ALGEBRA, LOGIC AND NUMBER THEORY
BANACH CENTER PUBLICATIONS, VOLUME 108
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2016

ON COHOMOLOGICAL SYSTEMS OF GALOIS REPRESENTATIONS

WOJCIECH GAJDA

Faculty of Mathematics and Computer Sciences, Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
E-mail: gajda@amu.edu.pl

SEBASTIAN PETERSEN

Universität Kassel, Fachbereich 10
Wilhelmshöher Allee 73, 34119 Kassel, Germany
E-mail: petersen@mathematik.uni-kassel.de

Abstract. The paper contains an expanded version of the talk delivered by the first author during the conference *ALANT3* in Będlewo in June 2014. We survey recent results on independence of systems of Galois representations attached to ℓ-adic cohomology of schemes. Some other topics ranging from the Mumford–Tate conjecture and the Geyer–Jarden conjecture to applications of geometric class field theory are also considered. In addition, we have highlighted a variety of open questions which can lead to interesting research in near future.

1. Geometric Galois representations. Let K be a field of characteristic $p \geq 0$, $\overline{K}$ an algebraic closure of K and X/K a smooth projective variety of dimension d. Let $\mathbb{L}$ be the set of all prime numbers. Grothendieck and M. Artin defined in [4] for every $j \in \mathbb{N}$ the étale cohomology groups $H^j(X_{\overline{K}}, \mathbb{Z}/n)$. These groups are $\mathbb{Z}/n$-modules. Unlike singular cohomology the cohomology groups of Grothendieck and Artin can be defined even if $p > 0$. In the need for cohomology groups which are vector spaces over a field of characteristic zero one defines (cf. [4]) for every $\ell \in \mathbb{L}$ the ℓ-adic étale cohomology groups

$$H^j(X_{\overline{K}}, \mathbb{Q}_\ell) = \Big(\varprojlim_{i \in \mathbb{N}} H^j(X_{\overline{K}}, \mathbb{Z}/\ell^i)\Big) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

There is a natural representation

$$\rho_{\ell,X}^{(j)} : G_K \to \mathrm{Aut}(H^j(X_{\overline{K}}, \mathbb{Q}_\ell))$$

2010 *Mathematics Subject Classification*: Primary 11G10; Secondary 14F20.
Key words and phrases: Galois representation, Mumford–Tate group, Abelian variety.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc108-0-5

of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ on them. Main topics of the paper centre around the families of ℓ-adic Galois representations $(\rho_{\ell,X}^{(j)})_{\ell\in\mathbb{L}}$ and their images.

1.1. Properties of ℓ-adic cohomology

Basic facts. We refer the reader to [13] and to Milne's lecture notes [24] for definitions and proofs of the facts discussed in this section. The groups $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ behave well for $\ell \neq \mathrm{char}\, K$ and they constitute a cohomology theory with properties which we know very well from algebraic topology.

- *Finiteness*: $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces. The groups vanish for $j > 2\dim X = 2d$ and the top non-zero group $H^{2d}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is identified canonically with $\mathbb{Q}_\ell$.
- *Poincare duality*: For $i + j = 2d$ there exists a perfect pairing of finite dimensional $\mathbb{Q}_\ell$-vector spaces
$$H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \times H^j(X_{\overline{K}}, \mathbb{Q}_\ell) \longrightarrow H^{2d}(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell.$$
In particular, $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is dual to $H^{2d-i}(X_{\overline{K}}, \mathbb{Q}_\ell)$, and they have the same dimension.
- *Künneth theorem*: For any $i > 0$ and two smooth projective varieties X and Y, there are canonical isomorphisms
$$H^i(X_{\overline{K}} \times Y_{\overline{K}}, \mathbb{Q}_\ell) \cong \bigoplus_{j+k=i} H^j(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes H^k(Y_{\overline{K}}, \mathbb{Q}_\ell).$$
- *Lefschetz trace formula*: One of the most important features of the ℓ-adic cohomology is a fixed point formula which says that for any morphism $f : X \longrightarrow X$ with isolated fixed points one can compute the number of fixed points of $f : X(\overline{K}) \to X(\overline{K})$, counted with multiplicities, by calculating traces of $\mathbb{Q}_\ell$-linear maps induced on cohomology by the morphism f (cf. [24], Thm. 25.1): If Γ is the graph of f and Δ the diagonal of X, then
$$(\Gamma \cdot \Delta) = \sum_i (-1)^i \,\mathrm{Tr}\big[f^* : H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i(X_{\overline{K}}, \mathbb{Q}_\ell)\big].$$

The natural action of the Galois group G_K on the vector space $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ defines a continuous homomorphism
$$\rho_{\ell,X}^{(j)} : G_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell} H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$$
the ℓ-adic geometric Galois representation attached to the variety X. The representation $\rho_{\ell,X}^{(j)}$ encodes a lot of important information, which is very interesting even in the case when K is a subfield of the field of complex numbers.

Classical special cases. We remark to illustrate the usefulness of ℓ-adic cohomology by describing three special cases of varieties X and their cohomology G_K-modules.

Remark. If $\dim X = 0$, then $X = \mathrm{Spec}(K[T]/(P(T)))$ for a polynomial $P \in K[T]$ and the set of $\overline{K}$-rational points $X(\overline{K}) = \{\alpha \in \overline{K} : P(\alpha) = 0\}$ of X is the set of roots of P. We identify the only ℓ-adic cohomology group $H^0(X_{\overline{K}}, \mathbb{Q}_\ell)$ with the free $\mathbb{Q}_\ell$-module $\coprod_{\alpha\in X(\overline{K})} \mathbb{Q}_\ell$ with basis $X(\overline{K})$. The representation $\rho_{\ell,X}^{(0)} : G_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(H^0(X_{\overline{K}}, \mathbb{Q}_\ell))$ is then induced by the permutation representation $G_K \to \mathrm{Sym}(X(\overline{K}))$ well known from basic courses in algebra and Galois theory.

Remark. Let X be a smooth projective geometrically connected curve over K. Let g be the genus of the curve X. Then $H^0(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, $H^1(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell^{2g}$, and $H^2(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$. The absolute Galois group G_K acts on $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ via the inverse of the cyclotomic character $\epsilon_\ell : G_K \longrightarrow \mathbb{Z}_\ell^\times$ defined by the action of G_K on ℓ-powers roots of unity. The action of G_K on $H^1(X_{\overline{K}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell^{2g}$ is closely related to the action of G_K on the Tate module of the Jacobian variety of X, compare the next example.

Remark. Let X be an abelian variety defined over K. Let

$$T_\ell(X) = \lim_{\leftarrow} X(\overline{K})[\ell^k] \cong \mathbb{Z}_\ell^{2g}$$

be the Tate module of X. Then $H^i(X_{\overline{K}}, \mathbb{Q}_\ell) = \bigwedge^i H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ is the i-th exterior power of the $\mathbb{Q}_\ell$-vector space $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$. Denote by $\hat{X}$ the dual abelian variety. Then there is an isomorphism of G_K-modules

$$T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong H^1(\hat{X}_{\overline{K}}, \mathbb{Q}_\ell)(1),$$

where for a $\mathbb{Q}_\ell$-vector space M which is a G_K-module, by $M(1)$ we denote the Tate twist $M \otimes \epsilon_\ell$. The above isomorphisms show the importance of the representation $\rho_{\ell,X}^{(1)}$ for an abelian variety X. Note that the Tate module representation $\sigma_\ell := \rho_{\ell,X}^{(1)} \otimes \epsilon_\ell^{-1}$ and its deformation theory was instrumental in the proof of Shimura–Taniyama modularity conjecture by Wiles, for a semistable elliptic curve X defined over the rationals.

1.2. Spectacular applications

Weil conjectures. Let X be a smooth projective variety over the finite field $K = \mathbb{F}_p$. Fix a rational prime $\ell \neq p$. Consider *the congruence zeta function for X*:

$$Z(X,T) := \exp\Big(\sum_{r\geq 1} |X(\mathbb{F}_{p^r})| \frac{T^r}{r}\Big)$$

which by definition is a formal power series in the variable T. Let $\mathrm{Frob} \in G_K = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ denote the Frobenius automorphism induced by the p-th power homomorphism. It follows easily by the Lefschetz formula in ℓ-adic cohomology that for every $r \geq 1$

$$|X(\mathbb{F}_{p^r})| = \sum_{i\geq 0} (-1)^i \operatorname{Tr}(\rho_{\ell,X}^{(i)}(\mathrm{Frob}^r)).$$

We denote by $P_i(T) := \det(1 - \rho_{\ell,X}^{(i)}(\mathrm{Frob})T)$ the characteristic polynomial of the $\mathbb{Q}_\ell$-linear map $\rho_{\ell,X}^{(i)}(\mathrm{Frob}) : H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$. It was proven by Grothendieck and completed in the important case of *Riemann conjecture for X* by Deligne [12] that the following statements hold.

- *Rationality of zeta*: The congruence zeta function
$$Z(X,T) = \frac{P_1(T)P_3(T)\dots P_{2d-1}(T)}{P_0(T)P_2(T)\dots P_{2d}(T)}$$
is a rational function in $\mathbb{Q}(T)$ which satisfies a functional equation.
- *Riemann conjecture*: For every $i \geq 0$ the polynomial $P_i(T)$ has integral coefficients, which are *independent of $\ell \neq p$*, and an analogue of Riemann conjecture holds for $X/\mathbb{F}_p$: The inverse roots of the polynomial $P_i(T)$ have absolute value $p^{i/2}$.

We have stressed the independence of ℓ of the coefficients of the polynomial $P_i(T)$. It is a first manifestation we would like to point out of the *independence of the prime* ℓ of cohomological properties of algebraic varieties.

Mordell conjecture. Let X be an abelian variety defined over a finitely generated extension K of $\mathbb{Q}$. In 1983 Faltings proved in the impressive paper [14] the following crucial properties of the Tate module representation

$$\sigma_\ell : G_K \longrightarrow \mathrm{Aut}(T_\ell(X) \otimes \mathbb{Q}_\ell) \cong \mathrm{Gl}_{2d}(\mathbb{Q}_\ell)$$

of the abelian variety X, which were conjectured in 70's by Tate.

- The representation σ_ℓ is semisimple.
- The natural homomorphism

  $$\mathrm{Hom}(X, X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(X))^{G_K}$$

  induced by sending any algebraic K-morphism $X \to X$ to the attached $\mathbb{Z}_\ell$-linear map $T_\ell(X) \to T_\ell(X)$, is an isomorphism of $\mathbb{Z}_\ell$-modules.

Using the above properties of representations σ_ℓ Faltings in [14] proved the conjecture of Mordell from 1922 which says that every smooth projective curve C of genus $g > 1$ defined over a number field F has finite number of F-rational points.

2. Mumford–Tate conjecture

2.1. On two conjectures of Serre. It is a natural and central question to what extent the ℓ-adic cohomology groups are independent of the auxiliary prime number ℓ. There is a variety of classical results on this question. For example, if $\mathbb{L}' = \mathbb{L} \setminus \{p\}$, then by a well-known corollary to the Riemann hypothesis for varieties over finite fields proved by Deligne, the function

$$\mathbb{L}' \to \mathbb{N}, \quad \ell \mapsto \dim_{\mathbb{Q}_\ell}(H^j(X_{\overline{K}}, \mathbb{Q}_\ell))$$

is constant.

Assume now that K is a finitely generated subfield of $\mathbb{C}$. It is then natural to consider in addition to the data involved so far the more classical singular cohomology groups: $H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}/n)$ for $n \in \mathbb{N}$, $H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ and the comparison theorem from [4] gives isomorphisms

$$H^j(X_{\overline{K}}, \mathbb{Z}/n) \cong H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}/n) \quad \text{and} \quad H^j(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

This can already be viewed as an ℓ-independence result, because the $\mathbb{Q}$-vector space $H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ does not depend on ℓ. To go further let us look at the images of the representations $\rho^{(j)}_{\ell,X}$ and ask for some kind of ℓ-independence. Let us put $V_\ell = H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$, $\rho_\ell = \rho^{(j)}_{X,\ell}$ for $\ell \in \mathbb{L}$ and define

$$\begin{aligned} G_\ell &:= \rho_\ell(G_K), \\ \underline{G}_\ell &:= \text{Zariski closure of } G_\ell \text{ in } \mathrm{GL}_{V_\ell}, \\ \underline{G}^\circ_\ell &:= \text{connected component of the neutral element in } \underline{G}_\ell. \end{aligned}$$

Note that $\underline{G}_\ell$ and $\underline{G}_\ell^\circ$ are algebraic groups over $\mathbb{Q}_\ell$. Serre proved (cf. [28]) that the kernel of

$$G_K \xrightarrow{\rho_\ell} \underline{G}_\ell(\mathbb{Q}_\ell) \to \underline{G}_\ell(\mathbb{Q}_\ell)/\underline{G}_\ell^\circ(\mathbb{Q}_\ell)$$

does not depend on ℓ. This implies the existence of a finite extension K_{conn}/K such that the Zariski closure of $\rho_\ell(G(\bar{K}/K_{\mathrm{conn}}))$ equals $\underline{G}_\ell^\circ$. We define $V := H^j_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q})$. As far as the algebraic groups $\underline{G}_\ell^\circ$ are concerned there is the following conjecture (cf. [30]).

Conjecture 2.1 (Mumford–Tate–Serre Conjecture). There exists a (connected) reductive subgroup $\underline{G}/\mathbb{Q}$ of $\mathrm{GL}_V/\mathbb{Q}$, not depending on ℓ, such that $\underline{G}_\ell^\circ = \underline{G} \times_{\mathbb{Q}} \mathrm{Spec}(\mathbb{Q}_\ell)$ for every $\ell \in \mathbb{L}$.

There is a natural candidate for such an algebraic group $\underline{G}/\mathbb{Q}$, namely the corresponding *motivic Galois group.* This "conjecture of Mumford–Tate type" is a wide open problem, though there is some evidence due to work of Serre, Ribet, Larsen–Pink, Zarhin–Moonen, Chin, Hall, Banaszak–Gajda–Krasoń [5], [6], Vasiu, Hui, Arias-de-Reyna–Gajda–Petersen [1], [2] and many others, mainly in the case of abelian varieties.

Let us assume now that there is an algebraic group $\underline{G}$ as above for the variety X and value j under consideration. Choose a $\mathbb{Z}$-lattice T in V ($T = H^j_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Z})/(\mathrm{Torsion})$ is a natural choice) such that $T_\ell := T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is a G_K-equivariant $\mathbb{Z}_\ell$-lattice in V_ℓ for every $\ell \in \mathbb{L}$, and define $\underline{G}_\ell^\circ(\mathbb{Z}_\ell) := \underline{G}_\ell^\circ(\mathbb{Q}_\ell) \cap \mathrm{GL}_{T_\ell}(\mathbb{Z}_\ell)$ where the intersection takes place inside $\mathrm{GL}_{V_\ell}(\mathbb{Q}_\ell)$. One can then consider the so-called adelic representation

$$\rho^{\mathrm{adelic}} : \mathrm{Gal}(\overline{K}/K_{\mathrm{conn}}) \xrightarrow{\rho} \prod_{\ell\in\mathbb{L}} \rho_\ell(\mathrm{Gal}(\overline{K}/K_{\mathrm{conn}})) \xrightarrow{i} \prod_{\ell\in\mathbb{L}} \underline{G}_\ell^\circ(\mathbb{Z}_\ell) \subset \underline{G}(\mathbb{Q}\otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$$

where ρ is induced by the restricted representations $\rho_\ell \mid \mathrm{Gal}(\overline{K}/K_{\mathrm{conn}}) \to \mathrm{GL}_{V_\ell}(\mathbb{Q}_\ell)$. It is natural to ask whether ρ^{adelic} has an open image (cf. [30, 10.6?]).

Conjecture 2.2 (Adelic Openness). The image of ρ^{adelic} is open if the corresponding motive is maximal (cf. [30, 10.6?], for the details).

The adelic openness conjecture is a wide open problem, but surprisingly there is a recent progress on the first task in the above setup, i.e., in considering the map ρ alone and proving unconditionally that it has a large image. Serre's recent paper [31] studied $\mathrm{im}(\rho)$ in the important case where K is a number field, continuing part of his work on abelian varieties over number fields back in the 80's. The paper [15] of Gajda and Petersen studied $\mathrm{im}(\rho)$ in the case $\mathrm{trdeg}(K/\mathbb{Q}) > 1$. Together this gave a solution to conjecture [30, 10.1?] of Serre in the case of motives attached to smooth projective varieties, and it answered a question of Serre and Illusie (cf. [31, Section 3] and [19, 5.5]) at the same time. Preprints of Böckle–Gajda–Petersen [7], Cadoret–Tamagawa [9] and the habilitation thesis of Petersen follow this line of thought begun by Serre even further.

2.2. Mumford–Tate for abelian varieties. For abelian varieties defined over number fields, or more generally over finitely generated fields over their prime fields, one can say more about images of the Galois representations and the Mumford–Tate conjecture. We illustrate this by quoting a few theorems from [5], [6] and [1], where the authors investigated the image of Galois ℓ-adic and mod ℓ representations associated with the

Tate module of a simple abelian variety of type I, II and III in the Albert classification list. We use Weil restriction functor $R_{E/K}$ of affine group schemes for a finite field extension E/K in order to describe the image of Galois representation in terms of linear algebraic groups. Denote by

$$\sigma_\ell : G_K \to GL(T_\ell(A)),$$
$$\sigma_\ell^0 : G_K \to GL(V_\ell(A))$$
$$\overline{\sigma}_\ell : G_K \to GL(A[\ell])$$

the Galois representations related to the Tate module $T_\ell = T_\ell(A)$ of the abelian varieties A over a finitely generated field K, where $V_\ell(A) = T_\ell \otimes \mathbb{Q}_\ell$ and $A[\ell] = T_\ell/\ell T_\ell$. We consider Zariski closure $\underline{G_\ell}$, of the image of the representation σ_ℓ^0 and the image $G(\ell)$ of the representation $\overline{\sigma}_\ell$. We have a classical theorem of Serre (cf. [29]):

Theorem 2.3. *Let A/K be an abelian variety over a finitely generated field of characteristic zero with* $\mathrm{End}(A) = \mathbb{Z}$ *and* $\dim(A) = 2, 6$ *or odd. Then for almost all ℓ the image of σ_ℓ is as large as possible, i.e., it equals the group of $\mathbb{Z}_\ell$-points of the group of symplectic similitudes $\underline{GSp}_{2g}$.*

Methods modeled on the approach of Serre and based on Lie algebra technics enabled proving in [5] the following results.

Theorem 2.4 ([5], Thm. 6.9). *Let A be a simple abelian variety over a number field K, of type I or II with the endomorphism ring $D = \mathrm{End}(A) \otimes \mathbb{Q}$ with centre E of degree $e = [E : \mathbb{Q}]$ and $d = [D : E]$ such that $2g = 2hed$ where h is odd. Then for ℓ big enough we have equalities of group schemes*:

$$(\underline{G_\ell})' = \prod_{\lambda|\ell} R_{E_\lambda/\mathbb{Q}_\ell}(\mathrm{Sp}_{2h})$$
$$(G(\ell))' = \prod_{\lambda|\ell} R_{k_\lambda/\mathbb{F}_\ell}(\mathrm{Sp}_{2h}),$$

where $\underline{G}'$ is the derived group scheme of $\underline{G}$.

Theorem 2.5 ([5], Thm. 6.16). *Let A be an abelian variety as in the last theorem. Then for all $\ell \gg 0$, we have*:

$$(\overline{\sigma_\ell}(G_K))' = \prod_{\lambda|\ell} \mathrm{Sp}_{2h}(k_\lambda),$$
$$\overline{(\sigma_\ell(G_K))'} = \prod_{\lambda|\ell} \mathrm{Sp}_{2h}(\mathcal{O}_\lambda)$$

where $\overline{(\sigma_\ell(G_K))'}$ denotes the closure of $(\sigma_\ell(G_K))'$ in the natural (λ-adic in each factor) topology of the group $\prod_{\lambda|\ell} \mathrm{Sp}_{2h}(\mathcal{O}_\lambda)$.

Remark. In [6] similar (but more technically involved) theorems about Galois images were obtained for a large class of abelian varieties of type III. The last two theorems enabled verification of the Mumford–Tate for the investigated class of abelian varieties. As a final application of the method an analogue of the open image theorem of Serre was proven.

There is yet another way developed recently by Chris Hall [17] to compute the image of the representation σ_ℓ for abelian varieties which is based on group theory, instead of theory of Lie algebras. Using Hall's method we proved in [1] together with Sara Arias-de-Reyna the following result which works over fields of arbitrary characteristic. In order to state it we need a definition. We denote by $\mathcal{N} \longrightarrow \operatorname{Spec} \mathcal{O}_v$ the Néron model of A and by $\mathcal{N}_v^\circ$ the connected component of its special fibre. We say that an abelian variety A/K has *semistable reduction of toric dimension one* at a place v of K, if there is an exact sequence of group schemes over the residue field $\kappa(v) = \mathcal{O}_v/m_v$

$$1 \longrightarrow T \longrightarrow \mathcal{N}_v^o \longrightarrow B \longrightarrow 0$$

where $T/\kappa(v)$ is a torus of dimension one and $B/\kappa(v)$ is an abelian variety.

EXAMPLE 2.6. Let $f \in \mathbb{Z}[x]$ be a monic square free polynomial of degree $\deg f = n \geq 5$. Denote by C_f the smooth projective curve with an affine part the hyperelliptic curve $y^2 = f(x)$. Let $A = \operatorname{Jac}(C_f)$ denote the Jacobian variety. It was proven by Zarhin [32] that if $\operatorname{Gal}(\operatorname{Spl}(f)/\mathbb{Q}) = S_n$, then $\operatorname{End} A = \mathbb{Z}$. Moreover the Jacobian variety A has semistable reduction of toric dimension one at a rational prime p, if the image $\bar{f}$ of f in $\mathbb{F}_p[x]$ admits a factorization $\bar{f} = f_1 f_2$ in $\mathbb{F}_p[x]$, with coprime polynomials $f_1,\ f_2 \in \mathbb{F}_p[x]$, such that $f_1 = (x-\alpha)^2$ with $\alpha \in \mathbb{F}_p$, and f_2 is a square-free polynomial of degree $n-2$. Note that Kowalski proved in [17] using sieves that most polynomials $f \in \mathbb{Z}[x]$ have a prime p with the above properties.

THEOREM 2.7 (Arias-de-Reyna, Gajda, Petersen [1]). *Let A/K be an abelian variety over a finitely generated field of arbitrary characteristic with* $\operatorname{End}(A) = \mathbb{Z}$ *and such that A has semistable reduction of toric dimension one. Then A has* big monodromy*, i.e., for almost all ℓ the image of σ_ℓ contains the group of $\mathbb{Z}_\ell$-points of the group $\underline{\mathrm{Sp}}_{2g}$.*

2.3. Application to Geyer–Jarden conjecture. We applied Theorem 2.7 to prove a new result towards the Geyer–Jarden conjecture of 1978. Denote by K_{sep} the separable closure of K. For a positive integer e and for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_e)$ in the group $G_K^e = G_K \times G_K \times \ldots \times G_K$, we denote by $K_{\mathrm{sep}}(\sigma)$ the subfield in K_{sep} fixed by $\sigma_1, \sigma_2, \ldots, \sigma_e$. There exists a substantial literature on arithmetic properties of the fields $K_{\mathrm{sep}}(\sigma)$. In particular, the Mordell–Weil groups $A(K_{\mathrm{sep}}(\sigma))$ have been already studied. We considered the torsion part of the groups $A(K_{\mathrm{sep}}(\sigma))$. In order to recall the Geyer–Jarden conjecture, we agree to say that a property $\mathcal{A}(\sigma)$ *holds for almost all* $\sigma \in G_K^e$, if $\mathcal{A}(\sigma)$ holds for all $\sigma \in G_K^e$, except for a set of measure zero with respect to the (unique) normalized Haar measure on the compact group G_K^e. In [16] Geyer and Jarden proposed the following conjecture on the torsion part of $A(K_{\mathrm{sep}}(\sigma))$.

CONJECTURE 2.8. Let K be a finitely generated field. Let A be an abelian variety defined over K.

(a) For almost all $\sigma \in G_K$ there are infinitely many prime numbers ℓ such that the group $A(K_{\mathrm{sep}}(\sigma))[\ell]$ of ℓ-division points is nonzero.
(b) Let $e \geq 2$. For almost all $\sigma \in G_K^e$ there are only finitely many prime numbers ℓ such that the group $A(K_{\mathrm{sep}}(\sigma))[\ell]$ of ℓ-division points is nonzero.

It is known due to the work of Jacobson and Jarden [20] that for all $e \geq 1$, almost all $\sigma \in G_K^e$ and all primes ℓ the group $A(K_{\mathrm{sep}}(\sigma))[\ell^\infty]$ is finite. This was formerly part (c) of the conjecture. Moreover Conjecture 2.8 is known for elliptic curves [16]. Part (b) is true provided $\mathrm{char}(K) = 0$ (see [20]). Zywina [33] proves part (a) in the special case where K is a number field, strengthening results of Geyer and Jarden. As for today, for an abelian variety A of dimension ≥ 2 defined over a finitely generated field of positive characteristic, parts (a) and (b) of Conjecture 2.8 are open and part (a) is open over a finitely generated transcendental extension of $\mathbb{Q}$. We prove the Geyer–Jarden conjecture for abelian varieties of Theorem 2.7.

THEOREM 2.9 (Arias-de-Reyna, Gajda, Petersen [2]). *Let K be a finitely generated field and A/K an abelian variety with* big monodromy. *Then Conjecture* 2.8 *of Geyer and Jarden is true for A/K.*

Surprisingly enough, the most difficult part in the proof of Theorem 2.9 is the case (a) of the Conjecture 2.8 for abelian varieties with big monodromy, when $\mathrm{char}(K) > 0$. The method of our proof relies in this case on the Borel–Cantelli Lemma of measure theory and on a delicate counting argument in the group $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$.

3. Main results

3.1. Statements. Our main results concern research initiated by Serre [31] and continued by us in [15], by Cadoret–Tamagawa [9], Böckle–Gajda–Petersen [7] and by the second author in his habilitation thesis [26]. In the sequel we call a family of homomorphisms $(\rho_\ell : G \to G_\ell)_{\ell \in \mathbb{L}'}$ of locally compact topological groups with index set $\mathbb{L}' \subset \mathbb{L}$ *almost independent* if there exists an open subgroup $H \subset G$ such that $\rho(H) = \prod_{\ell \in \mathbb{L}'} \rho_\ell(H)$ where $\rho : G \to \prod_{\ell \in \mathbb{L}'} G_\ell$ is the homomorphism induced by the ρ_ℓ. The main results are as follows.

THEOREM 3.1 (Serre, Gajda–Petersen, cf. [31], [15]). *Let K be a finitely generated field of characteristic zero. Let X/K be a separated algebraic scheme and $j \in \mathbb{N}$. For $\ell \in \mathbb{L}$ let $V_\ell = H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ and let*

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{V_\ell}(\mathbb{Q}_\ell)$$

be the corresponding Galois representation. Then the family $(\rho_\ell)_{\ell \in \mathbb{L}}$ is almost independent.

The following independence theorem concerns the case where the ground field is *a geometric function field*, i.e., a field K containing an algebraically closed subfield k such that K/k is finitely generated. It was obtained independently by Cadoret–Tamagawa [9] and Böckle–Gajda–Petersen [7]. Part b) carries a considerable amount of information about the images of the representations.

THEOREM 3.2 (Böckle–Gajda–Petersen, Tamagawa–Cadoret, cf. [7], [9]). *Let K be a geometric function field of characteristic $p \geq 0$ and $\mathbb{L}' = \mathbb{L} \setminus \{p\}$. Let X/K be a separated algebraic scheme. For $\ell \in \mathbb{L}'$ let ρ_ℓ be the representation of* $\mathrm{Gal}(\overline{K}/K)$ *on $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$. There exists a finite Galois extension K'/K and a constant $c \in \mathbb{N}$ with the following properties.*

(a) *The family* $(\rho_\ell)_{\ell\in\mathbb{L}'}$ *is almost independent.*
(b) *For every* $\ell \in \mathbb{L}'$ *the group* $H_\ell := \rho_\ell(\mathrm{Gal}(\overline{K}/K'))$ *is generated by its* ℓ*-Sylow subgroups and it has a normal series*

$$H_\ell \rhd N_\ell \rhd P_\ell \rhd \{e\}$$

with the following composition factors.

(i) *The group* H_ℓ/N_ℓ *is a finite product of finite simple groups of Lie type in characteristic* ℓ*.*

(ii) *The group* N_ℓ/P_ℓ *is abelian of order prime to* ℓ *and* $|N_\ell/P_\ell| \leq c$*.*

(iii) *The group* P_ℓ *is pro-*ℓ*.*

Illusie suggested some time ago to use the theory developed so far, combined with uniform constructability and semistability results recently proved by Orgogozo [25], in order to prove a relative version for families of sheaves over arithmetic schemes. Let $\mathcal{FTD}$ be the category of all separated reduced and irreducible Spec($\mathbb{Z}$)-schemes S which are of finite type and dominant over Spec($\mathbb{Z}$). Let $S \in \mathcal{FTD}$, let $K = R(S)$ be the field of rational functions on S and let $\bar{x} : \mathrm{Spec}(\overline{K}) \to S$ be the geometric generic point of S afforded by the choice of algebraic closure of $\overline{K}$. Let $\mathcal{F}_\ell$ be an étale sheaf of $\mathbb{F}_\ell$-vector spaces on S for every $\ell \in \mathbb{L}$. We call such a family $(\mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *almost independent*, if the family $(\mathcal{F}_{\ell,\bar{\S}})_{\ell\in\mathbb{L}}$ of stalks, viewed as a family of representations of $\mathrm{Gal}(\overline{K}/K)$, is almost independent in the above sense. We define a certain condition (Ram) for families $(\mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ as above which is roughly a "potential uniform constructibility" and "potential semistability" condition in a certain sense. The family $(\mathbb{F}_\ell)_{\ell\in\mathbb{L}}$ of constant sheaves on S does satisfy condition (Ram).

Theorem 3.3 (Petersen, cf. [26]). *Let* $S \in \mathcal{FTD}$*. For every* $\ell \in \mathbb{L}$ *let* $\mathcal{F}_\ell$ *be an étale sheaf of* $\mathbb{F}_\ell$*-vector spaces on* S*.*

(a) *If the family* $(\mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *satisfies condition* (Ram)*, then* $(\mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *is almost independent.*
(b) *If the family* $(\mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *satisfies condition* (Ram) *and* $f : S \to S'$ *is a morphism in* $\mathcal{FTD}$*, then for every* $j \in \mathbb{N}$ *the families* $(R^j f_* \mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *and* $(R^j f_! \mathcal{F}_\ell)_{\ell\in\mathbb{L}}$ *satisfy condition* (Ram)*.*

Remark. The main merit of Theorem 3.3 is to give more flexibility on the level of coefficients than Theorem 3.1. For example, if $f : S \to S'$ and $g : S' \to S''$ are two morphisms in $\mathcal{FTD}$, then for $i, j \in \mathbb{N}$ the family of E_2-terms of the Leray spectral sequence

$$(R^i g_* R^j f_* \mathbb{F}_\ell)_{\ell\in\mathbb{L}}$$

is almost independent by Theorem 3.3. It is an interesting question to prove a version of Theorem 3.3 for families of $\mathbb{Q}_\ell$-sheaves instead of families of $\mathbb{F}_\ell$-sheaves. The thesis [26] restricted attention to the technically simpler case of finite coefficients only. Furthermore it is desirable to prove a version that does not assume S to be dominant over $\mathbb{Z}$; for example S should well be allowed to be an $\mathbb{F}_p$-variety. The final result of this type should contain Theorems 3.1 and 3.2 as a special case.

Remark. The independence results mentioned so far have applications in field arithmetic. Some such applications have been worked out in [26], and it is possible that one can work out more. For example it seems promising to apply the new independence results on part (a) of the Geyer–Jarden conjecture about torsion of abelian varieties (cf. [16]). Part (a) of this conjecture has been solved by Zywina [33] for abelian varieties over number fields. It seems plausible that one can prove part (a) in the case of abelian varieties over finitely generated fields of characteristic zero combining the methods of Zywina [33] with Theorem 3.2. One can speculate that Theorem 3.2 might help to solve part (a) of the conjecture in positive characteristic.

3.2. Proof sketch of Theorem 3.2

Group theory. Let ℓ be a rational prime and let d be a positive integer. We define the set $\mathrm{Jor}(d)$ of *Jordanian groups* as the set of finite groups H that have a normal abelian subgroup N of index at most d. The subset of $\mathrm{Jor}(d)$ of those H for which N has order prime to ℓ is denoted by $\mathrm{Jor}_\ell(d)$. For a positive integer c define the set $\Sigma_\ell(c)$ of profinite groups M such that M has normal open subgroups $P \subset I \subset M$ such that P is a pro-ℓ group, I/P is a finite abelian group of order prime to ℓ with the index $[I : P] \leq c$, and M/I is a product of finite simple groups of Lie type in characteristic ℓ. As the starting point for the proof of Theorem 3.2 we use the following description of closed subquotients of $Gl_n(\overline{\mathbb{Q}_\ell})$, which we deduce from the main result of the paper [23] of Larsen and Pink.

Proposition 3.4. *For any positive integer n there exists a constant $J(n)$ such that for all rational primes ℓ and any closed subquotient G_ℓ of $Gl_n(\overline{\mathbb{Q}_\ell})$ there is a short exact sequence of profinite groups*

$$1 \longrightarrow M_\ell \longrightarrow G_\ell \longrightarrow H_\ell \longrightarrow 1$$

such that $M_\ell \in \Sigma_\ell(2^{n-1})$ and $H_\ell \in \mathrm{Jor}(J(n))$.

Bounding ramification. In the proof of Theorem 3.2 we need deep results on the ramification behavior of the representations ρ_ℓ. The following proposition summarizes the information we need.

Proposition 3.5. *Let k be a perfect field of characteristic $p \geq 0$, S/k a smooth k-variety, K the function field of S and X/K a separated algebraic scheme. For $\ell \in \mathbb{L} \setminus \{p\}$ let ρ_ℓ be the representation of $\mathrm{Gal}(K)$ on $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$. There exists a dense open subscheme U of S, a smooth proper k-variety $\overline{V}$, a dense open subscheme V of $\overline{V}$ and an alteration $V \to U$ such that, if we denote by E the function field of $\overline{V}$, the following holds.*

(a) *For all $\ell \in \mathbb{L} \setminus \{p\}$ the representation ρ_ℓ factors through $\pi_1(U)$ (and consequently $\rho_\ell \,|\, \mathrm{Gal}(E)$ factors through $\pi_1(V)$).*
(b) *For every $\ell \in \mathbb{L} \setminus \{p\}$ and every discrete valuation v of K which is localized at a codimension 1 point of $\overline{V}$ the image $\rho_\ell(I(v))$ of the inertia group $I(v)$ of v is a pro-ℓ group.*

This proposition is established in [7] making use of the base change theorems, de Jong's alteration technique, the different notions of tameness from Kerz–Schmidt [21] and two results of Deligne: from [11] and from [12].

Remark. If in the situation of Proposition 3.5 $\rho_\ell(\mathrm{Gal}(K))$ is of order prime to ℓ for every $\ell \in \mathbb{L} \setminus \{p\}$, then $\rho_\ell \,|\, \mathrm{Gal}(E)$ factors through $\pi_1(\overline{V})$ because of the purity of the branch locus.

Finiteness results for fundamental groups. Let us call a (possibly infinite) Galois extension Ω/K *Jordanian* if there exists an integer d and a family $(K_i)_{i\in I}$ of Galois extensions of K inside Ω such that $\mathrm{Gal}(K_i/K) \in \mathrm{Jor}(d)$ and $\Omega = \prod_{i\in I} K_i$. We shall make use of the following finiteness theorem for Jordanian extensions of function fields with restricted ramification.

Proposition 3.6. *Let k be a prime field. Let S/k be a smooth variety. If k is finite, assume that S/k is proper. Let K/k be the function field of S. Let Ω/K be a Jordanian extension which is unramified along S. Then $\overline{k}\Omega/\overline{k}K$ is a* finite *extension.*

The proof of Proposition 3.6 makes use of the fact that $\pi_1(S_{\overline{k}})$ is a finitely generated profinite group and of central results of Katz and Lang in geometric class field theory.

Putting things together. We shall now give a tentative sketch of proof for Theorem 3.2 in the special case where K is a finitely generated extension of $\overline{\mathbb{F}}_p$. This special case is particularly important. Theorem 3.2 in the case $p > 0$ is derived from this special case in [7], and the proof in the case $p = 0$ follows similar lines being in fact easier. In the special case under consideration there exists a finitely generated extension $K_0/\mathbb{F}_p$ and a separated algebraic K_0-scheme X_0 such that $K = \overline{\mathbb{F}}_p K_0$ and $X = X_0 \otimes \mathrm{Spec}(K)$. Hence ρ_ℓ extends to a representation of $\mathrm{Gal}(K_0)$ on $H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ and we denote this extension again by ρ_ℓ.

By Proposition 3.5 there exists, after replacing K_0 by a finite extension, a finite extension $k/\mathbb{F}_p$ inside K_0, a smooth proper geometrically connected k-variety $\overline{V}$ with function field K_0 and a dense open subscheme V in $\overline{V}$ such that ρ_ℓ factors through $\pi_1(V)$ and such that for every $\ell \in \mathbb{L} \setminus \{p\}$ and every discrete valuation v of K which is localized at a codimension 1 point of $\overline{V}$ the group $\rho_\ell(I(v))$ is a pro-ℓ group.

Let $G_\ell = \rho_\ell(\mathrm{Gal}(K_0))$ and let K_ℓ be the fixed field of G_ℓ inside $\overline{K}$. By Proposition 3.4 there exist constants $J, c \in \mathbb{N}$ (not depending on ℓ) such that for every $\ell \in \mathbb{L} \setminus \{p\}$ there exists a normal subgroup M_ℓ in G_ℓ such that $M_\ell \in \Sigma_\ell(c)$ and $G_\ell/M_\ell \in \mathrm{Jor}(J)$. For every $\ell > J$ the extension $K_\ell^{M_\ell}/K_0$ is unramified along $\overline{V}$. Let $\Omega = \prod_{\ell\in\mathbb{L}\setminus\{p\}} K_\ell^{M_\ell}$. It follows from Proposition 3.6 that $K' := \overline{k}\Omega$ is a *finite* extension of K. By construction we see that

$$\rho_\ell(\mathrm{Gal}(K')) \lhd M_\ell \in \Sigma_\ell(c) \quad \text{for all } \ell \in \mathbb{L}'.$$

Using Goursat's Lemma, one can prove that the class $\Sigma_\ell(c)$ is closed under taking normal subgroups, hence $\rho_\ell(\mathrm{Gal}(K')) \in \Sigma_\ell(c)$, for all $\ell \in \mathbb{L}'$.

After replacing K' by a finite extension we can assume in addition that $\rho_\ell(\mathrm{Gal}(K'))$ is pro-ℓ for all $\ell \notin \{2, 3\}$. An additional argument, based on the fact that $\pi_1(S_{\overline{k}})$ is finitely generated and some group theory proves, that we can also assume that $\rho_\ell(\mathrm{Gal}(K'))$ is generated by its ℓ-Sylow subgroups. This proves part (b) of Theorem 3.2 in the special case under consideration. Part (a) of this theorem follows then from its part (b) with the help of the following classical theorem of E. Artin from group theory. Let Lie_ℓ denote the class of all finite simple groups of Lie type in characteristic ℓ. For example the list of orders of groups in Lie_5 begins with

$$60,\ 7800,\ 126000,\ 976500,\ \ldots$$

and the list for Lie_7 begins with

$$168,\ 58800,\ 1876896,\ 56633616, \ldots$$

cf. [31], Section 6.

THEOREM 3.7 (E. Artin [3], [22]). *If ℓ_1 and ℓ_2 are two distinct rational primes bigger than* 3*, then the classes Lie_{ℓ_1} and Lie_{ℓ_2} are disjoint.*

3.3. Questions: Openness in families. Another line of investigation concerns the variation of the images of the corresponding representations in families of varieties. Let k be a Hilbertian field and S/k a (base) variety with function field $K = k(X)$. Let $\mathcal{X} \to S$ be a family of algebraic schemes. For $s \in S$ we denote by X_s the fibre of $\mathcal{X}$ over S. Fix $j \in \mathbb{N}$. Let $\rho_{\ell,s}$ be the representation of $G_{k(s)}$ on $H^j(X_{s,\overline{k(s)}}, \mathbb{Q}_\ell)$ where $G_{k(s)}$ is the Galois group of the residue field $k(s)$. Let ξ be the generic point of S, $X = X_\xi$ and $\rho_\ell = \rho_{\ell,\xi}$. It is natural to compare the cohomology of the fibres X_s to the cohomology of the generic fibre X; the generic base change theorem (Katz–Laumon, Illusie) tells us that there exists an open subset $U \subset S$ such that

$$H^j(X_{s,\overline{k(s)}}, \mathbb{Q}_\ell) \cong H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$$

for every $\ell \in \mathbb{L}'$ and every $s \in U$. Having this information at hand one may look for equivariant versions and compare the image of $\rho_{\ell,s}$ with the image of ρ_ℓ. We can view $\rho_{\ell,s}(G_{k(s)})$ as a subgroup of $\rho_\ell(G_K)$. We define the “good loci” for the specialization to be

$$S^{\mathrm{good}}(\mathcal{X}, \ell) = \{s \in S : \rho_{\ell,s}(G_{k(s)}) \text{ has finite index in } \rho_\ell(G_K)\}$$
$$S^{\mathrm{good}}(\mathcal{X}) = \{s \in S : \rho_{\ell,s}(G_{k(s)}) \text{ has finite index in } \rho_\ell(G_K) \text{ for all } \ell\}.$$

A classical Frattini argument sketched by Serre in [27] (in a special case, but the argument works in general) shows that $S^{\mathrm{good}}(\mathcal{X}, \ell)$ is a Hilbertian set. The motivic philosophy predicts that $S^{\mathrm{good}}(\mathcal{X}, \ell)$ should be independent of ℓ. As a consequence one would expect, that $S^{\mathrm{good}}(\mathcal{X})$ is Hilbertian as well. One may even look at the *good locus* of specialization for the adelic representations

$$S^{\mathrm{ad}}(\mathcal{X}) = \Big\{s \in S : \prod_\ell \rho_{\ell,s}(G_{k(s)}) \text{ is open in } \prod_\ell \rho_\ell(G_K)\Big\}.$$

The following catalogue of question about *variational adelic openness* seems very natural.

QUESTION 3.8. Assume that K is an infinite finitely generated field. Let $S^{(d)}$ be the set of all closed points of S of degree $\le d$.

(a) Can we prove that the good locus $S^{\mathrm{good}}(\mathcal{X}, \ell)$ is *big* in the sense that $S^{(d)} \setminus S^{\mathrm{good}}(\mathcal{X}, \ell)$ is finite for all d?
(b) Is it true that $S^{\mathrm{good}}(\mathcal{X}, \ell)$ does not depend on $\ell \in \mathbb{L}'$? This would in particular imply that $S^{(d)} \setminus S^{\mathrm{good}}(\mathcal{X})$ is finite for all d.
(c) What can we say about $S^{\mathrm{ad}}(\mathcal{X})$?

Remark. The important work of Cadoret and Tamagawa [10] answers question (a) affirmatively if $\mathrm{char}(k) = 0$ and $\dim(S) = 1$. Question (b), which may seem surprising at a first glance, has an affirmative answer if $\mathrm{char}(k) = 0$, $\dim(S) = 1$ and $\mathcal{X}$ is an abelian scheme due to work of Hui (cf. [18]). Recent work of Cadoret (cf. [8]) settles question (c) if $\mathrm{char}(k) = 0$, $\dim(S) = 1$ and $\mathcal{X}$ is an abelian scheme. One would like to establish analogous results in the special case where $\mathrm{char}(k) > 0$, and it seems plausible to us that this might be possible. The main problem in this area is to find a good replacement for the Mumford–Tate group in case when the base field is of positive characteristic. This is an interesting path of research which shall be followed in future.

Acknowledgments. W.G. thanks the Interdisciplinary Center for Scientific Computing (IWR) at Heidelberg University for hospitality during a research visit in January and February 2015 when part of this work was written. He was supported by the Alexander von Humboldt Foundation and by research grants UMO-2012/07/B/ST1/03541 and UMO-2014/15/B/ST1/00128 of the National Centre of Sciences of Poland. S.P. thanks the Mathematics Department at Adam Mickiewicz University for hospitality and support during several research visits. The authors thank the organizers of the meeting *ALANT3* in Będlewo for the invitation. Finally we want to thank an anonymous referee for careful reading of the manuscript and for helpful comments.

References

[1] S. Arias-de-Reyna, W. Gajda, S. Petersen, *Abelian varieties over finitely generated fields and the conjecture of Geyer and Jarden on torsion*, Math. Nachr. 286 (2013), 1269–1286.

[2] S. Arias-de-Reyna, W. Gajda, S. Petersen, *Big monodromy theorem for abelian varieties over finitely generated fields*, J. Pure Appl. Algebra 217 (2013), 218–229.

[3] E. Artin, *The orders of the classical simple groups*, Comm. Pure Appl. Math. 8 (1955), 455–472.

[4] M. Artin, A. Grothendieck, J.-L. Verdier (eds.), *Séminaire de géométrie algébrique 4 – Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math. 269, 270, 305, Springer, Berlin 1972, 1973.

[5] G. Banaszak, W. Gajda, P. Krasoń, *On the image of l-adic Galois representations for abelian varieties of type I and II*, Doc. Math. (2006), Extra vol. Coates, 35–75.

[6] G. Banaszak, W. Gajda, P. Krasoń, *On the image of Galois ℓ-adic representations for abelian varieties of type III*, Tohoku Math. J. (2) 62 (2010), 163–189.

[7] G. Böckle, W. Gajda, S. Petersen, *Independence of ℓ-adic representations of geometric Galois groups*, J. Reine Angew. Math., to appear; available at http://gajda.faculty.wmi.amu.edu.pl/index.php.

[8] A. Cadoret, *An open adelic image theorem for abelian schemes*, Int. Math. Res. Not. IMRN 2015 (2015), 10208–10242.

[9] A. Cadoret, A. Tamagawa, *A refined Jordan-Nori theorem and applications to representations of etale fundamental groups*, preprint.

[10] A. Cadoret, A. Tamagawa, *A uniform open image theorem for ℓ-adic representations II*, Duke Math. J. 162 (2013), 2301–2344.

[11] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, in: Modular Functions of one Variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math. 349, Springer, Berlin 1973, 501–597.

[12] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.

[13] P. Deligne, *Cohomologie étale*, Lecture Notes Math. 569, Springer, Berlin 1977.

[14] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73 (1983), 349–366.

[15] W. Gajda, S. Petersen, *Independence of ℓ-adic Galois representations over function fields*, Compos. Math. 149 (2013), 1091–1107.

[16] W. Geyer, M. Jarden, *Torsion points of elliptic curves over large algebraic extensions of finitely generated fields*, Israel J. Math. 31 (1978), 257–297.

[17] C. Hall, *An open-image theorem for a general class of abelian varieties. With an appendix by Emmanuel Kowalski*, Bull. Lond. Math. Soc. 43 (2011), 703–711.

[18] C.-Y. Hui, *Specialization of monodromy group and ℓ-independence*, C. R. Math. Acad. Sci. Paris 350 (2012), 5–7.

[19] L. Illusie, *Constructibilité générique et uniformité en ℓ*, preprint.

[20] M. Jacobson, M. Jarden, *Finiteness theorems for torsion of abelian varieties over large algebraic fields*, Acta Arith. 98 (2001), 15–31.

[21] M. Kerz, A. Schmidt, *On different notions of tameness in arithmetic geometry*, Math. Ann. 346 (2010), 641–668.

[22] W. Kimmerle, R. Lyons, R. Sandling, D. Teague, *Composition factors from the group ring and Artin's theorem on orders of simple groups*, Proc. London Math. Soc. (3) 60 (1990), 89–122.

[23] M. Larsen, R. Pink, *Finite subgroups of algebraic groups*, J. Amer. Math. Soc. 24 (2011), 1105–1158.

[24] J. Milne, *Lectures on Etale Cohomology*, available at www.jmilne.org, 2012.

[25] F. Orgogozo, *Sur les propriétiés d'uniformité des images directes en cohomologie étale*, preprint.

[26] S. Petersen, *Independence of ℓ-adic Galois representations attached to étale cohomology*, Habilitation thesis, 2014.

[27] J.-P. Serre, *Lettre à Ken Ribet du 7/3/1986*, in: Œuvres. Collected Papers IV, Springer, Berlin 2000.

[28] J.-P. Serre, *Algèbre et géométrie. Résumé des cours de 1984–1985*, in: Annuaire du Collège de France, 1985.

[29] J.-P. Serre, *Algèbre et géométrie. Résumé des cours de 1985–1986*, in: Œuvres. Collected Papers IV, Springer, Berlin 2000.

[30] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*, in: Motives (Seattle, 1991), Proc. Symp. Pure Math. 55, Part 1, Amer. Math. Soc., Providence, RI 1994, 377–400.

[31] J.-P. Serre, *Un critère d'indépendance pour une famille de représentations ℓ-adiques*, Comm. Math. Helv. 88 (2013), 541–554.

[32] Y. Zarhin, *Families of absolutely simple hyperelliptic Jacobians*, Proc. Lond. Math. Soc. (3) 100 (2010), 24–54.

[33] D. Zywina, *Abelian varieties over large algebraic fields with infinite torsion*, Israel Math. J. 211 (2016), 493–508.