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BANACH–MAZUR GAME PLAYED IN PARTIALLY ORDERED SETS

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Abstract. Concepts, definitions, notions, and some facts concerning the Banach–Mazur game are customized to a more general setting of partial orderings. It is applied in the theory of Fraïssé limits and beyond, obtaining simple proofs of universality of certain objects and classes.

1. Introduction. The Banach–Mazur game was discovered by S. Mazur. Its solution, given by S. Banach, is dated August 4, 1935 in the Scottish Book, see R. Telgársky [8] for more details. The game can be described as follows. Fix a set A contained in the unit interval $\mathbb{I} = [0, 1]$. Two players alternately choose non-degenerate intervals $J_0 \supseteq J_1 \supseteq \dots$ contained in $\mathbb{I}$. The second player wins if $\bigcap_{n\in\mathbb{N}} J_n \subseteq A$. Otherwise, the first player wins. Note that the definition of this game requires only the structure of non-empty open sets with the reverse inclusion $\supseteq$ as the partial ordering. In other words, the game can be played in any topological space X where two players alternately choose non-empty open sets $V_0 \supseteq V_1 \supseteq \dots$ so that the players build a decreasing sequence of sets V_n and the result of the game is its intersection. Variants of the game are detailed by setting targets to which players aspire. For example, if the second player wins whenever $\bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$, then the game is called *Choquet*, compare [3, p. 43].

In this note we develop a more abstract setting for the Banach–Mazur game. Namely, the family of all nonempty open sets in a fixed topological space can be regarded as a partially ordered set. In order to say who wins, one needs to distinguish a “winning” family of countably generated ideals of this poset. More precisely, one of the players wins if the ideal generated by the sequence resulted from a play belongs to our distinguished family.

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It turns out that one can reformulate and extend known results in this new setting. As an application, we discuss the Banach–Mazur game played with finitely generated models taken from a Fraïssé class, showing that one of the players has a winning strategy "leading to" the Fraïssé limit.

We remark that other generalizations of the Banach–Mazur game, still in the context of topological spaces, were considered by Morgan II [6] and Kubicki [4]. In both cases, the games are played with sets ordered by reversed inclusion, therefore they are particular instances of the Banach–Mazur game studied in this note.

2. Basic notions concerning partial orders. A partial order is a binary relation $\leqslant$ over a set P which is *reflexive* ($a \leqslant a$), *antisymmetric* ($a \leqslant b$ and $b \leqslant a$ implies $a = b$) and *transitive* ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$). When such a relation is fixed, P is called a *partially ordered* set, briefly *poset*, what is denoted as $\langle P, \leqslant \rangle$. We write $a < b$, whenever $a \leqslant b$ and $a \neq b$. A subset $D \subseteq P$ is *cofinal* in P if for every $p \in P$ there is $d \in D$ with $p \leqslant d$. An *ideal* on P is a subset $I \subseteq P$ satisfying the following conditions:

(I1) $(\forall\, x, y \in I)\ (\exists\, z \in I)\ \ x \leqslant z$ and $y \leqslant z$;
(I2) $(\forall\, x \in I)\ \ (\leftarrow, x] = \{y \in P : y \leqslant x\} \subseteq I$.

An ideal I is *countably generated* if it has a countable cofinal subset. So, I is countably generated if and only if there is an increasing sequence $\{a_n\}_{n\in\omega}$ such that $I = \{x \in P : (\exists\, n \in \omega)\ \ x \leqslant a_n\}$. Denote by σP the poset whose elements are all countably generated ideals on P and which is ordered by the inclusion. Thus, σP is a natural extension of P. Namely, each $p \in P$ can be identified with $(\leftarrow, p] \in \sigma P$. Every increasing sequence in σP has the supremum in σP. In fact, σP can be called the "sequential completion" of P, because of the following reason. If $f\colon P \to Q$ is order preserving and Q is sequentially complete (that is, every increasing sequence in Q has the supremum) then there is a unique extension $\tilde{f}\colon \sigma P \to Q$ of f to a sequentially continuous order preserving mapping, given by the formula

$$\tilde{f}(I) = \sup_{n\in\omega} f(a_n), \ \text{where } I = \{x \in P : (\exists\, n \in \omega)\ \ x \leqslant a_n\}.$$

Recall that a mapping g between sequentially complete posets is *sequentially continuous* if $\sup_{n\in\omega} g(x_n) = g(\sup_{n\in\omega} x_n)$ for every increasing sequence $\{x_n\}_{n\in\omega}$. Points $x, y \in P$ are *incompatible* if there is no $z \in P$ with $x \leqslant z$ and $y \leqslant z$; otherwise we say that x and y are *compatible*. A subset $A \subseteq P$ is called an *antichain* if it consists of pairwise incompatible elements. An antichain A is *maximal* if it cannot be extended to a bigger antichain, that is, every element of $P \setminus A$ is compatible with some element of A.

Given a topological space X, we denote by $\mathcal{T}^+(X)$ the collection of all nonempty open subsets of X. Recall that a π-*base* in X is a family $\mathcal{U} \subseteq \mathcal{T}^+(X)$ such that for every $V \in \mathcal{T}^+(X)$ there is $U \in \mathcal{U}$ satisfying $U \subseteq V$. Thus, a cofinal subset of $\langle \mathcal{T}^+(X), \supseteq \rangle$ is just a π-base of the topological space X. Let us admit that from a topological point of view the ordering is reversed. Therefore instead of an ideal (in our terminology) of open sets one can say a "filter of open sets". That is why we deal with countably generated filters of open sets instead of countably generated ideals. Disjoint sets are incompatible elements. Antichains are families of pairwise disjoint open sets, and so on. Below we prepare the framework for the Banach–Mazur game in posets.

In order to avoid confusion, we shall give names to the Players: *Eve* and *Odd*. The result of a play will be a sequence $u_0 \leqslant u_1 \leqslant u_2 \leqslant u_3 \leqslant u_4 \leqslant \ldots$ in P, where the u_n with n even are chosen by Eve, while the u_n with n odd are chosen by Odd. Fix a poset $\langle P, \leqslant\rangle$ and fix $W \subseteq \sigma P$. The *Banach–Mazur game* $\mathrm{BM}(P, W)$ is defined in the following way. Eve starts the game by choosing $u_0 \in P$. Odd responds by choosing $u_1 \in P$ with $u_0 \leqslant u_1$. Then Eve responds by choosing $u_2 \in P$ with $u_1 \leqslant u_2$. In general, if after one player's move we have a sequence $u_0 \leqslant \ldots \leqslant u_n$ then the other player (no matter whether it is Eve or Odd) responds by choosing $u_{n+1} \in P$ with $u_n \leqslant u_{n+1}$. We say that *Odd wins* if the ideal generated by $\{u_n\}_{n\in\omega}$ is an element of W; otherwise *Eve wins*.

It is clear how to define a *winning strategy* for any of the two players. A strategy is *stationary* (also called a *tactic*) if it depends only on the last move of the opponent. Note that even in the topological Banach–Mazur game, the existence of a winning strategy does not imply the existence of a stationary one, see [1].

The following simple example is taken from the foundations of set-theoretic forcing.

Example 1. Let $\langle P, \leqslant\rangle$ be a poset and let $\mathcal{D}$ be family of cofinal subsets of P. An ideal I of P is *$\mathcal{D}$-generic* if $I \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Let $W \subseteq \sigma P$ be the family of all $\mathcal{D}$-generic ideals (a priori, we do not assume that $W \neq \emptyset$). We claim that if $\mathcal{D}$ is countable then Odd has a winning strategy in $\mathrm{BM}(P, W)$.

Indeed, let $\mathcal{D} = \{D_n\}_{n\in\omega}$ and suppose p_{2n} was the last Eve's choice in a fixed play. Odd should choose $p_{2n+1} \in D_n$ so that $p_{2n+1} \geqslant p_{2n}$. Applying this strategy, it is evident that Odd wins.

The above example can be viewed as a strengthening of the well-known and simple Rasiowa–Sikorski lemma on the existence of generic ideals with respect to countably many cofinal sets. Let us admit that in set-theoretic forcing typically the ordering is reversed and instead of a "generic ideal" one uses the name "generic filter".

3. Rephrasing some classical results. Recall that a *tree* is a poset $\langle T, \leqslant\rangle$ such that for every $t \in T$ the set $\{x \in T : x < t\}$ is well-ordered. Its order type is the *height* of t in T. The set of all elements of T of a fixed height α is called the α-th *level* of T. The *height* of a tree T is the minimal ordinal δ such that δ-th level is empty, i.e. every element of T has the height $< \delta$. Maximal elements of a tree are called *leaves*. We are interested mainly in trees of height ω. Namely, given a tree T, let ∂T denote the set of all *branches* of T, that is, all maximal chains in T. Given $t \in T$, we set $t^+ = \{X \in \partial T : t \in X\}$. Then the family $\{t^+\}_{t\in T}$ is a basis of a topology on ∂T, and the set t^+ is clopen (i.e., closed and open) with respect to this topology. In particular, ∂T is zero-dimensional and also, when T has the height ω, the topology on ∂T is metrizable. Indeed, given $X, Y \in \partial T$, if $X \neq Y$ then we may define their distance $\varrho(X, Y)$ to be $1/n$ where n is the maximal level of T containing some element of $X \cap Y$. Then t^+ becomes the open ball centered at any fixed branch containing t and with radius $1/n$, where n is the height of t. Note that with this metric, ∂T is always complete.

In fact, if T is a tree of height ω, then ∂T corresponds to a closed subset of the *generalized Baire space* $\mathfrak{m}^\omega$, where $\mathfrak{m}$ is the cardinality of T. Indeed, let $\{t(\alpha) : \alpha < \mathfrak{m}\}$ enumerate all elements of the 0-th level of T. In general, if $t(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in T$ is fixed, then let $\{t(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha) : \alpha < \mathfrak{m}\}$ enumerate all elements of the n-th level of T such that $t(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) < t(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha)$. Note that it is possible that not all ordinals $< \mathfrak{m}$ are used in the above enumeration. The formula

$$\{t(\alpha_0, \alpha_1, \ldots, \alpha_n)\}_{n\in\omega} \mapsto \{\alpha_n\}_{n\in\omega}$$

gives the desired correspondence, whenever every leaf $t(\alpha_0, \alpha_1, \ldots, \alpha_n)$ corresponds to the sequence $\{\alpha_n\}_{n\in\omega}$ such that $\alpha_n = \alpha_{n+k}$ for $k \in \omega$.

The proof of the theorem below is actually a direct translation of Oxtoby's arguments, compare [3, p. 51].

THEOREM 2. *Let P be a poset, $W \subseteq \sigma P$, and suppose that Odd has a winning strategy in* $\mathrm{BM}(P, W)$. *Then there exists a tree $T \subseteq P$ of height ω with the following properties*:

(1) *Every level of T is a maximal antichain in P.*
(2) *For every $I \in \partial T$, the ideal generated by I in P is an element of W.*

Proof. Let A_0 be a maximal antichain in P consisting of Odd's responses to the first Eve's move. For each $a \in A_0$, choose a maximal antichain $A_1(a)$ in $[a, \to)$ consisting of Odd's responses to the second Eve's move after a (more formally, these are responses to 3-element sequences where a was the second element chosen by Eve). We set $A_1 = \bigcup_{a\in A_0} A_1(a)$ and we note that A_1 is a maximal antichain in P. Continuing this way, we obtain maximal antichains $\{A_n\}_{n\in\omega}$, where $A_{n+1} = \bigcup_{a\in A_n} A_{n+1}(a)$ and $A_{n+1}(a)$ is a maximal antichain above a consisting of Odd's responses to a suitable partial play. Finally, $T = \bigcup_{n\in\omega} A_n \subseteq P$ is a tree of height ω satisfying (1). Every branch I of T encodes a play of $\mathrm{BM}(P, W)$ where Odd was using his winning strategy, thus the ideal of P generated by I must be an element of W. ∎

We shall see that under some circumstances the converse is also true. Namely, let $\langle P, \leqslant\rangle$ be a poset and $T \subseteq P$ be a tree of height ω. Suppose $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n$ is a partial play with n even. Assume that Odd can choose elements a_k from the k-th level of T such that $a_0 < a_1 < \ldots < a_n$ and there exists $v \in P$ satisfying: $u_n \leqslant v$ and $a_n \leqslant v$. Then put $u_{n+1} = v$. In the case when all levels of T are maximal antichains, the above algorithm gives two sequences $\{a_n\}_{n\in\omega}$ and $\{u_n\}_{n\in\omega}$ such that the ideal generated by $\{a_n\}_{n\in\omega}$ is contained in the ideal generated by $\{u_n\}_{n\in\omega}$. The converse may not be fulfilled. For example, when P consists of open sets in the Sorgenfrey line[1] which are ordered by $\supseteq$ and T consists of open intervals, then it could be that $a_n = (-\frac{1}{n}, \frac{1}{n})$ and $u_n = [0, \frac{1}{n})$.

THEOREM 3. *Assume $\langle P, \leqslant\rangle$ is a poset and $T \subseteq P$ is a tree of height ω in which all levels are maximal antichains in P. If $W \subseteq \sigma P$ is such that for every $X \in W$, $I \in \partial T$, $X \subseteq I$ implies $I \in W$, then Odd has a winning strategy in* $\mathrm{BM}(P, W)$.

Proof. The above algorithm gives a suitable strategy. ∎

Note that, if we used a stronger assumption that T is cofinal in P, then we would get the same conclusion for W consisting of ideals generated by branches of T.

[1]The *Sorgenfrey line* is the real line equipped with the topology generated by intervals of the form $[a, b)$, with $a < b$ arbitrary. This space is a "canonical counter-example" to various concepts in general topology.

We say that a mapping of posets $\varphi\colon Q \to P$ is *order preserving* if $x \leqslant y$ implies $\varphi(x) \leqslant \varphi(y)$. An order preserving mapping $\varphi\colon Q \to P$ is *dominating* whenever the image $\varphi[Q]$ is cofinal in P, and for every $q \in Q$, for every $p \in P$ with $\varphi(q) \leqslant p$, there exists $q' \geqslant q$ in Q such that $p \leqslant \varphi(q')$. Notice that not necessarily $\varphi(q) \leqslant p \leqslant \varphi(q')$ implies $q \leqslant q'$ for an arbitrary order preserving mapping φ. So, the following result allows us to "move" the Banach–Mazur game from one poset to another, without changing its status.

THEOREM 4. *Let $\varphi\colon Q \to P$ be a dominating mapping of posets, let $W \subseteq \sigma P$ and let W^{φ} consist of all ideals I of Q such that the ideal generated by $\varphi[I]$ is in W. The following conditions are equivalent*:

(a) *Odd has a winning strategy in* $\mathrm{BM}(P, W)$.
(b) *Odd has a winning strategy in* $\mathrm{BM}(Q, W^{\varphi})$.

The same applies to Eve.

Proof. Suppose Odd has a winning strategy Σ in the game $\mathrm{BM}(P, W)$. We describe his winning strategy in the game $\mathrm{BM}(Q, W^{\varphi})$. Namely, suppose Eve has chosen $v_0 \in Q$. Odd first finds $u_1 \geqslant \varphi(v_0)$ according to Σ, then chooses $v_1 \in Q$ such that $v_0 \leqslant v_1$ and $u_1 \leqslant \varphi(v_1)$, using the fact that φ is dominating.

In general, given a sequence $v_0 \leqslant v_1 \leqslant \ldots \leqslant v_{n-1}$, where $n > 0$ is odd, we assume that we have the following sequence in P:

$$\varphi(v_0) \leqslant u_1 \leqslant \varphi(v_1) \leqslant \varphi(v_2) \leqslant u_3 \leqslant \ldots \leqslant u_{n-2} \leqslant \varphi(v_{n-2}) \leqslant \varphi(v_{n-1}),$$

where $u_k = \Sigma(\varphi(v_0), u_1, \varphi(v_2), u_3 \ldots, u_{k-2}, \varphi(v_{k-1}))$ for every odd $k < n$. Let

$$u_n = \Sigma(\varphi(v_0), u_1, \varphi(v_2), u_3, \varphi(v_4), \ldots, u_{n-2}, \varphi(v_{n-1})).$$

Again, Odd chooses $v_n \geqslant v_{n-1}$ in Q such that $u_n \leqslant \varphi(v_n)$. Thus, Odd wins the game since the ideal generated by $\{\varphi(v)_n\}_{n\in\omega}$ is in W. So, Odd's winning strategy in $\mathrm{BM}(Q, W^{\varphi})$ is defined.

Suppose now that Odd has a winning strategy Π in $\mathrm{BM}(Q, W^{\varphi})$. Assume Eve has started with $u_0 \in P$. First, Odd chooses $v_0 \in Q$ with $u_0 \leqslant \varphi(v_0)$. Next, he replies to v_0 according to Π, obtaining $v_1 \geqslant v_0$, and puts $u_1 = \varphi(v_1)$.

In general, given a sequence $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_{n-1}$ with n odd, we assume that there is a sequence $v_0 \leqslant v_1 \leqslant \ldots \leqslant v_{n-2}$ in Q such that $v_k = \Pi(v_0, v_1, \ldots, v_{k-1})$ and $u_k = \varphi(v_k) \leqslant u_{k+1}$ for every odd number $k < n$. Odd's response to $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_{n-1}$ is as follows. He finds $v_{n-1} \geqslant v_{n-2}$ such that $\varphi(v_{n-1}) \geqslant u_{n-1}$, then takes $v_n = \Pi(v_0, \ldots, v_{n-1})$ and puts $u_n = \varphi(v_n)$. The sequence $\{u_n\}_{n\in\omega}$ resulting from this strategy contains a cofinal subsequence which is the φ-image of $\{v_n\}_{n\in\omega}$ which was winning in $\mathrm{BM}(Q, W^{\varphi})$, therefore the ideal generated by $\{u_n\}_{n\in\omega}$ is in W. Thus, we have described Odd's winning strategy in $\mathrm{BM}(P, W)$.

The second part (when Eve has a winning strategy) is almost the same, as the rules for both players are identical. ∎

COROLLARY 5. *Let Q be a cofinal subset of a poset P and let $W \subseteq \sigma P$. Then Odd has a winning strategy in* $\mathrm{BM}(P, W)$ *if and only if he has a winning strategy in* $\mathrm{BM}(Q, W')$, *where $W' = \{I \cap Q : I \in W\}$.*

Proof. It suffices to notice that the identity mapping $\varphi\colon Q \to P$ is dominating. ∎

As a more concrete corollary, we see that Mazur was right by playing with nonempty open intervals instead of arbitrary open subsets of the real line. Let us now recall Oxtoby's theorem [7]:

THEOREM 6. *Let X be a metrizable space. Then Odd has a winning strategy in the Choquet game* $\mathrm{BM}(X)$ *if and only if X contains a dense subset which is homeomorphic to a closed subset of a generalized Baire space.*

Proof. Suppose first that $G \subseteq X$ is dense and homeomorphic to a closed subset of the generalized Baire space $\mathfrak{m}^\omega$. Then there exists a complete metric ϱ on G. We claim that Odd has a winning strategy. Namely, assuming U was the last Eve's move, Odd responds with a nonempty open set V satisfying the following two conditions: the closure of V is contained in U, and the ϱ-diameter of $V \cap G$ is (finite) smaller than half of the ϱ-diameter of $U \cap G$. By Cantor's theorem, the intersection of any sequence resulting from a play with this strategy is a singleton of G.

Now suppose Odd has a winning strategy in the Choquet game $\mathrm{BM}(X)$. As in the proof Theorem 2, we build a tree $T = \bigcup_{n\in\omega} A_n \subseteq P$ of height ω, choosing sets of diameters at most 2^{-n} for each level A_n. Then the set

$$G = \bigcap_{n\in\omega} \bigcup A_n$$

is as required, namely, it is dense in X and homeomorphic to ∂T, which is a closed subspace of a generalized Baire space (see the remarks before Theorem 2). ∎

We also have another variant of the Banach–Mazur game, for compact Hausdorff spaces. Namely, if the classical Banach–Mazur game is played in a compact Hausdorff space, then Odd has an obvious stationary (that is, depending only on the last move) winning strategy: he always chooses an open set whose closure is contained in the last set chosen by Eve. Now consider the Banach–Mazur game where the objective is to get a *single* point in the intersection of the chain of open sets. Let us call this game $\mathrm{BM}(X, \star)$, where X is the topological space in question. It turns out that there are non-metrizable compact Hausdorff spaces where Eve has a winning strategy in this game.

THEOREM 7. *Let K be a compact Hausdorff space. The following properties are equivalent*:

(a) *Odd has a winning strategy in* $\mathrm{BM}(K, \star)$.
(b) *K contains a dense G_δ metrizable subspace.*

Proof. The proof of (b) $\Longrightarrow$ (a) is like the one in Theorem 6. For the converse, we use the same tree T as above, noting that it induces a metrizable subspace without assuming that the entire space K is metrizable. The fact that a compact Hausdorff space is regular is needed to conclude that the completely metrizable space of branches of T is indeed dense in K. ∎

Let us recall the *double arrow* space $K = D(\mathbb{I})$. This is a compact Hausdorff space whose universe is $((0,1] \times \{0\}) \cup ([0,1) \times \{1\})$ endowed with the interval topology induced from the lexicographic ordering[2]. Let $p\colon K \to \mathbb{I}$ be the canonical projection. Eve's winning

[2]Note that the double arrow K consists of two disjoint copies the Sorgenfrey line, both being dense in K.

strategy in $\mathrm{BM}(K, \star)$ is as follows: She always chooses an interval U in K such that the closure of $p[U]$ is in the interior of $p[V]$, where V was the last choice of Odd. If we suppose that Odd wins while Eve plays this strategy, there would be a single point $x \in K$ in the intersection of $U_0 \supseteq U_1 \supseteq \ldots$ resulting from a play. Now observe that x is isolated from one side. For example, assume that $x = \langle y, 0 \rangle$, where $y < 1$. Then $\max p[U_n] = y$ from some point on (otherwise $\langle y, 1 \rangle$ would be in the intersection), but this contradicts Eve's strategy saying that the closure of $p[U_{n+1}]$ is contained in $p[U_n]$.

At this point it is worth recalling that there are separable metric spaces in which the Banach–Mazur game is not determined. Namely, recall that a *Bernstein set* in a metrizable space is a set S satisfying $S \cap P \neq \emptyset \neq S \setminus P$ for every perfect set P (a set is *perfect* if it is nonempty, completely metrizable, and has no isolated points). A Bernstein set in 2^ω can be easily constructed by a transfinite induction, by enumerating all perfect sets in 2^ω and knowing that each perfect set in 2^ω has cardinality continuum.

The following fact is well-known and easy to prove using Theorem 2.

PROPOSITION 8. *Let $X \subseteq 2^\omega$ be a Bernstein set. Then the Banach–Mazur game* $\mathrm{BM}(X)$ *is not determined. Namely, neither Eve nor Odd has a winning strategy in* $\mathrm{BM}(X)$.

4. Applications to model theory. We are now going to show that the Banach–Mazur game is determined when one considers the poset of all finitely generated structures of a fixed first order language, as long as some natural conditions are satisfied.

We now recall the concept of a *Fraïssé class*. Namely, this is a class $\mathfrak{K}$ of finitely generated models of a fixed language satisfying the following conditions:

(F1) For each $X, Y \in \mathfrak{K}$ there is $Z \in \mathfrak{K}$ such that both X and Y embed into Z.
(F2) Given embeddings $f \colon Z \to X$, $g \colon Z \to Y$ with $Z, X, Y \in \mathfrak{K}$, there exist $V \in \mathfrak{K}$ and embeddings $f' \colon X \to V$, $g' \colon Y \to V$ such that $f' \circ f = g' \circ g$.
(F3) For every $X \in \mathfrak{K}$, every finitely generated substructure of X is in $\mathfrak{K}$.
(F4) There are countably many isomorphic types in $\mathfrak{K}$.

Condition (F1) is called the *joint embedding property*, (F2) is called the *amalgamation property*. Condition (F3) says that $\mathfrak{K}$ is *hereditary* with respect to finitely generated substructures.

Fraïssé theorem [2] says that there exists a unique countably generated model $\mathbb{U}$ (called the *Fraïssé limit of* $\mathfrak{K}$) that can be presented as the union of a countable chain in $\mathfrak{K}$ and satisfies the following conditions:

(U) Every $X \in \mathfrak{K}$ embeds into $\mathbb{U}$.
(E) Given an isomorphism $h \colon A \to B$ between finitely generated substructures $A, B \subseteq \mathbb{U}$, there exists an automorphism $H \colon \mathbb{U} \to \mathbb{U}$ extending h.

Let us denote by $\mathrm{BM}(\mathfrak{K}, \mathbb{U})$ the Banach–Mazur game played in the poset[3] consisting of all structures $X \in \mathfrak{K}$, where the ordering $\leqslant$ is inclusion (more precisely, extension of structures) and the winning ideals are precisely the structures isomorphic to the Fraïssé

[3]We implicitly assume that all models "live" in a certain fixed set, by this way we avoid dealing with a proper class instead of a set. For example, if finitely generated structures are finite then we may assume that the universe of each of them is a subset of $\mathbb{N}$.

limit of $\mathfrak{K}$. In other words, Odd wins if and only if the resulting structure is isomorphic to $\mathbb{U}$. In general, $\mathfrak{K}$ can be an arbitrary class and $\mathbb{U}$ can be an arbitrarily fixed model that is presentable as the union of a countable chain of models from the class $\mathfrak{K}$. We shall later consider a more general version of this game, where a single model $\mathbb{U}$ is replaced by a family of models $\mathcal{U}$ and Odd wins if the union of the chain built by the two players is isomorphic to some $U \in \mathcal{U}$. We denote this game by $\mathrm{BM}(\mathfrak{K},\mathcal{U})$.

Recall that a strategy of a fixed player is *Markov* if his/her move depends only on the last move of the opponent and on the number of past moves.

THEOREM 9. *If $\mathfrak{K}$ is a Fraïssé class and $\mathbb{U}$ is the Fraïssé limit of $\mathfrak{K}$, then Odd has a Markov winning strategy in* $\mathrm{BM}(\mathfrak{K},\mathbb{U})$.

Proof. Let $\mathbb{U}$ be the Fraïssé limit of $\mathfrak{K}$ and write $\mathbb{U} = \bigcup_{n\in\omega} U_n$, where $U_n \in \mathfrak{K}$ for each $n \in \omega$. Odd's strategy is described as follows.

Supposing that last Eve's move was V_n (with n even) and having recorded an embedding $f_{n-1}\colon V_{n-1} \to \mathbb{U}$, Odd first chooses an embedding $g\colon V_n \to \mathbb{U}$ extending f_{n-1}. Next, he finds $V_{n+1} \in \mathfrak{K}$ with $V_n \leqslant V_{n+1}$ and an embedding $f_{n+1}\colon V_{n+1} \to \mathbb{U}$ extending g and such that U_n is contained in the range of f_{n+1}. In case $n = 0$, we assume that f_{-1} was the empty map.

It is clear that this strategy is winning for Odd, because after playing the game we obtain an isomorphism $f = \bigcup_{n\in\omega} f_{2n+1}$ of $\bigcup_{n\in\omega} V_n$ onto $\mathbb{U}$. The strategy depends only on the result of last Eve's move and on the number of previous moves. ∎

One of the most important features of the Fraïssé limit is that it is universal for the class of all countably generated structures obtained as unions of countable chains in $\mathfrak{K}$. Using the Banach–Mazur game, we can give a simple direct argument in a more general setting.

THEOREM 10. *Let $\mathfrak{K}$ be a class of finitely generated models with the amalgamation property. Let $\mathcal{U}$ be a class of countably generated models of the same language, such that Odd has a winning strategy in* $\mathrm{BM}(\mathfrak{K},\mathcal{U})$.

Then every countably generated model representable as the union of a countable chain in $\mathfrak{K}$ is embeddable into some $U \in \mathcal{U}$.

In case $\mathfrak{K}$ is a Fraïssé class, we can set $\mathcal{U} = \{\mathbb{U}\}$, where $\mathbb{U}$ is the Fraïssé limit of $\mathfrak{K}$.

Proof. Assume $X = \bigcup_{n\in\omega} X_n$, where $X_n \leqslant X_{n+1}$ and $X_n \in \mathfrak{K}$ for each $n \in \omega$. Let us play the game $\mathrm{BM}(\mathfrak{K},\mathcal{U})$, where Odd uses his winning strategy. We shall denote by $U_0 \leqslant U_1 \leqslant U_2 \leqslant \dots$ the concrete moves. We describe a strategy of Eve leading to an embedding of X into some $U \in \mathcal{U}$.

Namely, Eve starts with $U_0 = X_0$ and records $e_0 = \mathrm{id}_{X_0}$. Supposing that the last Odd's move was U_{2n-1} and Eve has recorded an embedding $e_{n-1}\colon X_{n-1} \to U_{2n-2}$, she uses the amalgamation property to find $U_{2n} \geqslant U_{2n-1}$ and an embedding $e_n\colon X_n \to U_{2n}$ such that the diagram

$$\begin{array}{ccccc} U_{2n-2} & \longrightarrow & U_{2n-1} & \longrightarrow & U_{2n} \\ {\scriptstyle e_{n-1}}\uparrow & & & & \uparrow{\scriptstyle e_n} \\ X_{n-1} & & \longrightarrow & & X_n \end{array}$$

commutes, where the horizontal embeddings are inclusions. By this way, e_n extends e_{n-1}.

After playing the game, knowing that Odd wins, we conclude that $U = \bigcup_{n\in\omega} U_n$ is an element of $\mathcal{U}$ and $e = \bigcup_{n\in\omega} e_n$ is an embedding of X into U. ∎

Let $\mathfrak{K}$ be a class of finitely generated models and let U be a countably generated model of the same language such that Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$. It is natural to ask what can be said about $\mathfrak{K}$ and U.

Clearly, $\mathfrak{K}$ has the joint-embedding property (F1), because Eve can play with any element of $\mathfrak{K}$, showing that U is universal for $\mathfrak{K}$ (one can also use Theorem 10, however this would be an overkill). Assuming that $\mathfrak{K}$ consists of finite substructures, we conclude that (F4) must hold too, because U is countable and therefore it has countably many isomorphic types of finite structures. Obviously, (F3) may fail. For example, let $\mathfrak{G}$ be a relational Fraïssé class (say, the class of all finite graphs), and let $\mathfrak{K}$ be the subclass of $\mathfrak{G}$ consisting of all $G \in \mathfrak{G}$ whose cardinality is a prime number. Let $\mathbb{U}$ be the Fraïssé limit of $\mathfrak{G}$. Then Odd has a winning strategy in the game $\mathrm{BM}(\mathfrak{K}, \mathbb{U})$, as he can improve his Markov winning strategy for $\mathrm{BM}(\mathfrak{G}, \mathbb{U})$ by enlarging his choices so that their cardinalities are always prime.

The following two examples from graph theory show that $\mathfrak{K}$ may fail the amalgamation property even when it satisfies (F1), (F3) and (F4).

Example 11 (Graphs with bounded degree). Let $N > 1$ be a fixed integer and let $\mathfrak{K}$ be the class of all finite graphs whose each vertex has degree $\leqslant N$. It is well-known and easy to prove that each finite connected graph $G \in \mathfrak{K}$ embeds into a connected graph $H \in \mathfrak{K}$ such that the degree of *every* vertex of H is precisely N. Let us call such a graph *N-complete*. For example, finite 2-complete graphs are precisely the cycles.

Clearly, $\mathfrak{K}$ is not a Fraïssé class, as it fails the amalgamation property.

Let us enumerate by $\{H_n\}_{n\in\omega}$ all finite N-complete graphs. Let $U = \bigoplus_{n\in\omega} U_n$, where "$\bigoplus$" means the disjoint sum (and no extra edges between the summands), $\{U_n : n \in \omega\} = \{H_n : n \in \omega\}$, and for each $k \in \omega$ the set $\{n \in \omega : U_n = H_k\}$ is infinite. In other words, U is the direct sum of an indexed family consisting of (countably) infinitely many copies of each H_n.

We claim that Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$. Indeed, after n-th move of Eve resulting in a graph $G_{2n} \in \mathfrak{K}$, Odd chooses G_{2n+1} of the form $\bigoplus_{i<k(n)} U_i$, knowing that each component of G_{2n} embeds into some U_i. He only needs to take care that $k(n) \to \infty$ while $n \to \infty$. By this way, the graph resulting from a single play is obviously isomorphic to U.

Example 12 (Cycle-free graphs). Let $\mathfrak{K}$ denote the class of all finite cycle-free graphs. Again, this is not a Fraïssé class, as it fails the amalgamation property. On the other hand, we claim that there is a countable cycle-free graph T such that Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, T)$.

Namely, let T be the (uniquely determined) countable connected cycle-free graph whose each vertex has infinite degree. The graph T is well-known as the complete infinitely-branching tree with a single root. The winning strategy of Odd is as follows. At stage n, after Eve's move G_{2n}, Odd chooses $G_{2n+1} \supseteq G_{2n}$ so that G_{2n+1} is connected, has at least n levels (when fixing the root), and that the degree of each vertex except those at the highest level is greater than or equal to n. It is clear that $\bigcup_{n\in\omega} G_{2n+1}$ is isomorphic to T.

Conclusion. Let, as above, $\mathfrak{K}$ be a countable class of finitely generated models of a fixed first order language and assume U is such that Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$ (obviously, U must be presentable as the union of a countable chain of models from $\mathfrak{K}$). In that case we say that U is *generic* over $\mathfrak{K}$ (see Example 1 for an inspiration). We have seen that Fraïssé limits are generic over their Fraïssé classes, however, there exist generic models that are not Fraïssé limits in the usual sense. There exists a category-theoretic generalization of Fraïssé limits [5], which in the case of models discards condition (F3) of being hereditary and possibly makes restrictions on embeddings. By this way, we can talk about *Fraïssé categories* instead of Fraïssé classes. It can be proved that if $\mathfrak{K}$ contains a Fraïssé subcategory $\mathfrak{L}$ that is *dominating* in the sense of [5] then Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$ if and only if U is the Fraïssé limit of $\mathfrak{L}$ (in the setting of [5]). We do not know whether the converse is true. In any case, generic objects seem to be a natural and applicable (see Theorem 10) generalization of Fraïssé limits.

Finally, let us note that in the Banach–Mazur game $\mathrm{BM}(\mathfrak{K}, U)$, the class $\mathfrak{K}$ can be just an abstract class of objects as long as the notion of an "embedding" is defined. It seems that the language of category theory is most suitable here. Namely, $\mathfrak{K}$ could be a fixed category and U could be a fixed object (typically in a bigger category containing $\mathfrak{K}$) that is isomorphic to the co-limit of some sequence in $\mathfrak{K}$. This approach will be explored elsewhere.

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