

*THE THERMODYNAMIC APPROACH TO MULTIFRACTAL
ANALYSIS FOR WEIGHTED TOPOLOGICAL ENTROPY*

BY

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Abstract. We give a general framework incorporating the most important multifractal spectra using the thermodynamic approach to establish the multifractal formalism for weighted topological entropy.

1. Introduction. Multifractal theory, situated at the interface between dynamical systems and fractal geometry, provides tools to describe non-uniform scaling behaviors of measures and sets by means of their dimension spectra. Since the seminal work of [HJ⁺86], where the pointwise dimension spectrum $\mathcal{D}_D$ was introduced, it has been observed that for many systems this spectrum can be expressed as the Legendre transform of a function $T(q)$ related to the Hentschel–Procaccia spectrum

$$(1.1) \quad \mathcal{D}_D(\alpha) = T^L(\alpha) := \inf_{q \in \mathbb{R}} (T(q) + q\alpha).$$

General properties of Legendre transforms [Roc70] imply that $\mathcal{D}_D(\alpha)$ is concave. In numerous situations, the function T is analytic and strictly convex in q , which ensures that $\mathcal{D}_D(\alpha)$ is analytic and strictly concave in α . The link between this spectrum and the Hentschel–Procaccia spectrum via Legendre transform was clarified in [HJ⁺86]. A fundamental task in multifractal analysis is to determine when such a formalism is valid for natural fractal measures. For further background and rigorous treatments of multifractal theory, see [Fal03, Ols95, Pes97].

Within the thermodynamic approach to multifractal analysis, Climenhaga [Cli10a, Cli11] established a framework that relates dimension spectra to topological pressure. He derived Bowen’s equation for the classical topological pressure, thereby connecting the dimension of a set with the pressure of an associated potential. In addition, he proved a partial multi-

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fractal formalism, showing that the multifractal spectrum observed in the dynamics is bounded above by a spectrum predicted from thermodynamic principles.

A significant development for self-affine systems was achieved by Feng and Huang [FH16], who introduced the concept of weighted topological pressure. Their construction extends the classical notion by incorporating a weight vector $a = (a_1, \dots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. They defined a -weighted Bowen balls, where the iteration length of the dynamical system is adjusted according to the weight parameters so as to reflect non-uniform scaling priorities. Furthermore, they proved a variational principle for weighted pressure, providing a theoretical foundation for analyzing hierarchical or non-uniform dynamical systems. This extension greatly broadens the applicability of pressure-based methods to multifractal problems. More recently, Mesón and Vericat [MV22] obtained a variational formula for the weighted multifractal spectrum of V -statistics.

Against this background, the purpose of the present paper is to develop a general multifractal framework that combines Feng and Huang's weighted topological pressure with Climenhaga's thermodynamic approach. Our goal is to establish a multifractal formalism adapted to systems where dynamical complexity is described by weighted pressure, thereby providing tools suitable for non-uniform scaling scenarios.

Main contributions. The principal results of this paper can be summarized as follows.

(1) *Weighted Bowen's equation.* We introduce the notion of weighted u -dimension, where $u : X_1 \rightarrow \mathbb{R}$ is a continuous weight function satisfying condition (P). We then prove a weighted analogue of Bowen's equation (Proposition 2.8), which links the weighted u -dimension of a set $Z \subset X_1$ with the weighted topological pressure of the potential $-tu$. This result generalizes the classical relation obtained in [Cli11].

(2) *Partial multifractal formalism.* We study mixed multifractal spectra describing the joint distribution of two dynamical quantities, typically given by the ratio of Birkhoff averages of continuous functions φ and ψ . For $\alpha \in \mathbb{R}$, we consider the level set

$$K_\alpha = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{S_n \varphi(x)}{S_n \psi(x)} = \alpha \right\},$$

and define the corresponding weighted spectrum $\mathcal{F}^a(\alpha) = \dim_u^a K_\alpha$. Following the thermodynamic prediction method of [Cli10a], we introduce a spectrum $\mathcal{S}^a(\alpha)$ derived from weighted pressure (3.5). We then establish an upper bound result (Proposition 3.4), showing that the observed spectrum $\hat{\mathcal{F}}^a(\alpha)$ (restricted to a physically relevant subset $\hat{X}_1$) does not exceed $\mathcal{S}^a(\alpha)$.

(3) *Exactness of the formalism.* Under additional assumptions, the observed and predicted spectra coincide. By employing Ruelle's derivative formula for weighted pressure (Proposition 3.7), we prove that if the equilibrium state associated with $\mathcal{S}^a(\alpha)$ has positive integrals of ψ and u , and if the pressure function is differentiable at the relevant parameter, then this equilibrium state yields equality of spectra (Theorem 3.5). This extends Climenhaga's multifractal results [Cli10a] to the weighted setting.

(4) *Conditional variational principle.* We provide a conditional variational principle (Theorem 4.3), which states that for most values of α (those in $\hat{I}$ where 0 lies in the interior of the set of integrals of $\varphi - \alpha\psi$ with respect to invariant measures), the predicted spectrum $\mathcal{S}^a(\alpha)$ equals the supremum of weighted u -dimensions of ergodic invariant measures supported on K_α . This establishes a probabilistic interpretation of the weighted multifractal spectrum.

(5) *Legendre transform connection.* Finally, we investigate the link between weighted spectra and Legendre transforms. In the special case $\psi = u$, the predicted spectrum $\mathcal{S}^a(\alpha)$ is shown to reduce to the Legendre transform of a convex pressure function (5.2). Moreover, we prove that $\hat{\mathcal{S}}^a|_{\hat{I}}$ is the concave hull of $\hat{\mathcal{F}}^a|_{\hat{I}}$ (Theorem 5.3), a structural property inherited from convexity of the underlying pressure function.

In summary, this paper extends the classical thermodynamic framework of [Cli10a, Cli11] by incorporating the weighted pressure of Feng and Huang [FH16], thereby developing a multifractal formalism suitable for non-uniform dynamical systems. The results not only preserve key thermodynamic principles but also enlarge the scope of multifractal analysis to systems with hierarchical structures or non-uniform scaling, offering new perspectives for the study of complex dynamical phenomena.

2. Definitions and results. We begin by recalling the setting of our study. Let $k \geq 2$, and suppose that for each $1 \leq i \leq k$ we are given a topological dynamical system (X_i, d_i, T_i) , where (X_i, d_i) is a compact metric space. Assume further that (X_{i+1}, T_{i+1}) is a factor of (X_i, T_i) for $1 \leq i \leq k-1$, with a factor map $\pi_i : X_i \rightarrow X_{i+1}$. In other words, $\pi_1, \dots, \pi_{k-1}$ are continuous maps making the following diagram commute:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \dots & \xrightarrow{\pi_{k-1}} & X_k \\ T_1 \downarrow & & T_2 \downarrow & & & & T_k \downarrow \\ X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \dots & \xrightarrow{\pi_{k-1}} & X_k \end{array}$$

For convenience, let π_0 denote the identity map on X_1 , and define

$$\tau_i = \pi_i \circ \pi_{i-1} \circ \dots \circ \pi_0 : X_1 \rightarrow X_{i+1}, \quad i = 0, 1, \dots, k-1.$$

We write $\mathcal{M}(X_i, T_i)$ for the space of all T_i -invariant Borel probability measures on X_i , endowed with the weak* topology. Fix a weight vector $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$.

DEFINITION 2.1 ([FH16]). For $x, y \in X_1$, $n \in \mathbb{N}$, and $\epsilon > 0$, set

$$d_n^{\mathbf{a}}(x, y) := \sup \{d_i(T_i^j \tau_{i-1} x, T_i^j \tau_{i-1} y) : 0 \leq j \leq \lceil (a_1 + \dots + a_i)n \rceil - 1, i = 1, \dots, k\},$$

and define the $\mathbf{a}$ -weighted Bowen ball

$$B_n^{\mathbf{a}}(x, \epsilon) := \{y \in X_1 : d_n^{\mathbf{a}}(x, y) < \epsilon\}.$$

Given $Z \subset X_1$, $s \geq 0$, $\epsilon > 0$, $\phi \in C(X_1)$ and $N \in \mathbb{N}$, define

$$\Lambda^{\mathbf{a}}(Z, \phi, s, \epsilon, N) = \inf \sum_j \exp \left(-sn_j + \sup_{x \in B_{n_j}^{\mathbf{a}}(x_j, \epsilon)} \frac{1}{a_1} S_{\lceil a_1 n_j \rceil} \phi(x) \right),$$

where the infimum is taken over all countable collections $\Gamma = \{B_{n_j}^{\mathbf{a}}(x_j, \epsilon)\}$ with $n_j \geq N$ and $\bigcup_j B_{n_j}^{\mathbf{a}}(x_j, \epsilon) \supset Z$. Since $\Lambda^{\mathbf{a}}(Z, \phi, s, \epsilon, N)$ is non-decreasing in N , the limit

$$\Lambda^{\mathbf{a}}(Z, \phi, s, \epsilon) = \lim_{N \rightarrow \infty} \Lambda^{\mathbf{a}}(Z, \phi, s, \epsilon, N)$$

exists. There is a critical value of s , denoted $P^{\mathbf{a}, B}(T_1, Z, \phi, \epsilon)$, such that

$$\Lambda^{\mathbf{a}}(Z, \phi, s, \epsilon) = \begin{cases} 0, & s > P^{\mathbf{a}, B}(T_1, Z, \phi, \epsilon), \\ \infty, & s < P^{\mathbf{a}, B}(T_1, Z, \phi, \epsilon). \end{cases}$$

Since $P^{\mathbf{a}, B}(T_1, Z, \phi, \epsilon)$ does not decrease with ϵ , the following limit exists:

$$P^{\mathbf{a}, B}(T_1, Z, \phi) := \lim_{\epsilon \rightarrow 0} P^{\mathbf{a}, B}(T_1, Z, \phi, \epsilon).$$

Following [FH16], this is called the $\mathbf{a}$ -weighted Bowen topological pressure of T_1 restricted to Z . When $Z = X_1$, we write simply $P^{\mathbf{a}, B}(T_1, \phi)$.

In the case $\phi = 0$, this reduces to the weighted topological entropy $h^{\mathbf{a}, B}(Z) = P^{\mathbf{a}, B}(T_1, Z, 0)$ introduced in [FH16].

REMARK 2.2. When $Z = X_1$, one may also compute the pressure via spanning sets [Ji25]. For $n \in \mathbb{N}$ and $\epsilon > 0$, define

$$\mathcal{Z}_n^{\mathbf{a}}(\phi, \epsilon) = \inf \left\{ \sum_{x \in E} \exp \left(\sup_{y \in B_n^{\mathbf{a}}(x, \epsilon)} \frac{1}{a_1} S_{\lceil a_1 n \rceil} \phi(y) \right) : B(E, n, \epsilon) = X_1 \right\},$$

where $B(E, n, \epsilon) := \bigcup_{x \in E} B_n^{\mathbf{a}}(x, \epsilon)$. Then

$$P^{\mathbf{a}, B}(T_1, X_1, \phi) = \lim_{\epsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{Z}_n^{\mathbf{a}}(\phi, \epsilon).$$

In what follows we assume $h_{\text{top}}^{\mathbf{a}, B}(X_1) < \infty$, otherwise the pressure function is identically infinite and uninformative.

DEFINITION 2.3. For $\epsilon > 0$ and $N \in \mathbb{N}$, let $\mathcal{P}^{\mathbf{a}}(Z, N, \epsilon)$ be the collection of countable sets $\{(x_i, n_i)\} \subset Z \times \{N, N+1, \dots\}$ with $Z \subset \bigcup_i B_{n_i}^{\mathbf{a}}(x_i, \epsilon)$.

DEFINITION 2.4. Let $u : X_1 \rightarrow \mathbb{R}$ be continuous. We require the following condition:

(P) For every $\epsilon > 0$ there exist covers $E_N \in \mathcal{P}^{\mathbf{a}}(X_1, N, \epsilon)$ such that

$$\lim_{N \rightarrow \infty} \inf_{(x, n) \in E_N} S_{\lceil a_1 n \rceil} u(x) = +\infty.$$

In [BS00] it is assumed that $u > 0$, which guarantees property (P). However, (P) may hold even if u is not strictly positive.

PROPOSITION 2.5. *If $u \in C(X_1)$ satisfies (P), then $\int_{X_1} u d\mu \geq 0$ for every $\mu \in \mathcal{M}(X_1, T_1)$.*

Proof. It suffices to prove the claim for ergodic measures. Suppose $\mu \in \mathcal{M}(X_1, T_1)$ is ergodic with $\int u d\mu < 0$. Let x be μ -generic. Then

$$\frac{1}{\lceil a_1 n \rceil} S_{\lceil a_1 n \rceil} u(x) \rightarrow \int u d\mu < 0.$$

Choose $\varepsilon > 0$ such that $S_{\lceil a_1 n \rceil} u(x) < -\lceil a_1 n \rceil \varepsilon$ for large n . By continuity of u , there exists $\delta > 0$ with $|u(z) - u(y)| < \varepsilon$ whenever $d_n^{\mathbf{a}}(y, z) < \delta$. Then for any $y \in X_1$ and large n with $x \in B_n^{\mathbf{a}}(y, \delta)$, one has

$$S_{\lceil a_1 n \rceil} u(y) \leq S_{\lceil a_1 n \rceil} u(x) + \lceil a_1 n \rceil \varepsilon \leq 0,$$

contradicting property (P). ■

DEFINITION 2.6. For $Z \subset X_1$, define

$$(2.1) \quad \begin{aligned} m_u^{\mathbf{a}}(Z, s, N, \epsilon) &= \inf_{\mathcal{P}^{\mathbf{a}}(Z, N, \epsilon)} \sum_{(x_i, n_i)} \exp \left(\sup_{x \in B_{n_i}^{\mathbf{a}}(x_i, \epsilon)} -\frac{s}{a_1} S_{\lceil a_1 n_i \rceil} u(x) \right), \\ m_u^{\mathbf{a}}(Z, s, \epsilon) &= \lim_{N \rightarrow \infty} m_u^{\mathbf{a}}(Z, s, N, \epsilon). \end{aligned}$$

This function is non-increasing in s and takes only the values ∞ and 0 except at a critical threshold. Define

$$\dim_u^{\mathbf{a}}(Z, \epsilon) = \inf \{s \in \mathbb{R} : m_u^{\mathbf{a}}(Z, s, \epsilon) = 0\}.$$

Thus $m_u^{\mathbf{a}}(Z, s, \epsilon)$ equals ∞ when $s < \dim_u^{\mathbf{a}}(Z, \epsilon)$, and equals 0 when $s > \dim_u^{\mathbf{a}}(Z, \epsilon)$. The *weighted u -dimension* of Z is then

$$\dim_u^{\mathbf{a}} Z = \lim_{\epsilon \rightarrow 0} \dim_u^{\mathbf{a}}(Z, \epsilon),$$

and the limit exists as in the pressure case.

REMARK 2.7. When $u \equiv 1$, this recovers the entropy $h^{\mathbf{a}, B}(Z)$.

Both weighted pressure and weighted u -dimension are examples of Carathéodory dimension characteristics, hence they satisfy countable stability:

for any countable family $\{Z_i\} \subset X_1$,

$$(2.2) \quad P^{\mathbf{a},B}\left(T_1, \bigcup_i Z_i, \phi\right) = \sup_i P^{\mathbf{a},B}(T_1, Z_i, \phi),$$

$$(2.3) \quad \dim_u^{\mathbf{a}}\left(\bigcup_i Z_i\right) = \sup_i \dim_u^{\mathbf{a}}(Z_i).$$

A central tool for computing u -dimension is the following weighted version of Bowen's equation.

PROPOSITION 2.8. *Let X_1 be compact, $T_1 : X_1 \rightarrow X_1$ continuous, and $u \in C(X_1)$ satisfy (P). Suppose $Z \subset X_1$ is such that $\lim_{n \rightarrow \infty} \frac{1}{|a_1 n|} S_{[a_1 n]} u(x) > 0$ for all $x \in Z$. Then*

$$(2.4) \quad \dim_u^{\mathbf{a}} Z = \inf \{t \in \mathbb{R} : P^{\mathbf{a},B}(T_1, Z, -tu) \leq 0\}.$$

To prove Proposition 2.8, we will repeatedly use the following lemma.

LEMMA 2.9. *Let $T_1 : X_1 \rightarrow X_1$ be continuous, $\phi \in C(X_1)$, and $Z \subset X_1$. Assume there exist $\alpha, \beta \in \mathbb{R}$ such that for every $x \in Z$,*

$$\alpha \leq \varliminf_{n \rightarrow \infty} \frac{1}{n} S_n \phi(x) \leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} S_n \phi(x) \leq \beta.$$

Then for all $t \in \mathbb{R}$ and $h > 0$,

$$(2.5) \quad P^{\mathbf{a},B}(T_1, Z, t\phi) + h\alpha \leq P^{\mathbf{a},B}(T_1, Z, (t+h)\phi) \leq P^{\mathbf{a},B}(T_1, Z, t\phi) + h\beta.$$

Proof. The argument follows the idea of [Cli11, Proposition 4.2]. Fix $\eta > 0$ and for $m \geq 1$ set

$$Z_m = \left\{ x \in Z : \frac{1}{n} S_n \phi(x) \in (\alpha - \eta, \beta + \eta) \text{ for all } n \geq m \right\}.$$

Clearly $Z = \bigcup_{m=1}^{\infty} Z_m$.

Take $t \in \mathbb{R}$, $h > 0$, and $N \geq m/a_1 + 1/a_1$. By the definition of Z_m , for any $\epsilon > 0$ and $s \in \mathbb{R}$,

$$\begin{aligned} \Lambda^{\mathbf{a}}(Z_m, (t+h)\phi, s, N, \epsilon) &= \inf_j \sum \exp \left(-sn_j + \sup_{x \in B_{n_j}^{\mathbf{a}}(x_j, \epsilon)} \frac{1}{a_1} S_{[a_1 n_j]}(t+h)\phi(x) \right) \\ &\geq \inf_j \sum \exp \left(-n_j \left(s - \sup \left(t \cdot \frac{1}{a_1 n_j} S_{[a_1 n_j]} \phi(x_j) + h(\alpha - \eta) \right) \right) \right) \\ &= \Lambda^{\mathbf{a}}(Z_m, t\phi, s - h(\alpha - \eta), N, \epsilon). \end{aligned}$$

Passing to the limits $N \rightarrow \infty$ and $\epsilon \rightarrow 0$, we get

$$P^{\mathbf{a},B}(T_1, Z_m, (t+h)\phi) \geq P^{\mathbf{a},B}(T_1, Z_m, t\phi) + h(\alpha - \eta).$$

Since weighted pressure is countably stable, that is, $P^{\mathbf{a},B}(T_1, Z, \varphi) = \sup_m P^{\mathbf{a},B}(T_1, Z_m, \varphi)$, it follows that

$$P^{\mathbf{a},B}(T_1, Z, (t+h)\phi) \geq P^{\mathbf{a},B}(T_1, Z, t\phi) + h(\alpha - \eta).$$

Letting $\eta \rightarrow 0$ gives the left inequality of (2.5). The right inequality follows in the same way, using $\beta + \eta$ instead of $\alpha - \eta$. ■

Proof of Proposition 2.8. For $Z \subset X_1$, note that

$$\begin{aligned} \Lambda^{\mathbf{a}}(Z, -tu, 0, \epsilon, N) &= \inf_j \sum \exp \left(\sup_{x \in B_{n_j}^{\mathbf{a}}(x_j, \epsilon)} -\frac{t}{a_1} S_{\lceil a_1 n_j \rceil} u(x) \right) \\ &= m_u^{\mathbf{a}}(Z, t, N, \epsilon), \end{aligned}$$

hence $\Lambda^{\mathbf{a}}(Z, -tu, 0, \epsilon) = m_u^{\mathbf{a}}(Z, t, \epsilon)$.

Define

$$t^* = \inf \{t \in \mathbb{R} : P^{\mathbf{a},B}(T_1, Z, -tu) \leq 0\}.$$

If $t < t^*$, then $P^{\mathbf{a},B}(T_1, Z, -tu, \epsilon) > 0$ for small $\epsilon > 0$, so $m_u^{\mathbf{a}}(Z, t, \epsilon) = \infty$ and $\dim_u^{\mathbf{a}} Z \geq t$. Thus $\dim_u^{\mathbf{a}} Z \geq t^*$.

For the reverse inequality, define

$$Z_m = \left\{ x \in Z : \lim_{n \rightarrow \infty} \frac{1}{n} S_n u(x) \geq \frac{1}{m} \right\}.$$

By assumption $\lim_{n \rightarrow \infty} \frac{1}{\lceil a_1 n \rceil} S_{\lceil a_1 n \rceil} u(x) > 0$ for all $x \in Z$, hence $Z = \bigcup_{m \geq 1} Z_m$. By countable stability (2.3), it suffices to show $\dim_u^{\mathbf{a}} Z_m \leq t^*$.

Fix m and take $t > t^*$. By Lemma 2.9,

$$\begin{aligned} P^{\mathbf{a},B}(T_1, Z_m, -tu) &\leq P^{\mathbf{a},B}(T_1, Z_m, -t^*u) - \frac{t - t^*}{m} \\ &\leq P^{\mathbf{a},B}(T_1, Z, -t^*u) - \frac{t - t^*}{m} < 0. \end{aligned}$$

Thus for small $\epsilon > 0$, $m_u^{\mathbf{a}}(Z_m, t, \epsilon) = 0$, implying $\dim_u^{\mathbf{a}} Z_m \leq t$. Since $t > t^*$ was arbitrary, we conclude $\dim_u^{\mathbf{a}} Z_m \leq t^*$, hence $\dim_u^{\mathbf{a}} Z \leq t^*$. This completes the proof. ■

REMARK 2.10. If $Z = \emptyset$, then $P^{\mathbf{a},B}(T_1, Z, -tu) = -\infty$ for all t , which is consistent with $\dim_u^{\mathbf{a}} Z = -\infty$. If $Z \neq \emptyset$, then there exists $x \in Z$ with $\lim_{n \rightarrow \infty} \frac{1}{\lceil a_1 n \rceil} S_{\lceil a_1 n \rceil} u(x) = \alpha > 0$. In this case, for $t < 0$,

$$P^{\mathbf{a},B}(T_1, Z, -tu) \geq P^{\mathbf{a},B}(T_1, \{x\}, -tu) \geq |t|\alpha > 0,$$

so only $t \geq 0$ are relevant in Bowen's equation.

The next step is to introduce weighted u -dimension for measures, which provides an effective tool to estimate $\dim_u^{\mathbf{a}} Z$ from below.

DEFINITION 2.11. For $\mu \in \mathcal{M}(X_1)$, the *weighted u -dimension* of μ is

(2.6) $\dim_u^{\mathbf{a}} \mu := \inf \{ \dim_u^{\mathbf{a}} Z : \mu(Z) = 1 \}.$

PROPOSITION 2.12. *Let $\mu \in \mathcal{M}_E(X_1, T_1)$ be ergodic with $\int u d\mu > 0$. Then*

$$(2.7) \quad \dim_u^{\mathbf{a}} \mu = \frac{h_\mu^{\mathbf{a}}(T_1)}{\int u d\mu}.$$

Proof. Let G_μ be the set of generic points for μ . Then $\frac{1}{n}S_n u(x) \rightarrow \int u d\mu$ for all $x \in G_\mu$, and $h^{\mathbf{a},B}(G_\mu) = h_\mu^{\mathbf{a}}(T_1)$ by [ZCZ18]. Applying Lemma 2.9, we obtain

$$P^{\mathbf{a},B}(T_1, G_\mu, -tu) = h_\mu^{\mathbf{a}}(T_1) - t \int u d\mu.$$

This vanishes exactly at $t = h_\mu^{\mathbf{a}}(T_1) / \int u d\mu$. By Proposition 2.8, $\dim_u^{\mathbf{a}} G_\mu = h_\mu^{\mathbf{a}}(T_1) / \int u d\mu$, hence $\dim_u^{\mathbf{a}} \mu \leq \dim_u^{\mathbf{a}} G_\mu$.

Conversely, if $Z \subset X_1$ satisfies $\mu(Z) = 1$, set

$$Z' = \left\{ x \in Z : \frac{1}{n}S_n u(x) \rightarrow \int u d\mu \right\}.$$

Then $\mu(Z') = 1$ and $h^{\mathbf{a},B}(Z') \geq h_\mu^{\mathbf{a}}(T_1)$. Lemma 2.9 gives

$$P^{\mathbf{a},B}(T_1, Z', -tu) \geq h_\mu^{\mathbf{a}}(T_1) - t \int u d\mu,$$

so by Proposition 2.8, $\dim_u^{\mathbf{a}} Z \geq \dim_u^{\mathbf{a}} Z' \geq h_\mu^{\mathbf{a}}(T_1) / \int u d\mu$. ■

REMARK 2.13. Formula (2.7) does not extend to non-ergodic measures. Indeed, $\mu \mapsto h_\mu^{\mathbf{a}}(T_1)$ and $\mu \mapsto \int u d\mu$ are affine, while from (2.6) one checks easily that for convex combinations,

$$\dim_u^{\mathbf{a}}(t\mu_1 + (1-t)\mu_2) = \max \{ \dim_u^{\mathbf{a}} \mu_1, \dim_u^{\mathbf{a}} \mu_2 \}, \quad 0 < t < 1.$$

3. Mixed multifractal spectra. Let $\varphi, \psi, u \in C(X_1)$ with u satisfying condition **(P)**. For $\alpha \in \mathbb{R}$, define the level set

$$(3.1) \quad K_\alpha = K_\alpha(\varphi, \psi) = \left\{ x \in X_1 : \lim_{n \rightarrow \infty} \frac{S_n \varphi(x)}{S_n \psi(x)} = \alpha \right\}.$$

The weighted multifractal spectrum associated with (φ, ψ, u) is

$$\mathcal{F}^{\mathbf{a}}(\alpha) = \mathcal{F}_u^{\mathbf{a}, \varphi, \psi}(\alpha) := \dim_u^{\mathbf{a}} K_\alpha(\varphi, \psi),$$

with the convention $\dim_u^{\mathbf{a}} \emptyset = -\infty$, so that $\mathcal{F}^{\mathbf{a}}(\alpha) = -\infty$ whenever $K_\alpha = \emptyset$.

Next, consider the subset

$$(3.2) \quad \hat{X}_1 := \left\{ x \in X_1 : \varliminf_{n \rightarrow \infty} S_n \psi(x) > 0 \text{ and } \varliminf_{n \rightarrow \infty} \frac{1}{n} \log S_n u(x) > 0 \right\},$$

and the restricted level sets $\hat{K}_\alpha := K_\alpha \cap \hat{X}_1$. We then study the modified spectrum

$$\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) := \dim_u^{\mathbf{a}} \hat{K}_\alpha(\varphi, \psi).$$

In the applications of interest, one typically finds $\hat{\mathcal{F}}^{\mathbf{a}} = \mathcal{F}^{\mathbf{a}}$ on the relevant domain, so the results obtained apply directly to $\mathcal{F}^{\mathbf{a}}$.

For each $\alpha \in \mathbb{R}$, define

$$(3.3) \quad R_\alpha^{\mathbf{a}}(q, t) := P^{\mathbf{a}, B}(T_1, X_1, q(\varphi - \alpha\psi) - tu),$$

and the function

$$(3.4) \quad T_\alpha^{\mathbf{a}}(q) := \inf \{t \in \mathbb{R} : R_\alpha^{\mathbf{a}}(q, t) \leq 0\}.$$

The thermodynamic prediction for the spectrum is then

$$(3.5) \quad \mathcal{S}^{\mathbf{a}}(\alpha) := \inf_{q \in \mathbb{R}} T_\alpha^{\mathbf{a}}(q),$$

and its natural domain is

$$(3.6) \quad I := \{\alpha \in \mathbb{R} : \mathcal{S}^{\mathbf{a}}(\alpha) > -\infty\}.$$

PROPOSITION 3.1. *For every $\alpha \in I$, one has $\mathcal{S}^{\mathbf{a}}(\alpha) \geq 0$.*

LEMMA 3.2. *If $\phi \in C(X_1)$ satisfies $P^{\mathbf{a}, B}(T_1, \phi) \leq 0$, then $P^{\mathbf{a}, B}(T_1, \lambda\phi) \leq 0$ for all $\lambda \geq 1$.*

Proof. For $\mu \in \mathcal{M}(X_1, T_1)$, the condition $P^{\mathbf{a}, B}(T_1, \phi) \leq 0$ implies

$$h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \phi d\mu \leq 0.$$

Thus $\int_{X_1} \phi d\mu \leq 0$, and for $\lambda \geq 1$,

$$h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \lambda\phi d\mu = \left(h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \phi d\mu\right) + (\lambda - 1) \int_{X_1} \phi d\mu \leq 0.$$

Taking the supremum over μ yields the claim. ■

Proof of Proposition 3.1. Suppose $\mathcal{S}^{\mathbf{a}}(\alpha) < 0$. Then there exist $q, t \in \mathbb{R}$ with $t < 0$ and

$$P^{\mathbf{a}, B}(T_1, q(\varphi - \alpha\psi) - tu) \leq 0.$$

By Lemma 3.2, $R_\alpha^{\mathbf{a}}(\lambda q, \lambda t) \leq 0$ for all $\lambda \geq 1$, hence $T_\alpha^{\mathbf{a}}(\lambda q) \leq \lambda t$. Since $t < 0$, this forces $\mathcal{S}^{\mathbf{a}}(\alpha) = -\infty$, contradicting $\alpha \in I$. Thus $\mathcal{S}^{\mathbf{a}}(\alpha) \geq 0$. ■

Our first main theorem establishes one side of the multifractal formalism.

LEMMA 3.3 ([Cli10a]). *If $\frac{S_n \varphi(x)}{S_n \psi(x)} \rightarrow \alpha$, then $\frac{1}{n} S_n(\varphi - \alpha\psi) \rightarrow 0$.*

THEOREM 3.4. *Let $\varphi, \psi, u \in C(X_1)$ with u satisfying $(\mathbf{P})$. Then*

$$(3.7) \quad \hat{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \mathcal{S}^{\mathbf{a}}(\alpha), \quad \forall \alpha \in \mathbb{R}.$$

In particular, $\hat{K}_\alpha = \emptyset$ whenever $\alpha \notin I$.

Proof. Fix $\alpha \in \mathbb{R}$ and $t < \hat{\mathcal{F}}^{\mathbf{a}}(\alpha)$. If $\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) = -\infty$, the claim is trivial. We assert that

$$(3.8) \quad R_\alpha^{\mathbf{a}}(q, t) = P^{\mathbf{a}, B}(T_1, q(\varphi - \alpha\psi) - tu) > 0, \quad \forall q \in \mathbb{R}.$$

This implies $T_\alpha^{\mathbf{a}}(q) \geq t$ and hence

$$\mathcal{S}^{\mathbf{a}}(\alpha) = \inf_q T_\alpha^{\mathbf{a}}(q) \geq t.$$

Since $t < \hat{\mathcal{F}}^{\mathbf{a}}(\alpha)$ was arbitrary, the inequality (3.7) follows.

To prove (3.8), note that $t < \dim_u^{\mathbf{a}} \hat{K}_\alpha$ and $\lim_{n \rightarrow \infty} \frac{1}{n} S_n u(x) > 0$ for $x \in \hat{K}_\alpha$ imply

$$P^{\mathbf{a},B}(T_1, \hat{K}_\alpha, -tu) > 0$$

via Bowen's equation (Proposition 2.8). Moreover, Lemmas 2.9 and 3.3 yield

$$P^{\mathbf{a},B}(T_1, \hat{K}_\alpha, q(\varphi - \alpha\psi) - tu) = P^{\mathbf{a},B}(T_1, \hat{K}_\alpha, -tu) > 0,$$

proving (3.8). ■

In all our examples, $\hat{\mathcal{F}}^{\mathbf{a}} = \mathcal{F}^{\mathbf{a}}$, so Theorem 3.4 applies directly to the spectrum $\mathcal{F}^{\mathbf{a}}$. Note also that (3.7) extends to $\alpha \in \partial I$, but not to $\alpha = \infty$.

A measure ν is called a *full measure* for $\hat{\mathcal{F}}^{\mathbf{a}}$ at α if ν is ergodic and invariant, $\nu(\hat{K}_\alpha) = 1$, and $\dim_u^{\mathbf{a}} \nu = \mathcal{S}^{\mathbf{a}}(\alpha)$.

THEOREM 3.5. *Let φ, ψ, u be as in Theorem 3.4. Fix $\alpha \in \mathbb{R}$ and assume $q = q(\alpha)$ realizes the infimum in (3.5), so that $\mathcal{S}^{\mathbf{a}}(\alpha) = T_\alpha^{\mathbf{a}}(q)$. Suppose further that*

- (1) *the potential $q(\alpha)(\varphi - \alpha\psi) - \mathcal{S}^{\mathbf{a}}(\alpha)u$ admits an equilibrium state ν_α with $\int \psi d\nu_\alpha > 0$ and $\int u d\nu_\alpha > 0$;*
- (2) *$R_\alpha^{\mathbf{a}}$ is differentiable at $(q(\alpha), \mathcal{S}^{\mathbf{a}}(\alpha))$.*

Then ν_α is a full measure for $\hat{\mathcal{F}}^{\mathbf{a}}$ at α , and in particular $\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) = \mathcal{S}^{\mathbf{a}}(\alpha)$.

REMARK 3.6. The equivalence between the two characterizations of $\mathcal{S}^{\mathbf{a}}(\alpha)$ is classical in cases where the multifractal formalism holds (see e.g. [BS01] in the non-weighted setting). Theorem 4.3 extends this equivalence even beyond those situations.

The proof relies on Ruelle's formula for derivatives of weighted pressure.

PROPOSITION 3.7. *Let $\eta, \phi \in C(X_1)$. If $q \mapsto P^{\mathbf{a},B}(T_1, X_1, \eta + q\phi)$ is differentiable at $q = 0$ and ν is an equilibrium state for η , then*

$$(3.9) \quad \left. \frac{d}{dq} P^{\mathbf{a},B}(T_1, X_1, \eta + q\phi) \right|_{q=0} = \int_{X_1} \phi d\nu.$$

Proof. Let $g(q) = P^{\mathbf{a},B}(T_1, X_1, \eta + q\phi)$. Then

$$g(q) = \sup_{\mu} \left\{ h^{\mathbf{a}}(\mu) + \int \eta d\mu + q \int \phi d\mu \right\}.$$

In particular,

$$g(q) \geq h^{\mathbf{a}}(\nu) + \int \eta d\nu + q \int \phi d\nu = g(0) + q \int \phi d\nu.$$

Thus $\frac{g(q) - g(0)}{q} \geq \int \phi d\nu$ for $q > 0$, yielding $g'(0) \geq \int \phi d\nu$, and the reverse inequality follows similarly for $q < 0$. Hence equality holds. ■

Proof of Theorem 3.5. Since $R_\alpha^{\mathbf{a}}(q, T_\alpha^{\mathbf{a}}(q)) = 0$ for all q , we have

$$0 = \frac{d}{dq} P^{\mathbf{a},B}(T_1, X_1, q(\varphi - \alpha\psi) - T_\alpha^{\mathbf{a}}(q)u).$$

At $q = q(\alpha)$, the fact that $\frac{d}{dq} T_\alpha^{\mathbf{a}}(q) = 0$ (as q realizes the infimum in (3.5)) implies via Proposition 3.7 that

$$\int (\varphi - \alpha\psi) d\nu_\alpha = 0,$$

hence $\alpha = \frac{\int \varphi d\nu_\alpha}{\int \psi d\nu_\alpha}$ with $\int \psi d\nu_\alpha > 0$. Ergodicity gives $\nu_\alpha(K_\alpha) = 1$, so

$$\mathcal{F}^{\mathbf{a}}(\alpha) \geq \dim_u^{\mathbf{a}} \nu_\alpha.$$

Since ν_α is an equilibrium state for $q(\varphi - \alpha\psi) - \mathcal{S}^{\mathbf{a}}(\alpha)u$, we also have

$$0 = P^{\mathbf{a},B}(T_1, X_1, q(\varphi - \alpha\psi) - \mathcal{S}^{\mathbf{a}}(\alpha)u) = h_{\nu_\alpha}^{\mathbf{a}}(T_1) - \mathcal{S}^{\mathbf{a}}(\alpha) \int u d\nu_\alpha,$$

whence

$$\mathcal{S}^{\mathbf{a}}(\alpha) = \frac{h^{\mathbf{a}}(\nu_\alpha)}{\int u d\nu_\alpha} = \dim_u^{\mathbf{a}} \nu_\alpha \leq \mathcal{F}^{\mathbf{a}}(\alpha).$$

Thus ν_α is a full measure for $\hat{\mathcal{F}}^{\mathbf{a}}$ at α , and $\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) = \mathcal{S}^{\mathbf{a}}(\alpha)$. ■

4. Conditional variational principle. As mentioned earlier, the predicted spectrum can be given in terms of a conditional variational principle for nearly every value of α .

Given $\alpha \in \mathbb{R}$, let

$$\mathcal{K}(\alpha) := \left\{ \int_{X_1} (\varphi - \alpha\psi) d\mu : \mu \in \mathcal{M}(X_1, T_1) \right\},$$

$$\hat{I} := \{\alpha : 0 \in \text{int } \mathcal{K}(\alpha)\},$$

$$\mathcal{M}_\alpha(X_1, T_1) := \left\{ \mu \in \mathcal{M}(X_1, T_1) : \int_{X_1} (\varphi - \alpha\psi) d\mu = 0, \int_{X_1} u d\mu > 0 \right\}.$$

PROPOSITION 4.1. *If there exists $\gamma > 0$ such that $\int_{X_1} \psi d\mu \geq \gamma h_\mu^{\mathbf{a}}(T_1)$ for all $\mu \in \mathcal{M}(X_1, T_1)$, then $\hat{I} = \text{int } I$.*

Proof. First, we show that $\hat{I} \subset \text{int } I$ without any hypotheses regarding $\int_{X_1} \psi d\mu$ or $h_\mu^{\mathbf{a}}(T_1)$. To this end, fix $\alpha \in \hat{I}$, so $0 \in \text{int } \mathcal{K}(\alpha)$. Thus there exist $\eta > 0$ and $\mu^\pm \in \mathcal{M}(X_1, T_1)$ such that $\int_{X_1} (\varphi - \alpha\psi) d\mu^\pm = \pm\eta$. Choosing $\varepsilon > 0$ sufficiently small, we have $\int_{X_1} (\varphi - \alpha'\psi) d\mu^+ > 0$ and $\int_{X_1} (\varphi - \alpha'\psi) d\mu^- < 0$ for all $\alpha' \in (\alpha - \varepsilon, \alpha + \varepsilon)$. It follows that $0 \in \text{int } \mathcal{K}(\alpha')$ for each such α' , so $\hat{I}$ is open. Furthermore, given $\alpha \in \hat{I}$ and $\eta > 0$, $\mu^\pm \in \mathcal{M}(X_1, T_1)$ as before, we have

$$\begin{aligned}
R_\alpha^{\mathbf{a}}(q, 0) &= P^{\mathbf{a}, B}(T_1, X_1, q(\varphi - \alpha\psi)) \\
&\geq \max \left(h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} q(\varphi - \alpha\psi) d\mu^\pm \right) \geq |q|\eta > 0
\end{aligned}$$

for all $q \neq 0$, whence $T_\alpha^{\mathbf{a}}(q) > 0$. It follows that $\mathcal{S}^{\mathbf{a}}(\alpha) \geq 0$, and so $\alpha \in I$. Now, since $\hat{I}$ is open and contained in I , we have $\hat{I} \subset \text{int } I$.

It remains to show that $\text{int } I \subset \hat{I}$. To this end, let $\gamma > 0$ be such that $\int_{X_1} \psi d\mu \geq \gamma h_\mu^{\mathbf{a}}(T_1)$ for all $\mu \in \mathcal{M}(X_1, T_1)$, and fix $\alpha \notin \hat{I}$. Then $0 \notin \text{int } \mathcal{K}(\alpha)$, and since $\mathcal{K}(\alpha)$ is an interval, we have either $\mathcal{K}(\alpha) \subset [0, +\infty)$ or $\mathcal{K}(\alpha) \subset (-\infty, 0]$.

First, consider the case $\mathcal{K}(\alpha) \subset [0, +\infty)$. Then $\int_{X_1} (\varphi - \alpha\psi) d\mu \geq 0$ for all $\mu \in \mathcal{M}(X_1, T_1)$, and so for every $\varepsilon < 0$ and $q < 0$, we have

$$\begin{aligned}
R_{\alpha+\varepsilon}^{\mathbf{a}}(q, t) &= P^{\mathbf{a}, B}(T_1, X_1, q(\varphi - (\alpha + \varepsilon)\psi) - tu) \\
&= \sup_{\mu \in \mathcal{M}(X_1, T_1)} \left(h_\mu^{\mathbf{a}}(T_1) + q \int_{X_1} (\varphi - \alpha\psi) d\mu - q\varepsilon \int_{X_1} \psi d\mu - t \int_{X_1} u d\mu \right) \\
&\leq \sup_{\mu \in \mathcal{M}(X_1, T_1)} (h_\mu^{\mathbf{a}}(T_1)(1 - q\varepsilon\gamma) + |t| \sup u).
\end{aligned}$$

For sufficiently small q and for $t < 0$ with $|t|$ sufficiently small, this gives $R_{\alpha+\varepsilon}^{\mathbf{a}}(q, t) \leq 0$, whence $\mathcal{S}^{\mathbf{a}}(\alpha + \varepsilon) \leq T_{\alpha+\varepsilon}^{\mathbf{a}}(q) \leq t < 0$, and so $\alpha + \varepsilon \notin I$. Since $\varepsilon < 0$ was arbitrary, we have $\alpha \notin \text{int } I$. The proof in the case $\mathcal{K}(\alpha) \subset (-\infty, 0]$ is similar, and we see that $\text{int } I = \hat{I}$. ■

REMARK 4.2. The hypothesis of Proposition 4.1 holds if ψ is uniformly positive and $h_\mu^{\mathbf{a}}(T_1) < \infty$; thus in this case Theorem 4.3 applies to all $\alpha \in \text{int } I$.

THEOREM 4.3. *Let φ, ψ, u be as in Theorem 3.4. Then for every $\alpha \in \hat{I}$, the infimum in (3.5) is achieved at a finite value of q , and the weighted predicted spectrum $\mathcal{S}^{\mathbf{a}}(\alpha)$ satisfies*

$$(4.1) \quad \mathcal{S}^{\mathbf{a}}(\alpha) = \sup_{\mu \in \mathcal{M}_\alpha(X_1, T_1)} \left(\frac{h_\mu^{\mathbf{a}}(T_1)}{\int_{X_1} u d\mu} \right).$$

Proof. We write

$$\tilde{\mathcal{S}}^{\mathbf{a}}(\alpha) := \sup \left\{ \frac{h_\mu^{\mathbf{a}}(T_1)}{\int_{X_1} u d\mu} : \mu \in \mathcal{M}_\alpha(X_1, T_1) \right\}.$$

We begin by showing that $\mathcal{S}^{\mathbf{a}}(\alpha) \geq \tilde{\mathcal{S}}^{\mathbf{a}}(\alpha)$ for every α . To this end, fix $t < \tilde{\mathcal{S}}^{\mathbf{a}}(\alpha)$. Then there exists $\mu \in \mathcal{M}_\alpha(X_1, T_1)$ such that $h_\mu^{\mathbf{a}}(T_1) > t \int_{X_1} u d\mu$, and therefore for every $q \in \mathbb{R}$ we have

$$P^{\mathbf{a}, B}(T_1, X_1, q(\varphi - \alpha\psi) - tu) \geq h_\mu^{\mathbf{a}}(T_1) + q \int_{X_1} (\varphi - \alpha\psi) d\mu - t \int_{X_1} u d\mu > 0.$$

So $T_\alpha^{\mathbf{a}}(q) > t$. This implies that $\mathcal{S}^{\mathbf{a}}(\alpha) \geq t$, and since $t < \tilde{\mathcal{S}}^{\mathbf{a}}(\alpha)$ was arbitrary, one direction is done.

It remains only to show that $\tilde{\mathcal{S}}^{\mathbf{a}}(\alpha) \geq \mathcal{S}^{\mathbf{a}}(\alpha)$ when $\alpha \in \hat{I}$. Fix $\alpha \in \hat{I}$, and for all $q, t \in \mathbb{R}$, write $\phi_{q,t} := q(\varphi - \alpha\psi) - tu$. Consider the collections

$$\begin{aligned}\mathcal{M}^+ &:= \left\{ \mu \in \mathcal{M}_\alpha(X_1, T_1) : \int_{X_1} (\varphi - \alpha\psi) d\mu \geq 0 \right\}, \\ \mathcal{M}^- &:= \left\{ \mu \in \mathcal{M}_\alpha(X_1, T_1) : \int_{X_1} (\varphi - \alpha\psi) d\mu \leq 0 \right\}\end{aligned}$$

of measures and define two functions $g^\pm : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g^\pm(q, t) := \sup \left\{ h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \phi_{q,t} d\mu : \mu \in \mathcal{M}^\pm \right\}.$$

Observe that $P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t}) = \max(g^+(q, t), g^-(q, t))$, and so letting

$$\begin{aligned}(4.2) \quad E^\pm &:= \{(q, t) \in \mathbb{R}^2 : P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t}) = g^\pm(q, t)\} \\ &= \{(q, t) \in \mathbb{R}^2 : g^\pm(q, t) \geq g^\mp(q, t)\},\end{aligned}$$

we have $E^+ \cup E^- = \mathbb{R}^2$. (The sets E^+ and E^- need not be disjoint.) It follows from the definitions that $g^-(q, t)$ is non-increasing in q and $g^+(q, t)$ is non-decreasing in q . Using the second characterization in (4.2), this implies that given any $(q, t) \in E^-$, we have $(q', t) \in E^-$ and $P^{\mathbf{a},B}(T_1, X_1, \phi_{q',t}) \geq P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t})$ for all $q' < q$. Similarly, given any $(q, t) \in E^+$, we have $(q', t) \in E^+$ and $P^{\mathbf{a},B}(T_1, X_1, \phi_{q',t}) \leq P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t})$ for all $q' > q$.

Now fix $t < \mathcal{S}^{\mathbf{a}}(\alpha)$ and let $E^\pm(t) := \{q \in \mathbb{R} : (q, t) \in E^\pm\}$. Because $0 \in \text{int } \mathcal{K}(\alpha)$ (which follows from $\alpha \in \hat{I}$), there exists $\mu \in \mathcal{M}(X_1, T_1)$ with $\int_{X_1} (\varphi - \alpha\psi) d\mu > 0$, whence $P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t}) > P^{\mathbf{a},B}(T_1, X_1, \phi_{0,t})$ for sufficiently large values of q ; for such q we have $q \in E^+(t)$, and so $E^+(t) \neq \emptyset$. A similar argument shows that $E^-(t) \neq \emptyset$.

Because both $E^-(t)$ and $E^+(t)$ are non-empty, there exists $q_0 \in \overline{E^-(t)} \cap \overline{E^+(t)}$. If $q_0 \in E^-(t)$, then we have $q_0 + \gamma \in E^+(t)$ for all $\gamma > 0$, while if $q_0 \in E^+(t)$, then $q_0 - \gamma \in E^-(t)$ for all $\gamma > 0$. Without loss of generality, we assume the former case; the proof in the latter case is similar.

Now we have $q^- := q_0 \in E^-(t)$ such that $q^+ \in E^+(t)$ for all $q^+ > q^-$. Let $\delta := \inf \{P^{\mathbf{a},B}(T_1, X_1, \phi_{q,t}) : |q - q^-| \leq 1\}$, and observe that $\delta > 0$. Choose $q^+ > q^-$ so that $q^+ - q^- < \frac{\delta}{2} \|\varphi - \alpha\psi\|^{-1}$. Then there exist $\mu^\pm \in \mathcal{M}^\pm$ such that

$$h_{\mu^\pm}^{\mathbf{a}}(T_1) + \int_{X_1} \phi_{q^\pm, t} d\mu^\pm \geq \delta/2.$$

Choose $s \in [0, 1]$ such that $\mu := s\mu^- + (1-s)\mu^+$ satisfies $\int_{X_1} (\varphi - \alpha\psi) d\mu = 0$. Then we have

$$\begin{aligned}
h_\mu^{\mathbf{a}}(T_1) - t \int_{X_1} u \, d\mu &= h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \phi_{q^-,t} \, d\mu \\
&= s h_{\mu^-}^{\mathbf{a}}(T_1) + (1-s) h_{\mu^+}^{\mathbf{a}}(T_1) + s \int_{X_1} \phi_{q^-,t} \, d\mu^- + (1-s) \int_{X_1} \phi_{q^-,t} \, d\mu^+ \\
&\geq \frac{\delta}{2} + (1-s) \int_{X_1} (\phi_{q^-,t} - \phi_{q^+,t}) \, d\mu^+ \\
&\geq \frac{\delta}{2} - (q^+ - q^-) \|\varphi - \alpha\psi\| > 0,
\end{aligned}$$

where the final inequality follows from the choice of q^+ . Let $q = q(\alpha)$ be such that $T_\alpha^{\mathbf{a}}(q) = \mathcal{S}^{\mathbf{a}}(\alpha)$ such a q exists because $0 \in \text{int } \mathcal{K}(\alpha)$ and observe that

$$\begin{aligned}
h_\mu^{\mathbf{a}}(T_1) - \mathcal{S}^{\mathbf{a}}(\alpha) \int u \, d\mu &= h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} \phi_{q,\mathcal{S}(\alpha)} \, d\mu \\
&\leq P^{\mathbf{a},B}(T_1, X_1, \phi_{q,\mathcal{S}(\alpha)}) \\
&= 0 < h_\mu^{\mathbf{a}}(T_1) - t \int_{X_1} u \, d\mu.
\end{aligned}$$

Hence $\int_{X_1} u \, d\mu > 0$, so $\mu \in \mathcal{M}_\alpha(X_1, T_1)$. Finally, the inequality

$$h_\mu^{\mathbf{a}}(T_1) - t \int_{X_1} u \, d\mu > 0$$

implies that $t < \frac{h_\mu^{\mathbf{a}}(T_1)}{\int_{X_1} u \, d\mu}$, and since $t < \mathcal{S}^{\mathbf{a}}(\alpha)$ was arbitrary, this yields $\tilde{\mathcal{S}}^{\mathbf{a}}(\alpha) \geq \mathcal{S}^{\mathbf{a}}(\alpha)$. ■

REMARK 4.4. The infimum in (3.5) may not be achieved by any finite value of q ; Theorem 4.3 gives conditions under which it is achieved. Even if it is achieved, however, the minimizing value of q may not be unique.

The measures in the conditional variational principle (4.1) need not be supported on K_α unless they are ergodic. Thus, in general, we have the following extended version of (3.7):

$$(4.3) \quad \sup \{ \dim_u^{\mathbf{a}} \mu : \mu \in \mathcal{M}_E(K_\alpha, T_1) \} \leq \hat{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \mathcal{S}^{\mathbf{a}}(\alpha).$$

It is of interest to know when all three quantities in (4.3) are equal. By Theorem 3.5, this is the case when $R_\alpha^{\mathbf{a}}$ is differentiable at $(q(\alpha), \mathcal{S}^{\mathbf{a}}(\alpha))$, provided an equilibrium state ν exists with $\int_{X_1} \psi \, d\nu > 0$ and $\int_{X_1} u \, d\nu > 0$; furthermore, in this case the supremum is achieved. Corollary 4.5 gives equality in (4.3) under weaker conditions, but in this case the supremum may not be achieved.

COROLLARY 4.5. *Let φ, ψ, u be as in Theorem 3.4. Fix $\alpha \in \mathbb{R}$, and suppose that there exist compact invariant sets $X_1^n \subset X_1$ such that the hypotheses of Theorem 3.5 hold at α for the restriction of the system to X_1^n . Then equality holds in (4.3). In particular, $\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) = \mathcal{S}^{\mathbf{a}}(\alpha)$.*

5. Legendre transform. When $\psi = u$, we obtain

$$R_\alpha^{\mathbf{a}}(q, t) = P^{\mathbf{a}, B}(T_1, X_1, q\varphi - (\alpha q + t)u),$$

so that

$$T_\alpha^{\mathbf{a}}(q) = \inf \{t \in \mathbb{R} : P^{\mathbf{a}, B}(T_1, X_1, q\varphi - (\alpha q + t)u) \leq 0\}.$$

In particular, the case $\alpha = 0$ gives

$$(5.1) \quad T_0^{\mathbf{a}}(q) = \inf \{t \in \mathbb{R} : P^{\mathbf{a}, B}(T_1, X_1, q\varphi - tu) \leq 0\},$$

and hence

$$T_\alpha^{\mathbf{a}}(q) = T_0^{\mathbf{a}}(q) - \alpha q.$$

Therefore,

$$(5.2) \quad \mathcal{S}^{\mathbf{a}}(\alpha) = \inf_{q \in \mathbb{R}} \{T_0^{\mathbf{a}}(q) - \alpha q\},$$

which shows that $\mathcal{S}^{\mathbf{a}}$ is the Legendre transform of $T_0^{\mathbf{a}}$. Geometrically, for each α , $\mathcal{S}^{\mathbf{a}}(\alpha)$ is the y -intercept of the subtangent line to the graph of $T_0^{\mathbf{a}}$ with slope α . Convexity ensures the subtangent is unique, although it need not touch the graph at a single point.

PROPOSITION 5.1. *Let $Q := \{q \in \mathbb{R} : T_0^{\mathbf{a}}(q) < \infty\}$. Then exactly one of the following holds:*

- (1) $Q = \mathbb{R}$;
- (2) $Q = (q_0, \infty)$ or $[q_0, \infty)$ for some $q_0 \geq 0$;
- (3) $Q = (-\infty, q_0)$ or $(-\infty, q_0]$ for some $q_0 \leq 0$.

Proof. This is an immediate consequence of the lemma below, valid even when $u \neq \psi$. ■

LEMMA 5.2. *For all $\lambda \geq 1$ and $\alpha, q \in \mathbb{R}$,*

$$T_\alpha^{\mathbf{a}}(\lambda q) \leq \lambda T_\alpha^{\mathbf{a}}(q).$$

In particular, if $T_\alpha^{\mathbf{a}}(q) < \infty$, then $T_\alpha^{\mathbf{a}}(\lambda q) < \infty$ for all $\lambda \geq 1$.

Proof. The condition $T_\alpha^{\mathbf{a}}(q) < \infty$ means that some $t < \infty$ satisfies

$$P^{\mathbf{a}, B}(T_1, X_1, q(\varphi - \alpha\psi) - tu) \leq 0.$$

By Lemma 3.2, this implies

$$P^{\mathbf{a}, B}(T_1, X_1, \lambda q(\varphi - \alpha\psi) - \lambda tu) \leq 0 \quad \text{for all } \lambda \geq 1,$$

hence $T_\alpha^{\mathbf{a}}(\lambda q) \leq \lambda T_\alpha^{\mathbf{a}}(q)$. ■

By convexity of $T_0^{\mathbf{a}}$, the one-sided derivatives

$$D^-T_0^{\mathbf{a}}(q) = \lim_{\varepsilon \rightarrow 0^+} \frac{T_0^{\mathbf{a}}(q) - T_0^{\mathbf{a}}(q - \varepsilon)}{\varepsilon}, \quad D^+T_0^{\mathbf{a}}(q) = \lim_{\varepsilon \rightarrow 0^+} \frac{T_0^{\mathbf{a}}(q + \varepsilon) - T_0^{\mathbf{a}}(q)}{\varepsilon}$$

exist for each $q \in \text{int } Q$. Define $J(q) := [D^-T_0^{\mathbf{a}}(q), D^+T_0^{\mathbf{a}}(q)]$. Then $T_0^{\mathbf{a}}$ is differentiable at q if and only if $\text{int } J(q) = \emptyset$. Moreover, convexity implies the intervals $\text{int } J(q)$ are disjoint for distinct q , so at most countably many q admit a non-trivial interval. On each such $J(q)$, $\mathcal{S}^{\mathbf{a}}$ is affine in α , while on intervals disjoint from every $\text{int } J(q)$ it is strictly concave.

For each q , the graph of $T_0^{\mathbf{a}}$ admits subtangents at slope α for all $\alpha \in J(q)$. Define

$$\tilde{I} := \bigcup_{q \in \text{int } Q} J(q),$$

the set of slopes corresponding to finite subtangents. If Q is open, then $\tilde{I} = I$. Otherwise, e.g. when $Q = [q_0, \infty)$ or $Q = (-\infty, q_0]$, this inclusion can be strict.

THEOREM 5.3. *Let φ, ψ, u be as in Theorem 3.4, with $u = \psi$. Then on $\tilde{I}$ the predicted spectrum is the concave hull of $\hat{\mathcal{F}}^{\mathbf{a}}$:*

$$\mathcal{S}^{\mathbf{a}}(\alpha) = \inf \{ \tilde{\mathcal{S}}^{\mathbf{a}}(\alpha) : \tilde{\mathcal{S}}^{\mathbf{a}} \geq \hat{\mathcal{F}}^{\mathbf{a}}, \tilde{\mathcal{S}}^{\mathbf{a}} \text{ concave on } \tilde{I} \} \quad \text{for all } \alpha \in \tilde{I}.$$

Proof. It suffices to show that for $q \in \text{int } Q$,

$$T_0^{\mathbf{a}}(q) = \sup_{\alpha \in \mathbb{R}} \{ \hat{\mathcal{F}}^{\mathbf{a}}(\alpha) + q\alpha \},$$

since then $\mathcal{S}^{\mathbf{a}}$ is the double Legendre transform of $\hat{\mathcal{F}}^{\mathbf{a}}$ on $\tilde{I}$, i.e. its concave hull. We check the inequalities:

(i) By Theorem 3.4,

$$\hat{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \mathcal{S}^{\mathbf{a}}(\alpha) \leq T_0^{\mathbf{a}}(q) - q\alpha,$$

so $T_0^{\mathbf{a}}(q) \geq \hat{\mathcal{F}}^{\mathbf{a}}(\alpha) + q\alpha$ for all α .

(ii) Conversely, if $t < T_0^{\mathbf{a}}(q)$, then the variational principle for $q\varphi - tu$ can be restricted to ergodic ν with $\int u \, d\nu > 0$. For such ν , let α satisfy $\nu(K_\alpha) = 1$. Then

$$(\dim_u^{\mathbf{a}} \nu + q\alpha - t) \int u \, d\nu > 0,$$

implying $t < \mathcal{F}^{\mathbf{a}}(\alpha) + q\alpha$. As $t < T_0^{\mathbf{a}}(q)$ was arbitrary, we obtain

$$T_0^{\mathbf{a}}(q) \leq \sup_{\alpha \in \mathbb{R}} \{ \hat{\mathcal{F}}^{\mathbf{a}}(\alpha) + q\alpha \}. \quad \blacksquare$$

REMARK 5.4. If $u \neq \psi$, the spectrum $\mathcal{S}$ need not be concave. Counterexamples in the unweighted case appear in [BS01, IK09].

Finally, the Legendre transform characterization of $\mathcal{S}$ aligns with the conditional variational principle in many cases, since Proposition 4.1 ensures that Theorem 4.3 applies throughout $\text{int } I$.

PROPOSITION 5.5. *Let X_1 be compact metric, $T_1 : X_1 \rightarrow X_1$ continuous, and $\varphi, \psi, u \in C(X_1)$ with $\psi \equiv u$ satisfying (P). Assume every ergodic μ with $h_\mu^{\mathbf{a}}(T_1) > 0$ also satisfies $\int u d\mu > 0$, and $\dim_u^{\mathbf{a}} X < \infty$. Then*

$$\int u d\mu \geq \frac{h_\mu^{\mathbf{a}}(T_1)}{\dim_u^{\mathbf{a}} X}, \quad \forall \mu \in \mathcal{M}(X_1, T_1).$$

Consequently, Proposition 4.1 and Theorem 4.3 yield (4.1) for all $\alpha \in \text{int } I$.

Proof. For ergodic μ , the assumption implies $h_\mu^{\mathbf{a}}(T_1) = 0$ whenever $\int u d\mu = 0$. For $\int u d\mu > 0$, Proposition 2.12 and (2.6) give

$$\dim_u^{\mathbf{a}} X \geq \dim_u^{\mathbf{a}} \mu = \frac{h_\mu^{\mathbf{a}}(T_1)}{\int u d\mu},$$

which proves the claim. ■

6. Multifractal spectrum and localized entropy. We temporarily restrict our attention to the case where $\psi = u \equiv 1$. In this case the predicted spectrum takes on a particularly simple form: indeed, we compute

$$\begin{aligned} T_0^{\mathbf{a}}(q) &= \inf \{t \in \mathbb{R} : P^{\mathbf{a},B}(T_1, X_1, q\varphi - t) \leq 0\} \\ &= \inf \{t \in \mathbb{R} : P^{\mathbf{a},B}(T_1, X_1, q\varphi) - t \leq 0\} \\ &= P^{\mathbf{a},B}(T_1, X_1, q\varphi), \end{aligned}$$

so that $\mathcal{S}^{\mathbf{a}}$ is the Legendre transform of the convex function $q \mapsto P^{\mathbf{a},B}(T_1, X_1, q\varphi)$. The corresponding level sets are

$$(6.1) \quad K_\alpha^\varphi = \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \varphi(x) = \alpha \right\},$$

and the multifractal spectrum coincides with the entropy spectrum for Birkhoff averages:

$$\mathcal{F}^{\mathbf{a}}(\alpha) = \mathcal{B}_E^{\mathbf{a}}(\alpha) := h^{\mathbf{a},B}(K_\alpha^\varphi).$$

Kucherenko and Wolf [KW15] introduced the notion of localized topological pressure, restricting the pressure to orbits whose Birkhoff averages satisfy prescribed constraints. Motivated by this, we introduce the localized weighted entropy.

Given a neighborhood $U \subset \mathbb{R}$ of α , define the approximate level set

$$(6.2) \quad G_n(U) := \left\{ x \in X : \frac{1}{\lceil a_1 n \rceil} S_{\lceil a_1 n \rceil} \varphi(x) \in U \right\},$$

and let

$$(6.3) \quad A_n^{\mathbf{a},\epsilon}(U) = \min \{ \#E : B^{\mathbf{a}}(E, n, \epsilon) \supset G_n(U) \},$$

where $B^{\mathbf{a}}(E, n, \epsilon) := \bigcup_{x \in E} B_n^{\mathbf{a}}(x, \epsilon)$.

We then define the *localized weighted entropy* by

$$\begin{aligned}\underline{\mathcal{F}}^{\mathbf{a}}(\alpha) &:= \inf_{U \ni \alpha} \lim_{\epsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\mathbf{a}, \epsilon}(U), \\ \overline{\mathcal{F}}^{\mathbf{a}}(\alpha) &:= \inf_{U \ni \alpha} \lim_{\epsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\mathbf{a}, \epsilon}(U).\end{aligned}$$

In this framework, we obtain the following strengthening of Theorem 3.4.

THEOREM 6.1. *Let $\varphi \in C(X_1)$. Then, for all $\alpha \in \mathbb{R}$,*

$$(6.4) \quad \mathcal{F}^{\mathbf{a}}(\alpha) \leq \underline{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \overline{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \mathcal{S}^{\mathbf{a}}(\alpha).$$

Proof. The proof splits into two parts. First, we compare the fine spectrum with the localized entropy, establishing $\mathcal{F}^{\mathbf{a}} \leq \underline{\mathcal{F}}^{\mathbf{a}}$. Second, we compare the localized entropy with the predicted spectrum, proving $\overline{\mathcal{F}}^{\mathbf{a}} \leq \mathcal{S}^{\mathbf{a}}$. (The middle inequality in (6.4) is immediate.)

STEP 1: $\mathcal{F}^{\mathbf{a}}(\alpha) \leq \underline{\mathcal{F}}^{\mathbf{a}}(\alpha)$. We first rewrite the exact level sets (6.1) in terms of the approximate sets (6.2):

$$K_\alpha^\varphi \subset \bigcap_{U \ni \alpha} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} G_n(U).$$

For each neighborhood $U \ni \alpha$, define

$$\begin{aligned}\tilde{G}_N(U) &= \bigcap_{n \geq N} G_n(U) = \left\{ x \in X : \frac{1}{\lceil a_1 n \rceil} S_{\lceil a_1 n \rceil} \varphi(x) \in U \ \forall n \geq N \right\}, \\ \tilde{G}(U) &= \bigcup_{N \in \mathbb{N}} \tilde{G}_N(U).\end{aligned}$$

Thus $K_\alpha^\varphi \subset \bigcap_{U \ni \alpha} \tilde{G}(U)$. In particular,

$$(6.5) \quad h^{\mathbf{a}, B}(K_\alpha) \leq h^{\mathbf{a}, B}(\tilde{G}(U)) = \sup_N h^{\mathbf{a}, B}(\tilde{G}_N(U)) \leq \sup_N \underline{Ch}^{\mathbf{a}}(\tilde{G}_N(U)),$$

where $\underline{Ch}^{\mathbf{a}}$ denotes the lower capacity entropy (see [WH19]).

Mimicking the definition of $\Lambda_n^{\mathbf{a}, \epsilon}(U)$, set

$$\tilde{\Lambda}_n^{\mathbf{a}, \epsilon}(U, N) := \inf \{ \#E : B(E, n, \epsilon) \supset \tilde{G}_N(U) \}.$$

Since $G_n(U) \supset \tilde{G}_N(U)$ for $n \geq N$, it follows that $\Lambda_n^{\mathbf{a}, \epsilon}(U) \geq \tilde{\Lambda}_n^{\mathbf{a}, \epsilon}(U, N)$. Consequently,

$$\lim_{\epsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\mathbf{a}, \epsilon}(U) \geq \sup_N \underline{Ch}^{\mathbf{a}}(\tilde{G}_N(U)) \geq h^{\mathbf{a}, B}(K_\alpha).$$

Taking the infimum over all neighborhoods $U \ni \alpha$ yields $\underline{\mathcal{F}}^{\mathbf{a}}(\alpha) \geq \mathcal{F}^{\mathbf{a}}(\alpha)$.

STEP 2: $\overline{\mathcal{F}}^{\mathbf{a}}(\alpha) \leq \mathcal{S}^{\mathbf{a}}(\alpha)$. Fix $\alpha \in \mathbb{R}$ and let U be a neighborhood of α . For $t < \overline{\mathcal{F}}^{\mathbf{a}}(\alpha)$ and $\varepsilon > 0$, there exist $\delta > 0$ and a sequence $n_k \rightarrow \infty$ such that $\Lambda_{n_k}^{\mathbf{a}, \delta}(U) \geq e^{n_k t}$ for all k , and $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

Let E be an $(\mathbf{a}, n_k, \delta)$ -spanning set for X_1 , and define

$$E' = \{x \in E : B_{n_k}^{\mathbf{a}}(x, \delta) \cap G_{n_k}(U) \neq \emptyset\}.$$

By construction, $\#E' \geq e^{n_k t}$. Hence,

$$\sum_{x \in E} e^{\frac{1}{a_1} S_{\lceil a_1 n_k \rceil}(q\varphi)(x)} \geq \sum_{x \in E'} e^{\frac{1}{a_1} S_{\lceil a_1 n_k \rceil}(q\varphi)(x)} \geq e^{n_k t} \cdot e^{\frac{1}{a_1} \lceil a_1 n_k \rceil q(\inf U - \varepsilon)}.$$

Therefore $P^{\mathbf{a}, B}(T_1, X_1, q\varphi) \geq t + q(\inf U - \varepsilon)$. Since $t < \overline{\mathcal{F}^{\mathbf{a}}}(\alpha)$, and $U \ni \alpha$, $\varepsilon > 0$ were arbitrary, we deduce

$$P^{\mathbf{a}, B}(T_1, X_1, q\varphi) \geq \overline{\mathcal{F}^{\mathbf{a}}}(\alpha) + q\alpha,$$

which proves the claim. ■

For clarity, we state the following corollary summarizing our main conclusions in the case $\psi = u = 1$ and when the entropy map is upper semicontinuous (ensuring existence of equilibrium states).

COROLLARY 6.2 (Entropy spectrum for Birkhoff averages). *Let X_1 be a compact metric space, $T_1 : X_1 \rightarrow X_1$ continuous, and $\varphi \in C(X_1)$. Assume the entropy map $\mathcal{M}(X_1, T_1) \rightarrow \mathbb{R}$ is upper semicontinuous. Define*

$$T_0^{\mathbf{a}}(q) = P^{\mathbf{a}, B}(T_1, X_1, q\varphi), \quad J(q) = [D^- T_0^{\mathbf{a}}(q), D^+ T_0^{\mathbf{a}}(q)].$$

Then the predicted spectrum is

$$\mathcal{S}^{\mathbf{a}}(\alpha) = \inf_{q \in \mathbb{R}} \{T_0^{\mathbf{a}}(q) - q\alpha\},$$

and the domain $I = \{\alpha : \mathcal{S}^{\mathbf{a}}(\alpha) \geq 0\}$ satisfies

$$(6.6) \quad \begin{aligned} I &= \left[\lim_{q \rightarrow -\infty} D^- T_0^{\mathbf{a}}(q), \lim_{q \rightarrow \infty} D^+ T_0^{\mathbf{a}}(q) \right] \\ &= \left\{ \int_{X_1} \varphi d\mu : \mu \in \mathcal{M}(X_1, T_1) \right\}. \end{aligned}$$

For $\alpha \notin I$, the level set $K_\alpha^\varphi = \emptyset$. For $\alpha \in \text{int } I \setminus \bigcup_{q \in \mathbb{R}} \text{int } J(q)$, we have

$$(6.7) \quad \begin{aligned} h^{\mathbf{a}, B}(K_\alpha^\varphi) &= \underline{\mathcal{F}^{\mathbf{a}}}(\alpha) = \overline{\mathcal{F}^{\mathbf{a}}}(\alpha) \\ &= \inf_{q \in \mathbb{R}} \{T_0^{\mathbf{a}}(q) - q\alpha\} \\ &= \sup \left\{ h_\mu(T_1) : \mu \in \mathcal{M}(X_1, T_1), \int_{X_1} \varphi d\mu = \alpha \right\} \\ &= \sup \{h_\mu(T_1) : \mu \in \mathcal{M}_E(K_\alpha^\varphi, T_1)\}. \end{aligned}$$

Proof. We first establish (6.6). The initial equality is direct: recall that $\alpha \in I$ if and only if $T_\alpha^{\mathbf{a}}(q) \geq 0$ for all q , which here is equivalent to requiring

$$P^{\mathbf{a}, B}(T_1, X_1, q\varphi) \geq q\alpha \quad \text{for all } q \in \mathbb{R}.$$

By convexity properties, this holds precisely when

$$\alpha \in \left[\lim_{q \rightarrow -\infty} D^- T_0^{\mathbf{a}}(q), \lim_{q \rightarrow +\infty} D^+ T_0^{\mathbf{a}}(q) \right] = \overline{\bigcup_{q \in \mathbb{R}} J(q)}.$$

For the second equality in (6.6), assume $\mu \in \mathcal{M}(X_1, T_1)$ with $\int_{X_1} \varphi d\mu = \alpha$. Then

$$P^{\mathbf{a}, B}(q\varphi) \geq h_{\mu}^{\mathbf{a}}(T_1) + \int_{X_1} q\varphi d\mu \geq q\alpha, \quad \forall q \in \mathbb{R}.$$

Conversely, fix $q \in \mathbb{R}$ and let $\alpha^+ = D^+ T_0(q)$. For $\gamma > 0$, let μ_{γ} be an equilibrium state for $(q + \gamma)\varphi$, and take a weak* limit μ of μ_{γ} as $\gamma \rightarrow 0$. Upper semicontinuity of the entropy ensures that μ is an equilibrium state for $q\varphi$, and moreover

$$\int_{X_1} \varphi d\mu = \lim_{\gamma \rightarrow 0} \int_{X_1} \varphi d\mu_{\gamma} = \alpha^+,$$

since $\int_{X_1} \varphi d\mu_{\gamma} \in J(q + \gamma)$ for all γ . Because $\alpha^+ \in \partial J(q)$, there exists an ergodic component ν^+ of μ with $\int_{X_1} \varphi d\nu^+ = \alpha^+$. A symmetric argument yields an ergodic ν^- with $\int_{X_1} \varphi d\nu^- = \alpha^-$. Taking convex combinations of $\nu^{\pm}$ shows that

$$\left\{ \int_{X_1} \varphi d\mu : \mu \in \mathcal{M}(X_1, T_1) \right\} \supset J(q), \quad \forall q \in \mathbb{R}.$$

Hence (6.6) holds. (Note also that the set $\{\int \varphi d\mu : \mu \in \mathcal{M}(X_1, T_1)\}$ is closed, being the continuous image of a compact set.)

The emptiness result $K_{\alpha}^{\varphi} = \emptyset$ for $\alpha \notin I$ follows from Theorem 3.4. If α is such that $J(q) = \{\alpha\}$ for some q , the first two lines of (6.7) are obtained from Theorems 3.5 and 6.1. If $\alpha \in \partial J(q)$ for some q , they follow from Theorem 6.1 together with the construction above of an ergodic equilibrium state ν satisfying $\int_{X_1} \varphi d\nu = \alpha$.

The third identity in (6.7) is a consequence of Theorem 4.3, while the fourth comes from the fact that we proved the first two by explicitly constructing a full measure. Finally, the last statement of Corollary 6.2 follows directly from Corollary 4.5. ■

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