ALGEBRA, LOGIC AND NUMBER THEORY
BANACH CENTER PUBLICATIONS, VOLUME 108
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2016

KAZIMIERZ SZYMICZEK (1939–2015)

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Kazimierz Szymiczek was born on the 20th of May, 1939, in Pietwałd in the Zaolzie district (now a part of the Czech Republic). His childhood years were marked by the difficult times of the Second World War, during which he was separated from his closest family that had to go into hiding to avoid persecution by the German occupant. He attended elementary and high schools in the region of Cieszyn Silesia, and in 1956 graduated from the Pedagogical High School in Cieszyn with the class profiled in physical education. He then became a student at the WSP Higher Pedagogical College in Katowice and obtained his M.Sc. in mathematics in 1960. Right after the graduation he was offered a position first of an adjunct professor, and then of an assistant professor at the WSP. He held that position until 1968, when the College became incorporated into the newly established University of Silesia. His initial scientific interests were predominantly concerned with number theory and he worked under the supervision of Antoni Wakulicz, at the time the head of the Algebra and Number Theory Group. He focused on problems in Diophantine equations and pseudoprimes inspired by the school of Wacław Sierpiński. As a result, in 1966 Szymiczek presented his Ph.D. dissertation entitled *On the distribution of prime factors of Mersenne numbers.* It is, perhaps, worth mentioning that around that time Antoni Wakulicz was the sole resident of the Upper Silesia region with a doctorate in mathematics (obtained in 1949 at the University of Warsaw under the supervision of Wacław Sierpiński), and the Algebra and Number Theory Group consisted, in fact, of two people: Wakulicz and Szymiczek.

Once the University of Silesia was established in 1968, Szymiczek was appointed as an assistant professor in the Institute of Mathematics and held this position until 1971.

2010 *Mathematics Subject Classification*: Primary: 01A70; Secondary: 11E04, 11E12.
Translated to English by Paweł Gładki.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc108-0-1

That year, as a fellow of the British Council Scholarship, he traveled to the University of Cambridge for a postdoctoral scholarship. At first, he was hoping to expand his interests in number theory, but soon was fascinated by the newly emerging algebraic theory of quadratic forms—an area of mathematics to which he remained faithful till the end of his scientific activity.

After returning from Cambridge Szymiczek was promoted to the position of associate professor and became the chair of the Algebra and Number Theory Group, a function that he held until his retirement in 2009. Since the very beginning of his chairing he managed to direct the interests of young researchers within the Group towards problems with which he had familiarized himself while in Cambridge. As an outcome, new scientific contacts with leading mathematicians in the field (such as T. Y. Lam from Berkeley, R. Elman from Los Angeles, A. Prestel from Konstanz, W. Scharlau from Münster and many more) were established, and a period of hard, yet very enthusiastic work has followed, that soon led to a number of new results that sparked interest in the international community. Szymiczek obtained his habilitation in 1977, and three of his students (M. Kula, A. Sładek, and L. Szczepanik) defended their Ph.D. theses in 1979. That year marked the end of the first period of research on quadratic forms in Katowice, although, of course, it continued in later years, and, in particular, in 1985 another student of Szymiczek (K. Kozioł) was granted a doctorate.

In the 1980s and early 1990s the research interests of Szymiczek gradually shifted in the general direction of quadratic forms over global fields. He became strongly influenced during his visits to the United States at the time: he was a visiting professor at the Southern Illinois University in Carbondale during the years 1980–1981, and at the Louisiana State University in Baton Rouge during the years 1985–1986 and the summer of 1992. The visits to Baton Rouge have proven to be particularly fruitful. Joint research carried out with leading specialists in algebraic number theory (P. E. Conner and R. Perlis) resulted in finding a complete description of the behavior of quadratic forms over global fields (see below for details). As a "side-effect" of this cooperation another student of Szymiczek (A. Czogała) defended his Ph.D. in 1988. These achievements were met with wide recognition both in Poland and abroad, and in 1989 the President of Poland awarded him the degree of Professor. In 1990 he was promoted to the position of full professor, and in 1996 the Senate of the University of Silesia decided to grant him the position of an ordinary professor (*professor ordinarius*), which is the most prestigious research position in the Polish university system.

The research on classification of global fields with respect to quadratic forms carried out by Szymiczek and his collaborators led to the development of new ideas that were later successfully applied to studies of quadratic forms over other classes of fields. In particular, Professor supervised the work of his next Ph.D. student (P. Koprowski), who defended his thesis in 2001.

From the early 2000s his interests moved towards the theory of quadratic forms over rings, and, supported by a grant received for that purpose from the Polish Committee of Scientific Research, he supervised the work of another student (M. Ciemała), who defended her thesis in 2004. Altogether he advised the research of seven Ph.D. students. His tenure as the chair of the Algebra and Number Theory Group was always characterized by a very friendly and warm atmosphere within the group. He was always willing to support his colleagues and often inspired them in their research, even if it was rather remote from his scientific interests at the time. Four of his Ph.D. students eventually obtained their habilitations, and, in turn, supervised works of their own Ph.D. students: that way Szymiczek became a "grandfather" of, up to date, nine doctorates. He was always extremely friendly for his students—not only he helped them in their scientific careers, but also extended his support to their teaching and student supervision, as well as offered a helping hand in solving whatever personal problems his students encountered.

He authored 67 papers published in peer reviewed journals, three textbooks and two research monographs. His book *Bilinear Algebra: An Introduction to the Algebraic Theory of Quadratic Forms* [20] became a standard text in the literature devoted to the algebraic theory of quadratic forms.

Szymiczek was an exceptional speaker. His public speeches, no matter whether they were difficult and important scientific talks given at conferences, or merely popular lectures for broad audiences, were always very well thought through. Every single mark left by him on a chalkboard served a carefully planned purpose. He delivered 48 conference presentations and invited talks at foreign universities. He held particularly close ties with mathematicians from Czech Republic and Slovakia. Every two years since 1973 a Czech and Slovak number theory conference has been organized, initially as the Summer School on Number Theory, and now as the Czech and Slovak International Conference on Number Theory. During the second conference in the series, which was held in 1975 in Kočovce, Szymiczek was the first international guest of the meeting. Since then he has been invited to all conferences in the series, and attended most of them. During one of his visits to the University of Ostrava, he suggested to organize a Czech, Polish and Slovak conference on number theory, with the special emphasis on providing opportunities for young researchers to establish scientific contacts. The first meeting in the series was organized in 1996, and since then every two years a conference is organized either by the University of Silesia or by the University of Ostrava. Since 2010 the conference was joined by another annual event, the Colloquiumfest on Algebra and Logic, and is now organized under the name ALANT (Algebra, Logic and Number Theory). Every meeting gathers an abundance of renowned mathematicians from around the world.

As a well known mathematician, Szymiczek was often engaged in reviewing doctoral theses, and applications for habilitations and professorships. He reviewed a number of Ph.D. dissertations and provided his expertise in the evaluation of 10 professorship applications. He was also refereeing papers for both Polish and foreign journals, as well as providing reviews for *Zentralblatt* and *Mathematical Reviews*.

Szymiczek was an excellent teacher. He often taught classes in algebra, number theory and geometry, some of them either highly non-standard or at an advanced graduate level. His teaching methods have always been (and will continue to be) a perfect model for his peers. He supervised works of 200 M.Sc. students, which is probably the best illustration of what a popular teacher he was.

Last but not least, one has to mention his committee work for both the Institute of Mathematics and the University. He was the vice-chair of the Institute in the years 1975–1981, and in 1993 he built from scratch the whole system of doctoral studies at the Institute, and remained its chair until 2002. He collected numerous awards for his achievements, the Award of the Minister of Higher Education (1965, 1977, 1978), the award of the Polish Mathematical Society for young mathematicians (1965), Medal of the Commission of National Education (2001), the Gold Cross of Merit (1977), and the Knight's Cross of the Order Polonia Restituta (1985).

His death on July 20th, 2015, filled with sadness the hearts of those who were fortunate enough to know him.

Algebraic theory of quadratic forms in the works of Kazimierz Szymiczek. As we have already mentioned, during the academic year 1971–1972 Szymiczek was given an opportunity to familiarize himself with the newly emerging area of the algebraic theory of quadratic forms. As a postdoctoral fellow at the University of Cambridge, he frequently attended lectures delivered by John Cassels. It is widely accepted that the theory was founded by Ernst Witt in his seminal 1937 paper [32]. Roughly speaking, he built a new algebraic structure, later called the Witt ring, that in a concise way encapsulates essential information concerned with the behavior of quadratic forms over an arbitrary field. It took next 30 years until suitable methods have been developed to apply the ideas of Witt to various classes of fields. A major breakthrough was made by Albrecht Pfister, who published a series of papers on that theme over the years 1965–1967. By 1971, when Szymiczek was starting his appointment in Cambridge, it became apparent that a new field of research had just opened up: a significant number of papers containing important results was being published, and a broad selection of intriguing problems waiting to be solved was being asked. It seemed like a perfect opportunity to join the rising tide of research in the area: already while in Cambridge, Szymiczek worked on his first paper in quadratic forms [8] devoted to the investigation of the structure of Grothendieck groups and rings of classes of quadratic forms over arbitrary fields of characteristic different from 2. This research was continued in later works [9], [10], [11], and [12].

Let K be a field of characteristic different from 2 and let (V_1, B_1), (V_2, B_2) be two finitely dimensional, nonsingular bilinear spaces over the field K. We say that they are *isometric* if there is a linear isomorphism $\varphi : V_1 \to V_2$, such that $B_1(u, v) = B_2(\varphi(u), \varphi(v))$ for all $u, v \in V_1$. For given bilinear spaces one can build

their *orthogonal sum* $(V_1, B_1) \perp (V_2, B_2) = (V_1 \oplus V_2, B_1 \oplus B_2)$

and their *tensor product* $(V_1, B_1) \otimes (V_2, B_2) = (V_1 \otimes V_2, B_1 \otimes B_2)$.

It turns out that these operations are compatible with isometry, so that the set $S(K)$ of classes of isometric bilinear spaces forms a commutative semiring with the binary operations $\perp$ and $\otimes$. This semiring can be embedded into its Grothendieck ring $G(K)$, whose elements are represented by formal differences of isometry classes of bilinear forms. For fields of characteristic different from 2 there is a bijective correspondence between bilinear and quadratic forms, and thus $G(K)$ is called the *Grothendieck ring* of quadratic forms of the field K. The quotient ring of $G(K)$ modulo the ideal generated by hyperbolic forms is called the *Witt ring* of the field K. In his first works on the algebraic theory of quadratic forms Szymiczek investigated some algebraic properties of the Grothendieck ring. In particular, he proved the following theorem:

If $\mathfrak{p}$ is a prime ideal of the ring $G(K)$, then $G(K)/\mathfrak{p} \cong \mathbb{Z}$ or $G(K)/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$. In the former case the ideal $\mathfrak{p}$ is minimal, while in the latter it is maximal.

He also gave a full characterization of nilpotent, torsion and unit elements of the ring $G(K)$.

Another group of results was concerned with the structure of the additive groups of Grothendieck and Witt rings of fields of characteristic different from 2. One of the results on that theme is the following one:

For any field K there exists a free abelian subgroup U of the group $G(K)$ such that $G(K) = U \oplus G^t(K)$. If the field K is non-real, then the group $W(K)$ is torsion. If the field K is formally real, then there exists a free subgroup V of $W(K)$ such that $W(K) = V \oplus W^t(K)$. (Here, $G^t(K), W^t(K)$ denote the torsion subgroups of the corresponding rings.)

In this context the natural question arises whether a further decomposition of the torsion parts of Grothendieck and Witt groups is possible, that is, a decomposition into a direct sum of cyclic subgroups. In [12] some partial positive answers to that question are to be found, for example for the class of non-real fields, or formally real fields with a finite Pythagoras number and others. To the best of our knowledge, this question is, in general, still open.

Szymiczek made substantial contributions to the investigation of the equivalence of fields with respect to quadratic forms. The notion of such equivalence was introduced by D. Harrison and expresses the intuition that the behavior of quadratic forms over equivalent fields should be identical. C. Cordes [28] observed that the Witt rings of equivalent fields are isomorphic. Szymiczek extended this notion by analyzing what properties of quadratic forms are preserved when Grothendieck or Witt groups or rings are isomorphic and when the corresponding isomorphisms satisfy some additional conditions. It turns out that in multiple cases these notions are equivalent, which is expressed by the following theorem:

For fields K and L the following conditions are equivalent:

1. *There is an isomorphism T of the rings $G(K)$ and $G(L)$ such that $T\langle -1\rangle = \langle -1\rangle$.*
2. *The Witt rings $W(K)$ and $W(L)$ are isomorphic.*
3. *There is a ring isomorphism between $W(K)$ and $W(L)$ that preserves dimensions of forms.*
4. *There is a ring isomorphism T between $G(K)$ and $G(L)$ that preserves dimensions of ring elements and such that $T\langle -1\rangle = \langle -1\rangle$.*
5. *The rings $W(K)/I^3(K)$ and $W(L)/I^3(L)$ are isomorphic ($I^3(\cdot)$ is the third power of the, so called, fundamental ideal of the Witt ring, that contains quadratic forms of even dimension).*

The last of the abovementioned conditions is especially interesting, as it shows, roughly speaking, that the information about properties of all quadratic forms over a given field is, in fact, contained in a small portion of its Witt ring.

While analyzing properties of fields equivalent with respect to quadratic forms one notices that certain parameters of these fields remain unchanged within a class of equivalent fields. These parameters are thus called invariants of equivalence. The simplest ones are square class number $q(K) = |K^*/K^{*2}|$, the number of orderings $r(K)$, and the Pfister index $s(K)$. They can be used in the classification of fields with respect to quadratic forms equivalence.

Craig Cordes classified all Witt groups of non-real fields with square class number no greater than 8. In the paper [9] Szymiczek fully classified all fields with square class number no greater than 8, as well as described the structure of their Grothendieck and Witt groups. In particular he obtained information on how the number of equivalence classes depends on square class number:

$\lvert K^*/K^{*2}\rvert$	1	2	4	8
the number of classes of formally real fields	–	1	2	7
the number of classes of non-real fields	1	2	4	10

The method of classification that was applied first eliminated these groups that have not possessed the desired properties. This was achieved by means of the investigation of known properties of Grothendieck and Witt groups. Szymiczek then searched for fields whose Grothendieck and Witt groups were isomorphic to the groups that were not eliminated. In the case of square class number less than or equal to 4 the classification was complete, but in the case of square class number equal to 8, four new conceivable Witt groups were isolated, for which no corresponding fields were found. It turned out that only after applying special constructions of fields it became possible to conclude the classification of fields with eight square classes, which was done in [6]. It is worth noting that thanks to this classification it became apparent that for fields of no more than 8 square classes the isomorphism of Witt groups implies the existence of the isomorphism of corresponding Witt rings. The question of classifying fields with respect to quadratic forms was investigated by Szymiczek's students and resulted in three doctorates (Andrzej Sładek, Mieczysław Kula and Lucyna Szczepanik, 1979).

The problems of describing the algebraic properties of Grothendieck and Witt rings and classifying fields with low square class numbers are now considered to be elementary questions in the algebraic theory of quadratic forms. It is, however, worth realizing that at the time when these questions were investigated, the modern methods in quadratic form theory were not available yet, as they were invented during the same period of time. Being a student of Szymiczek was a very inspiring and motivating experience—it was clear to us that we were participating in an actual process of creating new mathematics, and also competing with leading mathematicians in the field.

During his research on properties of quadratic forms over arbitrary fields, Szymiczek observed that there exist natural generalizations of local and Hilbertian fields known from number theory. This way he proposed the notion of n-Hilbert fields (meaning that the n-fold Pfister forms are universal or semi-universal) and n-local fields (that is such that there exists exactly one nonisotropic, torsion n-fold Pfister form). In [13] it has been shown, among other things, that if K is an n-local field, then it is an $(n-1)$-Hilbert field. For non-real fields the converse is also true. Moreover, the existence of such fields for every n was also demonstrated. These results were improved and led to another Ph.D. thesis by one of Szymiczek's students (Krzysztof Kozioł, 1985).

In a series of three papers [26], [15], [16] results related to rigid elements and basic parts of fields were published. These notions naturally appeared while the structure of Witt rings of fields was investigated in more detail. It is worth noting that the second of the papers cited above contained a substantial improvement of results previously obtained by Carson and Marshall [27]. It relied on presenting the basic parts of numerous classes of fields by an iterated value set of binary quadratic forms.

Quadratic forms over global fields in the works of Kazimierz Szymiczek. Since the 1980s the scientific interests of Szymiczek leaned towards the theory of quadratic forms over global fields[1].

Although the theory of quadratic forms over global fields was already developed in the beginning of the 20th century and led to discoveries such as the celebrated local-global principles due to H. Hasse and H. Minkowski, the interest in the algebraic theory of quadratic forms that flourished in the 1970s resulted in a number of new questions and problems.

In 1983, during the Czech and Slovak Conference on Number Theory in Chlébské, Szymiczek asked the following question: *Are there Witt equivalent but non-isomorphic number fields*? (see [14]). This question was repeated in 1985 during Szymiczek's stay in Baton Rouge, and during the same year, as a result of joint research with P. E. Conner and R. Perlis, an answer was found and examples of infinite families of non-isomorphic quadratic number fields that are Witt equivalent were given.

The analysis of the construction of these examples by its authors led to the introduction of the new notion of *reciprocity equivalence* (nowadays called *Hilbert-symbol equivalence*). It played an important role in the research on Witt rings of global fields.

A Hilbert-symbol equivalence (*HSE, for short*) *between two global fields K and L is a pair of maps*

$$T\colon \Omega(K) \to \Omega(L), \qquad t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2,$$

where T is bijective map between the sets of all primes of K and L and t is an isomorphism of square class groups preserving quadratic Hilbert symbols in the sense that

$$(a,b)_{\mathfrak{p}} = (ta, tb)_{T\mathfrak{p}}, \qquad \forall a, b \in \dot{K}/\dot{K}^2, \quad \forall \mathfrak{p} \in \Omega(K).$$

In the paper [7] the following theorem was given:

Two global field K, L are Witt equivalent if and only if they are Hilbert-symbol equivalent.

[1]Since all the results mentioned below concern global fields of characteristic different from 2, from now on we will restrict our considerations to such fields. Thus whenever we write “global field”, we mean “global field of characteristic $\neq 2$”.

Although the proof that HSE implies Witt equivalence is a straightforward consequence of the Witt equivalence criterion due to Harrison, the proof of the converse is highly nontrivial. Szymiczek, in addition to providing the proof in [7], gave two more, substantially different proofs of the same theorem in [17] and [21].

The key result regarding the Witt equivalence of global fields is the following *local-global principle for Witt equivalence* proven in [7]:

Two global fields are Witt equivalent if and only if their primes can be paired so that corresponding completions are Witt equivalent.

This principle fits very well in the classical set of Hasse–Minkowski local-global principles.

Szymiczek's question together with the paper [7], whose early, unpublished preprints were available already in 1986, marked a milestone in the research on quadratic forms over global fields. Szymiczek's role in this research is hard to overestimate.

Basing on the local-global principle for Witt equivalence, Szymiczek pointed out the existence of a *Witt equivalence invariant* for the Witt equivalence of number fields. For a number field F this invariant is given by the sequence of integers

$$S(F) = \big(n, r, s, g;\ (n_i, s_i),\ i = 1, \dots, g\big),$$

where n denotes the degree of the extension $F \supset \mathbb{Q}$, r is the number of real places, and s the number of the complex ones (assuming $s = 0$ when F is formally real). Further, g is the number of dyadic places of F and for every place n_i and s_i denote, respectively, the local dyadic degree and the local (dyadic) Pfister index.

Szymiczek showed in [18] that two global fields E and F are Witt equivalent if and only if their Witt equivalence invariants $S(E)$ and $S(F)$ are equal. Moreover, he proved that for every sequence $\big(n, r, s, g;\ (n_i, s_i),\ i = 1, \dots, g\big)$ satisfying some natural necessary conditions for being a Witt equivalence invariant, one can find a suitable number field for which it, indeed, is the invariant. These facts allow us to determine the number $w(n)$ of Witt equivalence classes of number fields of degree n (for example, according to [18], one has $w(2) = 7$, $w(3) = 8$, $w(4) = 29$, $w(9) = 365$) on the one hand, and on the other they constitute a starting point for further classifying fields of a given degree, or fields contained in a given family, up to Witt equivalence. Using these methods Szymiczek classified number fields of degree 3 in [18], and, together with R. Kučera in [5], cyclotomic fields of degree ≤ 400.

Szymiczek continued his work from [18] and in [19] investigated the number and the structure of Witt equivalence classes in passing from a field to its quadratic extension.

It is worth noting that Hilbert equivalence proved to be a useful tool not only in the classification of global fields up to Witt equivalence, but also of other classes of fields. Przemysław Koprowski, who defended his Ph.D. thesis under the supervision of Szymiczek, managed to carry the notion of Hilbert equivalence to algebraic function fields over a global field and algebraic function fields over a real closed field. These results were published in [30] and [31].

While introducing the Hilbert equivalence of fields, its inventors (Perlis, Conner and Szymiczek) distinguished between two types of equivalence: tame and wild.

The Hilbert-symbol equivalence (T,t) *of* K *and* L *is tame at a finite prime* $\mathfrak{p}$ *if*

$$\mathrm{ord}_{\mathfrak{p}} a \equiv \mathrm{ord}_{T\mathfrak{p}} ta \pmod 2$$

for all square classes $a \in \dot{K}/\dot{K}^2$*; otherwise* (T,t) *is* wild *at* $\mathfrak{p}$*. The set* $W(T,t)$ *of wild primes for* (T,t) *is called the wild set. When the wild set is empty, we say that* (t,T) *is tame.*

In [3] they discovered the relationship between 2-ranks of ideal class groups of K and L and the number of wild places in the set of wild Hilbert equivalences of these fields. This paper paved the way to the research on relationships between Witt equivalence of global fields and arithmetic properties of such fields.

Szymiczek remained involved in the research on that theme, and, among other results, proved the following theorem in [23]:

For each prime p *dividing* n*, every infinite class of Witt equivalent number fields of degree* $n \geq 2$ *contains a field with the* p*-rank of the ideal class group exceeding any given number.*

Another question regarding relationships between arithmetic properties of number fields and their Witt equivalence is the following *Conner's conjecture*:

If a number field K *satisfies the condition*

$$s(K) = 2 \text{ and } s(K_{\mathfrak{p}}) = 1 \text{ for each dyadic prime } \mathfrak{p},$$

known as the Conner's Level Condition (*CLC*)*, then every number field Witt equivalent to* K *has an even class number.*

It is relatively easy to verify this conjecture for fields of degree ≤ 3. Szymiczek together with S. Jakubec and F. Marko confirmed in [4] the conjecture for number fields of degree 4. Moreover, in [25] he found a partial answer to the question whether CLC for a field K implies the parity of class number of all fields Witt equivalent to K.

The tame Hilbert equivalence has a number of interesting properties investigated by numerous mathematicians. Szymiczek greatly contributed to this research. In [24] he proposed a generalization of the notion of tame equivalence in the context of semigroups with the divisor theory. He supervised the work of a Ph.D. student Alfred Czogała, who completely classified quadratic number fields with respect to tame Hilbert equivalence [29].

Already in the early stages of research on tame Hilbert equivalence, the following crucial property was discovered:

If the Hilbert equivalence (T,t) *of number fields* K*,* L *is tame, then it induces the following commutative diagram composed of the Knebusch–Milnor sequences of* K *and* L:

$$\begin{array}{ccccccccc}
0 \longrightarrow & W(\mathcal{O}_K) & \longrightarrow & WK & \xrightarrow{\partial_K} & \coprod_{\mathfrak{p}} W(\mathcal{O}_K/\mathfrak{p}) & \longrightarrow & C_K/C_K^2 & \longrightarrow 1 \\
& \downarrow & & \downarrow \phi_t & & \downarrow \overline{\phi}_t & & \downarrow & \\
0 \longrightarrow & W(\mathcal{O}_L) & \longrightarrow & WL & \xrightarrow{\partial_L} & \coprod_{\mathfrak{p}} W(\mathcal{O}_L/T\mathfrak{p}) & \longrightarrow & C_L/C_L^2 & \longrightarrow 1\,,
\end{array}$$

where $\mathcal{O}_K$ *is the ring of integers of* K*,* C_K *is the ideal class group of* K*, and* $W(K)$*,*

$W(\mathcal{O}_K)$, $W(\mathcal{O}_K/\mathfrak{p})$ are the Witt rings of K, $\mathcal{O}_K$ and $\mathcal{O}_K/\mathfrak{p}$, respectively. In the diagram the first three vertical arrows are ring isomorphisms, and the fourth one is a group isomorphism.

It turns out, as proven by Szymiczek, that this property fully characterizes the tame Hilbert equivalence of number fields. To be more specific, he showed in [22] that *the Hilbert equivalence (T, t) of the number fields K and L is tame if and only if it induces the above commutative diagram.*

The Witt ring $W(\mathcal{O}_K)$ that appears in the above diagram is a special case of the Witt ring of a commutative ring. The construction of such a ring was first proposed by Knebusch in 1970, and it is a natural generalization of the Witt ring of a field. Every ring homomorphism $R \to R'$ induces a homomorphism of their Witt rings $W(R) \to W(R')$. In particular, when R is an integral domain, one obtains a natural homomorphism $\varphi\colon W(R) \to W(K)$, where K is the field of fractions of R. Knebusch showed that in the case when R is a Dedekind domain, the homomorphism ϕ is injective. A natural question then arises: in what other cases is this homomorphism injective, and what can one say about its kernel when it is not injective? This problem has been taken up in a number of papers by Pardon, Ojanguren, Craven, Rosenberg and Ware. In the last years of his work, Szymiczek together with his Ph.D. student Marzena Ciemała actively participated in research on this problem. They were particularly interested in the case when R is an order of a number field. In [1] he and Ciemała showed that *if R is a non-maximal order of a number field, then the kernel of the homomorphism φ is a nilideal*, and in [2] they proved, among other results, that for every non-maximal order R of the Gaussian field $\mathbb{Q}(\sqrt{-1})$ the homomorphism φ is not injective.

The research conducted by Szymiczek occupies a prominent space in modern mathematics and provides a continual inspiration for both his students and mathematicians not closely related to him. He will always live in our memories thanks to his achievements that are testimony to his wide knowledge, erudition, passion and engagement in research.

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