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AN EXPLICIT SOLUTION FOR OPTIMAL INVESTMENT PROBLEMS WITH AUTOREGRESSIVE PRICES AND EXPONENTIAL UTILITY

Abstract. We calculate explicitly the optimal strategy for an investor with exponential utility function when the price of a single risky asset (stock) follows a discrete-time autoregressive Gaussian process. We also calculate its performance and analyse it when the trading horizon tends to infinity. Dependence of the asymptotic performance on the autoregression parameter is determined. This provides, to the best of our knowledge, the first instance of a theorem linking directly the memory of the asset price process to the attainable satisfaction level of investors trading in the given asset.

1. Introduction. Sequences of independent random variables have no memory at all, Markovian processes remember their past through their present value only. In the case of processes with longer memory the entire past may influence the current evolution of the given stochastic system, e.g. in the case of fractional Brownian motion and related processes.

Econometric time series exhibit various degrees of influence of the past on the present, depending on the sampling frequency. High-frequency volatility has long-range dependence while asset prices may or may not have this property [2]. The principal motivating question of our research is the following: how does the memory of an asset's price influence the satisfaction attainable from investing into this asset?

The present paper concentrates on a Markovian setting. It precisely characterizes the dependence of performance on memory in a concrete model

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class where the price follows a Gaussian autoregressive process. In the case of investors with exponential utility we find the optimal trading strategy for each finite time horizon and analyse what happens when the horizon tends to infinity. We determine the exact dependence of the asymptotic performance on the autoregression parameter and hence make the first step towards general results linking investment performance to the memory length of the underlying security price.

The present paper continues previous investigations of [3, 4, 5], where asymptotic arbitrage in the utility sense was considered, i.e. the speed of the expected utility growth when the time horizon tends to infinity. The first two references concentrated on continuous-time models, while [5] treated a model where borrowing and short-selling were forbidden and utility functions were defined on the positive axis only.

The possibly negative prices of the model we consider may be acceptable in certain contexts (e.g. futures trading). We nonetheless stress that our purpose is to exhibit a theoretical model whose qualitative conclusions are hoped to extend to a broader class of processes in the future so we are not bothered by the possible negativity of prices.

We stress that it occurs very rarely that optimal strategies can be determined in closed form for discrete-time investment problems. As far as we know, our paper is the first to have found an explicit solution for the case of autoregressive Gaussian processes.

In the present section we explain our model and the optimisation problem under consideration. In Section 2 we present our results, and Section 3 contains the proofs.

We will consider random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

We are working with a financial market in which two assets are traded: a riskless asset and a single risky asset whose price X_t is an $\mathbb{R}$-valued stochastic process governed by the equation

$$X_{t+1} = \alpha X_t + \sigma \varepsilon_{t+1}, \qquad t \geq 0, \tag{1}$$

where $\alpha \in \mathbb{R}$, $\sigma > 0$ are parameters and ε_t, $t \geq 1$ are i.i.d. standard Gaussian random variables, and the initial value X_0 (which may be random) is independent of the ε_t, $t \geq 1$. The price of the riskless asset is assumed to be constant 1 (which amounts to X_t being, in fact, discounted prices). It would also be easy to incorporate a nonzero (constant) interest rate.

Introducing $\beta := \alpha - 1$, we may rewrite (1) as

$$X_{t+1} - X_t = \beta X_t + \sigma \varepsilon_{t+1}, \qquad t \geq 0; \tag{2}$$

in this way we express the price change at time t as a function of the previous price and the new random shock ε_{t+1}. The information flow is given by the filtration

$$\mathcal{F}_t := \sigma(X_s, 0 \le s \le t), \quad t \ge 0. \tag{3}$$

We interpret α (or, equivalently, β) as a "memory parameter" indicating how the previous values of the process X influence its present value. Eventually, our purpose is to find the dependence of the maximal achievable utility on this parameter.

A trading strategy is described by the number of units in the risky asset at t, denoted by ϕ_t for $t \ge 1$. Trading strategies are assumed to be $(\mathcal{F}_t)_{t\ge 0}$-predictable $\mathbb{R}$-valued processes (i.e. ϕ_t is $\mathcal{F}_{t-1}$-measurable for all t), in particular, short-selling is allowed. The totality of trading strategies is denoted by Φ.

The wealth process corresponding to a given trading strategy $(\phi_t)_{t\ge 1}$ is

$$L_t^{\phi} := L_{t-1}^{\phi} + \phi_t(X_t - X_{t-1}), \quad t \ge 1, \tag{4}$$

where $L_0^{\phi} := L_0$ is the initial capital of the investor. In other words, the terminal wealth of the investor is given by

$$L_T^{\phi} = L_0 + \sum_{j=1}^{T} \phi_j (X_j - X_{j-1}), \tag{5}$$

where $T \ge 1$ is a time horizon.

We focus on a finite horizon utility maximization problem and look for the optimal strategy $(\phi_t^*)_{1\le t\le T}$ which satisfies

$$\sup_{\phi\in\Phi} \mathbb{E}U(L_T^{\phi}) = \mathbb{E}U(L_T^{\phi^*}), \tag{6}$$

where $U : \mathbb{R} \to \mathbb{R}$ is the utility function $U(x) = -e^{-x}$. Note that the expectations exist but may be $-\infty$. We are going to give an explicit solution for this problem.

In order to make a comparison, we also consider an investor who is not using the accumulated past information, i.e. we define Φ_0 as the set of trading strategies for which ϕ_t is $\mathcal{F}_0$-measurable for all $1 \le t \le T$. We will also find $\eta^* \in \Phi_0$ such that

$$\sup_{\phi\in\Phi_0} \mathbb{E}U(L_T^{\phi}) = \mathbb{E}U(L_T^{\eta^*}).$$

As we consider the exponential utility, we may and will suppose $L_0 = 0$ in what follows.

2. Results. We are now ready to present the main result of this paper.

THEOREM 2.1.

(i) *The optimal strategies for time horizon* $T \in \mathbb{N}$ *are* $(\phi_t^*)_{1\le t\le T} := (\hat{\phi}_t^T(X_{t-1}))_{1\le t\le T}$ *when past information is used and* $(\eta_t^*)_{1\le t\le T} := (\hat{\phi}_t^T(X_0))_{1\le t\le T}$ *when past information is ignored, where*

$$\hat{\phi}_t^T(z) = \frac{\beta z}{\sigma^2}\theta_t^T \quad \text{for all } 1 \le t \le T \text{ and } z \in \mathbb{R}, \tag{7}$$

and $\theta_t^T = 1 - (T-t)\beta$.

(ii) *Using these strategies, the maximal conditional expected utilities are*

$$\mathbb{E}[U(L_T^{\phi^*}) \mid X_0 = z] = -\frac{1}{\sqrt{\gamma_\beta(T)}} e^{-\frac{\beta^2 z^2}{2\sigma^2}T}, \tag{8}$$

and

$$\mathbb{E}[U(L_T^{\eta^*}) \mid X_0 = z] = -e^{-\frac{\beta^2 z^2}{2\sigma^2}T}, \tag{9}$$

respectively, where γ_β *is given by*

$$\gamma_\beta(T) = \begin{cases} \dfrac{\beta^{2T}\Gamma(1/\beta^2 + T)}{\Gamma(1/\beta^2)} & \text{if } \beta \ne 0, \\ 1 & \text{if } \beta = 0, \end{cases} \tag{10}$$

and Γ *is the well-known gamma function,* $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$.

Our conclusion is that using accumulated past information leads to a qualitatively better strategy (i.e. the expected utility using past information is the expected utility without past information times a factor tending to 0).

In order to make a meaningful comparison for various values of β we should normalize our process. We will assume stability (i.e. $|\alpha| < 1$) and choose X_0 standard normal in such a way that X_t, $t \in \mathbb{N}$, is stationary (this can clearly be done if the probability space is large enough).

COROLLARY 2.2. *For a stable autoregressive process (when* $|\alpha| < 1$, *or equivalently* $\beta \in (-2, 0)$*) assuming that* $\mathrm{var}(X_t) = 1$, $t \ge 0$, *and* X_0 *is* $N(0,1)$, *the maximal expected utilities are*

$$\mathbb{E}[U(L_T^{\phi^*})] = -\sqrt{\frac{\beta+2}{(2-(T-1)\beta)\gamma_\beta(T)}}, \tag{11}$$

$$\mathbb{E}[U(L_T^{\eta^*})] = -\sqrt{\frac{\beta+2}{(2-(T-1)\beta)}}. \tag{12}$$

REMARK 2.3. The asymptotic behaviour of γ_β is $\lim_{T\to\infty} \frac{\gamma_\beta(T)}{h_\beta(T)} = 1$ where

$$h_\beta(T) = \begin{cases} \Gamma\left(\frac{1}{\beta^2}\right)^{-1}(\beta^2)^{1-1/\beta^2}\sqrt{2\pi(T-1+1/\beta^2)}\left(\frac{1+(T-1)\beta^2}{e}\right)^{T-1+1/\beta^2} & \text{if } \beta \neq 0, \\ 1 & \text{if } \beta = 0. \end{cases} \tag{13}$$

Indeed, using $y(T) := T-1+1/\beta^2$, we get

$$\frac{\gamma_\beta(T)}{h_\beta(T)} = \frac{\beta^{2T}\Gamma(y(T)+1)}{(\beta^2)^{1-1/\beta^2}\sqrt{2\pi y(T)}\left(\frac{\beta^2 y(T)}{e}\right)^{T-1+1/\beta^2}} = \frac{\Gamma(y(T)+1)}{\sqrt{2\pi y(T)}\left(\frac{y(T)}{e}\right)^{y(T)}}, \tag{14}$$

and it is well-known (see e.g. [1, p. 257]) that this expression tends to 1 if T (and hence also $y(T)$) tends to infinity.

3. Computations and proofs. It is easily seen that there exists a measurable (even continuous) function $y_0,\dots,y_{T-1} \mapsto \ell(dy_{T-1},\dots,dy_1 \mid y_0)$ such that it is the density of $(X_{T-1},\dots,X_1)$ (with respect to the Lebesgue measure) when $X_0 = y_0$. This allows us to define a version of

$$\mathbb{E}[h(X_1,\dots,X_{T-1}) \mid X_0 = z] := \int_{\mathbb{R}^{T-1}} h(y_1,\dots,y_{T-1})\ell(y_{T-1},\dots,y_1|z)\,dy_1\cdots dy_{T-1}$$

for any positive (or negative) measurable h, for all $z \in \mathbb{R}$. As X is a homogeneous Markov chain, we also have

$$\mathbb{E}[h(X_2,\dots,X_T) \mid X_1 = z] = \int_{\mathbb{R}^{T-1}} h(y_1,\dots,y_{T-1})\ell(y_{T-1},\dots,y_1 \mid z)\,dy_1\cdots dy_{T-1}, \tag{15}$$

and even

$$\mathbb{E}[h(X_0,X_1,X_2,\dots,X_T) \mid X_1 = z,\, X_0 = w] = \int_{\mathbb{R}^{T-1}} h(w,z,y_1,\dots,y_{T-1})\ell(y_{T-1},\dots,y_1 \mid z)\,dy_1\cdots dy_{T-1}. \tag{16}$$

The following lemma will help to construct the optimal strategy ϕ^* inductively.

Lemma 3.1. *Let $\tilde{\phi}_t$, $t = 1,\dots,T-1$, be a trading strategy up to time $T-1$ given by $\tilde{\phi}_t = f_t(X_0,\dots,X_{t-1})$ with Borel functions f_t. If $\tilde{\phi}$ is optimal up to $T-1$, i.e. for all z and all strategies ϕ_t, $t = 1,\dots,T-1$, one has*

$$\mathbb{E}[U(L^{\phi}_{T-1}) \mid X_0 = z] \le \mathbb{E}[U(L^{\tilde{\phi}}_{T-1}) \mid X_0 = z], \tag{17}$$

then the strategy defined by $\bar{\phi}_t := f_{t-1}(X_1, \ldots, X_{t-1})$, $t = 2, \ldots, T$, *is optimal between* 2 *and* T, *i.e. for all strategies* ψ_t, $t = 2, \ldots, T$, *one has*

$$\mathbb{E}\Big[U\Big(\sum_{t=2}^{T} \psi_t(X_t - X_{t-1})\Big) \;\Big|\; \mathcal{F}_1\Big] \le \mathbb{E}\Big[U\Big(\sum_{t=2}^{T} \bar{\phi}_t(X_t - X_{t-1})\Big) \;\Big|\; \mathcal{F}_1\Big]. \tag{18}$$

Proof. Let $\psi_t = g_t(X_0, \ldots, X_{t-1})$, $t = 2, \ldots, T$, with some Borel functions g_t. Fix $w \in \mathbb{R}$ and define the strategy $\phi_t = g_{t+1}(w, X_0, \ldots, X_{t-1})$, $t = 1, \ldots, T-1$. By (17),

$$\begin{aligned} &\int_{\mathbb{R}^{T-1}} U\Big(\sum_{t=1}^{T-1} g_{t+1}(w, y_0, \ldots, y_{t-1})(y_t - y_{t-1})\Big)\ell(dy_{T-1}, \ldots, dy_1 \mid y_0) \\ &\quad \le \int_{\mathbb{R}^{T-1}} U\Big(\sum_{t=1}^{T-1} f_t(y_0, \ldots, y_{t-1})(y_t - y_{t-1})\Big)\ell(dy_{T-1}, \ldots, dy_1 \mid y_0) \end{aligned} \tag{19}$$

for all y_0, w. Note that, by (15), the right-hand side can be rewritten as

$$\begin{aligned} &\mathbb{E}\Big[U\Big(\sum_{t=2}^{T} \bar{\phi}_t(X_t - X_{t-1})\Big) \;\Big|\; X_1 = y_0\Big] \\ &\qquad = \mathbb{E}\Big[U\Big(\sum_{t=2}^{T} \bar{\phi}_t(X_t - X_{t-1})\Big) \;\Big|\; X_1 = y_0,\, X_0 = w\Big] \end{aligned} \tag{20}$$

for all y_0, w, recalling the Markov property. Similarly, by (16), the left-hand side of (19) equals

$$\mathbb{E}\Big[U\Big(\sum_{t=2}^{T} \psi_t(X_t - X_{t-1})\Big) \;\Big|\; X_1 = y_0,\, X_0 = w\Big]. \tag{21}$$

Now we obtain (18) from (19)–(21). ∎

After these preparations we are able to give an explicit solution for the optimal strategies of the wealth process in case the price is an autoregressive process.

In this section we prove Theorem 2.1. First we focus on the case where the investor uses past information. It suffices to establish that, for all $\phi \in \Phi$,

$$\sup_{\phi \in \Phi} \mathbb{E}[U(L_T^{\phi}) \mid \mathcal{F}_0] \le \mathbb{E}[U(L_T^{\phi^*}) \mid \mathcal{F}_0], \tag{22}$$

with ϕ^* defined in Theorem 2.1.

We consider the case $T = 1$, so the wealth process according to (4) takes the form

$$L_1^{\phi} = \phi_1(X_1 - X_0) = \phi_1(\beta X_0 + \sigma\varepsilon_1). \tag{23}$$

Note that $\phi_1 = m(X_0)$ for some measurable function $m : \mathbb{R} \to \mathbb{R}$. We have

$$\mathbb{E}[e^{-\phi_1(\beta X_0+\sigma\varepsilon_1)} \mid X_0 = z] = e^{-m(z)\beta z}\mathbb{E}[e^{-m(z)\sigma\varepsilon_1}] = e^{-m(z)\beta z}e^{m^2(z)\sigma^2/2}, \tag{24}$$

hence we get

$$\arg\min_{m(z)} e^{-m(z)\beta z+\frac{(m(z)\sigma)^2}{2}} = \arg\min_{m(z)} e^{\frac{1}{2}(m(z)\sigma-\frac{\beta z}{\sigma})^2-\frac{\beta^2 z^2}{2\sigma^2}} = \frac{\beta z}{\sigma^2} = \frac{\beta z}{\sigma^2}\theta_1^1 = \hat{\phi}_1^1(z),$$

because $\theta_1^1 = 1$. So we proved the first part of Theorem 2.1 for $T = 1$. Now let's assume that (7) is true for $T-1$, i.e.

$$\phi_t^* := \hat{\phi}_t^{T-1}(X_{t-1}) \text{ with } \hat{\phi}_t^{T-1}(z) = \frac{\beta z}{\sigma^2}\theta_t^{T-1} \quad 1 \le t \le T-1 \tag{25}$$

satisfies (22) for all $\phi \in \Phi$. We will prove that (7) also satisfies (22) for T. By Lemma 3.1, for all $\psi \in \Phi$,

$$\begin{aligned}\mathbb{E}[e^{-L_T^\psi} \mid \mathcal{F}_0] &= \mathbb{E}[e^{-\psi_1(X_1-X_0)}\mathbb{E}[e^{-\sum_{j=2}^T \psi_j(X_j-X_{j-1})} \mid \mathcal{F}_1] \mid \mathcal{F}_0] \\ &\ge \mathbb{E}[e^{-\psi_1(X_1-X_0)}\mathbb{E}[e^{-\sum_{j=2}^T \hat{\phi}_{j-1}^{T-1}(X_{j-1})(X_j-X_{j-1})} \mid \mathcal{F}_1] \mid \mathcal{F}_0] \\ &= \mathbb{E}[e^{-\psi_1(X_1-X_0)}\mathbb{E}[e^{-\sum_{j=2}^T \hat{\phi}_j^T(X_{j-1})(X_j-X_{j-1})} \mid \mathcal{F}_1] \mid \mathcal{F}_0],\end{aligned} \tag{26}$$

since $\hat{\phi}_{j-1}^{T-1}(z) = \hat{\phi}_j^T(z)$ for all z. Now define the trading strategy $\omega = (\phi, \hat{\phi}_2^T(X_1), \ldots, \hat{\phi}_T^T(X_{T-1}))$ and the function $Q_T : \mathbb{R}^{T+2} \to \mathbb{R}$ such that

$$Q_T(\phi, X_0, \varepsilon) := L_T^\omega, \quad \text{where } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_T)^{\mathbf{T}}. \tag{27}$$

Hence, according to (26), it remains to find ϕ which minimizes

$$\mathbb{E}[e^{-Q_T(\phi,X_0,\varepsilon)} \mid X_0 = z]. \tag{28}$$

If we prove that $\phi = \hat{\phi}_1^T(z)$ does the job then we will be able to conclude that the optimal strategy for time horizon T is indeed as given in (25) for $T-1$.

To compute the minimiser ϕ we will write $Q_T(\phi, X_0, \varepsilon)$ as a sum of a quadratic, a linear and a constant function of ε. We have

$$\begin{aligned}Q_T(\phi, X_0, \varepsilon) &= \phi(X_1 - X_0) + \sum_{j=2}^T \hat{\phi}_j^T(X_{j-1})(X_j - X_{j-1}) \\ &= \phi(X_1 - X_0) + \sum_{j=2}^T \frac{\beta\theta_j^T}{\sigma^2} \underbrace{X_{j-1}(X_j - X_{j-1})}_{A_j}.\end{aligned}$$

For A_j we obtain

$$\begin{aligned}
A_j &= \Big(\alpha^{j-1}X_0 + \sigma\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\Big)\\
&\quad\times\Big(\alpha^{j}X_0 + \sigma\sum_{i=1}^{j}\alpha^{j-i}\varepsilon_i - \alpha^{j-1}X_0 - \sigma\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\Big)\\
&= \Big(\alpha^{j-1}X_0 + \sigma\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\Big)\Big(\alpha^{j-1}\beta X_0 + \sigma\varepsilon_j + \sigma\sum_{i=1}^{j-1}\alpha^{j-i-1}\beta\varepsilon_i\Big)\\
&= \alpha^{2j-2}\beta X_0^2 + \sigma\beta X_0\sum_{i=1}^{j-1}\alpha^{2j-i-2}\varepsilon_i + \sigma X_0\alpha^{j-1}\varepsilon_j\\
&\quad + \sigma^2\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\varepsilon_j + \sigma\beta X_0\sum_{i=1}^{j-1}\alpha^{2j-i-2}\varepsilon_i + \sigma^2\beta\Big(\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\Big)^2\\
&= \underbrace{\alpha^{2j-2}\beta X_0^2}_{B_1(j)} + \underbrace{2\sigma\beta X_0\sum_{i=1}^{j-1}\alpha^{2j-i-2}\varepsilon_i}_{B_2(j)} + \underbrace{\sigma X_0\alpha^{j-1}\varepsilon_j}_{B_3(j)}\\
&\quad + \underbrace{\sigma^2\sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\varepsilon_j}_{B_4(j)} + \underbrace{\sigma^2\beta\sum_{i=1}^{j-1}\sum_{k=1}^{j-1}\varepsilon_i\varepsilon_k\alpha^{2j-i-k-2}}_{B_5(j)}.
\end{aligned}$$

We substitute this into $Q_T(\phi, X_0, \varepsilon)$:

$$\begin{aligned}
&Q_T(\phi, X_0, \varepsilon)\\
&\quad= \phi(X_1 - X_0) + \sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}(B_1(j) + B_2(j) + B_3(j) + B_4(j) + B_5(j))\\
&\quad= \phi(\beta X_0 + \sigma\varepsilon_1) + \underbrace{\sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}B_1(j)}_{C_1} + \underbrace{\sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}B_2(j)}_{C_2} + \underbrace{\sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}B_3(j)}_{C_3}\\
&\qquad + \underbrace{\sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}B_4(j)}_{C_4} + \underbrace{\sum_{j=2}^{T}\frac{\beta\theta_j^T}{\sigma^2}B_5(j)}_{C_5}
\end{aligned}$$

We compute each C_n separately:

$$C_1 = \frac{\beta^2 X_0^2}{\sigma^2}\sum_{j=2}^{T}\theta_j^T\alpha^{2j-2},$$

$$C_2 = \sum_{j=2}^{T} \frac{\beta\theta_j^T}{\sigma^2} 2\sigma\beta X_0 \sum_{i=1}^{j-1} \alpha^{2j-i-2}\varepsilon_i = \sum_{j=1}^{T-1}\sum_{i=1}^{j} \frac{2\beta^2 X_0}{\sigma}\theta_{j+1}^T \alpha^{2j-i}\varepsilon_i$$
$$= \sum_{i=1}^{T-1}\sum_{j=i}^{T-1} \frac{2\beta^2 X_0}{\sigma}\theta_{j+1}^T \alpha^{2j-i}\varepsilon_i = \sum_{i=1}^{T-1}\varepsilon_i\left(\sum_{j=i}^{T-1}\frac{2\beta^2 X_0}{\sigma}\theta_{j+1}^T\alpha^{2j-i}\right),$$
$$C_3 = \sum_{j=2}^{T} \frac{\beta\theta_j^T}{\sigma^2}\sigma X_0 \alpha^{j-1}\varepsilon_j = \sum_{j=2}^{T}\varepsilon_j\left(\frac{\beta X_0}{\sigma}\theta_j^T\alpha^{j-1}\right),$$
$$C_4 = \sum_{j=2}^{T} \frac{\beta\theta_j^T}{\sigma^2}\sigma^2 \sum_{i=1}^{j-1}\alpha^{j-i-1}\varepsilon_i\varepsilon_j = \sum_{k=2}^{T}\sum_{i=1}^{k-1}\varepsilon_i\varepsilon_k(\beta\theta_k^T\alpha^{k-i-1}),$$
$$C_5 = \sum_{j=2}^{T} \frac{\beta\theta_j^T}{\sigma^2}\sigma^2\beta\sum_{i=1}^{j-1}\sum_{k=1}^{j-1}\varepsilon_i\varepsilon_k\alpha^{2j-i-k-2} = \beta^2\sum_{j=1}^{T-1}\sum_{i=1}^{j}\sum_{k=1}^{j}\theta_{j+1}^T\varepsilon_i\varepsilon_k\alpha^{2j-i-k}$$
$$= \beta^2\sum_{i=1}^{T-1}\sum_{j=i}^{T-1}\sum_{k=1}^{j}\theta_{j+1}^T\varepsilon_i\varepsilon_k\alpha^{2j-i-k}$$
$$= \beta^2\sum_{i=1}^{T-1}\left(\sum_{k=1}^{i}\sum_{j=i}^{T-1}\theta_{j+1}^T\varepsilon_i\varepsilon_k\alpha^{2j-i-k} + \sum_{k=i+1}^{T-1}\sum_{j=k}^{T-1}\theta_{j+1}^T\varepsilon_i\varepsilon_k\alpha^{2j-i-k}\right)$$
$$= \beta^2\sum_{i=1}^{T-1}\left(\sum_{k=1}^{i}\varepsilon_i\varepsilon_k\sum_{j=i}^{T-1}\theta_{j+1}^T\alpha^{2j-i-k} + \sum_{k=i+1}^{T-1}\varepsilon_i\varepsilon_k\sum_{j=k}^{T-1}\theta_{j+1}^T\alpha^{2j-i-k}\right).$$

Hence, we can write

$$Q_T(\phi, X_0, \varepsilon) = \varepsilon^{\mathbf{T}}\left(\mathbf{A}_T - \frac{1}{2}\mathbf{I}\right)\varepsilon + \mathbf{b}^{\mathbf{T}}(\phi, X_0)\varepsilon + c(\phi, X_0), \tag{29}$$

where $\mathbf{A}_T = [a_{i,k}] \in \mathbb{R}^{T\times T}$ is a symmetric matrix with

$$a_{i,i} = \frac{1}{2} + \beta^2\sum_{j=i}^{T-1}\theta_{j+1}^T\alpha^{2j-2i}, \qquad 1 \le i \le T-1,$$
$$a_{T,T} = \frac{1}{2},$$
$$a_{i,k} = \frac{\beta\theta_i^T\alpha^{i-k-1}}{2} + \beta^2\sum_{j=i}^{T-1}\theta_{j+1}^T\alpha^{2j-i-k}, \qquad 1 \le k < i \le T-1,$$
$$a_{T,k} = \frac{\beta\alpha^{T-k-1}}{2}, \qquad 1 \le k \le T-1;$$

$\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^T$ is a vector with

$$b_1(\phi, X_0) = \phi\sigma + \frac{2\beta^2 X_0}{\sigma} \sum_{j=1}^{T-1} \theta_{j+1}^T \alpha^{2j-1},$$

$$b_i(\phi, X_0) = \frac{\beta X_0}{\sigma} \theta_i^T \alpha^{i-1} + \frac{2\beta^2 X_0}{\sigma} \sum_{j=i}^{T-1} \theta_{j+1}^T \alpha^{2j-i}, \qquad 2 \le i \le T-1,$$

$$b_T = \frac{\beta X_0}{\sigma} \alpha^{T-1};$$

and $c : \mathbb{R}^2 \to \mathbb{R}$ where

$$c(\phi, X_0) = \phi\beta X_0 + \frac{\beta^2 X_0^2}{\sigma^2} \sum_{j=2}^{T} \theta_j^T \alpha^{2j-2}.$$

We need to compute the conditional expected utility given by

$$\mathbb{E}[e^{-Q_T(\phi, X_0, \varepsilon)} \mid X_0 = z] = \frac{1}{(\sqrt{2\pi})^T} \int_{\mathbb{R}^T} e^{-\mathbf{x^T A}_T \mathbf{x} - \mathbf{b^T}(\phi,z)\mathbf{x} - c(\phi,z)}\, d\mathbf{x}. \tag{30}$$

For this we need some preparation.

LEMMA 3.2. *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric, positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$. Then*

$$\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-\mathbf{x^T A x} - \mathbf{b^T x}}\, d\mathbf{x} = \frac{1}{\sqrt{2^n \det \mathbf{A}}} e^{\frac{\mathbf{b^T A^{-1} b}}{4}}. \tag{31}$$

Proof. First, we consider the case when $\mathbf{A}$ is diagonal, with diagonal entries $d_1, \ldots, d_n$. We know that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2 - bx}\, dx = \frac{1}{\sqrt{2a}} e^{\frac{b^2}{4a}}, \tag{32}$$

for all $b \in \mathbb{R}$ and $a > 0$, so we obtain

$$\begin{aligned}
\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-\mathbf{x^T A x} - \mathbf{b^T x}}\, d\mathbf{x} &= \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n d_i x_i^2 - \sum_{i=1}^n b_i x_i}\, d\mathbf{x} \\
&= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-d_i x_i^2 - b_i x_i}\, dx_i \\
&= \prod_{i=1}^n \frac{1}{\sqrt{2d_i}} e^{\frac{b_i^2}{4d_i}} = \frac{1}{\sqrt{2^n \det \mathbf{A}}} e^{\frac{\mathbf{b^T A^{-1} b}}{4}}.
\end{aligned}$$

If $\mathbf{A}$ is symmetric, there is an orthonormal matrix $\mathbf{S}$, and a diagonal matrix $\mathbf{D}$ for which $\mathbf{SDS^{-1}} = \mathbf{SDS^T} = \mathbf{A}$ and $|\det \mathbf{S}| = 1$. From the diagonal case and setting $\mathbf{y} := \mathbf{S^T x}$ we get

$$\begin{aligned}
&\frac{1}{(\sqrt{2\pi})^n}\int\limits_{\mathbb{R}^n} e^{-\mathbf{x^T A x - b^T x}}\,d\mathbf{x}\\
&\quad= \frac{1}{(\sqrt{2\pi})^n}\int\limits_{\mathbb{R}^n} e^{-\mathbf{x^T S D S^T x - b^T x}}\,d\mathbf{x} = \frac{1}{(\sqrt{2\pi})^n}\int\limits_{\mathbb{R}^n} e^{-\mathbf{(S^T x)^T D S^T x - b^T x}}\,d\mathbf{x}\\
&\quad= \frac{1}{(\sqrt{2\pi})^n}\int\limits_{\mathbb{R}^n} e^{-\mathbf{y^T D y - b^T S y}}|\det \mathbf{S}|\,d\mathbf{y} = \frac{1}{(\sqrt{2\pi})^n}\int\limits_{\mathbb{R}^n} e^{-\mathbf{y^T D y - (S^T b)^T y}}\,d\mathbf{y}\\
&\quad= \frac{1}{\sqrt{2^n \det \mathbf{D}}} e^{\frac{\mathbf{(S^T b)^T D^{-1} S^T b}}{4}} = \frac{1}{\sqrt{2^n \det \mathbf{D}}} e^{\frac{\mathbf{b^T S D^{-1} S^T b}}{4}}\\
&\quad= \frac{1}{\sqrt{2^n \det \mathbf{A}}} e^{\frac{\mathbf{b^T A^{-1} b}}{4}},
\end{aligned}$$

since $\det \mathbf{D} = \det \mathbf{A}$ and $\mathbf{A}^{-1} = \mathbf{S}\mathbf{D}^{-1}\mathbf{S}^{\mathbf{T}}$. ■

Now we can compute the expression in (30) using Lemma 3.2:

$$\mathbb{E}[e^{-Q_T(\phi,X_0,\varepsilon)} \mid X_0 = z] = \frac{1}{\sqrt{2^T \det \mathbf{A}_T}} e^{\frac{\mathbf{b^T}(\phi,z)\mathbf{A}_T^{-1}\mathbf{b}(\phi,z)}{4} - c(\phi,z)}. \tag{33}$$

We proceed to examine the determinant of $\mathbf{A}_T$ to prove that $\mathbf{A}_T$ is positive definite (as (33) holds only in this case) and we will need to compute one element of the inverse matrix, $(\mathbf{A}^{-1})_{1,1}$.

First, it is easy to check that for θ_t^T (defined in Theorem 2.1) and for all $m \le n$,

$$\sum_{i=m}^{n} \theta_i^T \alpha^i = (T+1-m)\alpha^m - (T-n)\alpha^{n+1}. \tag{34}$$

We will use this equation very often in the calculations below.

Lemma 3.3. *For $\mathbf{A}_T = [a_{i,j}]$ we have*

$$a_{1,k} - \beta \sum_{i=2}^{T} a_{i,k} = 0 \quad \textit{for all } 2 \le k \le T. \tag{35}$$

Proof. First we consider the case $k = T$:

$$\begin{aligned}
a_{1,T} - \beta\sum_{i=2}^{T} a_{i,T} &= \frac{\beta\alpha^{T-2}}{2} - \beta\bigg(\frac{1}{2} + \sum_{i=2}^{T-1}\frac{\beta\alpha^{T-i-1}}{2}\bigg)\\
&= \frac{\beta}{2}\Big(\alpha^{T-2} - 1 - \beta\sum_{i=2}^{T-1}\alpha^{T-i-1}\Big)\\
&= \frac{\beta}{2}\bigg(\alpha^{T-2} - 1 - \sum_{i=2}^{T-1}\alpha^{T-i} + \sum_{i=2}^{T-1}\alpha^{T-i-1}\frac{\beta}{2}\bigg)
\end{aligned}$$

$$= \frac{\beta}{2}\Big(\alpha^{T-2} - 1 - \sum_{i=2}^{T-1} \alpha^{T-i} + \sum_{i=3}^{T} \alpha^{T-i}\Big)$$
$$= \frac{\beta}{2}(\alpha^{T-2} - 1 - \alpha^{T-2} + 1) = 0.$$

Then we consider the case $k \neq T$:

$$a_{1,k} - \beta \sum_{i=2}^{T} a_{i,k} = a_{1,k} - \beta \sum_{i=2}^{k-1} a_{i,k} - \beta a_{k,k} - \beta \sum_{i=k+1}^{T-1} a_{i,k} - \beta a_{T,k}. \tag{36}$$

We compute the sums separately:

$$\sum_{i=2}^{k-1} a_{i,k} = \sum_{i=2}^{k-1} \Big(\frac{\beta \theta_k^T \alpha^{k-i-1}}{2} + \beta^2 \sum_{j=k}^{T-1} \theta_{j+1}^T \alpha^{2j-k-i} \Big)$$
$$= \frac{\beta \theta_k^T}{2} \sum_{i=2}^{k-1} \alpha^{k-i-1} + \beta^2 \sum_{j=k}^{T-1} \theta_{j+1}^T \sum_{i=2}^{k-1} \alpha^{2j-k-i}$$
$$= \frac{\theta_k^T}{2}\Big(\sum_{i=2}^{k-1} \alpha^{k-i} - \sum_{i=2}^{k-1} \alpha^{k-i-1} \Big)$$
$$+ \beta \sum_{j=k}^{T-1} \theta_{j+1}^T \Big(\sum_{i=2}^{k-1} \alpha^{2j-k-i+1} - \sum_{i=2}^{k-1} \alpha^{2j-k-i} \Big)$$
$$= \frac{\theta_k^T}{2}(\alpha^{k-2} - 1) + \beta \sum_{j=k}^{T-1} \theta_{j+1}^T (\alpha^{2j-k-1} - \alpha^{2j-2k+1})$$
$$= \frac{\theta_k^T \alpha^{k-2}}{2} - \frac{\theta_k^T}{2} + \beta \sum_{j=k}^{T-1} \theta_{j+1}^T \alpha^{2j-k-1} - \beta \sum_{j=k}^{T-1} \theta_{j+1}^T \alpha^{2j-2k+1}$$

and

$$\sum_{i=k+1}^{T-1} a_{i,k} = \sum_{i=k+1}^{T-1} \Big(\frac{\beta \theta_i^T \alpha^{i-k-1}}{2} + \beta^2 \sum_{j=i}^{T-1} \theta_{j+1}^T \alpha^{2j-i-k} \Big)$$
$$= \frac{\beta \alpha^{-k-1}}{2} \sum_{i=k+1}^{T-1} \theta_i^T \alpha^i + \beta^2 \sum_{j=k+1}^{T-1} \sum_{i=k+1}^{j} \theta_{j+1}^T \alpha^{2j-i-k}$$
$$= \frac{\beta \alpha^{-k-1}}{2} \sum_{i=k+1}^{T-1} \theta_i^T \alpha^i + \beta \sum_{j=k+1}^{T-1} \theta_{j+1}^T \Big(\sum_{i=k+1}^{j} \alpha^{2j-i-k+1} - \sum_{i=k+1}^{j} \alpha^{2j-i-k} \Big)$$
$$= \frac{\beta \alpha^{-k-1}}{2}((T-k)\alpha^{k+1} - \alpha^T) + \beta \sum_{j=k+1}^{T-1} \theta_{j+1}^T (\alpha^{2j-2k} - \alpha^{j-k}) \quad \text{by (34)}$$

$$= \frac{\beta(T-k)}{2} - \frac{\beta\alpha^{T-k-1}}{2} + \beta\sum_{j=k+1}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} - \beta\sum_{j=k+2}^{T}\theta_{j}^{T}\alpha^{j-k-1}$$

$$= \frac{\beta(T-k)}{2} - \frac{\beta\alpha^{T-k-1}}{2} + \beta\sum_{j=k+1}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} - \beta\alpha(T-k-1) \quad \text{by (34).}$$

The other terms in (36) are

$$a_{1,k} = \frac{\beta\theta_k^T\alpha^{k-2}}{2} + \beta^2\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-k-1},$$

$$\beta a_{k,k} = \frac{\beta}{2} + \beta^3\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k},$$

$$\beta a_{T,k} = \frac{\beta^2\alpha^{T-k-1}}{2}.$$

Substituting these into (36) yields

$$\begin{aligned}
a_{1,k} - \beta\sum_{i=2}^{T}a_{i,k} &= \frac{\beta\theta_k^T\alpha^{k-2}}{2} + \beta^2\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-k-1} - \frac{\beta\theta_k^T\alpha^{k-2}}{2} \\
&\quad + \frac{\beta\theta_k^T}{2} - \beta^2\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-k-1} \\
&\quad + \beta^2\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k+1} - \frac{\beta}{2} - \beta^3\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} \\
&\quad - \frac{\beta^2(T-k)}{2} + \frac{\beta^2\alpha^{T-k-1}}{2} \\
&\quad - \beta^2\sum_{j=k+1}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} + \beta^2\alpha(T-k-1) - \frac{\beta^2\alpha^{T-k-1}}{2} \\
&= \frac{\beta-\beta^2(T-k)}{2} + \beta^2\sum_{j=k}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} - \frac{\beta}{2} - \frac{\beta^2(T-k)}{2} \\
&\quad - \beta^2\sum_{j=k+1}^{T-1}\theta_{j+1}^{T}\alpha^{2j-2k} + \beta^2\alpha(T-k-1) \\
&= \beta^2\theta_{k+1}^{T} - \beta^2(T-k) + \beta^2\alpha(T-k-1) \\
&= \beta^2(\theta_{k+1}^{T} - (T-k) + \alpha(T-k-1)) = 0. \ \blacksquare
\end{aligned}$$

DEFINITION 3.4. For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ let $\mathbf{A}(i,j) \in \mathbb{R}^{(n-1)\times(n-1)}$ denote the minor of $\mathbf{A}$ obtained by omitting the ith row and the jth column.

Lemma 3.5. *We have*

$$\mathbf{A}_T(1,1) = \mathbf{A}_{T-1}. \tag{37}$$

Proof. We denote the entries of $\mathbf{A}_T(1,1)$ and $\mathbf{A}_{T-1}$ by $u_{i,k}$ and $v_{i,k}$, respectively. Then

$$\begin{aligned}
u_{i,i} &= a_{i+1,i+1} = \frac{1}{2} + \beta^2 \sum_{j=i+1}^{T-1} \theta_{j+1}^{T} \alpha^{2j-2(i+1)} = \frac{1}{2} + \beta^2 \sum_{j=i}^{T-2} \theta_{j+2}^{T} \alpha^{2j-2i} \\
&= \frac{1}{2} + \beta^2 \sum_{j=i}^{T-2} \theta_{j+1}^{T-1} \alpha^{2j-2i} = v_{i,i}, \quad 1 \le i \le T-2, \\
u_{i,k} &= a_{i+1,k+1} = \frac{\beta\theta_{i+1}^{T} \alpha^{i-k-1}}{2} + \beta^2 \sum_{j=i+1}^{T-1} \theta_{j+1}^{T} \alpha^{2j-i-k-2} \\
&= \frac{\beta\theta_{i}^{T-1} \alpha^{i-k-1}}{2} + \beta^2 \sum_{j=i}^{T-2} \theta_{j+2}^{T} \alpha^{2j-i-k} \\
&= \frac{\beta\theta_{i}^{T-1} \alpha^{i-k-1}}{2} + \beta^2 \sum_{j=i}^{T-2} \theta_{j+1}^{T-1} \alpha^{2j-i-k} \\
&= v_{i,k}, \quad 1 \le k < i \le T-2, \\
u_{T-1,k} &= a_{T,k+1} = \frac{\beta\alpha^{T-(k+1)-1}}{2} \\
&= \frac{\beta\alpha^{(T-1)-k-1}}{2} = v_{T-1,k}, \quad 1 \le k \le T-2, \\
u_{T-1,T-1} &= a_{T,T} = 1/2 = v_{T-1,T-1}. \ \blacksquare
\end{aligned}$$

Lemma 3.6. *For all $T \ge 2$, we have*

$$\det \mathbf{A}_T = \frac{1+\beta^2(T-1)}{2} \det \mathbf{A}_{T-1}. \tag{38}$$

Proof. We construct a matrix $\mathbf{B}_T$ in such a way that we subtract the rows of $\mathbf{A}_T$ multiplied by β from the first row. Then, according to Lemma 3.3, in the first row of $\mathbf{B}_T$ all elements except the first one ($b_{1,1}$) are zero. Hence, by Lemma 3.5,

$$\begin{aligned}
\det \mathbf{B}_T = b_{1,1} \det \mathbf{B}_T(1,1) &= \Big(a_{1,1} - \beta \sum_{i=2}^{T} a_{i,1}\Big) \det \mathbf{A}_T(1,1) \\
&= \Big(a_{1,1} - \beta \sum_{i=2}^{T} a_{i,1}\Big) \det \mathbf{A}_{T-1}.
\end{aligned}$$

We need to check that

$$a_{1,1} - \beta \sum_{i=2}^{T} a_{i,1} = \frac{1+\beta^2(T-1)}{2}. \tag{39}$$

Indeed,

$$\begin{aligned}
\beta \sum_{i=2}^{T} a_{i,1} &= \beta \sum_{i=2}^{T-1} \left(\frac{\beta \theta_i^T \alpha^{i-2}}{2} + \beta^2 \sum_{j=i}^{T-1} \theta_{j+1}^T \alpha^{2j-i-1} \right) + \frac{\beta^2 \alpha^{T-2}}{2} \\
&= \frac{\beta^2 \alpha^{-2}}{2} \sum_{i=2}^{T-1} \theta_i^T \alpha^i + \beta^3 \sum_{j=2}^{T-1} \sum_{i=2}^{j} \theta_{j+1}^T \alpha^{2j-i-1} + \frac{\beta^2 \alpha^{T-2}}{2} \\
&= \frac{\beta^2 \alpha^{-2}}{2} ((T-1)\alpha^2 - \alpha^T) + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-1} (\alpha - 1) \sum_{i=2}^{j} \alpha^{-i} \\
&\quad + \frac{\beta^2 \alpha^{T-2}}{2} \quad \text{by (34)} \\
&= \frac{\beta^2(T-1)}{2} + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-1} \Big(\sum_{i=2}^{j} \alpha^{-i+1} - \sum_{i=2}^{j} \alpha^{-i} \Big) \\
&= \frac{\beta^2(T-1)}{2} + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-1} (\alpha^{-1} - \alpha^{-j}) \\
&= \frac{\beta^2(T-1)}{2} + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-2} - \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{j-1} \\
&= \frac{\beta^2(T-1)}{2} + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-2} - \beta^2 \sum_{j=3}^{T} \theta_j^T \alpha^{j-2} \\
&= \frac{\beta^2(T-1)}{2} + \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-2} - \beta^2 (T-2)\alpha \quad \text{by (34).}
\end{aligned}$$

We substitute this into (39):

$$\begin{aligned}
a_{1,1} - \beta \sum_{i=2}^{T} a_{i,1} &= \frac{1}{2} + \beta^2 \sum_{j=1}^{T-1} \theta_{j+1}^T \alpha^{2j-2} - \frac{\beta^2(T-1)}{2} \\
&\quad - \beta^2 \sum_{j=2}^{T-1} \theta_{j+1}^T \alpha^{2j-2} + \beta^2 (T-2)\alpha \\
&= \frac{1}{2} + \beta^2 \theta_2^T - \frac{\beta^2(T-1)}{2} + \beta^2 (T-2)\alpha
\end{aligned}$$

$$= \frac{1}{2} + \beta^2(T-1-(T-2)\alpha) - \frac{\beta^2(T-1)}{2} + \beta^2(T-2)\alpha$$
$$= \frac{1+\beta^2(T-1)}{2}. \blacksquare$$

From the above lemma and from the fact that $\mathbf{A}_1 = 1/2$ we obtain the following.

Corollary 3.7. *$\mathbf{A}_T$ is positive definite and*

$$\det \mathbf{A}_T = \frac{1}{2^T} \prod_{i=0}^{T-1} (1+\beta^2 i) \quad \text{for all } T \geq 1. \blacksquare \tag{40}$$

We can express $\det \mathbf{A}_T$ in terms of the Γ function.

Corollary 3.8. *We have* $\det \mathbf{A}_T = \gamma_\beta(T)/2^T$, *where γ_β is the function defined in* (10). $\blacksquare$

Later on we will need the value of $p := (\mathbf{A}_T^{-1})_{1,1}$. Using Lemmas 3.5 and 3.6 we get

$$p = \frac{\det \mathbf{A}_T(1,1)}{\det \mathbf{A}_T} = \frac{\det \mathbf{A}_{T-1}}{\det \mathbf{A}_T} = \frac{2}{1+\beta^2(T-1)}. \tag{41}$$

We are finally ready to compute the minimiser of

$$\frac{1}{\sqrt{2^T \det \mathbf{A}_T}} e^{\frac{\mathbf{b}^\mathbf{T}(\phi,z)\mathbf{A}_T{}^{-1}\mathbf{b}(\phi,z)}{4} - c(\phi,z)}$$

(see (33)). Note that here only the exponent depends on ϕ, so we can focus on it. Let

$$f(\phi,z) := \frac{\mathbf{b}^\mathbf{T}(\phi,z)\mathbf{A}_T^{-1}\mathbf{b}(\phi,z)}{4} - c(\phi,z). \tag{42}$$

By the rules of differential calculus (note that the first term of $f(\phi,z)$ is $\mathbf{b}(\phi,z)$ plugged into a quadratic form),

$$(\partial_\phi f)(\phi,z) = \frac{(\partial_\phi \mathbf{b}^\mathbf{T})(\phi,z)\mathbf{A}_T^{-1}\mathbf{b}(\phi,z)}{2} - (\partial_\phi c)(\phi,z),$$

so we need to solve

$$\frac{(\partial_\phi \mathbf{b}^\mathbf{T})(\phi,z)\mathbf{A}_T^{-1}\mathbf{b}(\phi,z)}{2} - (\partial_\phi c)(\phi,z) = 0. \tag{43}$$

From the definition of $\mathbf{b}(\phi,z)$ and $c(\phi,z)$,

$$(\partial_\phi \mathbf{b}^\mathbf{T})(\phi,z) = (\sigma, 0, \dots, 0), \tag{44}$$
$$(\partial_\phi c)(\phi,z) = \beta z. \tag{45}$$

Note that we can write

$$\mathbf{b}(\phi,z) = \left(\sigma\phi - \frac{\alpha z}{\sigma}\right)\mathbf{e}_1 + \frac{2\alpha z}{\sigma}\mathbf{A}_T(:,1), \tag{46}$$

where $\mathbf{e}_1 = (1, 0, \dots, 0)^{\mathbf{T}}$, and $\mathbf{A}_T(:, 1)$ is the first column of $\mathbf{A}_T$. Let us substitute (44)–(46) into (43):

$$\begin{aligned}
&(\sigma, 0, \dots, 0)\mathbf{A}_T^{-1}\left[\left(\sigma\phi - \frac{\alpha z}{\sigma}\right)\mathbf{e}_1 + \frac{2\alpha z}{\sigma}\mathbf{A}_T(:, 1)\right] - 2\beta z = 0,\\
&\sigma\mathbf{A}_T^{-1}(1, :)\left[\left(\sigma\phi - \frac{\alpha z}{\sigma}\right)\mathbf{e}_1 + \frac{2\alpha z}{\sigma}\mathbf{A}_T(:, 1)\right] - 2\beta z = 0,\\
&(\sigma^2\phi - \alpha z)p + 2\alpha z - 2\beta z = 0,\\
&\sigma^2\phi - \alpha z = -\frac{2z}{p},\\
&\sigma^2\phi - \alpha z = -(1 + \beta^2(T-1))z \quad \text{by (41), hence}\\
&\phi = \frac{\beta z}{\sigma^2}(1 - (T-1)\beta),\\
&\phi = \frac{\beta z}{\sigma^2}\theta_1^T.
\end{aligned}$$

We can see from the above calculation that this ϕ is a global minimiser of f for a given z. Hence the minimiser ϕ for (28) is

$$\phi = \hat{\phi}_1^T(z) = \frac{\beta z}{\sigma^2}\theta_1^T,$$

and we have proved the first part Theorem 2.1 in the case of using past information. As we have found explicit optimal strategies for the expected utility problem, we can now turn to (8) and (11).

First we compute the maximal conditional expected utility, which is

$$\mathbb{E}[U(L_T^{\phi^*}) \mid X_0 = z] = -\frac{1}{\sqrt{\gamma_\beta(T)}}e^{f(\hat{\phi}_1^T(z), z)} \tag{47}$$

by (33), Corollary 3.8 and (42).

Let

$$\hat{f}(z) := f(\hat{\phi}_1^T(z), z), \quad \hat{\mathbf{b}}(z) := \mathbf{b}(\hat{\phi}_1^T(z), z), \quad \hat{c}(z) := c(\hat{\phi}_1^T(z), z).$$

Then

$$\begin{aligned}
\hat{\mathbf{b}}^{\mathbf{T}}(z)\mathbf{A}_T^{-1}\hat{\mathbf{b}}(z) &= \left[\frac{2\alpha z}{\sigma}\mathbf{A}_T(:, 1) - \frac{z}{\sigma}(1 + \beta^2(T-1))\mathbf{e}_1\right]^{\mathbf{T}}\mathbf{A}_T^{-1}\\
&\quad\cdot\left[\frac{2\alpha z}{\sigma}\mathbf{A}_T(:, 1) - \frac{z}{\sigma}(1 + \beta^2(T-1))\mathbf{e}_1\right] \quad \text{by (46)}\\
&= \left[\frac{2\alpha z}{\sigma}(1, 0, \dots, 0) - \frac{z}{\sigma}(1 + \beta^2(T-1))\mathbf{A}_T^{-1}(1, :)\right]\\
&\quad\cdot\left[\frac{2\alpha z}{\sigma}\mathbf{A}_T(:, 1) - \frac{z}{\sigma}(1 + \beta^2(T-1))\mathbf{e}_1\right]
\end{aligned}$$

$$= \frac{z^2}{\sigma^2}((1+\beta^2(T-1))^2 p - 4\alpha(1+\beta^2(T-1)) + 4\alpha^2 a_{1,1})$$
$$= \frac{z^2}{\sigma^2}(2(1+\beta^2(T-1)) - 4\alpha(1+\beta^2(T-1)) + 4\alpha^2 a_{1,1}) \quad \text{by (41)}.$$

Moreover,

$$\hat{f}(z) = \frac{\hat{\mathbf{b}}^{\mathbf{T}}(z)\mathbf{A}_T^{-1}\hat{\mathbf{b}}(z)}{4} - \hat{c}(z)$$
$$= \frac{z^2}{\sigma^2}\bigg(\frac{(1+\beta^2(T-1))}{2} - \alpha(1+\beta^2(T-1)) + \alpha^2 a_{1,1} \\ -\beta^2(1-(T-1)\beta) - \beta^2 \textstyle\sum_{j=2}^T \theta_j^T \alpha^{2j-2}\bigg)$$
$$= \frac{z^2}{\sigma^2}\bigg(\frac{(1+\beta^2(T-1))}{2} - \alpha(1+\beta^2(T-1)) + \frac{\alpha^2}{2} + \beta^2 \sum_{j=1}^{T-1} \theta_{j+1}^T \alpha^{2j}\bigg)$$
$$+ \frac{z^2}{\sigma^2}\bigg(-\beta^2(1-(T-1)\beta) - \beta^2 \sum_{j=2}^{T} \theta_j^T \alpha^{2j-2}\bigg)$$
$$= \frac{z^2}{\sigma^2}\bigg(\frac{1+\beta^2(T-1)}{2} - \beta - \beta^3(T-1) - 1 - \beta^2(T-1) \\ + \frac{(\beta+1)^2}{2} - \beta^2 - \beta^3(T-1)\bigg)$$
$$= -\frac{\beta^2 z^2}{2\sigma^2} T.$$

Hence the maximal achievable conditional expected utility is

$$\mathbb{E}[U(L_T^{\phi^*}) \mid X_0 = z] = -\frac{1}{\sqrt{\gamma_\beta(T)}} e^{-\frac{\beta^2 z^2}{2\sigma^2}T}, \tag{48}$$

and we have proved (8).

Now we focus on the case where the strategies depend only on the initial value X_0 of the autoregressive process. In this case using the strategy $\eta = (\eta_1, \dots, \eta_T)$ we get

$$L_T^\eta = \sum_{j=1}^T \eta_j (X_j - X_{j-1})$$
$$= \sum_{j=1}^T \eta_j \Big(\alpha^{j-1}\beta X_0 + \sigma\varepsilon_j + \sigma\beta \sum_{k=1}^{j-1} \alpha^{j-1-k}\varepsilon_k\Big)$$
$$= \beta X_0 \sum_{j=1}^T \eta_j \alpha^{j-1} + \sigma \sum_{j=1}^T \eta_j \varepsilon_j + \sigma\beta \sum_{j=1}^T \sum_{k=1}^{j-1} \eta_j \alpha^{j-k-1}\varepsilon_k$$
$$= \beta X_0 \sum_{j=1}^T \eta_j \alpha^{j-1} + \sigma \sum_{j=1}^T \eta_j \varepsilon_j + \sigma\beta \sum_{k=1}^T \varepsilon_k \sum_{j=k+1}^T \eta_j \alpha^{j-k-1}.$$

Let $c : \mathbb{R}^T \to \mathbb{R}$ and $\mathbf{b} : \mathbb{R}^T \to \mathbb{R}^T$ be given by

$$c(\eta) = \beta X_0 \sum_{j=1}^{T} \eta_j \alpha^{j-1}, \quad b_k(\eta) = \sigma \eta_k \varepsilon_k + \sigma\beta \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1}, \quad 1 \le k \le T.$$

Using the notation $L_T^\eta = c(\eta) + \mathbf{b}^{\mathbf{T}}(\eta)\varepsilon$ we deduce from Lemma 3.2 that

$$\mathbb{E}[U(L_T^\eta) \mid X_0] = -e^{g(\eta)}, \quad \text{where} \quad g(\eta) = -c(\eta) + \frac{\mathbf{b}^{\mathbf{T}}(\eta)\mathbf{b}(\eta)}{2}. \tag{49}$$

We need to solve the system of equations $\nabla g(\eta) = \mathbf{0}$. We denote these equations by (E_k), where $1 \le k \le T$:

$$0 = \frac{\partial g}{\partial \eta_k}(\eta) = -X_0 \beta \alpha^{k-1} + \mathbf{b}^{\mathbf{T}}(\eta) \frac{\partial \mathbf{b}}{\partial \eta_k}(\eta). \tag{E_k}$$

The partial derivatives of $\mathbf{b}$ are

$$\left(\frac{\partial \mathbf{b}}{\partial \eta_k}(\eta) \right)_l = \begin{cases} \sigma\beta\alpha^{k-l-1} & \text{if } l < k, \\ \sigma & \text{if } l = k, \\ 0 & \text{if } l > k. \end{cases}$$

Hence

$$\begin{aligned} \mathbf{b}^{\mathbf{T}}(\eta) \frac{\partial \mathbf{b}}{\partial \eta_k}(\eta) &= \beta\sigma \sum_{l=1}^{k-1} \alpha^{k-l-1} \left(\sigma\eta_l + \beta\sigma \sum_{j=l+1}^{T} \eta_j \alpha^{j-l-1} \right) \\ &\quad + \sigma^2 \eta_k + \beta\sigma^2 \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1} \\ &= \beta\sigma^2 \sum_{l=1}^{k-1} \eta_l \alpha^{k-l-1} + \beta^2\sigma^2 \sum_{l=1}^{k-1} \sum_{j=l+1}^{T} \eta_j \alpha^{j+k-2l-2} + \sigma^2 \eta_k \\ &\quad + \beta\sigma^2 \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1}, \end{aligned}$$

with the convention that the last sum is 0 when $k = T$. Therefore the equations (E_k) take the form

$$\beta \sum_{l=1}^{k-1} \eta_l \alpha^{k-l-1} + \beta^2 \sum_{l=1}^{k-1} \sum_{j=l+1}^{T} \eta_j \alpha^{j+k-2l-2} + \eta_k + \beta \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1} = \frac{X_0}{\sigma^2} \beta \alpha^{k-1}. \tag{50}$$

Let $k \in \{1, \ldots, T-1\}$. Then for (E_{k+1}) we have

$$\beta \sum_{l=1}^{k} \eta_l \alpha^{k-l} + \beta^2 \sum_{l=1}^{k} \sum_{j=l+1}^{T} \eta_j \alpha^{j+k-2l-1} + \eta_{k+1} + \beta \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2} = \frac{X_0}{\sigma^2} \beta \alpha^{k}. \tag{51}$$

We define equations (F_k) for $1 \le k \le T-1$ by multiplying (50) by α (to obtain (αE_k)) and taking the difference between (αE_k) and (51) to get

$$
\begin{aligned}
0 &= \beta\eta_k + \beta^2 \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1} + \eta_{k+1} - \alpha\eta_k \\
&\quad + \beta \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2} - \beta \sum_{j=k+1}^{T} \eta_j \alpha^{j-k}, \\
0 &= \eta_{k+1} - \eta_k + \beta\Big(\sum_{j=k+1}^{T} \eta_j \alpha^{j-k} - \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1} \Big) \\
&\quad + \beta \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2} - \beta \sum_{j=k+1}^{T} \eta_j \alpha^{j-k}, \\
0 &= \eta_{k+1} - \eta_k - \beta \sum_{j=k+1}^{T} \eta_j \alpha^{j-k-1} + \beta \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2}, \\
0 &= \eta_{k+1} - \eta_k - \beta\eta_{k+1} + \beta \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2}(1-\alpha), \\
(F_k) \qquad 0 &= (1-\beta)\eta_{k+1} - \eta_k - \beta^2 \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2}.
\end{aligned}
$$

Lemma 3.9. *For the solutions of the system (F_k), $k = 1, \ldots, T-1$, we have*

$$
\eta_k = \theta_k^T \eta_T = (1 - (T-k)\beta)\eta_T \tag{52}
$$

for all $k = 1, \ldots, T-1$.

Proof. First we consider the equation (F_{t-1}),

$$
\begin{aligned}
0 &= (1-\beta)\eta_T - \eta_{T-1}, \\
\eta_{T-1} &= \theta_{T-1}^T \eta_T.
\end{aligned}
$$

Assume that (52) holds for $l = k+1, \ldots, T-1$. Considering the equation (F_k), and using (34), we get

$$
\begin{aligned}
0 &= (1-\beta)\eta_{k+1} - \eta_k - \beta^2 \sum_{j=k+2}^{T} \eta_j \alpha^{j-k-2}, \\
0 &= (1-\beta)(1-(T-k-1)\beta)\eta_T - \eta_k - \beta^2 \sum_{j=k+2}^{T} \theta_j^T \eta_T \alpha^{j-k-2}, \\
0 &= (1-\beta)(1-(T-k-1)\beta)\eta_T - \eta_k - \beta^2 (T-k-1)\eta_T,
\end{aligned}
$$

$$\eta_k = (1 - (T - k - 1)\beta - \beta)\eta_T,$$
$$\eta_k = \theta_k^T \eta_T. \ \blacksquare$$

Because of $\theta_T^T = 1$, $\eta_T = \theta_T^T \eta_T$ also holds. To finish the proof of Theorem 2.1 for the case without using past information, we only need to show that $\eta_T = \beta X_0/\sigma^2$.

Substituting (52) into (E_T) (i.e. into (50) for $k = T$), we get

$$\beta \sum_{l=1}^{T-1} \theta_l^T \eta_T \alpha^{T-l-1} + \beta^2 \sum_{l=1}^{T-1} \sum_{j=l+1}^{T} \theta_j^T \eta_T \alpha^{j+T-2l-2} + \eta_T = \frac{X_0}{\sigma^2} \beta \alpha^{T-1},$$

$$\beta \Big(\sum_{l=1}^{T-1} \theta_l^T \alpha^{T-l-1} + \beta \sum_{l=1}^{T-1} \sum_{j=l+1}^{T} \theta_j^T \alpha^{j+T-2l-2} + 1 \Big) \eta_T = \frac{X_0}{\sigma^2} \beta \alpha^{T-1}. \tag{53}$$

We compute the two sums in parentheses in (53):

$$\begin{aligned}
\sum_{l=1}^{T-1} \theta_l^T \alpha^{T-l-1} &= \sum_{l=1}^{T-1} (T + 1 - l - (T - l)\alpha)\alpha^{T-l-1} \\
&= \sum_{l=1}^{T-1} (T + 1 - l)\alpha^{T-l-1} - \sum_{l=1}^{T-1} (T - l)\alpha^{T-l} \\
&= \sum_{l=2}^{T} (T + 2 - l)\alpha^{T-l} - \sum_{l=1}^{T-1} (T - l)\alpha^{T-l} \\
&= \sum_{l=2}^{T} 2\alpha^{T-l} - (T - 1)\alpha^{T-1}
\end{aligned}$$

and

$$\begin{aligned}
\beta \sum_{l=1}^{T-1} \sum_{j=l+1}^{T} \theta_j^T \alpha^{j+T-2l-2} &= \beta \sum_{l=1}^{T-1} \alpha^{T-2l-2} \sum_{j=l+1}^{T} \theta_j^T \alpha^j \\
&= \beta \sum_{l=1}^{T-1} (T - l)\alpha^{T-l-1} \qquad \text{by (34)} \\
&= \sum_{l=1}^{T-1} (T - l)\alpha^{T-l} - \sum_{l=1}^{T-1} (T - l)\alpha^{T-l-1} \\
&= \sum_{l=1}^{T-1} (T - l)\alpha^{T-l} - \sum_{l=2}^{T} (T - l + 1)\alpha^{T-l} \\
&= -\sum_{l=2}^{T} \alpha^{T-l} + (T - 1)\alpha^{T-1}.
\end{aligned}$$

Hence, from (53),

$$\Big(1+\beta\sum_{l=2}^{T}\alpha^{T-l}\Big)\eta_T=\frac{X_0}{\sigma^2}\beta\alpha^{T-1},$$
$$\Big(1+\sum_{l=2}^{T}\alpha^{T-l+1}-\sum_{l=2}^{T}\alpha^{T-l}\Big)\eta_T=\frac{X_0}{\sigma^2}\beta\alpha^{T-1},$$
$$\alpha^{T-1}\eta_T=\frac{X_0}{\sigma^2}\beta\alpha^{T-1},$$
$$\eta_T=\beta\frac{X_0}{\sigma^2}.$$

Now we prove (9). First we compute the entries of $\hat{\mathbf{b}}:=\mathbf{b}(\hat{\eta})$, using (34):

$$\hat{b}_k=\frac{\beta X_0}{\sigma}\theta_k+\frac{\beta^2X_0}{\sigma}\sum_{j=k+1}^{T}\theta_j\alpha^{j-k-1} \tag{54}$$
$$=\frac{\beta X_0}{\sigma}(1-(T-k)\beta)+\frac{\beta^2X_0}{\sigma}(T-k)=\frac{\beta X_0}{\sigma}.$$

Therefore $\hat{\mathbf{b}}^{\mathbf{T}}\hat{\mathbf{b}}=\frac{\beta^2X_0^2}{\sigma^2}T$, and

$$\hat{c}:=c(\hat{\eta})=\frac{\beta^2X_0^2}{\sigma^2}\sum_{j=1}^{T}\theta_j\alpha^{j-1}=\frac{\beta^2X_0^2}{\sigma^2}T,$$

by (34). Hence, using (49), we get

$$\mathbb{E}[U(L_T^{\hat{\eta}})\mid X_0]=e^{-\frac{\beta^2X_0^2}{2\sigma^2}T},$$

which proves (9) and hence Theorem 2.1.

Proof of Corollary 2.2. For stable processes, in the case of $\mathrm{var}(X_t)=1$ it is straightforward to verify $\sigma^2=1-\alpha^2$. As X_0 is $N(0,1)$, the maximal expected utility can be found using (32):

$$\mathbb{E}[U(L_T^{\phi^*})]=\mathbb{E}\big[\mathbb{E}[U(L_T^{\phi^*})\mid X_0]\big]=\mathbb{E}\Big[-\frac{1}{\sqrt{\gamma_\beta(T)}}e^{-\frac{\beta^2X_0^2}{2\sigma^2}T}\Big]$$
$$=-\frac{1}{\sqrt{2\pi\gamma_\beta(T)}}\int_{\mathbb{R}}e^{-\frac{\beta^2x^2}{2\sigma^2}T-\frac{x^2}{2}}\,dx=-\frac{1}{\sqrt{\gamma_\beta(T)(\frac{\beta^2T}{1-\alpha^2}+1)}}$$
$$=-\sqrt{\frac{\beta+2}{(2-(T-1)\beta)\gamma_\beta(T)}},$$

so we have proved (11). Based on the same calculation that we did to get (11) from (8), we get (12) from (9). ■

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