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LEVELS OF RINGS — A SURVEY

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Abstract. Let R be a ring with $1 \neq 0$. The level $s(R)$ of R is the least integer n such that -1 is a sum of n squares in R provided such an integer exists, otherwise one defines the level to be infinite. In this survey, we give an overview on the history and the major results concerning the level of rings and some related questions on sums of squares in rings with finite level. The main focus will be on levels of fields, of simple noncommutative rings, in particular division rings, and of arbitrary commutative rings. We also address several variations of the notion of level that have been studied in the literature.

1. Introduction. All rings we consider henceforth are assumed to contain a multiplicative identity $1 \neq 0$, and unless stated otherwise, they will be associative. The group of multiplicative units of a ring R will be denoted by R^*.

The study of sums of squares and ring invariants pertaining to sums of squares has a very long history. In this survey, we try to give a reasonably comprehensive account on rings in which -1 is a sum of squares, more precisely on the question, how many squares are needed in that case to represent -1 as a sum of squares, and what this information tells us about those elements that are sums of squares in such a ring. We will also say a few words about variants of these questions in which sums of squares are replaced by sums of products of squares (for noncommutative rings), sums of so-called hermitian squares (for rings with an involution), or sums of higher (even) powers.

We first set up the notation and definitions we will use in the sequel.

- $\sum_n R^2 = \{x_1^2 + \ldots + x_n^2 \mid x_i \in R\}$, all elements in R that can be written as sums of n squares, $n \in \mathbb{N}$;

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- $\sum R^2 = \bigcup_{n=1}^{\infty} \sum_n R^2$, all elements in R that can be written as sums of squares;
- we define the *length* of an element $x \in R$ as $\ell_R(x) = \infty$ if $x \notin \sum R^2$, otherwise

$$\ell_R(x) = \min\{n \,|\, x \in \textstyle\sum_n R^2\};$$

(we write $\ell(x)$ for short if no confusion regarding R can arise).

- the *level* s of R is defined to be

$$s(R) = \ell_R(-1);$$

- the *Pythagoras number* p of R is defined to be

$$p(R) = \sup\{\ell_R(x) \,|\, x \in \textstyle\sum R^2\}.$$

If $p(R)$ is finite, one has $p(R) = \min\{n \,|\, \sum R^2 = \sum_n R^2\}$.

So in this survey and with these conventions, we are interested mainly in rings with $s(R) < \infty$. By slight abuse of terminology, we will call such rings *nonreal*. Of course, the notion of Pythagoras number of rings makes perfect sense also if $s = \infty$, and there is a vast amount of literature on that topic too, but we decided not to include that case and instead to focus primarily on nonreal rings to keep the exposition within a manageable length. Thus, results on Pythagoras numbers are only mentioned in passing and only in situations where the levels are finite. Readers who want to learn more about Pythagoras numbers and their higher power generalizations are referred to [Beck82], [Beck84], [CDLR82], [Sch92], [Le10], [Pfi95].

Earlier surveys on the levels of rings can be found in [Lew87b], [Lew99], and although we will recall most results mentioned there, it seems appropriate to provide an updated survey because of the many new results obtained since their publications. In view of the many varied results on levels of rings, it is not feasible to cite all that is known. In many cases, we will restrict ourselves to providing a few samples or referring to the original literature.

2. Fields. Of course, the level of a field of characteristic 2 is always 1. Therefore, from now on, we assume all fields to be of characteristic not 2.

In 1927, Artin and Schreier [AS27] published a seminal paper in which they showed that a field F has an ordering if and only if -1 is not a sum of squares, i.e. $s(F) = \infty$. Here, an ordering can be understood as a multiplicative subgroup $P \leq F^*$ of index 2 that is closed under addition. Fields with $s(F) = \infty$ are called *formally real* or *real* for short. It then becomes a rather natural question to ask which levels are possible for nonreal fields.

LEMMA 2.1. *Let F be a field of characteristic not 2. Then $s(F) < \infty$ iff $F = \sum F^2$. In this case, $p(F) \in \{s(F), s(F)+1\}$.*

Proof. If $s(F) < \infty$, by definition we have $p(F) \geq s(F)$. Consider the diagonal quadratic form $\sigma_m = \sum_{i=1}^m x_i^2$ for which we use the standard notation for diagonalized forms (here with 1's in the diagonal), $\langle \underbrace{1, \ldots, 1}_{m} \rangle$. Then $s(F) = m$ is equivalent to σ_m being anisotropic and σ_{m+1} being isotropic. But σ_{m+1} being isotropic means that it splits off orthogonally a hyperbolic plane $\langle 1, -1 \rangle$. Now such a hyperbolic plane (and thus σ_{m+1}) represents all elements $x \in F^*$: $x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2$. In particular, $F = \sum F^2$ and $p(F) \leq m+1$. The converse is trivial. ∎

In the problems and answers section of the *Jahresberichte der Deutschen Mathematiker Vereinigung* of 1932, van der Waerden [vdW32] poses the following problem no. 144:

Wenn in einem Körper die Zahl -1 *Summe von* 3 *Quadraten ist, so auch von* 2 *Quadraten*; *wenn von* 5, 6 *oder* 7*, so auch von* 4; *wenn von* 15 *oder weniger, so auch von* 8.

Loosely translated: In a field, if -1 is a sum of 3 squares, then also of 2 squares; if of 5, 6 or 7, then also of 4; if of 15 or fewer, then also of 8. In our terminology, the question was to show that $s(F) \notin \{3, 5, 6, 7, 9, 10, \ldots, 15\}$. This suggested that s, if finite, might necessarily be a power of 2. The first result in that direction was due to H. Kneser [Kn34] who showed that for a field F, if $s(F)$ is finite then it is of the form 2^n, $n \leq 3$, or a multiple of 16 that is not of type $g2^{8h} + 16h$. In the proof, Kneser uses results by Radon on composition formulas for sums of squares. One should remark that it was Kneser in that article who coined the notion of *Stufe* for what in English is called the level and which later on gave also rise to the use of the letter s. For completeness' sake, let us note that in 1953, Kaplansky [Kap53] published a short paper in which he introduces three field invariants: A, the cardinality of the square class group of the field F; B, which is our level s; and C, what is now called the u-invariant, i.e. the supremum of the dimensions of anisotropic quadratic forms over F. Kaplansky derives some relations among these invariants that have inspired many mathematicians to extend, improve and generalize his results. In particular, Kaplansky showed that the level (his B) of a field, if finite, is of shape 1, 2, 4 or a multiple of 8, which of course is not an improvement on Kneser's results. It is worth mentioning that Kaplansky conjectured in that note that the level, if finite, is always a 2-power.

The full solution to the level problem was given in a celebrated paper published in 1965 by Pfister [Pfi65]:

THEOREM 2.2 (Pfister).

(i) *Let F be a field. Then there exists an $n \geq 0$ with $s(F) = 2^n$.*
(ii) *To any nonnegative integer n, there exists a field F_n with $s(F_n) = 2^n$.*

Nowadays, one can give a quick proof of these results with the theory of Pfister forms and the generic methods for quadratic forms that also go back to Pfister but which he mainly developed only after his original proof of the level problem. Since we will refer to it later, let us at this point give Pfister's example of a field of level 2^n. Let m be any integer with $2^n \leq m < 2^{n+1}$. Consider any real field K, for example $K = \mathbb{R}$. Then the polynomial $1 + X_1^2 + \ldots + X_m^2$ will be irreducible in $K[X_1, \ldots, X_m]$, so we have an integral domain

$$A_m(K) = \frac{K[X_1, \ldots, X_m]}{(1 + X_1^2 + \ldots + X_m^2)}\,.$$

Now take L to be the quotient field of $A_m(K)$. By construction, $s(L) \leq m$, and invoking part (i) of the theorem, we must have $s(L) \leq 2^n$. Indeed, one can show that one cannot have $s(L) < 2^n$, hence $F_n := L$ is a field with the desired property $s(F_n) = 2^n$.

In view of this theorem and Lemma 2.1, let us remark that to each n, there exist fields F'_n and F''_n with $2^n = s(F'_n) = p(F'_n) = s(F''_n) = p(F''_n) - 1$. For example, one can simply take $F''_n = F_n(X)$, the rational function field in one variable over the above F_n, in which case we will have $\ell_{F_n(X)}(X) = 2^n + 1$. The construction of an example F'_n is not quite so straightforward, but it can be gotten for example as a certain algebraic extension of the above F_n, see, e.g., [Pfi95, Ch. 7, Prop. 1.5].

Although our focus is on nonreal fields, let us remark that the situation for the Pythagoras number of real fields is quite more involved. As an example, let us just mention that to each positive integer n there exists a field F with $p(F) = n$, see Hoffmann [H99].

Let us now turn to examples concerning local and global fields.

Facts and Examples 2.3.

(i) Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements for an odd prime p. Then

$$s(\mathbb{F}_q) = \begin{cases} 1 & \text{if } q \equiv 1 \bmod 4 \\ 2 & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

This is an easy exercise about finite fields.

(ii) Let K be a local field with residue field κ (not necessarily finite). If $\mathrm{char}(\kappa) \neq 2$, then $s(K) = s(\kappa)$. This follows from (i) and an application of Hensel's lemma.

(iii) Let K be a finite extension of $\mathbb{Q}_2$. Then

$$s(K) = \begin{cases} 1 & \text{if } \sqrt{-1} \in K \\ 2 & \text{if } [K : \mathbb{Q}_2] \text{ is even and } \sqrt{-1} \notin K \\ 4 & \text{if } [K : \mathbb{Q}_2] \text{ is odd.} \end{cases}$$

See, e.g., [L05, Ch. XI, Ex. 2.4(7)].

(iv) Let K be a global number field. Let $K_{\mathfrak{p}}$ denote the completion at a place $\mathfrak{p}$ of K. Then

$$s(K) = \max\{s(K_{\mathfrak{p}}) \,|\, \mathfrak{p} \text{ place of } K\} \in \{1, 2, 4, \infty\}.$$

This is an easy consequence of the Hasse–Minkowski theorem. But the first published proof (more generally about elements that can be represented as sums of squares in number fields) is due to Siegel [Si21] preceding the Hasse–Minkowski theorem for arbitrary number fields. In his paper, Siegel mentions Hilbert who had announced earlier (without proof) that any totally positive element in a number field K can be written as a sum of four squares, see [Hi03, § 38]. Siegel's results imply this statement. Before that, Landau [La19] had proven Hilbert's claim for quadratic number fields.

(v) Let K be a nonreal global number field. Suppose the ramification in K of the place (2) in $\mathbb{Q}$ is given by

$$(2) = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_n^{e_n}$$

with respective completions $K_{\mathfrak{p}_i}$, residue class fields $\kappa_{\mathfrak{p}_i}$, and residue class degrees

$f_i := [\kappa_{\mathfrak{p}_i} : \mathbb{F}_2]$, so that $[K_{\mathfrak{p}_i} : \mathbb{Q}_2] = e_i f_i$. Then

$$s(K) = \begin{cases} 1 & \text{if } \sqrt{-1} \in K \\ 2 & \text{if all products } e_i f_i \text{ are even and } \sqrt{-1} \notin K \\ 4 & \text{if at least one of the products } e_i f_i \text{ is odd.} \end{cases}$$

This (and also (iii)) was shown by various authors more or less independently at about the same time: [FGS71], [Bar72], [Co72], [Fu73]. See also [L05, Ch. XI, Prop. 2.11].

(vi) Let $d > 0$ be a square free integer. Then

$$s(\mathbb{Q}(\sqrt{-d})) = \begin{cases} 1 & \text{if } d = 1 \\ 2 & \text{if } d > 1 \text{ and } d \not\equiv 7 \bmod 8 \\ 4 & \text{if } d \equiv 7 \bmod 8. \end{cases}$$

This follows from (v) since the ramification theory of quadratic number fields is well understood. It can also be shown directly by elementary arguments. Various proofs can be found in the literature, for example in [Mo70], [FGS71], [Bar72], [Ra75], [Sm77], [Sm86]. See also [L05, Ch. XI, Th. 2.11, Rem. 2.10].

(vii) Let $m \geq 3$ be an integer and consider the m-th cyclotomic field $K_m := \mathbb{Q}(e^{2\pi i/m})$. Clearly, for odd m, we have $K_m = K_{2m}$, and $\sqrt{-1} \in K_m$, i.e. $s(K_m) = 1$, iff 4 divides m. For odd $m \geq 3$, let $o(2)$ be the order of the element 2 in $(\mathbb{Z}/m\mathbb{Z})^*$. Then

$$s(K_m) = \begin{cases} 1 & \text{if } 4 \,|\, m \\ 2 & \text{if } m \text{ is odd and } o(2) \text{ is even} \\ 4 & \text{if } m \text{ is odd and } o(2) \text{ is odd.} \end{cases}$$

Here again, the ramification theory is known from number theory and one can apply (v). Partial results towards (vii) were obtained in [Ch69a], [Ch69b], [CC70], and in [Mo70] (also regarding (iii)), and proofs of the full result in [FGS71], [Bar72], [Mo73].

Let us remark that the fact that $s(K) \leq 4$ for number fields K and the precise determination of the value for specific types of fields use in general quite a bit of nontrivial algebraic number theory. More elementary arguments are known to work for imaginary quadratic number fields or certain cyclotomic fields (see the references mentioned previously), and for quartic number fields as well, see [PAR91].

There are also estimates on s for certain types of fields that are of interest in arithmetic geometry such as nonreal and nonalgebraic finitely generated extensions of $\mathbb{Q}$ or $\mathbb{R}$, but there are still many open problems. Here are some samples:

Facts and Examples 2.4.

(i) Let K be a real closed field (e.g. $K = \mathbb{R}$), and let L/K be a finitely generated nonreal extension of transcendence degree $n \geq 1$. Then $s(K) \leq 2^n$. This was shown by Pfister [Pfi67] who uses this to show that if L/K is any finitely generated extension (real or nonreal) of transcendence degree n, then $p(L) \leq 2^n$.

(ii) If X is a smooth surface over $\mathbb{R}$ with no $\mathbb{R}$-rational points and if $L = \mathbb{R}(X)$ is its function field, then L is nonreal and by (i), $s(L) \leq 4$.

The value 4 can occur. For example, let $f = 1 + x^2y^4 + x^4y^2 - 3x^2y^2$ be the so-called Motzkin polynomial, then for $L = \mathbb{R}(x, y)[z]/(z^2 + f)$ one has $s(L) = 4$. However, if X is rational, i.e. $X_{\mathbb{C}}$ is birational to $\mathbb{P}^2_{\mathbb{C}}$, then $s(\mathbb{R}(X)) = 2$, see [PS91].

(iii) Let X be a geometrically integral variety over a number field K and suppose that $K(X)$ is nonreal. If the transcendence degree of $K(X)/K$ is $n = 1, 2$, then $s(K(X)) \leq 2^{n+1}$. In both situations, one can realize both $s = 2$ and $s = 4$. However, it seems to be open if for $n = 2$ one can realize $s = 8$. For $n \geq 3$, the question what values s can occur seems to be wide open. See [Kat86], [CTJ91], [J92], [JS02].

Problem 2.5. For $n \geq 3$, do there exist finitely generated nonreal extensions K of $\mathbb{R}$ of transcendence degree n such that $s(K) = 2^n$?

For fields F with a finite number of square classes, one can get relations between the values of s and of $q = |F^*/F^{*2}|$. Here is how these estimates have improved over time.

- Pfister [Pfi66] showed that if $s = 2^k$, then $q \geq 2^{k(k+1)/2}$.
- Using combinatorial arguments, Chang [Cha74] found the following improvements: if $s = 4$ then $q \geq 8$ (note that $\mathbb{Q}_2$ is a field with $s = 4$ and $q = 8$); if $s = 8$ then $q \geq 2^7$; if $s \geq 16$ then $q \geq 2^{s+4}s^{-2}$.
- Djoković [Dj73] improved upon the bound for $s \geq 16$ by showing that then $q \geq 2^{s+1}s^{-1}$.
- Improvements for $s = 8, 16$ were found by Becher [Bech01]: $s = 8$ implies $q \geq 2^9$; $s = 16$ implies $q \geq 2^{15}$.

At this point in time, it is not known if there are any nonreal fields with finite q such that $s \geq 8$, which leads immediately to the following

Problem 2.6. Do there exist nonreal fields with finite q such that $s \geq 8$?

Becher's result shows that if such a field existed, it would have $q \geq 2^9$. Now the *Level Conjecture* claims that there are indeed no such fields. The Level Conjecture itself is implied by one of the great conjectures in the algebraic theory of quadratic forms, namely the *Elementary Type Conjecture* which, if true, would provide a deep insight into the structure theory of Witt rings of fields with finite q. For more on the Elementary Type Conjecture we refer to Marshall's survey [Mar04].

3. Simple rings and division rings. When working with noncommutative rings, the study of sums of squares seems less natural. Nevertheless, the level is still an interesting invariant in the sense that it is a test how well we understand the arithmetic in a ring. But in some sense, as we will see later, sums of products of squares are a more natural object to study. Thus, for a ring R, we introduce a new invariant, the *product level* $s_\pi(R)$, as follows. Let $\prod R^2 = \{x_1^2x_2^2 \ldots x_n^2 \mid n \in \mathbb{N},\ x_i \in R\}$ be the set of all products of squares in R. We put $s_\pi(R) = \infty$ if -1 is not a sum of elements in $\prod R^2$. Otherwise, we put

$$s_\pi(R) = \min\big\{n \mid \exists\, x_1, \ldots, x_n \in \prod R^2 :\ x_1 + \ldots + x_n = -1\big\}.$$

Clearly, we have $s(R) \geq s_\pi(R)$ with equality if R is commutative.

Of course, it makes sense to narrow down our study to certain types of rings that are of particular interest in algebra.

3.1. Simple rings. There are (at least) two natural generalizations of the notion of field to the noncommutative setting. A simple ring R is a ring that has no nontrivial two-sided ideals. A division ring (or skew-field) D is a ring in which the nonzero elements form a multiplicative group, i.e. $D^* = D \setminus \{0\}$. In the commutative case, both types of rings are exactly the fields. Generally, division rings are obviously simple but not conversely.

The center $K = Z(R)$ of a simple ring R is a field and thus R becomes a K-algebra. The simple rings R with center (isomorphic to the field) K and with $\dim_K R < \infty$ are called central simple K-algebras. The Artin–Wedderburn theory tells us that these are exactly the rings R that are isomorphic to matrix rings $M_n(D)$ for some central division K-algebra D, in which case D (up to K-algebra isomorphism) and n are uniquely determined by the isomorphism type of R. Furthermore, $\dim_K D$ is always a square, say, d^2. Then $\dim_K R = \dim_K M_n(D) = n^2d^2$, and one defines the *degree* $\deg R = nd$, and the *index* $\operatorname{ind} R = \deg D = d$. We refer to [S85, Ch. 8] for any facts regarding central simple algebras which we use without further reference.

The "simplest" simple rings are thus matrix rings $M_n(K)$ for some field K and some positive integer n. Since we are interested in the level, we may assume $\operatorname{char}(K) \neq 2$. Note that we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which immediately implies that for any K, we have $s(M_n(K)) = 1$ for n even, and $s(M_n(K)) \leq 3$ for n odd. We can be more precise. Richman [Ri85] proved the following nice result.

THEOREM 3.1. *Let $A \in M_n(K)$ with $n \geq 2$. Then $\ell_{M_n(K)}(A) \leq 3$. Furthermore, $\ell_{M_n(K)}(A) = 3$ if and only if all three of the following conditions are satisfied:* (i) *n is odd*; (ii) *$A = \lambda I_n$ with $\lambda \in K$ and I_n the $n \times n$ identity matrix*; (iii) $\ell_K(\lambda) \geq 3$.

COROLLARY 3.2. *Let $m \geq 1$. Then*

$$p(M_{2m}(K)) = 2, \quad p(M_{2m+1}(K)) = \begin{cases} 2 & \text{if } K \text{ is nonreal and } p(K) \leq 2 \\ 3 & \text{otherwise} \end{cases}$$

$$s(M_{2m}(K)) = 1, \quad s(M_{2m+1}(K)) = \min\{s(K), 3\}$$

$$s_\pi(M_{2m}(K)) = 1, \quad s_\pi(M_{2m+1}(K)) = \min\{s(K), 2\}.$$

Proof. Note that $-I_{2m+1}$ is not a product of squares if $s(K) \geq 2$ because of determinant considerations. By using the matrix equations above, the statement about s then follows immediately from Richman's result, and so does the statement about p since, for $n \geq 2$, there are always matrices in $M_n(K)$ that are not squares (exercise!). Clearly, we then also have $s_\pi(M_{2m}(K)) = 1$ and $s_\pi(M_{2m+1}(K)) \geq \min\{s(K), 2\}$. Thus, it suffices to show that $-I_{2m+1}$ is always a sum of two products of squares. We write down such an equation using only invertible matrices. We do this using equations for 3×3, resp. 2×2, blocks that then can be put together to get the result for arbitrary $(2m+1)\times(2m+1)$ matrices for $m \geq 1$ (recall that we always assume $\mathrm{char}(K) \neq 2$):

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \left[\begin{pmatrix} 1 & -3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^2 \right] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 \left[\begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 \right] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \blacksquare$$

The following has also been observed by Lewis [Lew87b] and Denert, Tignol, Van Geel and Vast [DTVGV90], but only in the case of central division K-algebras.

Corollary 3.3. *Let A be a central simple K-algebra of odd degree n. Then $s(A) \geq \min\{s(K), 3\}$ and $s_\pi(A) \geq \min\{s(K), 2\}$, with equality if A is not division.*

Proof. Every central simple K-algebra A of degree n and index d has a splitting field L/K with $[L : K] = d$, i.e. $A \otimes_K L \cong M_n(L)$. Furthermore, if d is odd, then Springer's theorem (see, e.g. [L05, Ch. VII. Th. 2.7]) implies that $s(K) = s(L)$. In our situation, n and hence d are assumed to be odd, so by Corollary 3.2, we have

$$s(A) \geq s(A \otimes_K L) = s(M_n(L)) = \min\{s(L), 3\} = \min\{s(K), 3\}$$

and

$$s_\pi(A) \geq s_\pi(A \otimes_K L) = s_\pi(M_n(L)) = \min\{s(L), 2\} = \min\{s(K), 2\}.$$

If A is not division, then $A \cong M_m(D)$ with D a central division K-algebra of degree d and with $n = md$. In particular, m is odd and we can apply Corollary 3.2 directly to deduce the desired equality. ■

In contrast, the product level is of not much interest for central simple algebras of even degree.

Proposition 3.4. *Let A be a central simple K-algebra of even degree n. Then $s_\pi(A) = 1$.*

Proof. Write $A \cong M_m(D)$ with D a central division K-algebra of degree d and with $n = md$. If m is even we can apply Corollary 3.2 to get $s_\pi(A) \leq s_\pi(M_m(K)) = 1$. If d is even, we get $s_\pi(A) \leq s_\pi(D)$, and Wadsworth [W90] showed that in this case $s_\pi(D) = 1$. ■

Remark 3.5. Wadsworth's proof uses a clever argument to reduce the general case of an even degree division algebra to that of a quaternion division algebra (which we will study in more detail later on). In such a quaternion division algebra, there are always anticommuting nonzero elements. More generally, let R be any ring with anticommuting elements $x, y \in R^*$, i.e. $xy = -yx$. Then $s_\pi(R) = 1$: $-1 = x^2(x^{-1}y)^2(y^{-1})^2$.

3.2. Division rings. Let D be a division ring. We introduce a new type of level: the *sublevel* $\underline{s}$ defined as follows. We put $\underline{s}(D) = \infty$ if 0 is not a sum of nonzero squares in D. Otherwise, we put

$$\underline{s}(D) = \min\{n \,|\, \exists\, x_1, \ldots, x_{n+1} \in D^* :\ x_1^2 + \ldots + x_{n+1}^2 = 0\}.$$

We clearly have $s_\pi(D) \leq \underline{s}(D) \leq s(D)$, and we have equality if $D = K$ is a field.

Szele [Sz52] showed that the product level is the “correct” notion when extending the Artin–Schreier theorem to division rings. However, the product level for division rings does not at all show the same behavior as the level for fields concerning the values it can take. As in the field case, we define an ordering in the division ring D to be a multiplicative subgroup of index 2 in D^* that is closed under addition.

Theorem 3.6.

(i) (Szele [Sz52]) *The division ring D has an ordering if and only if $s_\pi(D) = \infty$.*
(ii) (Scharlau und Tschimmel [ST83]) *Let $n \geq 1$ be an integer. Then there exists a division ring D with $s_\pi(D) = n$.*

Remark 3.7.

(i) Albert [A40] proved that any noncommutative division ring that has an ordering is necessarily infinite-dimensional over its center, so by our definition, central simple division algebras cannot possess an ordering.

(ii) It should also be noted that the examples constructed in [ST83] are certain twisted Laurent series fields that are infinite-dimensional over their center. Most of the work on (product) levels of division rings has been done in the case of central simple algebras, but to our knowledge, there is no analogous statement to (ii) above with “division ring” replaced by “central division K-algebra”.

As for product levels of noncommutative central division K-algebras D (i.e. in the case $\operatorname{ind}(D) > 1$), the only values that have been realized at this point are $s_\pi(D) = 1$, 2, or 4. Thus, we pose the following problem (see also [Lew99, Open Question 4]):

Problem 3.8. For which $n \geq 1$ does there exist a field K and a central division K-algebra D of index > 1 with $s_\pi(D) = n$?

From now on, let us only consider central division K-algebras, and let $\operatorname{tr} : D \to K$ be the reduced trace map. We thus get a quadratic form $T_D : D \to K$: $x \mapsto \operatorname{tr}(x^2)$ which we call the *trace form* of D. We have the following result.

Theorem 3.9. *Let D be a central division K-algebra with trace form T_D. Then the following are equivalent.*

(i) $\underline{s}(D) < \infty$;
(ii) $s(D) < \infty$;
(iii) $D = \sum D^2$;
(iv) *T_D is weakly isotropic, i.e. there exists an $n \in \mathbb{N}$ such that $n \times T_D = \underbrace{T_D \perp \ldots \perp T_D}_{n}$ is isotropic.*

The equivalences (i) ⇔ (ii) ⇔ (iii) and (i) ⇒ (iv) are due to Leep, Shapiro and Wadsworth [LSW85], (iv) ⇒ (i) is due to Lewis [Lew85]. It should be mentioned that the proof of (i) ⇔ (ii) ⇔ (iii) given in [LSW85] does not yield a quantitative version relating s, $\underline{s}$, p except for the obvious relation $\underline{s} \leq s \leq p$. We will provide more precise relations in certain cases later on, for example for quaternion division algebras.

In the case of division rings that are infinite-dimensional over their center, we still have the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), but it is not known if (i) $\Rightarrow$ (iii) holds. Thus, we pose the following problems.

PROBLEM 3.10. Let D be a division ring with $\underline{s}(D) < \infty$.

(i) (see also Open Problem in [LSW85, p. 161]) Suppose that D is infinite-dimensional over its center. Does one always have $D = \sum D^2$?

(ii) Suppose that D is of finite dimension n over its center. Find upper bounds for s, p in terms of $\underline{s}$ (and perhaps n), resp. for p in terms of s (and perhaps n).

3.3. Quaternion algebras. Central simple K-algebras of degree 2 are called quaternion algebras. We still assume that all our fields are of characteristic not 2. Then such a quaternion algebra Q is generated by two elements i, j subject to the relations $i^2 = a$, $j^2 = b$, $ij = -ji$ for some $a, b \in K^*$ and we use the notation $Q = (a, b)_K$. Q then has a K-basis $1, i, j, k := ij$, and for $\zeta = x + yi + zj + wk$, $x, y, z, w \in K$, we have the standard involution called *conjugation* given by $\zeta \mapsto \overline{\zeta} = x - yi - zj - wk$. We call $\zeta' = yi + zj + wk$ the pure part of ζ. The classical example is Hamilton's quaternions $\mathcal{H} = (-1, -1)_{\mathbb{R}}$ for which we obviously have $s(\mathcal{H}) = 1$.

The norm map

$$N_Q : Q \to K\colon\ \zeta \mapsto \zeta\overline{\zeta} = x^2 - ay^2 - bz^2 + abw^2$$

is a nondegenerate quadratic form with diagonalization $N_Q = \langle 1, -a, -b, ab\rangle$ with respect to the orthogonal basis $1, i, j, k$. Q is a division algebra iff N_Q is anisotropic. If R is another quaternion algebra over K, then $Q \cong R$ as K-algebras iff $N_Q \cong N_R$, i.e. iff the respective norm forms are isometric. We refer to [L05, Ch. III] for all this and more on quaternions.

We also have the trace $\operatorname{tr}(\zeta) = \zeta + \overline{\zeta}$ giving rise to the trace form

$$T_Q(\zeta) = \frac{1}{2}\operatorname{tr}(\zeta^2) = x^2 + ay^2 + bz^2 - abw^2,$$

a nondegenerate quadratic form with diagonalization $T_Q = \langle 1, a, b, -ab\rangle$ with respect to the orthogonal basis $1, i, j, k$. We define its pure part $T'_Q = \langle a, b, -ab\rangle$. Here are a few facts.

FACTS AND EXAMPLES 3.11. Let Q be a division quaternion algebra over K.

(i) $s(Q) \leq 2^m - 1$ if and only if either $\langle 1\rangle \perp (2^m - 1) \times T'_Q$ or $2^m \times T'_Q$ is isotropic, see Lewis [Lew87a].

(ii) $s(Q) \leq 2^m$ if and only if either $(2^m + 1) \times \langle 1\rangle \perp (2^m - 1) \times T'_Q$ or $\langle 1\rangle \perp 2^m \times T'_Q$ is isotropic, see Lewis [Lew89] for the "only if" part, and Leep [Le90] for the "if" part.

(iii) It is easy to see that -1 is a sum of squares of n pure quaternions iff $\langle 1\rangle \perp n \times T'_Q$ is isotropic. For any positive integer m, Koprowski [Ko98] constructed an example of such a Q with $s(Q) = 2^m$ but where -1 is not a sum of 2^m squares of pure quaternions.

(iv) The first examples of Q with $s(Q) = 2^m$, resp. $s(Q) = 2^m + 1$, were given by Lewis [Lew89] and somewhat modified by Tignol and Vast [TV87]. The general idea is as follows. Let K_0 be an arbitrary field with some $a \in K_0^* \setminus K_0^{*2}$. Then, over $K = K_0((T))$, $(a, T)_K$ is division. Consider the quadratic form $q = \langle 1, -a\rangle$. In the terminology of [BGM13], we define $\underline{s}_q(K_0)$ to be the smallest integer n such that $(n + 1) \times q$ is isotropic provided such an n exists, and we put $\underline{s}_q(K_0) = \infty$ otherwise. Using the theory of Pfister forms, one can show that if finite, then $\underline{s}_q(K_0) = 2^r$ for some r. Tignol and Vast [TV87] showed that then $s((a,T)_L) = \min\{\underline{s}_q(K_0)+1, s(K_0(\sqrt{a}))\}$.

We apply this to $K_0 = \mathbb{R}(X_1,\dots,X_n)$ for some $n \in \mathbb{N}$. Let m be such that $2^m \le n < 2^{m+1}$. With $a = \sum_{i=1}^n X_i^2$, $Q_1 = (a,T)_K$ and $Q_2 = (-a,T)_K$, and with the use of the above result, it was shown in [TV87] that

$$\begin{aligned} n = 2^m: &\quad s(Q_1) = 2^{m-1}+1 \quad\text{and}\quad s(Q_2) = 2^m,\\ 2^m < n < 2^{m+1}: &\quad s(Q_1) = 2^m+1 \quad\text{and}\quad s(Q_2) = 2^m. \end{aligned}$$

A closer inspection of the arguments used in the proofs also yields $\underline{s}(Q_1) = s(Q_1) - 1$ and $\underline{s}(Q_2) = s(Q_2)$.

(v) The first examples of quaternion division algebras with sublevel not a 2-power were given by Krüskemper and Wadsworth [KW91] who showed that there are quaternion division algebras of sublevel 3. In all these examples the level is 3. This follows more generally from results by Leep [Le90] who showed that if $\underline{s}(Q) = 2^m - 1$, resp. $s(D) = 2^m$, for some $m \ge 2$, then $s(Q) = \underline{s}(Q)$, and if $s(Q) = 2^m + 1 \ge 3$ then $\underline{s}(Q) = 2^m$ or $2^m + 1$.

(vi) Using generic methods involving function fields of quadratic forms, Laghribi and Mammone [LM01] also constructed quaternion division algebras of level 2^m, resp. 2^m+1, for any integer $m \ge 0$.

In view of the above, Lewis [Lew87a] and Leep [Le90] asked if the level can take values other than of type 2^m, 2^m+1, and if the level and the sublevel always differ at most by 1. There is a partial answer to the first question, and a full answer to the second question.

THEOREM 3.12 (Hoffmann [H08]). *To each integer $m \ge 0$ there exists a quaternion division algebra Q with*

$$m+1 \le s(Q) \le m+1+\left[\frac{m}{3}\right].$$

While this result does not yield any specific new values, it shows that they must exist. For example, for $m = 5$ we get $s(Q) \in \{6,7\}$. For any $k \ge 4$, the result shows the existence of at least two new values in the interval $[2^k+2, 2^{k+1}-1]$. The idea is to use the fact that for $m < n$, we have $s(Q) \in [m+1, n]$ provided $\langle 1\rangle \perp m \times T_Q$ is anisotropic and $\langle 1\rangle \perp n \times T'_Q$ is isotropic. To construct such a situation with the given bound on n, one uses certain generic methods which require some deep results on isotropy of quadratic forms over function fields of quadrics which we will not elaborate upon in the context of this survey. Instead, we refer the interested reader to the original article.

As mentioned previously, for any division ring with finite (sub)level, we trivially have $\underline{s} \le s \le p$. For quaternion algebras, we can do better.

THEOREM 3.13 (Hoffmann [H10] for s). *Let Q be a quaternion division algebra over the field K with $\underline{s}(Q) = n < \infty$. Then $s(Q) \le n+1$ and $p(Q) \le n+2$.*

Proof. Let $Q = (a,b)_K$ with basis $1, i, j, k$ as usual. In [H10], it was shown that $s(Q) = \ell_Q(-1) \le n+1$. Now only a tiny modification in the proof there is needed to actually show that for *any* $\lambda \in K$, we have $\ell_Q(\lambda) \le n+1$. So let $\zeta = x+yi+zj+wk$, $x,y,z,w \in K$. Let $\mu = \frac{1}{2} + yi + zj + wk$. Then we have $\mu^2 - T_D(\mu) + x = \zeta$. Now with $\lambda = -T_D(\mu) + x \in K$ we have

$$\ell_Q(\zeta) = \ell_Q(\mu^2+\lambda) \le 1 + \ell_Q(\lambda) \le n+2. \ \blacksquare$$

Remark 3.14.

(i) Combining the previous two theorems, we deduce that to any $m \geq 1$, there exist quaternion division algebras Q with

$$m \leq \underline{s}(Q) \leq m + 1 + \left[\frac{m}{3}\right],$$

yielding also the existence of values for $\underline{s}$ that are greater than 4 and not of shape 2^k (or $2^k + 1$). This has independently been observed (without invoking Theorem 3.13) by O'Shea [O10]. For certain values of m, he also gave somewhat better upper bounds than those in Theorem 3.12.

(ii) The proofs of Theorems 3.12 and 3.13 given in [H08], [H10] and of many of the other previously mentioned results on quaternion algebras can be adapted without major changes to work also for octonions, see O'Shea [O07], [O10], and even more generally, for algebras obtained by the Cayley–Dickson process. Algebras of this latter type are always of dimension 2^t for some $t \geq 2$, and they are nonassociative for $t \geq 3$ (for $t = 2$ these are quaternions, for $t = 3$ octonions), see Flaut [Fl11], [Fl13]. Here are a few samples:

- (see [O10]) To each $n \geq 1$, there exists an octonion division algebra $\mathcal{O}$ over a suitable field K with $n - \left[\frac{n}{8}\right] \leq s(\mathcal{O}) \leq n$ and $n - \left[\frac{n+7}{8}\right] \leq \underline{s}(\mathcal{O}) \leq n$. In particular, there exist octonion division algebras of level 6, resp. 7.
- (see [Fl13]) Let D be a division algebra obtained by the Cayley–Dickson process. Then $\underline{s}(D) \leq s(D) \leq \underline{s}(D) + 1$.
- (see [Fl13]) Let $n \geq 1$. Then to any $t \geq 2$, there exists an algebra A obtained by the Cayley–Dickson process and of dimension 2^t over a suitable field K such that $n - \left[\frac{n}{2^t}\right] \leq s(A) \leq n$. In particular, for $2^t > n$, there are such algebras with $s(A) = n$. For $t \leq 3$, these can be constructed to be division algebras, but for $t \geq 4$, the constructed examples may not be division algebras.
- (Sub)levels of composition algebras over function fields in one variable over local nondyadic fields were studied by O'Shea and Van Geel [OVG08]. In this situation, the (sub)level turns out to be 1 or 2, and conditions are derived for $s = 1$, resp. $\underline{s} = 1$, to hold.

In view of the results on quaternion algebras mentioned above, one of the obvious remaining questions is the following.

Problem 3.15. Which values n can be realized as level, resp. sublevel, of a quaternion algebra? More generally, which pairs (n, n), $(n, n+1)$ can be realized as $(\underline{s}, s)$ for quaternion algebras?

3.4. Central division algebras of higher degree. Results on the level, sublevel and product level on central division algebras of higher degree (of odd degree for the product level) are much less complete. Cyclic division algebras were studied by Denert, Tignol, Van Geel and Vast [DTVGV90]. Recall that a cyclic algebra $A = (L/K, \sigma, a)$ is given by a cyclic Galois extension L/K with Galois group generated by σ, say with $[L : K] = n$, and some $a \in K^*$ so that $A = \bigoplus_{i=0}^{n-1} Kz^i$ with multiplication given by $z\lambda = \sigma(\lambda)z$ for all $\lambda \in L$ and $z^n = a$. A is then a central simple K-algebra of degree $[L : K] = n$. In [DTVGV90], a new invariant is introduced: $t(L/K)$, the least integer t for which there exist $a_1, \ldots, a_t \in L$ with $-1 = \sum_{i=1}^{t} a_i\sigma(a_i)$, or ∞ if there is no such integer.

THEOREM 3.16. *Let $A = (L/K, \sigma, a)$ be a cyclic division algebra of degree $n = 2^m k$ with k odd. If $m \geq 1$, let M be the unique quadratic extension of K inside L. Put $t = t(L/K)$ and, for $m \geq 1$, $t' = t(L/M)$. Then*

	$m \geq 2$	$m = 1$	$m = 0$
$s(A)$	$\leq \min\{s(L), 2\}$	$\leq \min\{s(L), t+1, t'+1\}$	$\leq \min\{s(L), t+1\}$
$\underline{s}(A)$	1	$\leq \min\{t, t'\}$	$\leq t$
$s_\pi(A)$	1	1	$\leq t$

In general, it seems not so easy to get a good grip on the value $t(L/K)$. However, for $n = 3$ one has good control. Now it is well known that central simple algebras of degree 3 are always cyclic. Thus, one can deduce the following (see [DTVGV90], also cf. Corollary 3.3, Proposition 3.4).

COROLLARY 3.17. *Let A be a central division K-algebra of degree 3. Then*

$$s(A) = \underline{s}(A) = s_\pi(A) = s(K) \quad \text{if} \quad s(K) \leq 2$$
$$s(A) - 1 = \underline{s}(A) = s_\pi(A) = 2 \quad \text{if} \quad s(K) > 2.$$

For local and global base fields K, the levels of central division K-algebras are known.

THEOREM 3.18. *Let K be a local or global field and let D be a central division K-algebra of degree > 1. If K is global and $\mathfrak{p}$ is a place of K, then let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$ and we put $D_\mathfrak{p} = D \otimes_K K_\mathfrak{p}$.*

(i) *Let K be local or global and* $\deg(D)$ *odd. Then*

$$s(D) = \min\{3, s(K)\}, \quad \underline{s}(D) = s_\pi(D) = 2.$$

(ii) *Let K be local and* $\deg(D)$ *even. Then $s(D) = \underline{s}(D) = s_\pi(D) = 1$.*

(iii) *Let K be global and* $\deg(D)$ *even. Then D contains a subfield L with $s(L) \leq 2$. In particular, $s_\pi(D) = 1$ and $1 \leq \underline{s}(D) \leq s(D) \leq 2$. We have*

$$s(D) = 1 \iff \operatorname{ind}(D_\mathfrak{p}) \textit{ divides } \tfrac{1}{2}\deg(D) \textit{ for all places } \mathfrak{p} \textit{ with } s(K_\mathfrak{p}) = 1,$$
$$\underline{s}(D) = 1 \iff s(D) = 1, \textit{ or } 4 \textit{ divides } \deg(D), \textit{ or } D_\mathfrak{p} \textit{ is split at all infinite } \mathfrak{p}.$$

These results are all due to Leep, Tignol and Vast [LTV89] except for the result on L in (iii) which is due to Denert and Van Geel [DVG89]. Note that the result on s_π in the even degree case holds over arbitrary base fields by Proposition 3.4.

4. Commutative rings. We now turn our attention to arbitrary commutative rings. We want to develop the theory of levels in all generality without assuming, for example, that 2 is invertible, or that we have no zero divisors or nilpotent elements. The definition of the level need not be changed, and commutativity implies that $s = s_\pi$. But the sublevel as previously defined will not lead to interesting results, for example, in the presence of nilpotent elements. It has therefore become customary to modify the definition of sublevel. We will explain this now based on a more geometric approach that has been developed by Hoffmann and Leep in [HL15] and which we will sketch below.

Throughout this section, all rings are assumed to be commutative.

In the sequel, 'bilinear' shall always mean 'symmetric bilinear'. A *metabolic plane* $\mathbb{M}_a$, $a \in R$, is a free rank 2 module M together with a bilinear form $b : M \times M \to R$ such that with respect to a suitable basis $\mathbf{x}, \mathbf{y}$ of M, we have $b(\mathbf{x}, \mathbf{x}) = 0$, $b(\mathbf{x}, \mathbf{y}) = 1$, $b(\mathbf{y}, \mathbf{y}) = a$, i.e. we have the Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right)$. $\mathbb{H} = \mathbb{M}_0$ is called a *hyperbolic plane.*

Now let (M, b) be an arbitrary bilinear module over R. We say that (M, b) is *isotropic* if it contains a metabolic plane, i.e. if it contains a free rank 2 submodule N such that $(N, b|_N) \cong \mathbb{M}_a$ for some $a \in R$, in which case we write $\mathbb{M}_a \subset (M, b)$. We also define the value sets $D_R(b) = \{b(\mathbf{x}, \mathbf{x}) \,|\, \mathbf{x} \in M \backslash \{0\}\} \subseteq D_R^0(b) = \{b(\mathbf{x}, \mathbf{x}) \,|\, \mathbf{x} \in M\}$. We are interested in sums of squares, so in our context we consider the free rank n bilinear spaces σ_n with an orthonormal basis $\mathbf{x}_i$, $1 \le i \le n$, i.e. $\sigma_n(\mathbf{x}_i, \mathbf{x}_j) = \delta_{ij}$, which we also write in the usual diagonal notation $\sigma_n = \langle \underbrace{1, \dots, 1}_{n} \rangle$. So σ_n describes nothing but the usual scalar or dot product: $\sigma_n(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. In particular, we have $D_R(\sigma_n) = \sum_n R^2$. We now (re)define the following levels:

- the *level* s: $s(R) = \infty$ if there is no n such that $-1 \in D_R(\sigma_n)$, otherwise
$$s(R) = \min\{n \,|\, -1 \in D_R(\sigma_n)\};$$
- the *metalevel* s_m: $s_m(R) = \infty$ if there are no integer $n \ge 1$ and no $a \in R$ such that $\mathbb{M}_a \subset \sigma_n$, otherwise
$$s_m(R) = \min\{n \,|\, \exists a \in R: \ \mathbb{M}_a \subset \sigma_{n+1}\};$$
- the *hyperlevel* s_h: $s_h(R) = \infty$ if there is no integer $n \ge 1$ such that $\mathbb{H} \subset \sigma_n$, otherwise
$$s_h(R) = \min\{n \,|\, \mathbb{H} \subset \sigma_{n+1}\}.$$

We have $s_m(R) \le s$ since if $s = n < \infty$, then $\mathbb{M}_1 \cong \langle -1, 1 \rangle \subset \sigma_{n+1}$, and obviously also $s_m(R) \le s_h(R)$. Our s_m coincides with what in earlier literature has been called the sublevel (often denoted as before by $\underline{s}$) for general commutative rings. It turns out that this finer distinction between s_m and s_h bears great advantages when dealing with arbitrary commutative rings that are not necessarily integral domains and in which 2 need not be invertible.

Remark 4.1.
(i) If $2 \in R^*$, then for all $a \in R$ one has $\mathbb{M}_a \cong \mathbb{M}_0 = \mathbb{H}$ and hence $s_m(R) = s_h(R)$.
(ii) One easily sees that $D_R^0(\mathbb{H}) = 2R$, so if $2 \notin R^*$, then $1 \in D_R(\sigma_2)$ but $1 \notin D_R(\mathbb{H})$, hence $\mathbb{H} \ncong \sigma_2$ and thus $s_h(R) \ge 2$. For example, for $R = \mathbb{F}_2$ the field with two elements, one has $s_m(\mathbb{F}_2) = 1$ and $s_h(\mathbb{F}_2) = 2$.

We have the following result that is the commutative analogue to Theorem 3.9.

Theorem 4.2 (Hoffmann and Leep [HL15]). *Let R be a commutative ring. Then the following are equivalent*:

(i) $s(R) < \infty$;
(ii) $s_m(R) < \infty$;
(iii) $s_h(R) < \infty$;
(iv) $\sum R^2 = R^2 + 2R = \{a^2 + 2b \,|\, a, b \in R\}$.

If one and hence every of these statements holds, and if $n \geq 1$ is an integer such that $s_m(R) \in \{2n-1, 2n\}$, then

$$s_m(R) \leq s(R) \leq p(R) \leq s_h(R) + 1 \leq 2n + 1.$$

If, in addition, $2 \in R^*$, then $\sum R^2 = R$ and

$$s_m(R) = s_h(R) \leq s(R) \leq p(R) \leq s_m(R) + 1.$$

We mention that Joly [Jo70a] noted that if $s(R)$ is finite, then $s(R) \leq p(R) \leq s(R)+2$, and Peters [Pe74] strengthened this by adding that if $s(R)$ is even, then in fact $s(R) \leq p(R) \leq s(R)+1$. All these results are subsumed and generalized in the previous theorem.

Using the well known 2, 4, and 8-square identities, it is not difficult to show the following, see also Dai and Lam [DL84].

PROPOSITION 4.3. *If $s(R) \in \{1, 2, 4, 8\}$ or $s_m(R) \in \{1, 3, 7\}$, then $s_m(R) = s(R)$.*

4.1. Generic rings. It becomes an obvious question which values can be realized for s, s_m and s_h. As in the case of fields with prescribed level, there are certain generic rings that provide good candidates for yielding a given value as level.

For any ring R and any integer $n \geq 1$, we define the following quotient rings of polynomial rings over R.

(a) $A_n(R) = R[X_1, \dots, X_n]/I_A$ with
$$I_A = (X_1^2 + \dots + X_n^2 + 1);$$
(b) $B_n(R) = R[X_0, \dots, X_n, Y_0, \dots, Y_n]/I_B$ with
$$I_B = (X_0^2 + \dots + X_n^2, X_0Y_0 + \dots + X_nY_n - 1);$$
(c) $C_n(R) = R[X_0, \dots, X_n, Y_0, \dots, Y_n]/I_C$ with
$$I_C = (X_0^2 + \dots + X_n^2, Y_0^2 + \dots + Y_n^2, X_0Y_0 + \dots + X_nY_n - 1).$$

It is not difficult to show that if $2 \in R^*$, then $B_n(R) \cong C_n(R)$. Of particular interest are the cases $R = \mathbb{R}$ (which were studied extensively in [DL84]) and $R = \mathbb{Z}$. The generic property of these rings for $R = \mathbb{Z}$ follows from the following fairly obvious fact.

LEMMA 4.4. *Let R be a ring.*

(a) *$s(R) \leq n$ iff there exists a ring homomorphism $A_n(\mathbb{Z}) \to R$.*
(b) *$s_m(R) \leq n$ iff there exists a ring homomorphism $B_n(\mathbb{Z}) \to R$.*
(c) *$s_h(R) \leq n$ iff there exists a ring homomorphism $C_n(\mathbb{Z}) \to R$.*

It was a spectacular breakthrough at the time when Dai, Lam and Peng [DLP80] showed that $s(A_n(\mathbb{R})) = n$, thus proving that contrary to the case of fields, every integer n can be realized as the level of an integral domain. What made the result so remarkable was that the proof uses in an ingenious way the Borsuk–Ulam theorem from topology. Subsequently, further connections of levels of rings and topology (levels of topological spaces) have been studied, see [DL84] or, for an introductory survey, [Pfi95, Chs. 3, 10]. The fact that $s(A_n(\mathbb{R})) = n$ can also be deduced from the following result by Arason and Pfister [AP82] that was proved purely algebraically.

Theorem 4.5. *Let K be a field, φ and ψ be quadratic forms over K with* $\dim\varphi = n > \dim\psi$ *and with φ regular. Let $X = (X_1, \ldots, X_n)$ be an n-tuple of variables, $a, b \in K^*$, and let $A = K[X]/(\varphi(X) - a)$. If b is represented by ψ over A, then b is already represented by ψ over K.*

Corollary 4.6. $s(K[X_1, \ldots, X_n]/(1 + X_1^2 + \ldots + X_n^2)) = \min\{s(K), n\}$. *In particular,* $s(A_n(\mathbb{R})) = n$.

Proof. Let $A = K[X_1, \ldots, X_n]/(1 + X_1^2 + \ldots + X_n^2)$. Clearly, one has $s(A) \leq m := \min\{s(K), n\}$. Suppose $m \geq 2$. In Theorem 4.5, put $\varphi = n \times \langle 1 \rangle$, $\psi = (m-1) \times \langle 1 \rangle$, $a = b = -1$. By assumption, ψ does not represent -1 over K since $\dim\psi = m-1 < s(K)$, and therefore, by the theorem, also not over A since $\dim\psi = m - 1 < n = \dim\varphi$. Hence $s(A) \geq m$. ∎

Note that since $2 \in \mathbb{R}^*$, we have $B_n(\mathbb{R}) \cong C_n(\mathbb{R})$ and $s_m(R) = s_h(R)$ for $R = A_n(\mathbb{R})$, $B_n(\mathbb{R})$ or $C_n(\mathbb{R})$. The following was shown by Dai and Lam [DL84].

Theorem 4.7.

$$\begin{aligned} s(A_n(\mathbb{R})) = s_m(A_n(\mathbb{R})) &= n \quad \textit{for all } n \geq 1, \\ s(B_n(\mathbb{R})) = s_m(B_n(\mathbb{R})) &= n \quad \textit{for } n \in \{1, 3, 7\}, \\ s(B_n(\mathbb{R})) = s_m(B_n(\mathbb{R})) + 1 &= n + 1 \quad \textit{for } n \geq 2 \textit{ and } n \neq 3, 7. \end{aligned}$$

For $\mathbb{Z}$ instead of $\mathbb{R}$, the results are less complete, see [HL15].

Theorem 4.8.

		s_m	s_h	s
$A_n(\mathbb{Z})$	$n = 1, 3$	n	$n + 1$	n
	n *odd*, $n \geq 5$	n	[n *or* $n + 1$]	n
	n *even*	n	n	n
$B_n(\mathbb{Z})$	$n = 1, 3$	n	$n + 1$	n
	$n = 7$	7	[7 *or* 8]	7
	n *odd*, $n \neq 1, 3, 7$	n	[n *or* $n + 1$]	[$n + 1$ *or* $n + 2$; $n + 1$ *if* $s_h = n$]
	n *even*	n	n	$n + 1$
$C_n(\mathbb{Z})$	$n = 1, 3, 7$	n	n	n
	$n \neq 1, 3, 7$	n	n	$n + 1$

(*the exact values in the frames are not known to us*).

It should be noted that there are also results on the Pythagoras number p for these rings with coefficients in $\mathbb{R}$, see Choi, Dai, Lam and Reznick [CDLR82], and with coefficients in $\mathbb{Z}$, see [HL15]. We refrain from adding these (also incomplete) results here since our focus is on levels, and we refer to these papers instead.

Problem 4.9. What are the exact values in the frames in Theorem 4.8?

4.2. Integral domains and semi-local rings. In the sequel, if R is an integral domain, we denote its quotient field $\mathrm{Quot}(R)$ by K_R. Note that in this case we will clearly have $s(K_R) = s_m(K_R) \leq s_m(R)$. Here is a sample of facts about levels of integral domains and semi-local rings.

Facts and Examples 4.10.

(i) Let R be an arbitrary commutative ring. Then $s(R) < \infty$ iff $s(K_{R/\mathfrak{p}}) < \infty$ for all prime ideals $\mathfrak{p} \subset R$. This "local-global principle" was noted by Bröcker, Dress and Scharlau [BDS82] and others, see also Lam [L84, Th. 1.5].

(ii) Let R be a Prüfer domain which can be characterized, for example, by saying that for all maximal ideals $\mathfrak{m} \subset R$ the localization $R_\mathfrak{m}$ is a valuation ring for K_R. Then $s(R) < \infty$ iff $s(K_R) < \infty$, see [L84, Cor. 1.8].

(iii) Let R be a nonreal Dedekind domain. Then there exists an integer $n \geq 0$ such that

$$2^n = s(K_R) \leq s_m(R) \leq s(R) \leq 2^n + 1.$$

If, in addition, R is semi-local, then one has $s(K_R) = s_m(R) = s(R)$. For s, this was shown by Baeza [B79]. For s_m, we can apply Theorem 4.2.

(iv) If R is a nonreal principal ideal domain, then there exists an integer $n \geq 0$ such that

$$2^n = s(K_R) = s_m(R) \leq s(R) \leq 2^n + 1.$$

This follows from (iii) and Arason and Baeza [AB11, Prop. 3.5].

(v) For any integer $n \geq 1$, there is a principal ideal domain R with $s_m(R) = 2^n$ and $s(R) = 2^n + 1$, see [AB11, Prop. 3.4]. Note that by Proposition 4.3 such examples cannot exist for $n = 0$.

(vi) Let R be a nonreal semi-local ring. Then there exists an integer $n \geq 1$ such that

$$2^n - 1 \leq s_m(R) = s(R) \leq s_h(R) \leq 2^n.$$

The fact that s is bounded as above was shown by Baeza [B75]. The fact that $s_m = s$ was mentioned without proof in [AB11]. A proof based on a result on cancellation of bilinear forms due to Knebusch is given in [HL15].

(vii) Let K be a real closed field and let R be a K-algebra of transcendence degree d over K without any real point. Then $s(R) \leq 2^{d+1} - 1$ if $d \leq 4$, and $s(R) \leq 2^{d+1} + d - 5$ if $d \geq 5$. If, in addition, R is semi-local, then $s(R) \leq 2^d$, see Mahé [Mah90].

In all the above examples, we have not addressed s_h. Of course, with the above values and bounds for s_m, Theorem 4.2 gives obvious bounds for s_h, but to get the precise values, a further study is required. Also note that $s = 2^n$ can be realized for local rings (just work with fields!). $\mathbb{Z}/4\mathbb{Z}$ is a local ring with $s = 3$. However, no examples are known of local rings with $s = 2^n - 1$ for $n \geq 3$.

Problem 4.11. Are there (semi-)local rings R with $s(R) = 2^n - 1$ for $n \geq 3$?

4.3. Rings of algebraic integers. For number theorists, rings of algebraic integers are of particular interest. So let K be an algebraic number field. We denote by $\mathcal{O}_K$ the ring of algebraic integers in K. If R is any order in K (and thus a subring of $\mathcal{O}_K$ with

$\mathbb{Q}R = K$), then for $s(R)$ to be finite it is necessary that $s(K)$ be finite, i.e. that K be totally imaginary and thus, by Examples 2.3 (iv), $s(K) \in \{1, 2, 4\}$. We then have the following.

THEOREM 4.12. *Let K be a totally imaginary number field and let R be an order in K and let $\mathcal{O}_K$ be the ring of algebraic integers (and thus the maximal order) in K. Then*

(i) $s(R) \leq 4$, *and* $s(R) \leq 3$ *if* $s(K) \leq 2$ *and* $s(R \otimes_{\mathbb{Z}} \mathbb{Z}_2) \leq 3$.
(ii) $s(\mathcal{O}_K) = 1$ *iff* $s(K) = 1$; $2 \leq s(\mathcal{O}_K) \leq 3$ *iff* $s(K) = 2$; $s(\mathcal{O}_K) = 4$ *iff* $s(K) = 4$.

Proof. Part (i) is due to M. Kneser, and proofs can be found in Draxl [Dr71] and Peters [Pe72]. In [Dr71], part (ii) was deduced from (i), but since $\mathcal{O}_K$ is a Dedekind domain with quotient field K, this also follows from the more general Facts and Examples 4.10 (iii). ∎

Particular interest has been devoted to the case of an imaginary quadratic number field K. In this situation, $s(\mathcal{O}_K)$ was determined by Moser [Mo70], [Mo71] (Singh [Sin76] considered various special cases).

The following are well known facts from number theory. Let m be a square free positive integer, and let $K = \mathbb{Q}(\sqrt{-m})$. Then $\mathcal{O}_K$ has $\mathbb{Z}$-basis $1, \omega$ with

$$\omega = \begin{cases} \dfrac{1+\sqrt{-m}}{2} & \text{if } m \equiv 3 \bmod 4 \\ \sqrt{-m} & \text{if } m \equiv 1, 2 \bmod 4. \end{cases}$$

The discriminant Δ_K of $\mathcal{O}_K$ is then given by

$$\Delta_K = \begin{cases} -m & \text{if } m \equiv 3 \bmod 4 \\ -4m & \text{if } m \equiv 1, 2 \bmod 4. \end{cases}$$

For any positive integer f, we get an order R_f with $\mathbb{Z}$-basis $1, f\omega$, and we get all orders in K in this way. Peters [Pe72], [Pe74] showed the following:

THEOREM 4.13. *Let m, K, $\Delta = \Delta_K$, f and R_f be as above.*

(i) *If f is even, then*

$$s(R_f) = \begin{cases} 3 & \textit{if } f \equiv 2 \bmod 4 \textit{ and } \Delta \equiv 5 \bmod 8, \\ 4 & \textit{otherwise.} \end{cases}$$

(ii) *If f is odd, then*

- $\Delta = -4$: $s(R_f) = 1$ *iff* $f = 1$. *Otherwise,* $s(R_f) = 3$.
- $\Delta \equiv 0 \bmod 4$, $\Delta \neq -4$: $s(R_f) = 2$ *iff* $x^2 - mf^2y^2 = -1$ *has a solution over* $\mathbb{Z}$. *Otherwise,* $s(R_f) = 3$.
- $\Delta \equiv 1 \bmod 8$: $s(R_f) = 4$.
- $\Delta \equiv 5 \bmod 8$: $s(R_f) = 2$ *iff* $x^2 - mf^2y^2 = -2$ *has a solution over* $\mathbb{Z}$. *Otherwise,* $s(R_f) = 3$.

Again, it would be interesting to also study the invariants s_m and s_h in these situations.

5. Other types of levels. There are various ways in which the notions of level, sublevel, Pythagoras number, etc., may be generalized. We present a few of these generalizations and results.

5.1. Quadratic form levels. We consider a field K of characteristic not 2 and a finite-dimensional regular quadratic form (or *form* for short) q over K. Berhuy, Grenier-Boley and Mahmoudi [BGM13] defined the q-(sub)level s_q (resp. $\underline{s}_q$) as follows:

$$s_q(K) = \min\{n \,|\, n \times q \text{ represents } -1\}$$

if such an integer $n \geq 1$ exists, otherwise $s_q(K) = \infty$, and

$$\underline{s}_q(K) = \min\{n \,|\, (n+1) \times q \text{ is isotropic}\}$$

if such an integer $n \geq 0$ exists, otherwise $\underline{s}_q(K) = \infty$.

Note that for $q = \langle 1 \rangle$, we get the usual (sub)level: $s(K) = s_q(K)$, $\underline{s}(K) = \underline{s}_q(K)$. The following facts are easily verified:

- If q is isotropic, then $\underline{s}_q(K) = 0$ and $s_q(K) = 1$.
- If q is anisotropic, then $s_q(K) \leq \underline{s}_q(K) + 1$.
- If q is anisotropic and $1 \in D_K(q)$, then $\underline{s}_q(K) \leq s_q(K) \leq \underline{s}_q(K) + 1$.
- It is possible to have $\underline{s}_q(K) > s_q(K)$, for example, for $q = \langle -1 \rangle$ over $K = \mathbb{R}$, we have $\underline{s}_q(K) = \infty$, $s_q(K) = 1$.

Here is a sample of a few of the many results shown in [BGM13].

FACTS AND EXAMPLES 5.1.

(i) There exists a field K such that to each integer $n \geq 1$ there exists a form q_n over K with $s_{q_n}(K) = n$.

(ii) Let q be a round form over K, i.e. for any $a \in K^*$ we have $q \cong aq$ iff $a \in D_K(q)$. Then $s_q(K)$ is either infinite or a power of 2. The fact that all powers of two can be realized by round forms follows for example from Pfister's result on the level, using the fact that $\langle 1 \rangle$ is round.

(iii) Let q be a form over K and $\varphi \cong \langle 1 \rangle \perp q$. Then $s_q(K) < \infty$ iff $\underline{s}_\varphi(K) < \infty$.

(iv) Let K be a real field and q be a Pfister form over K, i.e. a product of binary forms representing 1. Then $s_q(K) = \infty$ iff $\underline{s}_q(K) = \infty$ iff q is definite with respect to at least one ordering of K.

5.2. Higher power levels. Let R be a ring. Instead of sums of squares, one can also consider sums of higher powers, which then leads to the notion of power levels or higher levels (and higher Pythagoras numbers, etc.). Let us again just focus on higher levels. For any integer $d \geq 2$, we define the d-th higher (or power) level of R as follows:

$$s_d(R) = \min\{n \,|\, \exists\, x_1, \ldots x_n \in R :\ x_1^d + \ldots + x_n^d = -1\}$$

or $s_d(R) = \infty$ if no such integer n exists. Obviously, only even d are of interest here. Joly [Jo70b] was perhaps the first to systematically study these higher levels for arbitrary fields. Of course, in number theory questions regarding sums of higher powers have a much longer history. As a key word it suffices to mention Waring's problem. Here are some results.

Facts and Examples 5.2.

(i) For a field K, one has

$$s(K) < \infty \iff s_d(K) < \infty \text{ for some even } d \geq 2$$
$$\iff s_d(K) < \infty \text{ for all even } d \geq 2.$$

This was proved by Joly [Jo70b]. For a different approach, see Becker [Beck79], [Beck84], and also Parnami, Agrawal and Rajwade [PAR90]. Joly actually shows that there is a function $S : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $s_d(K) \leq S(s(K), d)$. For a more explicit description of this function, see [CV05].

(ii) If A is a local henselian ring with residue field κ and $d \geq 2$ is an (even) integer, then $s_d(A) < \infty$ iff $s_d(\kappa) < \infty$, see [Jo70b].

(iii) If K is a field of characteristic 0 and level s, then $s_4(K) \leq 2^s + 2s + 2$, see Parnami, Agrawal and Rajwade [PAR90] who also derive an explicit (and more complicated) bound on s_6.

(iv) Parnami, Agrawal and Rajwade [PAR81] showed that if $K/\mathbb{Q}$ is an imaginary quadratic extension, then $s_4(K) \leq 15$. They also showed that $s_4(\mathbb{Q}(\sqrt{-2})) = 6$ and, for $m \equiv 7 \bmod 8$, that $s_4(\mathbb{Q}(\sqrt{-m})) = 15$, so in contrast to the level $s = s_2$, we have that s_4 need not be a 2-power, nor does it have to be even when $s_4 > 2$.

(v) Parnami et al. [PAR81] also showed for a finite field $\mathbb{F}_q$ with $q = p^n$ elements, p an odd prime:

$$s_4(\mathbb{F}_q) = \begin{cases} 1 & \text{if } q \equiv 1 \bmod 8; \\ 3 & \text{if } q = 29; \\ 4 & \text{if } q = 5; \\ 2 & \text{otherwise.} \end{cases}$$

s_6, s_8 and s_{10} for finite fields $\mathbb{F}_q$ were determined by Pall and Rajwade [PR83]. Note that for any odd prime p, one has $s_{p-1}(\mathbb{F}_p) = p - 1$, so for example $s_{10}(\mathbb{F}_{11}) = 10$.

If $q = p^n$ as above, let us write $q - 1 = 2^h m$ with m odd. Completing earlier results by Amice and Kahn [AK92], Revoy [Re98] showed that if $n \geq 2$ and for any integer $r \geq h$, we have $s_{2^r}(\mathbb{F}_q) = s_{2^h}(\mathbb{F}_q) = 2$. Apparently, in the case $n = 1$, it is still not known if there is a upper bound for $s_{2^h}(\mathbb{F}_p)$ that is independent of p.

(vi) In [PAR86], Parnami et al. show that if p is a prime and $K/\mathbb{Q}_p$ is a finite extension, and if κ is the residue field of K, then $s_4(K) = s_4(\kappa)$ for odd p, in which case one can apply (v).

For $p = 2$, they show that $s_4(K) \leq 15$ with more precise results depending on the ramification index e and the residue class degree f. In particular, they show that $s_4(K) = 15$ iff $e = f = 1$, i.e. iff $K = \mathbb{Q}_2$.

(vii) Let $d \geq 4$ be even. s_d for finite fields $\mathbb{F}_p$ of prime order p, for local fields $\mathbb{Q}_p$ and local rings $\mathbb{Z}_p$, and for rings of type $\mathbb{Z}/p^\ell\mathbb{Z}$ were determined by Becker and Canales [BC99]. The results are somewhat technical to state and we refer the reader to the original article, in which one can also find a good overview on other results regarding $s_d(\mathbb{F}_q)$ for finite fields $\mathbb{F}_q$ and higher d.

(viii) For any integers $n \geq 0$ and $m \geq 1$ with m odd there exists a field K with $s_{2m}(K) = 2^n$. In fact, take any field K_0 with $s(K_0) = 2^n$ and let K be an odd closure of K_0, i.e. a maximal algebraic extension K/K_0 such that any finite intermediate extension L/K_0 has odd degree. Then $s(K) = 2^n$ as well because of Springer's theorem (see, e.g. [L05, Ch. VII. Th. 2.7]). Now obviously $2^n = s(K) \leq s_{2m}(K)$. On the other hand, if in K we have $-1 = \sum_i a_i^2$, $a_i \in K$, then there exist $b_i \in K$ with $b_i^m = a_i$ since K is an odd closure of K_0. Hence, $-1 = \sum_i b_i^{2m}$ and thus $2^n = s(K) = s_{2m}(K)$.

In analogy to Pfister's result on the level $s = s_2$ for fields, the following problem seems natural.

Problem 5.3. Let d be an even integer ≥ 4. What are the integers $n \geq 1$ for which there exists a field K with $s_d(K) = n$?

For noncommutative rings R, one has the notion of higher product levels. For this, let $\Pi_n R$, $n \geq 1$, be the set of products that are words whose letters are elements of R such that each letter appears a multiple of n times in such a word. Note that if R is a division ring, then $\Pi_n R \setminus \{0\}$ is the subgroup of R^* generated by products of nonzero n-th powers and by nonzero commutators. Then one defines

$$ps_n(R) = \min\{k \mid \exists\, x_0, \ldots, x_n \in \Pi_n R \setminus \{0\} :\ 0 = x_0 + \ldots + x_n\}.$$

Let us also define $s_{\pi,n}$ like the usual product level but with sums of products of squares replaced by sums of products of n-th powers. So if R is a division ring, we obviously have $s_{\pi,n}(R) \geq ps_n(R)$. Here are some results.

Facts and Examples 5.4.

(i) Let R be an associative ring. Then $ps_n(M_k(R)) = 1$ if k is even, and $ps_n(M_k(R)) \leq 2$ if $k \geq 3$ is odd, see Cimprič [Ci01] (also cf. Corollary 3.2).

(ii) If R is noetherian and $ps_n(R) < \infty$, then $ps_{n\ell}(R) < \infty$ for any odd ℓ, [Ci01].

(iii) To each even $m \geq 2$ there exists a division ring D such that $ps_m(D) = 1$ but $ps_{2m}(D) = \infty$, [Ci01].

(iv) To each $n = 2m \geq 4$ there exists a division ring D with $ps_n(D) \leq m - 1$ and $s_{\pi,n}(D) = \infty$, see Cimprič [Ci05].

(v) If R is an Ore domain with quotient division ring D then $ps_n(D) = ps_n(R)$ for all n, see Cimprič and Velušček [CV05].

For further results on such generalized product levels and also on the corresponding notion of generalized Pythagoras numbers, we refer to the literature cited in the examples, and also to Klep and Velušček [KB08], Velušček [V10].

5.3. Hermitian levels. Let now R be a ring with an involution σ, i.e. an anti-automorphism of order two. An element $\sigma(x)x$ with $x \in R$ is called a hermitian square, and we define the *hermitian level* $s_h(R, \sigma)$ as the least integer n such that -1 is a sum of n hermitian squares in R provided such an n exists. Otherwise, one puts $s_h(R, \sigma) = \infty$. (The distinction from the hyperlevel $s_h(R)$ should be clear from the context and because of the inclusion of the symbol σ.)

In analogy to Theorem 3.9, Lewis [Lew88] proved:

Proposition 5.5. *Let* (R,σ) *be a division ring with involution and with* $2 \neq 0$. *The following are equivalent.*

(i) 0 *is a nontrivial sum of hermitian squares in* (R,σ);
(ii) $s_h(R,\sigma) < \infty$;
(iii) $x \in R$ *is a sum of hermitian squares iff* $x \in \mathrm{Sym}(R,\sigma) = \{x \in R \mid \sigma(x) = x\}$.

Here are a few further results that mimic previous results on levels of rings and fields.

Facts and Examples 5.6.

(i) To each integer $n \geq 1$ there exists a commutative ring R with nontrivial involution σ such that $s_h(R,\sigma) = n$. This was shown by Lewis [Lew88]. In fact, his example is the group ring $A_n(\mathbb{R})[C_3]$ with $C_3 = \{1, g, g^2\}$ the cyclic group of order 3 and involution induced by $\sigma(g) = g^{-1}$ ($A_n(\mathbb{R})$ is the ring in Theorem 4.7).

(ii) Let K be a field of characteristic not 2. Let either $R = K(\sqrt{a})$ with $a \in K^* \setminus K^{*2}$ and involution σ given by $\sigma(\sqrt{a}) = -\sqrt{a}$, or let $R = (a,b)_K$ be a quaternion division algebra as in Subsection 3.3 with standard involution given by $\sigma(\zeta) = \overline{\zeta}$. We then have the norm form $N : R \to K$: $x \mapsto \sigma(x)x$ which, for $R = K(\sqrt{a})$, is the quadratic 1-fold Pfister form $N = \langle 1, -a\rangle$, and for $R = (a,b)_K$ the quadratic 2-fold Pfister form $N = \langle 1, -a, -b, ab\rangle$. Then, with the notation from Subsection 5.1, $s_h(R,\sigma) = s_N(K)$. In particular, since Pfister forms are round forms, it follows from Facts and Examples 5.1 (ii) that in these cases, $s_h(R,\sigma)$ is always a power of 2. This is essentially due to Lewis [Lew88].

In particular, one easily shows that if $s(K) \geq 4$, then $R = (-1,-1)_K$ is division and $s_h(R,\sigma) = \frac{s(K)}{4}$, which allows us to realize all 2-powers as hermitian levels of quaternion division algebras with standard involution.

(iii) If R is a quaternion division algebra over a field K of characteristic not 2 with *arbitrary* nontrivial involution σ of the first or second kind, then $s_h(R,\sigma)$, if finite, is a power of 2, see Serhir [Se97], [Se02]. Similar and further results on hermitian levels of quaternion algebras can be found in dos Santos [dS11].

(iv) Pumplün and Unger [PU02], and to some extent also dos Santos [dS11], obtained results in a similar spirit for octonion algebras with involution.

(v) Let R be a commutative semi-local ring and σ be an involution on R such that there exists an element $x \in R$ with $x + \sigma(x) = 1$. Assume furthermore that for each maximal ideal $\mathfrak{m}$ we have $|A/\mathfrak{m}| > 2$ and, if $\mathfrak{m} = \sigma(\mathfrak{m})$ then $|A/\mathfrak{m}| > 4$. Then $s_h(R,\sigma)$, if finite, is a power of 2, see Pfalzgraf [Pfa88].

Let us finish with another apparently wide open question regarding hermitian levels.

Problem 5.7. Which integers can be realized as hermitian levels of central simple division algebras with involution? More precisely, what can be said regarding realizable values in the case of orthogonal, symplectic, or unitary involutions, respectively?

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