(X'W_2) = \sigma(W_1) - \sigma(W_2) \neq 0.$$

So $X'W_1, X'W_2$ are the desired sequences and Theorem 1.3 is proved in this case.

CASE 2: $\sum_{j=1}^{k-1}(|A_j| - 1) + 1 < p + 1 < \sum_{j=1}^{k}(|A_j| - 1) + 1$. Let $m = \sum_{j=1}^{k}(|A_j| - 1) + 1 - (p+1) \geq 1$. From the hypothesis, we have $|A_k| - 1 \geq m+1$, so $s - m = |A_1| - m \geq |A_k| - m \geq 2$. Set

$$B = A_1 \cdot \prod_{i=s-m+1}^{s} g_i^{-1} = \prod_{i=1}^{s-m} g_i.$$

Then $|B| = s - m$ and $g_1 g_2 \mid B$.

Let $U = B \prod_{j=2}^{k} A_j$. By the definition of m, we have $|U| = \sum_{j=1}^{k} |A_j| - m = p + k$ and $\mathsf{h}(U) = k$. By Lemma 3.3, there exist two subsequences U_1, U_2 of U such that $|U_1| = |U_2| = k$ and

$$\sigma(U_1) - \sigma(U_2) \in pC_{p^2} \setminus \{0\}. \tag{3.3}$$

Recalling that $g_1 - g_2 \notin pC_{p^2}$ and $1 < k \leq h_2 \leq p-1$, we have $\sigma(g_1^k) - \sigma(g_2^k) = k(g_1 - g_2) \notin pC_{p^2}$ and so $\{U_1, U_2\} \neq \{g_1^k, g_2^k\}$. Set

$$U_3 = \prod_{g \in C_{p^2}} g^{\max\{\mathsf{v}_g(U_1), \mathsf{v}_g(U_2)\}}.$$

Then $U_3 \neq g_1^k g_2^k \mid U$. Thus there is an element y in the sequence $U_3^{-1}U$ such that $y = g_1$ or g_2.

Consider the subsequence

$$Y = y \prod_{i=s-m+1}^{s} g_i \cdot \prod_{i=k+1}^{h_1} A_i$$

of S. It follows from $s - m \geq 2$ and $y \in \{g_1, g_2\}$ that the sequence $y \prod_{i=s-m+1}^{s} g_i$ is squarefree. Hence we have $|\varphi(Y)| = |Y| = 1 + |S| - |U| = 2p - k - 1$ and $\mathsf{h}(\varphi(Y)) = \mathsf{h}(Y) \leq h_1 - k + 1 \leq p - k$. By Lemma 2.2, $\sum_{p-k}(\varphi(Y)) = C_{p^2}/pC_{p^2}$ and so there is $Y' \mid Y$ such that $|Y'| = p - k$ and

$$\sigma(\varphi(Y')) = -\sigma(\varphi(U_1)) \in C_{p^2}/pC_{p^2},$$

or equivalently,

$$\sigma(Y'U_1) \in \ker(\varphi) = pC_{p^2}. \tag{3.4}$$

Now, we get two subsequences $Y'U_1, Y'U_2$ of S satisfying $|Y'U_1| = p = |Y'U_2|$. It follows from (3.3) and (3.4) that

$$\begin{aligned}\sigma(Y'U_2) &= \sigma(Y'U_1) - \sigma(U_1) + \sigma(U_2) \in pC_{p^2},\\ \sigma(Y'U_1) - \sigma(Y'U_2) &= \sigma(U_1) - \sigma(U_2) \neq 0.\end{aligned}$$

Therefore, $Y'U_1, Y'U_2$ are the desired sequences in this case. ∎

4. Final remarks. (1) We point out that, through replacing [5, Lemma 3.7] by our Theorem 1.3, we can give a shorter and more unified proof of [5, Theorem 1.3].

(2) The combinatorial constant s_p^* has a connection to p-adic theory. Here we introduce some notations from [7]. Given a prime p, we denote by $\mathbb{Q}_p$ the field of p-adic numbers, which can be written as

$$\mathbb{Q}_p = \Big\{ \sum_{i=v}^{\infty} a_i p^i \;\Big|\; v \in \mathbb{Z},\, a_i \in [0, p-1] \Big\}.$$

The power series $x = \sum_{i=v}^{\infty} a_i p^i$ is called the *p-adic expansion* (of the p-adic number x). And if v is the smallest integer i such that $a_i \neq 0$ in the p-adic expansion of x, then v is called the *p-adic valuation* of x. The p-adic valuation of 0 ($\in \mathbb{Q}_p$) is set to ∞. For any integer v, let $p^v\mathbb{Z}_p \subset \mathbb{Q}_p$ be the set of all p-adic numbers of p-adic valuation no less than v, where

$$\mathbb{Z}_p = \Big\{ \sum_{i=0}^{\infty} a_i p^i \;\Big|\; a_i \in [0, p-1] \Big\}$$

is called the *ring of p-adic integers*. Then $p^v\mathbb{Z}_p \backslash p^{v+1}\mathbb{Z}_p$ is the set of all p-adic numbers of p-adic valuation v.

By the Erdős–Ginzburg–Ziv Theorem ($\mathsf{s}(C_n) = 2n - 1$), $2p - 1$ is the smallest integer l such that, out of any given l p-adic numbers in $\mathbb{Z}_p$, one can always select p p-adic numbers such that the sum of those numbers belongs to $p\mathbb{Z}_p$. But the p-adic valuation of the sum cannot be precisely determined.

To control the p-adic valuation of the sum, we need more than $2p - 1$ p-adic numbers, and s_p^* is the smallest integer l such that, out of any given l p-adic numbers of p-adic valuation $v \neq \infty$, one can always select p p-adic numbers whose sum has p-adic valuation $v + 1$. Therefore, Conjecture 1.2 makes sense when translated into p-adic theory.

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