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# ON AN ITERATIVE METHOD FOR UNCONSTRAINED OPTIMIZATION

*Abstract.* We present a local and a semi-local convergence analysis of an iterative method for approximating zeros of derivatives for solving univariate and unconstrained optimization problems. In the local case, the radius of convergence is obtained, whereas in the semi-local case, sufficient convergence criteria are presented. Numerical examples are also provided.

**1. Introduction.** In this study we are concerned with the problem of approximating a solution $x^*$ of the equation

$$F'(x) = 0, \tag{1.1}$$

where $F$ is a differentiable function defined on a convex subset $D$ of $S$ where $S$ is $\mathbb{R}$ or $\mathbb{C}$. Many problems in computational sciences such as univariate minimization problems and unconstrained optimization problems, and in other disciplines, can be brought in a form like equation (1.1) using mathematical modeling [1, 2, 5, 16]. The solutions of equation (1.1) can be found in closed form only in special cases. That is why most methods of solving these equations are usually iterative. In particular, the practice of numerical functional analysis for finding such solutions involves variants of Newton-like methods. The study of convergence of iterative procedures is usually based of two types: semi-local and local convergence analysis. The semi-local convergence analysis is based on the information around an initial point, to give criteria ensuring the convergence of the method, while the local analysis is based on the information around a solution, to find estimates of the radii of convergence balls. There are many methods for solving equation (1.1)

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[12, 13, 14, 16, 18, 19]. It is well known that higher dimensional problems involve steps of searching for extrema along certain derivatives in $\mathbb{R}^m$. So, finding a step size $\mu_k$ along the directional vector $v_i$ involves solving the subproblem

$$\min F(x_{n+1}) = F(x_n + \mu_n d_n), \tag{1.2}$$

which is a unidimensional problem. Consequently, the unidimensional search methods are important for any iterative procedure in higher dimensions.

Newton's method for solving nonlinear, univariate and unconstrained optimization problems is defined by

$$x_{n+1} = x_n - \frac{F'(x_n)}{F''(x_n)} \quad \text{for } n = 0, 1, 2, \dots, \tag{1.3}$$

where $x_0 \in D$ is an initial point. With suitable assumptions, this method converges to $x^*$ with an order of convergence at least two [2, 5, 16, 17]. In particular, its variants may achieve a third order of convergence [11, 10, 15]. In this paper we focus on a class of second derivative free methods. Notice that the evaluation of the second derivatives can be avoided if replaced by

$$F''(x_{n+1}) \approx \frac{F'(x_n) - F'(x_{n-1})}{x_n - x_{n-1}}, \tag{1.4}$$

leading to the secant-method defined by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})F'(x_n)}{F'(x_n) - F'(x_{n-1})} \quad \text{for } n = 0, 1, 2, \dots, \tag{1.5}$$

where $x_{-1}, x_0 \in D$ are initial points. The secant-method is $Q$-superlinearly convergent with order $p = \frac{1}{2}(\sqrt{5}+1) \approx 1.618$ [2, 5, 16].

A more efficient replacement of the second derivative was given by Kahya and Chen [14]:

$$F''(x_{n+1}) \approx A_n = \frac{2\left\{F'(x_n) - \frac{F(x_n)-F(x_{n-1})}{x_n - x_{n-1}}\right\}}{x_n - x_{n-1}}, \tag{1.6}$$

leading to the method

$$x_{n+1} = x_n - A_n^{-1} F'(x_n) \quad \text{for } n = 0, 1, 2, \dots. \tag{1.7}$$

Suppose that $F'(x^*) = 0$, $F''(x^*) \neq 0$, $F'''(x^*) \neq 0$. Then method (1.7) converges with the same order of convergence as the secant method (1.5), but the leading error is $-\frac{1}{3}F'''(x_n)(x_n - x_{n-1})$, which is better approximation than $-\frac{1}{2}F'''(x_n)(x_n - x_{n-1})$ given for the secant method [14]. Although the convergence order of method (1.7) was established in [14], we have no information about the radius of convergence in the local case, and we do not have sufficient semi-local convergence criteria. The contribution of this prepare is twofold: First we provide a local convergence analysis and secondly we present a semi-local convergence analysis. It is also worth noticing that we make no assumptions on the third derivative.

The paper is organized as follows: In Section 2 we present the local and semi-local convergence analysis for method (1.7). The numerical examples are presented in Section 3.

**2. Convergence analysis.** In this section we present first the local convergence and then the semi-local convergence analysis of method (1.7). Let $U(w,\rho)$, $\overline{U}(w,\rho)$ denote the open and closed intervals in $S$ with center $w \in S$ and of radius $\rho > 0$.

THEOREM 2.1. *Let $F : D \subseteq S \to S$ be a twice differentiable function. Suppose that there exist $x^* \in D$ and $\ell > 0$ such that*

$$F'(x^*) = 0, \qquad F''(x^*) \neq 0, \tag{2.1}$$

$$|F''(x^*)^{-1}(F''(x) - F''(x^*))| \leq \ell |x - x^*| \quad \text{for each } x \in D, \tag{2.2}$$

$$\overline{U}(x^*, r) \subseteq D, \tag{2.3}$$

*where*

$$r = \frac{6}{23\ell}. \tag{2.4}$$

*Then the sequence $\{x_n\}$ generated by method (1.7) for $x_{-1}, x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = -1, 0, 1, 2, \ldots$ and converges to $x^*$. Moreover,*

$$|x_{n+1} - x^*| \leq \frac{\alpha_n}{\beta_n} |x_n - x^*| < |x_n - x^*| < r, \tag{2.5}$$

*where*

$$\alpha_n = 2\ell \int_0^1\!\!\int_0^1 |x_{n-1} + \theta(x_n - x_{n-1}) - x^* + s(1-\theta)(x_n - x_{n-1})|(1-\theta)\, d\theta\, ds + \tfrac{1}{2}|x_n - x^*|\ell, \tag{2.6}$$

$$\beta_n = 1 - 2\ell \int_0^1\!\!\int_0^1 |x_{n-1} + \theta(x_n - x_{n-1}) - x^* + s(1-\theta)(x_n - x_{n-1})|(1-\theta)\, d\theta\, ds. \tag{2.7}$$

*Furthermore, if there exists $T \in [r, 2/L)$ such that $\overline{U}(x^*, T) \subseteq D$, then the limit point $x^*$ is the only solution of the equation $F'(x) = 0$ in $\overline{U}(x^*, T)$.*

*Proof.* By hypothesis $x_{-1}, x_0 \in U(x^*, r)$. Suppose that $x_{k-1}, x_k \in U(x^*, r)$ for all $k \leq n$, where $n$ is an integer and $x_{k-1} \neq x_k$. We shall show that $A_k$ is invertible. Using the definition of $A_k$, conditions (2.1)–(2.4), and (2.7), we get

$$
\begin{aligned}
&|F''(x^*)^{-1}(A_k - F''(x^*))| \\
&= 2\left| F''(x^*)^{-1}\left[\frac{F'(x_k) - \frac{F(x_k)-F(x_{k-1})}{x_k-x_{k-1}}}{x_k - x_{k-1}} - \tfrac{1}{2}F''(x^*)\right]\right| \\
&= \frac{2|F''(x^*)^{-1}[F'(x_k)(x_k-x_{k-1})-(F(x_k)-F(x_{k-1}))-\frac{1}{2}F''(x^*)(x_k-x_{k-1})^2]|}{|x_k-x_{k-1}|^2} \\
&= \frac{2|F''(x^*)^{-1}[\int_0^1 (F'(x_k)-F'(x_k(\theta)))(x_k-x_{k-1})\,d\theta-\frac{1}{2}F''(x^*)(x_k-x_{k-1})^2]|}{|x_k-x_{k-1}|^2} \\
&= \frac{2|\int_0^1\int_0^1 F''(x^*)^{-1}[F''(x_k(\theta)+s(1-\theta)(x_k-x_{k-1}))-F''(x^*)(1-\theta)(x_k-x_{k-1})^2\,d\theta\,ds]|}{|x_k-x_{k-1}|^2} \\
&\le \frac{2\ell|x_k-x_{k-1}|^2\int_0^1\int_0^1 |x_{k-1}+(\theta(x_k-x_{k-1})-x^*+s(1-\theta)(x_k-x_{k-1}))|(1-\theta)\,d\theta\,ds}{|x_k-x_{k-1}|^2} \\
&\le 2\ell\int_0^1\int_0^1 \big[(1-\theta)|x_{k-1}-x^*| + \theta|x_k - x^*| + s(1-\theta)(|x_{k-1}-x^*| + |x_k - x^*|)\big] \\
&\qquad\qquad \times (1-\theta)\,d\theta\,ds \\
&< 2\ell\int_0^1\int_0^1 [r + 2s(1-\theta)r](1-\theta)\,d\theta\,ds = \tfrac{5}{3}\ell r < 1.
\end{aligned}
$$

Here,

$$
x_k(\theta) = x_{k-1} + \theta(x_k - x_{k-1}). \tag{2.8}
$$

It follows from (2.8) and the Banach lemma on invertible functions that $A_k$ is invertible and

$$
|A_k^{-1}F''(x^*)| \le \frac{1}{\beta_k} < \frac{1}{1-\frac{5}{3}\ell r}. \tag{2.9}
$$

Hence, $x_{k+1}$ is well defined by method (1.7).

Next, we need an estimate on $A_k(x_k - x^*) - F'(x_k)$. In view of the definition of $A_k$, we have

$$
\begin{aligned}
&A_k(x_k - x^*) - F'(x_k) \\
&= 2\left[\frac{F'(x_k) - \frac{F(x_k)-F(x_{k-1})}{x_k-x_{k-1}}}{x_k - x_{k-1}}(x_k - x^*) - \tfrac{1}{2}F'(x_k)\right] \\
&= 2\left[\frac{(F'(x_k) - \int_0^1 F'(x_k(\theta)))(x_k - x^*)\,d\theta}{x_k - x_{k-1}} - \tfrac{1}{2}F'(x_k)\right] \\
&= \frac{2}{x_k - x_{k-1}}\left[\int_0^1 (F'(x_k)-F'(x_k(\theta)))(x_k-x^*)\,d\theta-\tfrac{1}{2}F'(x_k)(x_k-x_{k-1})\right]
\end{aligned} \tag{2.10}
$$

$$= \frac{2\left[\int_0^1\int_0^1\big(F''(x_k(\theta)+s(1-\theta)(x_k-x_{k-1}))-F''(x^*)\big)(1-\theta)(x_k-x_{k-1})(x_k-x^*)\,d\theta\,ds\right]}{x_k-x_{k-1}}$$
$$+\frac{1}{x_k-x_{k-1}}\left[F''(x^*)(x_k-x_{k-1})(x_k-x^*)-F'(x_k)(x_k-x_{k-1})\right]$$
$$= \frac{2\left[\int_0^1\int_0^1\big(F''(x_k(\theta)+s(1-\theta)(x_k-x_{k-1}))-F''(x^*)\big)(1-\theta)(x_k-x_{k-1})(x_k-x^*)\,d\theta\,ds\right]}{x_k-x_{k-1}}$$
$$+\frac{F''(x^*)(x_k-x_{k-1})(x_k-x^*)-\int_0^1 F''(x^*+\theta(x_k-x^*))(x_k-x^*)(x_k-x_{k-1})\,d\theta}{x_k-x_{k-1}}$$
$$= 2\int_0^1\int_0^1\big(F''(x_k(\theta)+s(1-\theta)(x_k-x_{k-1}))-F''(x^*)\big)(1-\theta)(x_k-x^*)\,d\theta\,ds$$
$$+\int_0^1\big(F''(x^*)-F''(x^*+\theta(x_k-x^*))\big)(x_k-x^*)\,d\theta.$$

Using (2.1)–(2.5), and (2.10), we obtain

$$|F''(x^*)^{-1}(x^*)[A_k(x_k-x^*)-F'(x_k)]| \tag{2.11}$$
$$\le \ell\Big[2\int_0^1\int_0^1|x_k(\theta)-x^*+s(1-\theta)(x_k-x_{k-1})|(1-\theta)\,d\theta\,ds+\tfrac{1}{2}|x_k-x^*|\Big]|x_k-x^*|$$
$$\le \alpha_k|x_k-x^*| < \frac{13\ell r}{6}|x_k-x^*|.$$

In view of the identity

$$\begin{aligned} x_{k+1}-x^* &= x_k-x^*-A_k^{-1}F'(x_k) \\ &= [A_k^{-1}F''(x^*)][F''(x^*)^{-1}(A_k(x_k-x^*)-F'(x_k))], \end{aligned} \tag{2.12}$$

(2.9) and (2.11), we have

$$\begin{aligned} |x_{k+1}-x^*| &\le |A_k^{-1}F''(x^*)|\,|F''(x^*)^{-1}(A_k(x_k-x^*)-F'(x_k))| \\ &\le \frac{\alpha_k}{\beta_k}|x_k-x^*| < \frac{13\ell r}{6(1-5/3)}|x_k-x^*| = |x_k-x^*| < r, \end{aligned}$$

so (2.5) holds for all $k$ and $x_{k+1}\in U(x^*,r)$. Moverover, from the estimate

$$|x_{k+1}-x^*| < |x_k-x^*|,$$

we deduce that $\lim_{k\to\infty} x_k = x^*$.

Finally, to show the uniqueness part, let $y^*\in \overline{U}(x^*,T)$ with $F'(y^*)=0$. Define $T=|x^*-y^*|$ and $Q=\int_0^1 F''(y^*+\theta(x^*-y^*))\,d\theta$. Using (2.2) we get

$$|F''(x^*)^{-1}(Q-F'(x^*))| \le \int_0^1 \ell|y^*+\theta(x^*-y^*)|\,d\theta \le \frac{\ell}{2}|x^*-y^*| = \frac{\ell}{2}T < 1.$$

It follows that $Q$ is invertible. Then, from the identity $0 = F'(x^*) - F'(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. ∎

THEOREM 2.2. *Let $F : D \subseteq S \to S$ be a twice differentiable function. Suppose that there exist $x_{-1}, x_0 \in D$, $L_0, L > 0$, $b, c \geq 0$, and $q \in [0,1)$ such that*

$$F''(x_0) \neq 0, A_0 \text{ is invertible,} \tag{2.13}$$

$$|x_{-1} - x_0| \leq b, \tag{2.14}$$

$$|A_0^{-1} F'(x_0)| \leq c, \tag{2.15}$$

$$|F''(x_0)^{-1}(F''(x) - F''(x_0))| \leq L_0 |x - x_0| \quad \text{for all } x \in D, \tag{2.16}$$

$$|F''(x_0)^{-1}(F''(x) - F''(y))| \leq L|x - y| \quad \text{for all } x, y \in D, \tag{2.17}$$

$$g(q) \leq 0, \tag{2.18}$$

$$\overline{U}(x_0, R) \subseteq D, \tag{2.19}$$

$$R < \frac{3}{5L_0}, \tag{2.20}$$

*where $g(t) = (5 - 3L_0 b)t^2 - (3(1 + 3Lb) - 5L_0(b + c))t + 9L(b + c)$ and*

$$R = \frac{3q}{9L + 5L_0 q}. \tag{2.21}$$

*Then the sequence $\{x_n\}$ generated by method (1.7) is well defined, remains in $\overline{U}(x_0, R)$ for all $n = 0, 1, 2, \ldots$, and converges to a solution $x^* \in \overline{U}(x_0, R)$ of equation (1.1). Moveover, for each $n = 0, 1, 2, \ldots$,*

$$|x_{n+2} - x_{n+1}| \leq \frac{\gamma_n}{\delta_n}, \tag{2.22}$$

*where*

$$\gamma_n = L\big(\tfrac{1}{2}|x_{n+1} - x_n| + |x_n - x_{n-1}|\big)|x_{n+1} - x_n| \tag{2.23}$$

*and*

$$\delta_n = 1 - 2L_0 \int_0^1\int_0^1 \big|x_{n-1} + \theta(x_n - x_{n-1}) - x_0 + s(1-\theta)(x_n - x_{n-1})\big|(1-\theta)\, d\theta\, ds. \tag{2.24}$$

*Furthermore, if there exists $R_1 \in [R, 2/L_0 - R)$ such that $\overline{U}(x_0, R_1) \subseteq D$, then the limit point $x^*$ is the only solution of the equation $F'(x) = 0$ in $\overline{U}(x_0, R_1)$.*

*Proof.* By the definition of $R$, (2.18), (2.19) and (2.21), we have

$$b + \frac{c}{1-q} \leq \frac{3q}{9L + 5L_0 q} = R.$$

It then follows that $x_{-1}, x_0 \in \overline{U}(x_0, R)$, and $x_1$ is well defined, since $A_0$ is invertible. Suppose that $x_{k-1}, x_k \in \overline{U}(x_0, R)$ for all $k \leq n$, where $n$ is an integer and $x_{k-1} \neq x_k$. As in (2.8), (2.10) but replacing $x_0$, $\beta_n$ by $x^*$, $\delta_n$, respectively and using (2.16) instead of (2.2), we find that $A_{k+1}$ is invertible and

$$|A_{k+1}^{-1} F''(x_0)| \leq \frac{1}{\delta_{k+1}}. \tag{2.25}$$

Hence, $x_{k+1}$ is well defined by method (1.7). Using (1.7) we obtain

$$\begin{aligned}
(2.26)\quad & F'(x_{k+1}) = F'(x_{k+1}) - F'(x_k) - A_k(x_{k+1} - x_k) \\
&= [F'(x_{k+1}) - F'(x_k) - F''(x_k)(x_{k+1} - x_k)] + (F''(x_k) - A_k)(x_{k+1} - x_k) \\
&= \int_0^1 \big(F''(x_k + \theta(x_{k+1} - x_k)) - F''(x_k)\big)(x_{k+1} - x_k)\, d\theta \\
&\quad + \left(F''(x_k) - \frac{2\big(F'(x_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))\big)}{(x_k - x_{k-1})^2}\right)(x_{k+1} - x_k) \\
&= \int_0^1 \big(F''(x_k + \theta(x_{k+1} - x_k)) - F''(x_k)\big)(x_{k+1} - x_k)\, d\theta \\
&\quad - 2\left(\frac{F'(x_k) - \int_0^1 F'(x_{k-1} + \theta(x_k - x_{k-1}))\, d\theta}{x_k - x_{k-1}} - \frac{F''(x_k)}{2}\right)(x_{k+1} - x_k) \\
&= \int_0^1 \big(F''(x_k + \theta(x_{k+1} - x_k)) - F''(x_k)\big)(x_{k+1} - x_k)\, d\theta \\
&\quad - 2\int_0^1\int_0^1 \big(F''(x_k(\theta) + s(1-\theta)(x_k - x_{k-1})) - F''(x_k)\big)(1-\theta)(x_{k+1} - x_k)\, d\theta\, ds,
\end{aligned}$$

where $x_k(\theta)$ in the last equality is defined by (2.8).

Using (2.17) and (2.26), we get

$$\begin{aligned}
(2.27)\quad & |F''(x_0)^{-1} F'(x_{k+1})| \\
&\leq \frac{L}{2}|x_{k+1} - x_k|^2 + 2L\int_0^1\int_0^1 (1+s)(1-\theta)\, d\theta\, ds\, |x_k - x_{k-1}|\, |x_{k+1} - x_k| = \gamma_{k+1}.
\end{aligned}$$

Then, using method (1.7), (2.25) and (2.27), we get

$$|x_{k+2} - x_{k+1}| \leq |A_{k+1}^{-1} F''(x_0)|\, |F''(x_0)^{-1} F(x_{k+1})| \leq \frac{\gamma_{k+1}}{\delta_{k+1}},$$

which completes the induction for (2.22).

Using (2.22) and the definition of $R$, we have

$$
\begin{aligned}
&|x_{k+2}-x_{k+1}| \\
&\le \frac{L\big(\frac{1}{2}(|x_{k+1}-x_0|+|x_k-x_0|)+|x_k-x_0|+|x_{k-1}-x_0|\big)|x_{k+1}-x_k|}{1-2L_0\int_0^1\int_0^1\big((1-\theta)|x_{k-1}-x_0|+\theta|x_k-x_0|+s(1-\theta)(|x_k-x_0|+|x_{k-1}-x_0|)\big)(1-\theta)\,d\theta\,ds} \\
&\le \frac{L\big(\frac{1}{2}(2R)+2R\big)}{1-\frac{5}{3}L_0R}|x_{k+1}-x_k| = \frac{3LR}{1-\frac{5}{3}L_0R}|x_{k+1}-x_k| \\
&\le q|x_{k+1}-x_k| \le q^{k+1}|x_1-x_0| \le q^{k+1}c,
\end{aligned}
$$

so

$$
\begin{aligned}
|x_{k+2}-x_0| &\le |x_{k+2}-x_{k+1}|+|x_{k+1}-x_k|+\cdots+|x_1-x_0|+|x_0-x_{-1}| \\
&\le b+c+cq+\cdots+cq^{k+1} \\
&= b+c\frac{1-q^{k+2}}{1-q} < b+\frac{c}{1-q} \le \frac{3q}{9L+5L_0q} = R,
\end{aligned}
$$

which implies that $x_{k+2}\in\overline{U}(x_0,R)$.

Let $m\ge 0$. Then

$$
\begin{aligned}
\text{(2.28)}\quad &|x_{m+k}-x_k| \\
&\le |x_{m+k}-x_{m+k-1}|+|x_{m+k-1}-x_{m+k-2}|+\cdots+|x_{k+1}-x_k| \\
&\le (q^{m+k-1}+q^{m+k-2}+\cdots+q^k)c = \frac{1-q^m}{1-q}q^kc.
\end{aligned}
$$

It follows that $\{x_k\}$ is a Cauchy sequence in $S$, and as such it converges to some $x^*\in\overline{U}(x_0,R)$ (since $\overline{U}(x_0,R)$ is a closed interval). By letting $k\to\infty$ in (2.27), (2.28), we get $F'(x^*)=0$ and

$$
|x^*-x_k| \le \frac{q^kc}{1-q} < R. \tag{2.29}
$$

In particular, if $k=0$, we infer from (2.29) that $x^*\in\overline{U}(x_0,R)$.

For uniqueness, as in the case of Theorem 2.1 but using (2.16) instead of (2.2) for $y^*\in\overline{U}(x_0,R_1)$ with $F'(y^*)=0$, we get

$$
\begin{aligned}
|F''(x_0)^{-1}(Q-F'(x_0))| &\le \int_0^1 L_0|y^*+\theta(x^*-y^*)-x_0|\,d\theta \\
&\le L_0\int_0^1[\theta|x^*-x_0|+(1-\theta)|y^*-x_0|]\,d\theta \\
&\le \frac{L_0}{2}(R+R_1) < 1.
\end{aligned}
$$

It follows again that $Q$ is invertible and $x^*=y^*$. ∎

REMARK 2.3. (a) It follows from (2.22) that there exist constants $c_1, c_2 \geq 0$ such that

$$|x_{n+2} - x_{n+1}| \leq c_1|x_{n+1} - x_n|^2 + c_2|x_{n+1} - x_n|\,|x_n - x_{n-1}|. \tag{2.30}$$

It is well known that (2.30) implies that the convergence order of method (1.7) is $p = \frac{1}{2}(\sqrt{5}+1) \approx 1.618$, which is the same order of convergence as that of the secant method (1.5). However, as already noted in the Introduction, the leading error is a better approximation than the one obtained in [14].

(b) Notice that $L_0 \leq L$ and $L/L_0$ can be arbitrarily large [3, 4, 7, 8, 9].

**3. Numerical examples.** In this section we present two numerical examples.

EXAMPLE 3.1. Let $S = \mathbb{R}$, $D = \overline{U}(0, 1)$ and define a function $F$ on $D$ by

$$F(x) = \exp(x) - x. \tag{3.1}$$

Then, using (3.2), we see that conditions (2.1)–(2.3) are satisfied if $x^* = 0$, $\ell = e - 1$ and

$$r = \frac{6}{23(e-1)} = 0.15192001.$$

Then, according to Theorem 2.2, method (1.7) converges to the root 0 if $x_1, x_0 \in U(0, 0.15192001)$. Concerning the uniqueness ball, since

$$T \in [r, 2/L) = [0.15192001, 1.163953414), \qquad \overline{U}(0, T) \subseteq D,$$

we can choose $T = 1$.

EXAMPLE 3.2. Let $S = \mathbb{R}$, $x_0 = 0.578$, $R_0 = 0.03$, $D = \overline{U}(x_0, R) = [0.548, 0.608]$ and define a function $F$ on $D$ by

$$F(x) = x^3 - x. \tag{3.2}$$

Then by (2.17), (2.18) and (3.2) we get $L_0 = L = 1.7301038606$. Let $x_{-1} = 0.573$. Then

$$A_0 = 2(2x_0 + x_{-1}) = 3.458, \qquad A_0^{-1} = 0.2891845,$$
$$b = 0.005, \qquad c = 0.000651243,$$
$$g(t) = 4.974048443t^2 - 3.028968486t + 0.050861191,$$
$$q = 0.017282051, \quad g(q) = 0, \quad R = 0.027583573 < \frac{3}{5L_0} = 0.3468$$

and $\bar{U}(x_0, R) \subset \bar{U}(x_0, R_0)$. Hence, all hypotheses of Theorem 2.2 are satisfied. Therefore, method (1.7) converges to $x^* = \sqrt{1/3} = 0.577350268 \in \bar{U}(x_0, R)$.

Finally, concerning the uniqueness ball, since

$$R_1 \in [R, 2/L_0 - R) = [0.027583573, 1.128416427)$$

and $\overline{U}(x_0, R_1) \subseteq D = \overline{U}(x_0, 1 - \alpha)$, we can choose $R_1 = 0.03$.

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