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ON THE COMPOUND $\alpha(t)$-MODIFIED POISSON DISTRIBUTION

Abstract. In this paper we introduce compound $\alpha(t)$-modified Poisson distributions. We obtain the compound Delaporte distribution as the special case of the compound $\alpha(t)$-modified Poisson distribution. The characteristics of $\alpha(t)$-modified Poisson and some compound distributions with gamma, exponential and Panjer summands are presented.

1. Introduction. Much literature is devoted to compound distributions. Among other things, the compound Poisson distribution is extensively studied. It has been shown to have wide ranging applications, for example in insurance, queuing theory and biology (cf. Klugman et al. [KPW], Sundt and Vernic [SV], and Nadarajah and Withers [NW]).

The aim of our investigation is the compound $\alpha(t)$-modified Poisson distribution, i.e. the distribution of the random sum

$$S_N = X_1 + \cdots + X_N, \tag{1.1}$$

where $\{X_i,\, i \geq 1\}$ is an i.i.d. sequence, N is a random variable with $\alpha(t)$-modified Poisson distribution and N is independent of $\{X_i,\, i \geq 1\}$.

First we recall the definitions of the classical distributions used in this paper: the geometric distribution $G(q)$

$$P(X = x) = q^x(1-q), \qquad x \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ 0 < q < 1; \tag{1.2}$$

the negative binomial distribution $NB(t,q)$

$$P(X = x) = \binom{t+x-1}{x} q^x (1-q)^t, \qquad x \in \mathbb{N}_0,\ 0 < q < 1,\ t > 0; \tag{1.3}$$

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and the Poisson distribution

$$P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda},\qquad x\in\mathbb{N}_0,\ \lambda>0. \tag{1.4}$$

It is known that the binomial, Poisson, negative binomial and geometric distributions belong to the *Panjer class*, i.e. they satisfy the recursion

$$p_k=\left(a+\frac{b}{k}\right)p_{k-1},\qquad k=1,2,\ldots,\ a<1,\ a+b>0 \tag{1.5}$$

(cf. Klugman et al. [KPW, p. 221] and Sundt and Vernic [SV, p. 38]). We write $X\in\mathscr{P}(a,b)$ if X has the probability function given by (1.5). We see that for $a=0$, $b>0$, we have $X\sim P(b)$; for $a=q$, $b=0$, $X\sim G(q)$; for $a=q$, $b=(t-1)q$, $X\sim NB(t,q)$; and for $a=-p/q$, $b=(m+1)p/q$, $X\sim B(m,p)$, i.e.

$$P(X=x)=\binom{m}{x}p^xq^{m-x},\qquad x=0,1,\ldots,m,\ m\in\mathbb{N},\ p>0,\ p+q=1.$$

The concept of α-modified Poisson distribution was introduced by Berg and Jaworski [BJ]. The probability function of a random variable X having the *α-modified Poisson distribution*, written $X\sim MP(\psi,\lambda)$, is given by

$$P(X=x)=\frac{(\lambda+\alpha\psi)^x}{x!}(1-\psi)e^{-\lambda},\qquad x\in\mathbb{N}_0, \tag{1.6}$$

where $\lambda>0$ and ψ are parameters such that $|\psi|<1$, $\lambda+\psi\geq0$ and α is Riordan's symbol

$$\alpha^k\equiv\alpha_k=k!,\qquad k=0,1,2,\ldots \tag{1.7}$$

(cf. Riordan [R]).

Corresponding to (1.6), the *$\alpha(t)$-modified Poisson distribution*, written $X\sim MP(t,\psi,\lambda)$, is given by

$$P(X=x)=\frac{(\lambda+\psi\alpha(t))^x}{x!}(1-\psi)^te^{-\lambda},\qquad x\in\mathbb{N}_0, \tag{1.8}$$

where $\lambda>0$ and ψ are parameters such that $|\psi|<1$, $\lambda+\psi\geq0$, and

$$\alpha^l(t)=\begin{cases}\binom{l+t-1}{l}l! & \text{for } t>0,\ l\in\mathbb{N}_0,\\ 0 & \text{for } t=0\end{cases} \tag{1.9}$$

(cf. Riordan [R], Berg and Nowicki [BN], and Chakraborty [C]).

If X and Y are independent and X has the geometric distribution (1.2) with $q:=\psi$, $0<\psi<1$, and Y has the Poisson distribution (1.4), then $Z=X+Y\sim MP(\psi,\lambda)$ (cf. Berg and Jaworski [BJ]). A generalization of this fact is known, namely if X and Y are independent random variables and X has the negative binomial distribution (1.3) with $q=\beta/(1+\beta)$, $\beta>0$, and Y has the Poisson distribution (1.4), then $Z=X+Y$ is the Delaporte distribution (cf. Johnson et al. [JKK, p. 242]). The probability function of the *Delaporte distribution* ($X \sim \mathrm{Del}(t,\beta,\lambda)$) is given by

$$P(X=x)=\sum_{i=0}^{x}\frac{\Gamma(t+i)\beta^i\lambda^{x-i}e^{-\lambda}}{\Gamma(t)i!(1+\beta)^{t+i}(x-i)!},\qquad x\in\mathbb{N}_0, \tag{1.10}$$

where $t,\beta,\lambda>0$ (cf. Vose [V, pp. 618–619]).

Formula (1.10) can be written in terms of $\alpha(t)$ as follows:

$$P(X=x)=\frac{\left(\lambda+\frac{\beta}{1+\beta}\alpha(t)\right)^x}{x!}\left(\frac{1}{1+\beta}\right)^t e^{-\lambda}.$$

Therefore $\mathrm{Del}(t,\beta,\lambda)=MP\left(t,\frac{\beta}{1+\beta},\lambda\right)$, which confirms that the convolution of the negative binomial distribution and the Poisson distribution is the $\alpha(t)$-modified Poisson distribution (cf. Chakraborty [C]).

Now we give the moments of the $\alpha(t)$-modified Poisson distribution ($X\sim MP(t,\psi,\lambda)$). We shall use the following notation:

$$\begin{aligned}
{}_\alpha m_{(r)}&={}_\alpha EX(X-1)\cdot\ldots\cdot(X-r+1),\ \text{the } r\text{th factorial moment, } r\in\mathbb{N},\\
{}_\alpha m_r&={}_\alpha EX^r,\ \text{the } r\text{th uncorrected moment or briefly the } r\text{th moment},\\
{}_\alpha \mu_r&={}_\alpha E(X-EX)^r,\ \text{the } r\text{th central moment.}
\end{aligned}$$

Chakraborty [C] gave a formula for the probability generating function of $X\sim MP(t,\psi,\lambda)$ which we write as

$${}_\alpha G_X(s)=\left(\frac{1-\psi}{1-\psi s}\right)^t e^{\lambda(1-s)},$$

and for the factorial moment

$$\begin{aligned}
{}_\alpha m_{(r)}&=\sum_{k=0}^{r}\binom{r}{k}\left(\frac{\psi}{1-\psi}\right)^k t^{[k]}\lambda^{r-k}\\
&=\left(\lambda+\alpha(t)\frac{\psi}{1-\psi}\right)^r,
\end{aligned}$$

where $t^{[n]}=t(t+1)\cdot\ldots\cdot(t+n-1)$ is the nth rising factorial power and $\alpha(t)$ is given by (1.9).

Hence and by the relations between the moments (cf. Johnson et al. [JKK, p. 52]) we have

$${}_\alpha m_l=\sum_{k=0}^{l}S(l,k)\,{}_\alpha m_{(k)}, \tag{1.11}$$

$${}_\alpha\mu_r={}_\alpha E(X-EX)^r=\sum_{l=0}^{r}\binom{r}{l}(-{}_\alpha m_1)^{r-l}\,{}_\alpha m_l, \tag{1.12}$$

and we get

$$ {}_{\alpha}m_r = \sum_{k=0}^{r} S(r,k) \sum_{i=0}^{k} \binom{k}{i} t^{[i]} \Big(\frac{\psi}{1-\psi} \Big)^i \lambda^{k-i} = \sum_{k=0}^{r} S(r,k) \Big(\lambda + \alpha(t) \frac{\psi}{1-\psi} \Big)^k $$

and

$$ \begin{aligned} {}_{\alpha}\mu_r &= \sum_{l=0}^{r} \binom{r}{l} \Big(-\lambda - \frac{t\psi}{1-\psi} \Big)^{r-l} \sum_{k=0}^{l} S(l,k) \sum_{i=0}^{k} \binom{k}{i} t^{[i]} \Big(\frac{\psi}{1-\psi} \Big)^i \lambda^{k-i} \\ &= \sum_{l=0}^{r} \binom{r}{l} \Big(-\lambda - \frac{t\psi}{1-\psi} \Big)^{r-l} \sum_{k=0}^{l} S(l,k) \Big(\lambda + \alpha(t) \frac{\psi}{1-\psi} \Big)^k \end{aligned} $$

(for $t = 1$ see Steliga and Szynal [SS]).

In particular, we obtain the rth factorial, rth uncorrected and rth central moments of the Delaporte distribution $\big(X \sim MP\big(t, \frac{\beta}{1+\beta}, \lambda\big)\big)$:

$$ \begin{aligned} {}_{\alpha}m_{(r)} &= \sum_{k=0}^{r} \binom{r}{k} \beta^k t^{[k]} \lambda^{r-k} = (\lambda + \beta\alpha(t))^r, \\ {}_{\alpha}m_r &= \sum_{k=0}^{r} S(r,k) \sum_{i=0}^{k} \binom{k}{i} \beta^i t^{[i]} \lambda^{k-i} \\ &= \sum_{k=0}^{r} S(r,k)(\lambda + \beta\alpha(t))^k, \\ {}_{\alpha}\mu_r &= \sum_{l=0}^{r} \binom{r}{l} (-\lambda - \beta t)^{r-l} \sum_{k=0}^{l} S(l,k) \sum_{i=0}^{k} \binom{k}{i} \beta^i t^{[i]} \lambda^{k-i} \\ &= \sum_{l=0}^{r} \binom{r}{l} (-\lambda - \beta t)^{r-l} \sum_{k=0}^{l} S(l,k)(\lambda + \beta\alpha(t))^k. \end{aligned} $$

Hence we obtain

$$ \begin{aligned} {}_{\alpha}m_1 &= \lambda + \beta t, \\ {}_{\alpha}m_2 &= \lambda + \lambda^2 + \beta t(1 + 2\lambda) + \beta^2 t^{[2]}, \\ {}_{\alpha}m_3 &= \lambda + 3\lambda^2 + \lambda^3 + \beta t(1 + 6\lambda + 3\lambda^2) + 3\beta^2 t^{[2]}(1 + \lambda) + \beta^3 t^{[3]}, \\ {}_{\alpha}m_4 &= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 + \beta t(1 + 14\lambda + 18\lambda^2 + 4\lambda^3) \\ &\quad + \beta^2 t^{[2]}(7 + 18\lambda + 6\lambda^2) + \beta^3 t^{[3]}(6 + 4\lambda) + \beta^4 t^{[4]}, \\ {}_{\alpha}\mu_2 &= \lambda + \beta t(1 + \beta), \\ {}_{\alpha}\mu_3 &= \lambda + \beta t(1 + 3\beta + 2\beta^2), \\ {}_{\alpha}\mu_4 &= \lambda + 3\lambda^2 + \beta t(1 + 6\lambda + 6\lambda\beta + 7\beta + 12\beta^2 + 6\beta^3 \\ &\quad + 3\beta t + 6\beta^2 t + 3\beta^3 t), \end{aligned} \tag{1.13} $$

and we can compute some characteristics

$$V = ({}_\alpha\mu_2)^{1/2}/{}_\alpha m_1, \qquad \gamma = {}_\alpha\mu_3/({}_\alpha\mu_2)^{3/2}, \qquad \kappa = {}_\alpha\mu_4/({}_\alpha\mu_2)^2. \tag{1.14}$$

Vose [V, p. 619] gave the mean, variance, and the coefficients of skewness γ and kurtosis κ, which are as in (1.13) and (1.14).

If we set $\lambda := s_0$, $\beta := 1/a$ and $t := b$ in (1.13), then we get the list of moments and central moments for $r = 1, 2, 3$, given by Delaporte [D], [DP].

2. The compound Delaporte distribution. In this section we consider characteristics of the compound $\alpha(t)$-modified Poisson distribution ($N \sim MP(t, \psi, \lambda)$). We call $N \sim MP\big(t, \frac{\beta}{1+\beta}, \lambda\big)$ the *compound Delaporte distribution.*

First, we will focus on the moments of the compound $\alpha(t)$-modified Poisson distribution with Panjer summands ($N \sim MP(t, \psi, \lambda)$, $X \in \mathscr{P}(a, b)$).

We shall use the following notation:

${}_\alpha ES_N^{(r)} = {}_\alpha ES_N(S_N - 1) \cdot \ldots \cdot (S_N - r + 1)$, the rth factorial moment,

${}_\alpha ES_N^r$, the rth uncorrected moment or briefly the rth moment,

${}_\alpha E(S_N - ES_N)^r$, the rth central moment, $r \in \mathbb{N}$.

THEOREM 2.1. *The probability generating function, the rth factorial moment, the rth uncorrected moment and the $(r+1)$th central moment of the compound $\alpha(t)$-modified Poisson distribution with Panjer summands are given by*

$$(2.1)\quad {}_\alpha G_{S_N}(s) = \begin{cases} (1-\psi)^t\big(1 - \psi\big(\frac{1-a}{1-sa}\big)^{1+b/a}\big)^{-t} \exp\big\{-\lambda\big(1 - \big(\frac{1-a}{1-sa}\big)^{1+b/a}\big)\big\} & \text{if } a \neq 0, \\ (1-\psi)^t(1 - \psi\exp[-b(1-s)])^{-t} \exp\{-\lambda(1 - \exp[-b(1-s)])\} & \text{if } a = 0, \end{cases}$$

$$(2.2)\quad {}_\alpha ES_N^{(r)} = \begin{cases} \big(\frac{a}{1-a}\big)^r \sum_{k=1}^{r} \big(\lambda + \alpha(t)\frac{\psi}{1-\psi}\big)^k \sum_{i=0}^{k-1} (-1)^i \frac{1}{i!(k-i)!} \big((k-i)\big(1 + \frac{b}{a}\big)\big)^{[r]} & \text{if } a \neq 0, \\ b^r \sum_{k=1}^{r} S(r,k)\big(\lambda + \alpha(t)\frac{\psi}{1-\psi}\big)^k & \text{if } a = 0, \end{cases}$$

$$(2.3)\quad {}_\alpha ES_N^r = \begin{cases} \sum_{n=1}^{r} S(r,n)\big(\frac{a}{1-a}\big)^n \sum_{k=1}^{n} \big(\lambda + \alpha(t)\frac{\psi}{1-\psi}\big)^k \sum_{i=0}^{k-1} (-1)^i \frac{1}{i!(k-i)!} \big((k-i)\big(1 + \frac{b}{a}\big)\big)^{[n]} & \text{if } a \neq 0, \\ \sum_{n=1}^{r} S(r,n) b^n \sum_{k=1}^{n} S(n,k)\big(\lambda + \alpha(t)\frac{\psi}{1-\psi}\big)^k & \text{if } a = 0, \end{cases}$$

(2.4) $\quad {}_{\alpha}E(S_N - ES_N)^{r+1} =$

$$\begin{cases} \frac{1}{1-a}\sum_{l=1}^{r}\binom{r}{l}\big(-\frac{a+b}{1-a}\big(\lambda+\frac{t\psi}{1-\psi}\big)\big)^{r-l}\sum_{n=1}^{l}S(l,n)\big(\frac{a}{1-a}\big)^{n}\big\{\sum_{k=1}^{n}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k} \\ \cdot\sum_{i=0}^{k-1}(-1)^{i}\frac{1}{i!(k-i)!}\big(n+\big(k-i-\lambda-\frac{t\psi}{1-\psi}\big)(a+b)\big)\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[n]} \\ +\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{n+1}\sum_{i=0}^{n}(-1)^{i}\frac{(n+1-i)(a+b)+na}{i!(n+1-i)!}\big((n+1-i)\big(1+\frac{b}{a}\big)\big)^{[n]}\big\} \\ \hfill \textit{if } a\neq 0, \\ \sum_{l=1}^{r}\binom{r}{l}\big(-\lambda b-\frac{bt\psi}{1-\psi}\big)^{r-l}\sum_{n=1}^{l}S(l,n)b^{n}\sum_{k=1}^{n}S(n,k) \\ \cdot\big\{\big(n+bk-\lambda b-\frac{bt\psi}{1-\psi}\big)\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}+b\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k+1}\big\} \quad \textit{if } a=0. \end{cases}$$

Proof. It is known that the probability generating function (pgf) of S_N satisfies

$$G_{S_N}(s) = G_N(G_X(s)), \tag{2.5}$$

where $G_N(s)$ and $G_X(s)$ denote the pgf of N and X, respectively (cf. Klugman et al. [KPW, p. 237]). For $N \sim MP(t,\psi,\lambda)$ and $X \in \mathscr{P}(a,b)$ the probability generating functions are

$${}_{\alpha}G_N(s) = \left(\frac{1-\psi}{1-s\psi}\right)^{t}\exp\{-\lambda(1-s)\},$$

and

$$G_X(s) = \begin{cases} \big(\frac{1-a}{1-sa}\big)^{1+b/a} & \text{if } a\neq 0, \\ \exp\{-b(1-s)\} & \text{if } a=0, \end{cases}$$

which after inserting in (2.5) gives (2.1).

The rth derivative of ${}_{\alpha}G_{S_N}(s)$ in (2.1) is given by

(2.6) $\quad \frac{d^r}{ds^r}\,{}_{\alpha}G_{S_N}(s) =$

$$\begin{cases} \Big(\frac{1-\psi}{1-\psi(\frac{1-a}{1-sa})^{1+b/a}}\Big)^{t}\exp\big\{-\lambda\big(1-\big(\frac{1-a}{1-sa}\big)^{1+b/a}\big)\big\}\big(\frac{a}{1-sa}\big)^{r}\sum_{k=1}^{r}\frac{1}{k!}\big(\frac{1-a}{1-sa}\big)^{k(1+b/a)} \\ \cdot\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi(\frac{1-a}{1-sa})^{1+b/a}}\Big)^{k}\sum_{i=0}^{k-1}\binom{k}{i}(-1)^{i}\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[r]} \qquad \text{if } a\neq 0, \\ \Big(\frac{1-\psi}{1-\psi\exp[-b(1-s)]}\Big)^{t}\exp\{-\lambda(1-\exp[-b(1-s)])\}b^{r}\sum_{k=1}^{r}S(r,k) \\ \cdot\exp\{-bk(1-s)\}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi\exp[-b(1-s)]}\Big)^{k} \qquad \text{if } a=0. \end{cases}$$

Setting $s=1$ in (2.6) we obtain (2.2).

Now taking into account the relations between moments and factorial moments (cf. (1.11))

$$ {}_{\alpha}ES_N^r = \sum_{k=0}^{r} S(r,k)\, {}_{\alpha}ES_N^{(k)} \tag{2.7} $$

and (2.2) we get (2.3). For the central moments (cf. (1.12)) we have

$$ {}_{\alpha}E(S_N - ES_N)^r = \sum_{l=0}^{r} \binom{r}{l} (-{}_{\alpha}ES_N)^{r-l}\, {}_{\alpha}ES_N^l \tag{2.8} $$

or

$$ \begin{aligned} &{}_{\alpha}E(S_N - ES_N)^{r+1} \\ &\quad = (-{}_{\alpha}ES_N)^{r+1} + \sum_{l=1}^{r+1} \binom{r+1}{l} (-{}_{\alpha}ES_N)^{r+1-l} \sum_{n=0}^{l} S(l,n)\, {}_{\alpha}ES_N^{(n)}. \end{aligned} $$

Now the property

$$ \binom{r+1}{l} = \binom{r}{l} + \binom{r}{l-1} \tag{2.9} $$

leads to

$$ \begin{aligned} &{}_{\alpha}E(S_N - ES_N)^{r+1} \\ &\quad = (-{}_{\alpha}ES_N)^{r+1} + \sum_{l=1}^{r} \binom{r}{l} (-{}_{\alpha}ES_N)^{r+1-l} \sum_{n=0}^{l} S(l,n)\, {}_{\alpha}ES_N^{(n)} \\ &\quad + \sum_{l=0}^{r} \binom{r}{l} (-{}_{\alpha}ES_N)^{r-l} \sum_{n=0}^{l+1} S(l+1,n)\, {}_{\alpha}ES_N^{(n)}. \end{aligned} $$

Since the Stirling numbers of the second kind satisfy

$$ S(l+1,n) = S(l,n-1) + nS(l,n), \tag{2.10} $$

after some evaluations we obtain

$$ \begin{aligned} &{}_{\alpha}E(S_N - ES_N)^{r+1} \\ &\quad = \sum_{l=1}^{r} \binom{r}{l} (-{}_{\alpha}ES_N)^{r-l} \sum_{n=1}^{l} S(l,n)[(n - {}_{\alpha}ES_N)\,{}_{\alpha}ES_N^{(n)} + {}_{\alpha}ES_N^{(n+1)}]. \end{aligned} \tag{2.11} $$

From (2.3) we have

$$ {}_{\alpha}ES_N = \begin{cases} \frac{a+b}{1-a}\left(\lambda + \frac{t\psi}{1-\psi}\right) & \text{if } a \neq 0, \\ b\left(\lambda + \frac{t\psi}{1-\psi}\right) & \text{if } a = 0. \end{cases} \tag{2.12} $$

Plugging (2.2) and (2.12) in (2.11) we obtain

$$
{}_{\alpha}E(S_N - ES_N)^{r+1} =
\begin{cases}
\sum_{l=1}^{r}\binom{r}{l}\big(-\frac{a+b}{1-a}\big(\lambda+\frac{t\psi}{1-\psi}\big)\big)^{r-l}\sum_{n=1}^{l}S(l,n)\big(\frac{a}{1-a}\big)^{n}\big\{\big[n-\frac{a+b}{1-a}\big(\lambda+\frac{t\psi}{1-\psi}\big)\big] \\
\cdot\sum_{k=1}^{n}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}\sum_{i=0}^{k-1}(-1)^{i}\frac{1}{i!(k-i)!}\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[n]} \\
+\frac{1}{1-a}\sum_{k=1}^{n}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}\sum_{i=0}^{k-1}(-1)^{i}\frac{(k-i)(a+b)+na}{i!(k-i)!}\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[n]} \\
+\frac{1}{1-a}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{n+1}\sum_{i=0}^{n}(-1)^{i}\frac{(n+1-i)(a+b)+na}{i!(n+1-i)!}\big((n+1-i)\big(1+\frac{b}{a}\big)\big)^{[n]}\big\} \\
\hfill \text{if } a\neq 0, \\
\sum_{l=1}^{r}\binom{r}{l}\big(-b(\lambda+\frac{t\psi}{1-\psi})\big)^{r-l}\sum_{n=1}^{l}S(l,n)b^{n}\big[\big(n-b\big(\lambda+\frac{t\psi}{1-\psi}\big)\big)\sum_{k=1}^{n}S(n,k) \\
\cdot\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}+b\sum_{k=1}^{n+1}S(n,k-1)\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k} \\
+b\sum_{k=1}^{n}kS(n,k)\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}\big] \hfill \text{if } a=0.
\end{cases}
$$

Hence we have

$$
{}_{\alpha}E(S_N - ES_N)^{r+1} =
\begin{cases}
\frac{1}{1-a}\sum_{l=1}^{r}\binom{r}{l}\big(-\frac{a+b}{1-a}\big(\lambda+\frac{t\psi}{1-\psi}\big)\big)^{r-l}\sum_{n=1}^{l}S(l,n)\big(\frac{a}{1-a}\big)^{n}\big\{\sum_{k=1}^{n}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k} \\
\cdot\sum_{i=0}^{k-1}(-1)^{i}\frac{1}{i!(k-i)!}\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[n]}\big(n-na-(a+b)\big(\lambda+\frac{t\psi}{1-\psi}\big)\big) \\
+\sum_{k=1}^{n}\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}\sum_{i=0}^{k-1}(-1)^{i}\frac{1}{i!(k-i)!}\big((k-i)\big(1+\frac{b}{a}\big)\big)^{[n]}((k-i)(a+b)+na) \\
+\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{n+1}\sum_{i=0}^{n}(-1)^{i}\frac{(n+1-i)(a+b)+na}{i!(n+1-i)!}\big((n+1-i)\big(1+\frac{b}{a}\big)\big)^{[n]}\big\} \\
\hfill \text{if } a\neq 0, \\
\sum_{l=1}^{r}\binom{r}{l}\big(-\lambda b-\frac{bt\psi}{1-\psi}\big)^{r-l}\sum_{n=1}^{l}S(l,n)b^{n}\sum_{k=1}^{n}S(n,k) \\
\cdot\big\{\big(n+bk-\lambda b-\frac{bt\psi}{1-\psi}\big)\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k}+b\big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\big)^{k+1}\big\} \hfill \text{if } a=0,
\end{cases}
$$

which leads to (2.4). ∎

For $\psi = \beta/(1+\beta)$, i.e. for the compound Delaporte distribution with Panjer summands, we have

THEOREM 2.2. *The probability generating function, the rth factorial moment, the rth uncorrected moment and the $(r+1)$th central moment of the compound Delaporte distribution with Panjer summands ($N \sim MP\big(t, \frac{\beta}{1+\beta}, \lambda\big)$ and $X \in \mathscr{P}(a,b)$) are given by*

$$
{}_\alpha G_{S_N}(s) = \begin{cases} \left(1+\beta\left[1-\left(\frac{1-a}{1-sa}\right)^{1+b/a}\right]\right)^{-t}\exp\left\{-\lambda\left(1-\left(\frac{1-a}{1-sa}\right)^{1+b/a}\right)\right\} & \textit{if } a\neq 0,\\ (1+\beta[1-\exp\{-b(1-s)\}])^{-t}\exp\{-\lambda(1-\exp[-b(1-s)])\} & \textit{if } a=0, \end{cases}
$$

$$
{}_\alpha ES_N^{(r)} = \begin{cases} \left(\frac{a}{1-a}\right)^r \sum\limits_{k=1}^{r}(\lambda+\beta\alpha(t))^k \sum\limits_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[r]} & \textit{if } a\neq 0,\\ b^r\sum\limits_{k=1}^{r}S(r,k)(\lambda+\beta\alpha(t))^k & \textit{if } a=0, \end{cases}
$$

$$
{}_\alpha ES_N^{r} = \begin{cases} \sum\limits_{n=1}^{r}S(r,n)\left(\frac{a}{1-a}\right)^n \sum\limits_{k=1}^{n}(\lambda+\beta\alpha(t))^k \sum\limits_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[n]} \\ \hfill \textit{if } a\neq 0,\\ \sum\limits_{n=1}^{r}S(r,n)b^n\sum\limits_{k=1}^{n}S(n,k)(\lambda+\beta\alpha(t))^k & \textit{if } a=0, \end{cases}
$$

$$
{}_\alpha E(S_N-ES_N)^{r+1} = \begin{cases} \frac{1}{1-a}\sum\limits_{l=1}^{r}\binom{r}{l}\left(-\frac{a+b}{1-a}(\lambda+\beta t)\right)^{r-l}\sum\limits_{n=1}^{l}S(l,n)\left(\frac{a}{1-a}\right)^n \\ \cdot\Big\{\sum\limits_{k=1}^{n}(\lambda+\beta\alpha(t))^k\sum\limits_{i=0}^{k-1}(-1)^i\frac{n+(k-i-\lambda-\beta t)(a+b)}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[n]} \\ +(\lambda+\beta\alpha(t))^{n+1}\sum\limits_{i=0}^{n}(-1)^i\frac{(n+1-i)(a+b)+na}{i!(n+1-i)!}\left((n+1-i)\left(1+\frac{b}{a}\right)\right)^{[n]}\Big\} \\ \hfill \textit{if } a\neq 0,\\ \sum\limits_{l=1}^{r}\binom{r}{l}(-\lambda b-bt\beta)^{r-l}\sum\limits_{n=1}^{l}S(l,n)b^n\sum\limits_{k=1}^{n}S(n,k) \\ \cdot\{(n+bk-\lambda b-\beta bt)(\lambda+\beta\alpha(t))^k+b(\lambda+\beta\alpha(t))^{k+1}\} & \textit{if } a=0. \end{cases}
$$

For $\psi = 0$ in Theorem 2.1, we have

THEOREM 2.3. *The probability generating function, the rth factorial moment, the rth uncorrected moment and the $(r+1)$th central moment of the compound Poisson distribution with Panjer summands ($N\sim P(\lambda)$ and $X\in\mathscr{P}(a,b)$) are given by*

$$
G_{S_N}(s) = \begin{cases} \exp\left\{-\lambda\left(1-\left(\frac{1-a}{1-sa}\right)^{1+b/a}\right)\right\} & \textit{if } a\neq 0,\\ \exp\{-\lambda(1-\exp[-b(1-s)])\} & \textit{if } a=0, \end{cases}
$$

$$
ES_N^{(r)} = \begin{cases} \left(\frac{a}{1-a}\right)^r \sum\limits_{k=1}^{r}\lambda^k \sum\limits_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[r]} & \textit{if } a\neq 0,\\ b^r\sum\limits_{k=1}^{r}S(r,k)\lambda^k & \textit{if } a=0, \end{cases}
$$

$$ES_N^r = \begin{cases} \sum_{n=1}^{r} S(r,n)\left(\frac{a}{1-a}\right)^n \sum_{k=1}^{n} \lambda^k \sum_{i=0}^{k-1}(-1)^i \frac{1}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[n]} & \text{if } a \neq 0, \\ \sum_{n=1}^{r} S(r,n) b^n \sum_{k=1}^{n} S(n,k)\lambda^k & \text{if } a = 0, \end{cases}$$

$$E(S_N - ES_N)^{r+1} = \begin{cases} \frac{1}{1-a}\sum_{l=1}^{r}\binom{r}{l}\left(-\frac{a+b}{1-a}\lambda\right)^{r-l}\sum_{n=1}^{l} S(l,n)\left(\frac{a}{1-a}\right)^n \\ \cdot\left\{\sum_{k=1}^{n}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{n+(k-i-\lambda)(a+b)}{i!(k-i)!}\left((k-i)\left(1+\frac{b}{a}\right)\right)^{[n]}\right. \\ \left. + \lambda^{n+1}\sum_{i=0}^{n}(-1)^i\frac{(n+1-i)(a+b)+na}{i!(n+1-i)!}\left((n+1-i)\left(1+\frac{b}{a}\right)\right)^{[n]}\right\} & \text{if } a \neq 0, \\ \sum_{l=1}^{r}\binom{r}{l}(-\lambda b)^{r-l}\sum_{n=1}^{l} S(l,n) b^n \sum_{k=1}^{n} S(n,k)\lambda^k (n+bk) & \text{if } a = 0, \end{cases}$$

which is an extension of Theorem 4.1 *in Steliga and Szynal* [SKS] *with some equivalent changes.*

3. The compound $\alpha(t)$-modified Poisson-Gamma distribution

3.1. The compound Poisson distribution with gamma summands. First, we consider the distribution of the random sum (1.1), where N has a Poisson distribution (1.4) and the i.i.d. summands $\{X_i,\ i \geq 1\}$ have a gamma distribution with the probability density function

$$f(x) = \frac{\theta^\rho}{\Gamma(\rho)} x^{\rho-1} e^{-\theta x}, \qquad x, \theta, \rho > 0. \tag{3.1}$$

We denote by

$M_{S_N}(s)$ the moment generating function (mgf) of S_N,

ES_N^r the rth uncorrected moment, or briefly the rth moment,

$E(S_N - ES_N)^r$ the rth central moment,

$B_r(y) = \sum_{k=1}^{r} B_{rk}(y)$ the rth Bell polynomial.

Nadarajah and Withers [NW] studied the compound Poisson-gamma distribution. Among other things, they obtained the following results.

THEOREM 3.1 ([NW]). *The moment generating function, the rth uncorrected moments and the rth central moments of the compound Poisson distribution with gamma summands are given by*

$$M_{S_N}(s) = \exp\Big\{\sum_{i=1}^{\infty}\frac{\lambda\rho^{[i]}(s/\theta)^i}{i!}\Big\}, \tag{3.2}$$

$$ES_N^r = \theta^{-r}B_r(\lambda\rho^{[1]}, \lambda\rho^{[2]}, \ldots), \tag{3.3}$$

$$E(S_N - ES_N)^r = \theta^{-r}B_r(0, \lambda\rho^{[2]}, \lambda\rho^{[3]}\ldots). \tag{3.4}$$

We derive counterparts of formulae (3.2)–(3.4) for the compound Poisson-gamma distribution.

THEOREM 3.2. *The moment generating function, the* r*th uncorrected moments and the* $(r+1)$*th central moments of the compound Poisson distribution with gamma summands are given by*

$$M_{S_N}(s) = \exp\Big\{-\lambda\Big[1 - \Big(1 - \frac{s}{\theta}\Big)^{-\rho}\Big]\Big\}, \tag{3.5}$$

$$ES_N^r = \theta^{-r}\sum_{k=1}^{r}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[r]}, \tag{3.6}$$

$$\begin{aligned} E(S_N - ES_N)^{r+1} &= \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l} \\ &\quad\cdot\Big\{\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{(k-i-\lambda)\rho+l}{i!(k-i)!}(\rho(k-i))^{[l]} \\ &\quad + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{\rho(l+1-i)+l}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}\Big\}. \end{aligned} \tag{3.7}$$

Proof. It is known that the moment generating function of S_N satisfies

$$M_{S_N}(s) = G_N(M_X(s)), \tag{3.8}$$

where $G_N(s)$ and $M_X(s)$ denote the pgf of N and the mgf of X, respectively. In the case of the compound Poisson-gamma distribution taking $G_N(s) = \exp\{\lambda(s-1)\}$ and $M_X(s) = (1-s/\theta)^{-\rho}$ in (3.8) yields (3.5). The rth derivative of (3.5) is given by

$$\begin{aligned} &\frac{d^r}{ds^r}M_{S_N}(s) \\ &\quad = M_{S_N}(s)\theta^{-r}\sum_{k=1}^{r}\frac{\lambda^k}{k!}\Big(1 - \frac{s}{\theta}\Big)^{-\rho k - r}\sum_{i=0}^{k-1}\binom{k}{i}(-1)^i(\rho(k-i))^{[r]}. \end{aligned} \tag{3.9}$$

Letting $s = 0$ in (3.9) gives (3.6).

Using the relation between moments (cf. (2.8)) and (3.6) we get

$$\begin{aligned}(3.10)\quad & E(S_N - ES_N)^r \\ &= \theta^{-r}\Big[(-\lambda\rho)^r + \sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l}\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]}\Big].\end{aligned}$$

Taking $r+1$ instead of r in (3.10) and using (2.9) we obtain

$$\begin{aligned}E(S_N - ES_N)^{r+1} &= \theta^{-r-1}\Big[\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r+1-l}\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]} \\ &\quad + \sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l}\sum_{k=1}^{l+1}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l+1]}\Big],\end{aligned}$$

which we write as

$$\begin{aligned}E(S_N - ES_N)^{r+1} &= \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l}\Big\{-\lambda\rho\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]} \\ &\quad + \sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l+1]} \\ &\quad + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{1}{i!(l+1-i)!}(\rho(l+1-i))^{[l+1]}\Big\}\end{aligned}$$

or

$$\begin{aligned}E(S_N - ES_N)^{r+1} &= \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l}\Big\{-\lambda\rho\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]} \\ &\quad + \sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]}(\rho(k-i)+l) \\ &\quad + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{1}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}(\rho(l+1-i)+l)\Big\}.\end{aligned}$$

Hence

$$E(S_N - ES_N)^{r+1} = \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l}\Big\{\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{-\lambda\rho}{i!(k-i)!}(\rho(k-i))^{[l]} + \sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{\rho(k-i)+l}{i!(k-i)!}(\rho(k-i))^{[l]} + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{\rho(l+1-i)+l}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}\Big\},$$

which leads to

$$E(S_N - ES_N)^{r+1} = \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda\rho)^{r-l} \cdot\Big\{\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}[-\lambda\rho+\rho(k-i)+l](\rho(k-i))^{[l]} + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{\rho(l+1-i)+l}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}\Big\},$$

and ends the proof of (3.7). ∎

In particular, from (3.6)–(3.7) (in the notation of Nadarajah and Withers [NW]), we have

$$\begin{aligned} ES_N\theta &= \lambda\rho,\\ ES_N^2\theta^2 &= \lambda\rho^{[2]}+\lambda^2\rho^2,\\ ES_N^3\theta^3 &= \lambda\rho^{[3]}+3\lambda^2\rho\rho^{[2]}+\lambda^3\rho^3,\\ ES_N^4\theta^4 &= \lambda\rho^{[4]}+\lambda^2[3(\rho^{[2]})^2+4\rho\rho^{[3]}]+6\lambda^3\rho^2\rho^{[2]}+\lambda^4\rho^4,\\ ES_N^5\theta^5 &= \lambda\rho^{[5]}+5\lambda^2[\rho\rho^{[4]}+2\rho^{[2]}\rho^{[3]}]+5\lambda^3[2\rho^2\rho^{[3]}+3\rho(\rho^{[2]})^2]\\ &\quad+10\lambda^4\rho^3\rho^{[2]}+\lambda^5\rho^5, \end{aligned}$$

and

$$\begin{aligned} E(S_N-ES_N)^2\theta^2 &= \lambda\rho^{[2]},\\ E(S_N-ES_N)^3\theta^3 &= \lambda\rho^{[3]},\\ E(S_N-ES_N)^4\theta^4 &= \lambda\rho^{[4]}+3\lambda^2(\rho^{[2]})^2,\\ E(S_N-ES_N)^5\theta^5 &= \lambda\rho^{[5]}+10\lambda^2\rho^{[2]}\rho^{[3]},\\ E(S_N-ES_N)^6\theta^6 &= \lambda\rho^{[6]}+15\lambda^3(\rho^{[2]})^3+5\lambda^2[3\rho^{[2]}\rho^{[4]}+2(\rho^{[3]})^2]. \end{aligned}$$

These formulae are as those in Nadarajah and Withers [NW] except $ES_N^5\theta^5$ where we have $2\rho^2\rho^{[3]}$ instead of the erroneous $2\rho\rho^{[3]}$ in [NW].

Taking into account the above formulae it is easy to compute the following characteristics:

$$\begin{aligned}V &= (E(S_N - ES_N)^2)^{1/2}/ES_N,\\ \gamma &= E(S_N - ES_N)^3/(E(S_N - ES_N)^2)^{3/2},\\ \kappa &= E(S_N - ES_N)^4/(E(S_N - ES_N)^2)^2.\end{aligned}$$

In particular for $\rho = 1$ in Theorem 3.2 we have formulae for the compound Poisson distribution with exponential summands ($N \sim P(\lambda)$ and $f(x) = \theta e^{-\theta x}$).

Theorem 3.3. *The moment generating function, the rth uncorrected moments and the $(r+1)$th central moments of the compound Poisson distribution with exponential summands are given by*

$$M_{S_N}(s) = \exp\left\{-\lambda\left[1 - \left(1 - \frac{s}{\theta}\right)^{-1}\right]\right\}, \tag{3.11}$$

$$ES_N^r = \theta^{-r}\sum_{k=1}^{r}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(k-i)^{[r]}, \tag{3.12}$$

$$\begin{aligned}E(S_N - ES_N)^{r+1} = \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda)^{r-l}\\ \cdot\left\{\sum_{k=1}^{l}\lambda^k\sum_{i=0}^{k-1}(-1)^i\frac{k-i-\lambda+l}{i!(k-i)!}(k-i)^{[l]}\right.\\ \left. + \lambda^{l+1}\sum_{i=0}^{l}(-1)^i\frac{2l+1-i}{i!(l+1-i)!}(l+1-i)^{[l]}\right\}.\end{aligned} \tag{3.13}$$

Hence we get

$$\begin{aligned}&ES_N = \frac{\lambda}{\theta}, \quad ES_N^2 = \frac{2\lambda+\lambda^2}{\theta^2}, \quad ES_N^3 = \frac{6\lambda+6\lambda^2+\lambda^3}{\theta^3},\\ &ES_N^4 = \frac{24\lambda+36\lambda^2+12\lambda^3+\lambda^4}{\theta^4},\\ &E(S_N - ES_N)^2 = \frac{2\lambda}{\theta^2}, \quad E(S_N - ES_N)^3 = \frac{6\lambda}{\theta^3},\\ &E(S_N - ES_N)^4 = \frac{24\lambda+12\lambda^2}{\theta^4}.\end{aligned} \tag{3.14}$$

Buishand [B, p. 17] and Öztürk [O] found the formulae of (3.14) for ES_N and $E(S_N - ES_N)^r$ for $r = 2, 3, 4$.

3.2. The compound $\alpha(t)$-modified Poisson distribution with gamma summands. Now we generalize the compound Poisson-gamma distribution to the compound $\alpha(t)$-modified Poisson-gamma distribution. We consider the random sum (1.1), where N has an $\alpha(t)$-modified Poisson distribution $MP(t,\psi,\lambda)$ and the i.i.d. summands $\{X_i,\ i\geq 1\}$ have a gamma distribution with the probability density function (3.1).

THEOREM 3.4. *The moment generating function, the rth uncorrected moments and the $(r+1)$th central moments of the compound $\alpha(t)$-modified Poisson distribution with gamma summands are given by*

$$(3.15)\quad {}_{\alpha}M_{S_N}(s)=\exp\Big\{-\lambda\Big[1-\Big(1-\frac{s}{\theta}\Big)^{-\rho}\Big]\Big\}\Big[\frac{1-\psi}{1-\psi\big(1-\frac{s}{\theta}\big)^{-\rho}}\Big]^{t},$$

$$(3.16)\quad {}_{\alpha}ES_N^r=\theta^{-r}\sum_{k=1}^{r}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^{k}\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[r]},$$

$$(3.17)\quad {}_{\alpha}E(S_N-ES_N)^{r+1}$$
$$=\theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}\Big[-\rho\Big(\lambda+t\frac{\psi}{1-\psi}\Big)\Big]^{r-l}\Big\{\sum_{k=1}^{l}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^{k}$$
$$\cdot\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}\Big[\rho\Big(k-i-\lambda-t\frac{\psi}{1-\psi}\Big)+l\Big](\rho(k-i))^{[l]}$$
$$+\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^{l+1}\sum_{i=0}^{l}(-1)^i\frac{\rho(l+1-i)+l}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}\Big\}.$$

Proof. Taking the pgf of the $\alpha(t)$-modified Poisson distribution

$${}_{\alpha}G_N(s)=\exp\{-\lambda(1-s)\}\Big(\frac{1-\psi}{1-\psi s}\Big)^{t}$$

and the mgf of the gamma distribution $M_X(s)=(1-s/\theta)^{-\rho}$ in (3.8) we obtain (3.15).

The rth derivative of ${}_{\alpha}M_{S_N}(s)$ in (3.15) is given by

$$(3.18)\quad \frac{d^r}{ds^r}\,{}_{\alpha}M_{S_N}(s)={}_{\alpha}M_{S_N}(s)\theta^{-r}\sum_{k=1}^{r}\frac{1}{k!}\Big(1-\frac{s}{\theta}\Big)^{-\rho k-r}\sum_{j=0}^{k}\binom{k}{j}t^{[j]}\lambda^{k-j}$$
$$\cdot\Big[\frac{\psi}{1-\psi\big(1-\frac{s}{\theta}\big)^{-\rho}}\Big]^{j}\sum_{i=0}^{k-1}\binom{k}{i}(-1)^i(\rho(k-i))^{[r]}.$$

Setting $s=0$ in (3.18) we obtain (3.16). Using the relation (2.8) between moments and (3.16) we get

$$(3.19)\qquad {}_{\alpha}E(S_N-ES_N)^r=\theta^{-r}\Big\{\Big[-\rho\Big(\lambda+t\frac{\psi}{1-\psi}\Big)\Big]^r+\sum_{l=1}^{r}\binom{r}{l}\Big[-\rho\Big(\lambda+t\frac{\psi}{1-\psi}\Big)\Big]^{r-l}\sum_{k=1}^{l}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^k\cdot\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]}\Big\},$$

Replacing r by $r+1$ in (3.19) and using (2.9) we get

$$\begin{aligned}{}_{\alpha}E(S_N-ES_N)^{r+1}&=\theta^{-r-1}\Big\{\sum_{l=1}^{r}\binom{r}{l}\Big[-\rho\Big(\lambda+t\frac{\psi}{1-\psi}\Big)\Big]^{r+1-l}\sum_{k=1}^{l}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^k\\&\cdot\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l]}+\sum_{l=1}^{r}\binom{r}{l}\Big[-\rho\Big(\lambda+t\frac{\psi}{1-\psi}\Big)\Big]^{r-l}\\&\cdot\sum_{k=1}^{l+1}\Big(\lambda+\alpha(t)\frac{\psi}{1-\psi}\Big)^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[l+1]}\Big\}.\end{aligned}$$

Next, in order to prove (3.17) we follow the proof of (3.7). ∎

For $\psi=\beta/(1+\beta)$ in Theorem 3.4 we get the following important conclusion.

THEOREM 3.5. *The moment generating function, the rth uncorrected moments and the $(r+1)$th central moments of the compound Delaporte distribution with gamma summands are given by*

$$ {}_{\alpha}M_{S_N}(s)=\exp\Big\{-\lambda\Big[1-\Big(1-\frac{s}{\theta}\Big)^{-\rho}\Big]\Big\}\Big[1+\beta\Big(1-\Big(1-\frac{s}{\theta}\Big)^{-\rho}\Big)\Big]^{-t},$$

$$ {}_{\alpha}ES_N^r=\theta^{-r}\sum_{k=1}^{r}(\lambda+\beta\alpha(t))^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(\rho(k-i))^{[r]},$$

$$\begin{aligned}{}_{\alpha}E(S_N-ES_N)^{r+1}&=\theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}[-\rho(\lambda+\beta t)]^{r-l}\Big\{\sum_{k=1}^{l}(\lambda+\beta\alpha(t))^k\\&\cdot\sum_{i=0}^{k-1}(-1)^i\frac{\rho(k-i-\lambda-\beta t)+l}{i!(k-i)!}(\rho(k-i))^{[l]}\\&+(\lambda+\beta\alpha(t))^{l+1}\sum_{i=0}^{l}(-1)^i\frac{\rho(l+1-i)+l}{i!(l+1-i)!}(\rho(l+1-i))^{[l]}\Big\},\end{aligned}$$

respectively.

Hence, one can get

$$
\begin{aligned}
{}_{\alpha}ES_N &= \frac{\rho}{\theta}(\lambda + t\beta),\\
{}_{\alpha}ES_N^2 &= \frac{\rho}{\theta^2}\big(\lambda(\rho+1) + \lambda^2\rho + t\beta[\rho + 1 + 2\lambda\rho] + t^{[2]}\beta^2\rho\big),\\
{}_{\alpha}ES_N^3 &= \frac{\rho}{\theta^3}\big(\lambda(\rho+1)^{[2]} + 3\lambda^2\rho^{[2]} + \lambda^3\rho^2 + t\beta[(\rho+1)^{[2]} + 6\lambda\rho^{[2]} + 3\lambda^2\rho^2]\\
&\quad + t^{[2]}\beta^2[3\rho^{[2]} + 3\lambda\rho^2] + t^{[3]}\beta^3\rho^2\big),\\
{}_{\alpha}ES_N^4 &= \frac{\rho}{\theta^4}\big(\lambda(\rho+1)^{[3]} + \lambda^2[3\rho(\rho+1)^2 + 4\rho^{[3]}] + 6\lambda^3\rho\rho^{[2]} + \lambda^4\rho^3\\
&\quad + t\beta((\rho+1)^{[3]} + 2\lambda[3\rho(\rho+1)^2 + 4\rho^{[3]}] + 18\lambda^2\rho\rho^{[2]} + 4\lambda^3\rho^3)\\
&\quad + t^{[2]}\beta^2(3\rho^{[2]}(\rho+1) + 4\rho^{[3]} + 18\lambda\rho\rho^{[2]} + 6\lambda^2\rho^3)\\
&\quad + t^{[3]}\beta^3(6\rho\rho^{[2]} + 4\lambda\rho^3) + t^{[4]}\beta^4\rho^3\big),\\
{}_{\alpha}E(S_N - ES_N)^2 &= \frac{\rho}{\theta^2}\big(\lambda(\rho+1) + t\beta(\rho + 1 + \beta\rho)\big),\\
{}_{\alpha}E(S_N - ES_N)^3 &= \frac{\rho}{\theta^3}\big(\lambda(\rho+1)^{[2]} + t\beta((\rho+1)^{[2]} + 2\beta^2\rho^2 + 3\beta\rho^{[2]})\big),\\
{}_{\alpha}E(S_N - ES_N)^4 &= \frac{\rho}{\theta^4}\big(\lambda(\rho+1)^{[3]} + 3\lambda^2\rho^{[2]}(\rho+1) + \beta t((\rho+1)^{[3]}\\
&\quad + 6\lambda\rho^{[2]}(\rho+1)) + \beta^2 t(6\lambda\rho + 3t(\rho+1) + 11 + 7\rho)\rho^{[2]}\\
&\quad + 6\beta^3(t^{[2]} + t)\rho^{[2]}\rho + 3\beta^4(t^{[2]} + t)\rho^3\big).
\end{aligned}
$$

For $\rho = 1$ in Theorem 3.4 we get the important conclusion.

THEOREM 3.6. *The moment generating function, the rth uncorrected moments and the $(r+1)$th central moments of the compound $\alpha(t)$-modified Poisson distribution with exponential summands are given by*

$$
{}_{\alpha}M_{S_N}(s) = \exp\left\{-\lambda\left[1 - \left(1 - \frac{s}{\theta}\right)^{-1}\right]\right\}\left[\frac{1-\psi}{1 - \psi\left(1 - \frac{s}{\theta}\right)^{-1}}\right]^t,
$$

$$
{}_{\alpha}ES_N^r = \theta^{-r}\sum_{k=1}^{r}\left(\lambda + \alpha(t)\frac{\psi}{1-\psi}\right)^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(k-i)^{[r]},
$$

$$
\begin{aligned}
&{}_{\alpha}E(S_N - ES_N)^{r+1}\\
&\quad = \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}\left[-\left(\lambda + t\frac{\psi}{1-\psi}\right)\right]^{r-l}\left\{\sum_{k=1}^{l}\left(\lambda + \alpha(t)\frac{\psi}{1-\psi}\right)^k\right.\\
&\qquad\cdot\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}\left(k - i - \lambda - t\frac{\psi}{1-\psi} + l\right)(k-i)^{[l]}\\
&\qquad\left. + \left(\lambda + \alpha(t)\frac{\psi}{1-\psi}\right)^{l+1}\sum_{i=0}^{l}(-1)^i\frac{2l+1-i}{i!(l+1-i)!}(l+1-i)^{[l]}\right\}.
\end{aligned}
$$

Hence for $\psi = \beta/(1+\beta)$ we have formulae for the compound Delaporte-exponential distribution.

THEOREM 3.7. *The moment generating function, the rth uncorrected moments and the $(r+1)$th central moments of the compound Delaporte distribution with exponential summands are given by*

$$ {}_{\alpha}M_{S_N}(s) = \exp\left\{-\lambda\left[1-\left(1-\frac{s}{\theta}\right)^{-1}\right]\right\}\left[1+\beta\left(1-\left(1-\frac{s}{\theta}\right)^{-1}\right)\right]^{-t}, $$

$$ {}_{\alpha}ES_N^r = \theta^{-r}\sum_{k=1}^{r}(\lambda+\beta\alpha(t))^k\sum_{i=0}^{k-1}(-1)^i\frac{1}{i!(k-i)!}(k-i)^{[r]}, $$

$$ \begin{aligned} {}_{\alpha}E(S_N - ES_N)^{r+1} &= \theta^{-r-1}\sum_{l=1}^{r}\binom{r}{l}(-\lambda-\beta t)^{r-l} \\ &\quad\cdot\left\{\sum_{k=1}^{l}(\lambda+\beta\alpha(t))^k\sum_{i=0}^{k-1}(-1)^i\frac{k-i-\lambda-\beta t+l}{i!(k-i)!}(k-i)^{[l]}\right. \\ &\quad\left.+(\lambda+\beta\alpha(t))^{l+1}\sum_{i=0}^{l}(-1)^i\frac{2l+1-i}{i!(l+1-i)!}(l+1-i)^{[l]}\right\}. \end{aligned} $$

Letting $\psi = 0$ in Theorem 3.4 we obtain (3.5)–(3.7). Setting $\psi = 0$ in Theorem 3.6 we get (3.11)–(3.13).

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