

## Symmetric Lie models of a triangle

by

**Urtzi Buijs** (Málaga), **Yves Félix** (Louvain-la-Neuve),  
**Aniceto Murillo** (Málaga) and **Daniel Tanré** (Lille)

**Abstract.** R. Lawrence and D. Sullivan have constructed a Lie model for an interval from the geometrical idea of flat connections and flows of gauge transformations. Their model supports an action of the symmetric group  $\Sigma_2$  reflecting the geometrical symmetry of the interval. In this work, we present a Lie model of the triangle with an action of the symmetric group  $\Sigma_3$  compatible with the geometrical symmetries of the triangle. We also prove that the model of a graph consisting of a circuit with  $k$  vertices admits a Maurer–Cartan element stable by the automorphisms of the graph.

**1. Introduction.** In rational homotopy theory, the existence of algebraic models of topological spaces in terms of Lie algebras was first established by D. Quillen [8] for simply connected spaces. His work is based on a succession of couples of adjoint functors between the category of simply connected spaces and the category of differential Lie algebras, positively graded. The complexity of the Quillen functors contrasts with the simple couple of functors  $(A_{PL}, \langle - \rangle)$  introduced by D. Sullivan [10] between the category of simplicial sets and the category of commutative differential graded algebras (cdga for short). Sullivan’s construction is based on a contractible simplicial cdga  $\Omega_\bullet$  defined by

$$\Omega_n = \mathbb{Q}[t_0, \dots, t_n] \otimes \wedge(dt_0, \dots, dt_n) / \left( \sum t_i - 1, \sum dt_i \right),$$

with the  $t_i$ ’s in degree zero and the  $dt_i$ ’s in degree one. The cdga  $A_{PL}(X)$  associated to a simplicial set  $X$  is defined from the set of simplicial maps from  $\Omega_\bullet$  to  $X$ .

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A natural question is the existence of a similar construction in the context of Lie algebras. We solve it in [1] from the cocommutative, coassociative, infinity coalgebra structure on the rational singular chains and a transfer process. For a finite simplicial set,  $K$ , this gives a complete differential graded Lie algebra  $\mathfrak{L}(K)$  which contains information on the rational homotopy type of  $K$ , like the Bousfield–Kan completion of its fundamental group for instance. The departure point is the construction of a family  $\mathfrak{L}_k = \mathfrak{L}(\Delta^k)$  which extends the construction by R. Lawrence and D. Sullivan [6] of a complete differential Lie algebra  $\mathfrak{L}_1$  in the case of the interval. We recall it in the first section. Their model is symmetric for the action of the symmetric group  $\Sigma_2$  deduced from the geometry of the interval.

In Section 4, we recall the construction of a Lie model  $\mathfrak{L}_2$  of the triangle  $\Delta^2$ . Being based on a vertex of  $\Delta^2$ , this model does not reflect the symmetries of a triangle. We modify it to a Lie model with an action of the symmetric group  $\Sigma_3$  compatible with the different symmetries. For that, we consider the geometrical action of  $\Sigma_3$  on the generators of  $\mathfrak{L}_2$  and prove the existence of a Maurer–Cartan element stable by any element of  $\Sigma_3$ . Even though the differential in  $\mathfrak{L}_2$  is, to some extent, uniquely determined (Proposition 4.1), there are many Maurer–Cartan elements invariant by  $\Sigma_3$ .

The Maurer–Cartan elements play the role of points and are recalled in Section 2. A Maurer–Cartan element invariant by the action of  $\Sigma_3$  is an algebraic representation of the barycentre of the triangle. After the interval of Lawrence and Sullivan, this is the second stage in the research of a Lie  $\Sigma_n$ -model of a simplex  $\Delta^n$ .

Our first step is to prove that the Lie model  $\mathfrak{L}(\Gamma)$ , of a graph  $\Gamma$  consisting of a circuit with  $k$  vertices, contains a Maurer–Cartan element invariant by the automorphisms of the graph.

In all this work, vector spaces and chain complexes are  $\mathbb{Z}$ -graded and we denote by  $|x|$  the degree of an element. The free Lie algebra on a rational vector space  $V$  is represented by  $\mathbb{L}(V)$ . The vector subspace  $\mathbb{L}^i(V)$  is the set of elements of  $\mathbb{L}(V)$  of bracket length  $i$ . The objects considered in this work are complete free Lie algebras, defined by

$$\widehat{\mathbb{L}}(V) = \varprojlim_i \mathbb{L}(V) / \mathbb{L}^{\geq i}(V).$$

Endowed with a Lie differential, they are called complete differential graded Lie algebras (henceforth cdgl), and if specifying the vector space  $V$  is not necessary, we denote also  $L = \widehat{\mathbb{L}}(V)$ . If  $x \in L$ , the map  $\text{ad}_x: L \rightarrow L$  is the adjoint derivation  $y \mapsto [x, y]$ .

**2. Background.** The *Baker–Campbell–Hausdorff product* (henceforth BCH-product) of two elements  $x$  and  $y$  of degree 0 in a free complete Lie

algebra  $\widehat{\mathbb{L}}(V)$  is defined by

$$x * y = \log(e^x e^y).$$

This product satisfies the following properties (see [1], [5] or [6] for instance) for any  $x, y, z$  in  $L_0$  and  $r, s$  in  $\mathbb{Q}$ :

- (i)  $x * (y * z) = (x * y) * z$ ,
- (ii)  $(rx) * (sx) = (r + s)x$ ; in particular, the inverse of  $x$  is  $-x$ , i.e.,  $x * (-x) = 0$ ,
- (iii)  $-(y * x) = (-x) * (-y)$ ,
- (iv)  $\text{ad}_{x*y} = \text{ad}_x * \text{ad}_y$  and  $e^{\text{ad}_{x*y}} = e^{\text{ad}_x} \circ e^{\text{ad}_y}$ ,
- (v)  $e^{\text{ad}_x}(y) = x * y * (-x)$  and  $e^{\text{ad}_x}(y) * e^{\text{ad}_x}(z) = e^{\text{ad}_x}(y * z)$ ,
- (vi)  $r(x * y) * x = x * r(y * x)$  and  $r(y * x) * (-x) = (-x) * r(x * y)$ .

As the BCH-product is associative, we can introduce the following notation, for any family  $(w_i)_{1 \leq i \leq r}$  of elements in  $L_0$ ,

$$\underset{i=1}{*}^r w_i = w_1 * \cdots * w_r.$$

The next property follows from the Dynkin formula for the BCH-product, see [9].

LEMMA 2.1. *Let  $(w_i)_{1 \leq i \leq r}$  be elements of degree zero in  $\widehat{\mathbb{L}}(V)$  that coincide in the quotient  $\widehat{\mathbb{L}}(V)/\widehat{\mathbb{L}}^{>n}(V)$ . Then, for any rational numbers  $\lambda_i$ , we have an equality of classes modulo  $\widehat{\mathbb{L}}^{>n+1}(V)$ ,*

$$\left[ \underset{i=1}{*}^r \lambda_i w_i \right]^{<n+1} = \left[ \sum_{i=1}^r \lambda_i w_i \right]^{<n+1}.$$

Given a cdgl  $(L, d)$ , a *Maurer–Cartan element* (for short a MC-element) is an element  $a \in L_{-1}$  such that

$$(2.1) \quad da + \frac{1}{2}[a, a] = 0.$$

If  $a$  is a MC-element in a cdgl  $(L, d)$ , the derivation defined by

$$d_a x = \text{ad}_a(x) + dx, \quad x \in L,$$

is a differential.

Denote by  $\text{MC}(L)$  the set of MC-elements of  $L$ . There is an action of the complete Lie algebra of elements of degree 0,  $L_0$ , on  $\text{MC}(L)$ , called *gauge action*, and defined as follows:

$$(2.2) \quad x\mathcal{G}a = e^{\text{ad}_x}(a) - \frac{e^{\text{ad}_x} - 1}{\text{ad}_x}(dx) = \sum_{i \geq 0} \frac{\text{ad}_x^i(a)}{i!} - \sum_{i \geq 0} \frac{\text{ad}_x^i(dx)}{(i+1)!},$$

with  $x \in L_0$ ,  $a \in \text{MC}(L)$  [2, Proposition 1.2]. If  $L = \widehat{\mathbb{L}}(V)$ , the linear and

the quadratic parts of the gauge action can be explicitly determined:

$$(2.3) \quad (x\mathcal{G}a)_1 = a_1 - (dx)_1,$$

$$(2.4) \quad (x\mathcal{G}a)_2 = a_2 + [x_1, a_1] - \frac{1}{2}[x_1, d_1x_1] - d_2x_1 - d_1x_2,$$

where the decompositions according to the bracket length are denoted  $a = \sum_{i \geq 1} a_i$ ,  $x = \sum_{i \geq 1} x_i$ ,  $d = \sum_{i \geq 1} d_i$  with  $d_i(V) \subset \mathbb{L}^i(V)$ .

The gauge action is compatible with the BCH-product in  $L_0$ : for any  $x, y \in L_0$  and  $a \in \text{MC}(L)$ , we have

$$(2.5) \quad x\mathcal{G}y\mathcal{G}a = (x * y)\mathcal{G}a.$$

(In the context of models of the interval, this property is a consequence of [6, Theorem 2] but it also appears in different works: see [4, Appendix B] for instance.) We denote by  $\widehat{\text{MC}}(L)$  the set of equivalence classes for the gauge action.

In [6], R. Lawrence and D. Sullivan have defined a Lie model  $\mathfrak{L}_1$  of the interval  $\Delta^1$ :

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a, b, x), d)$$

with  $|a| = |b| = -1$ ,  $|x| = 0$ ,  $da = -\frac{1}{2}[a, a]$ ,  $db = -\frac{1}{2}[b, b]$  and

$$(2.6) \quad dx = \text{ad}_x a + \frac{\text{ad}_{-x}}{e^{-\text{ad}_x} - 1}(b - a) = \frac{\text{ad}_x}{1 - e^{\text{ad}_x}} a + \frac{\text{ad}_x}{1 - e^{-\text{ad}_x}} b$$

$$(2.7) \quad = \text{ad}_x b + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}(b - a).$$

The cdgl  $\mathfrak{L}_1$  is called the *LS-interval*. The elements  $a$  and  $b$  are MC-elements. The set of MC-elements of  $\mathfrak{L}_1$  has been characterized in [2] and the unicity of such a model is established in [7]. As was observed by Lawrence and Sullivan, an element  $x$  of degree 0 in a cdgl  $(\widehat{\mathbb{L}}(V), d)$  defines a flow by

$$\frac{dv}{dt} = dx - \text{ad}_x(v), \quad |v| = -1.$$

The flow associated to  $x$ , from  $a$  to  $b$ , in time 1, corresponds to

$$(2.8) \quad (-x)\mathcal{G}a = b$$

which determines the expression (2.6) of the differential of  $x$ . Such an  $x$  is called a *path* from  $a$  to  $b$ . The differential  $dx$  is clearly stable by the symmetry  $(\tau: a \leftrightarrow b, \tau x = -x)$  which generates the symmetric group  $\Sigma_2$ . We also mention the existence of a symmetric model of a 2-gon in [5], where the geometrical action of  $\Sigma_2$  is reflected on the differential of its model.

The next proposition recalls the well known relation between  $d_a$  and  $d_b$  in  $\mathfrak{L}_1$ . We give the proof for completeness.

**PROPOSITION 2.2.** *Let  $(L, d)$  be a cdgl containing a LS-interval  $(\widehat{\mathbb{L}}(a, b, x), d)$ . Then, for any  $w \in L$ , we have*

$$d_a e^{\text{ad}_x}(w) = e^{\text{ad}_x}(d_b w).$$

*Proof.* From [6, Lemma 1], we have

$$d_b(e^{-\text{ad}_x} w) = e^{-\text{ad}_x}(d_b w) + (-1)^{|w|} e^{-\text{ad}_x} \text{ad}_w \frac{e^{-\text{ad}_x} - 1}{\text{ad}_x}(d_b x).$$

Replacing  $d_b x$  by its value (2.7) gives

$$d_b(e^{-\text{ad}_x} w) = e^{-\text{ad}_x}(d_b w + (-1)^{|w|} \text{ad}_w(b - a)) = e^{-\text{ad}_x}(d_a w). \blacksquare$$

**3. Lie model of a graph consisting of a circuit.** In this section we consider  $\text{cdgl}$ 's endowed with an action of a group  $G$ .

**DEFINITION 3.1.** A  $G$ - $\text{cdgl}$  is a  $\text{cdgl}$   $(L, d)$ , with a group action  $G \times L \rightarrow L$ , compatible with the differential and the Lie bracket, i.e., for any  $v, w \in L$  and  $\varphi \in G$ , we have

$$\varphi[v, w] = [\varphi(v), \varphi(w)] \quad \text{and} \quad \varphi(dv) = d(\varphi(v)).$$

Let  $\Gamma$  be a graph consisting of a circuit with  $k$  vertices. Then  $\Gamma$  is the union of  $k$  intervals and its Lie model is a concatenation of LS-intervals [6], i.e.,

$$\mathfrak{L}(\Gamma) = \left( \widehat{\mathbb{L}} \left( \bigoplus_{i=1}^k (\mathbb{Q}\mathbf{v}_i \oplus \mathbb{Q}\mathbf{x}_i) \right), d \right),$$

where the  $\mathbf{v}_i$  are MC-elements and the sub- $\text{dgl}$ 's  $(\widehat{\mathbb{L}}(\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{x}_i), d)$  are LS-intervals. Here, by convention, we set  $\mathbf{v}_{k+1} = \mathbf{v}_1$ ,  $\mathbf{x}_{k+1} = \mathbf{x}_1$ .

The group  $G$  of automorphisms of  $\Gamma$  is the dihedral group  $D_{2k}$  generated by the cyclic permutation  $\sigma$  and the involution  $\tau$  on the first vertex,

$$G = \langle \sigma, \tau; \sigma^k, \tau^2, \sigma\tau = \tau\sigma^{-1} \rangle.$$

The  $\text{cdgl}$   $\mathfrak{L}(\Gamma)$  is a  $G$ - $\text{cdgl}$ . The actions of  $\sigma$  and  $\tau$  are defined by

$$\sigma(\mathbf{x}_i) = \mathbf{x}_{i+1}, \quad \sigma(\mathbf{v}_i) = \mathbf{v}_{i+1}, \quad \tau(\mathbf{v}_i) = \mathbf{v}_{k-i+2}, \quad \tau(\mathbf{x}_i) = -\mathbf{x}_{k-i+1}.$$

Denote by  $V$  the graded vector space generated by the  $\mathbf{x}_i$  and the  $\mathbf{v}_i$ . We decompose the differential  $d$  in  $\mathfrak{L}(\Gamma)$  in the form  $d = \sum_{i \geq 1} d_i$  with  $d_i(V) \subset \mathbb{L}^i(V)$ .

**LEMMA 3.2.** *With the above notation, the following properties are satisfied.*

- (i) Any  $d_1$ -cycle of degree 0 in  $\widehat{\mathbb{L}}^{\geq 2}(V)$  is equal to 0.
- (ii) An element  $w$  of degree 0 is such that  $w\mathcal{G}_{\mathbf{v}_1} = \mathbf{v}_1$  if and only if  $w = \lambda(\mathbf{x}_1 * \cdots * \mathbf{x}_k)$  for some rational number  $\lambda$ .

*Proof.* (i) Since  $H(\widehat{\mathbb{L}}(V), d_1) = \widehat{\mathbb{L}}(H(V, d_1))$ , we are reduced to the determination of the homology of the 1-dimensional simplicial complex defined by the graph  $\Gamma$ . We obtain

$$H(\widehat{\mathbb{L}}(V), d_1) = \widehat{\mathbb{L}}(\mathbf{v}_1, \mathbf{x}_1 + \cdots + \mathbf{x}_k),$$

and any cycle of degree 0 having a zero linear part is a boundary. Since  $\widehat{\mathbb{L}}(V)$  has no element of strictly positive degree, this cycle must be 0.

(ii) First, from (2.5), we have  $\lambda(\mathbf{x}_1 * \cdots * \mathbf{x}_k) \mathcal{G}\mathbf{v}_1 = \mathbf{v}_1$ . Now, let  $w$  be such that  $w \mathcal{G}\mathbf{v}_1 = \mathbf{v}_1$ . Using (2.3), we get  $\mathbf{v}_1 - d_1 w_1 = \mathbf{v}_1$  and  $d_1 w_1 = 0$ . This implies the existence of  $\lambda \in \mathbb{Q}$  such that  $w_1 = \lambda(\sum_{i=1}^k \mathbf{x}_i)$ . The element  $z = (-\lambda(\mathbf{x}_1 * \cdots * \mathbf{x}_k)) * w$  satisfies  $z \mathcal{G}\mathbf{v}_1 = \mathbf{v}_1$  and has a zero linear part. From (2.4), we deduce that its quadratic part satisfies  $d_1 z_2 = 0$ . From (i), this implies  $z_2 = 0$ . By induction, we have  $z_i = 0$  for all  $i$  and  $w = \lambda(\mathbf{x}_1 * \cdots * \mathbf{x}_k)$ . ■

LEMMA 3.3. *With the previous notation, any  $w \in \widehat{\mathbb{L}}(\bigoplus_{i=1}^k \mathbb{Q}\mathbf{x}_i)$  such that  $\sigma(d_1 w) = d_1 w$  is invariant by  $\sigma$ , i.e.,  $\sigma(w) = w$ .*

*Proof.* Since  $\sigma$  preserves the bracket length, we can suppose that  $w \in \mathbb{L}^\ell(\bigoplus_{i=1}^k \mathbb{Q}\mathbf{x}_i)$  for some  $\ell \geq 1$ . In the enveloping algebra,  $T(\bigoplus_{i=1}^k \mathbb{Q}\mathbf{x}_i)$ ,  $w$  can be written in a unique way as

$$w = \sum_{i=1}^k \mathbf{x}_i X_i \quad \text{with} \quad X_i \in T^{\ell-1}\left(\bigoplus_{i=1}^k \mathbb{Q}\mathbf{x}_i\right).$$

Denoting  $\nu_i = d_1 \mathbf{x}_i = \mathbf{v}_{i+1} - \mathbf{v}_i$ , we have  $\sum_{i=1}^k \nu_i = 0$  and

$$\begin{aligned} d_1 w &= \sum_{i=1}^{k-1} \nu_i X_i - \left(\sum_{i=1}^{k-1} \nu_i\right) X_k + \sum_{i=1}^k \mathbf{x}_i d_1 X_i \\ &= \sum_{i=1}^{k-1} \nu_i (X_i - X_k) + \sum_{i=1}^k \mathbf{x}_i d_1 X_i, \\ \sigma(d_1 w) &= \sum_{i=1}^{k-1} \nu_i (\sigma X_{i-1} - \sigma X_{k-1}) + \sum_{i=1}^k \mathbf{x}_{i+1} \sigma(d_1 X_i). \end{aligned}$$

The hypothesis  $\sigma(d_1 w) = d_1 w$  implies

$$X_i - X_k = \sigma X_{i-1} - \sigma X_{k-1}.$$

Thus, the element  $\sigma X_{i-1} - X_i$  does not depend on  $i$ . Denoting this common value by  $E = \sigma X_{i-1} - X_i$ , we have  $\sigma X_i = X_{i+1} + E$ , which implies

$$\sigma(w) = \sum_{i=1}^k \mathbf{x}_{i+1} \sigma X_i = \sum_{i=1}^k \mathbf{x}_{i+1} X_{i+1} + \left(\sum_{i=1}^k \mathbf{x}_{i+1}\right) E = w + \left(\sum_{i=1}^k \mathbf{x}_i\right) E.$$

Since  $w$  and  $\sigma(w)$  are Lie elements, the vector  $(\sum_{i=1}^k \mathbf{x}_i) E$  must be primitive. Therefore,  $E = 0$  and  $\sigma(w) = w$ . ■

THEOREM 3.4. *The cdgl  $\mathfrak{L}(\Gamma)$  contains a MC-element invariant by the actions of  $\sigma$  and  $\tau$ .*

*Proof.* The first step of the proof consists in a construction by induction of a MC-element stable by the action of  $\sigma$ . We define

$$(3.1) \quad P[1] = - \left( \sum_{i=1}^{k-1} \frac{k-i}{k} \mathbf{x}_i \right) \mathcal{G} \mathbf{v}_1.$$

Using the property (2.3) of the gauge action, one finds the linear part  $P[1]_1$  of  $P[1]$ :

$$P[1]_1 = \mathbf{v}_1 + \sum_{i=1}^{k-1} \frac{k-i}{k} d_1 \mathbf{x}_i = \frac{1}{k} \sum_{i=1}^k \mathbf{v}_i.$$

Therefore,  $P[1]$  is a MC-element whose linear part is  $\sigma$ -invariant. By induction, suppose we have constructed MC-elements  $P[i]$  for  $i = 1, \dots, n-1$ , such that

- (a)  $P[i] - P[i-1] \in \mathbb{L}^{\geq i}(V)$ ,
- (b)  $\sigma P[i]_j = P[i]_j$  for any  $j \leq i$ , where  $P[i]_j$  denotes the component of  $P[i]$  in the Lie brackets of length  $j$ .

As the graph  $\Gamma$  is connected, from [3, Theorem] there is only one non-trivial equivalence class of MC-elements. Thus, there is an element  $T$  of degree 0 satisfying

$$(3.2) \quad T \mathcal{G} P[n-1] = \sigma P[n-1].$$

Formula (2.3) gives the following equality between linear parts:

$$P[n-1]_1 - d_1 T_1 = \sigma P[n-1]_1.$$

Since  $P[n-1]_1 = \sigma P[n-1]_1$ , it follows that  $d_1 T_1 = 0$  and  $T_1$  is  $\sigma$ -invariant. We prove by induction on  $j$  that  $T_j$  is  $\sigma$ -invariant for  $j < n$ . So suppose  $T_1, \dots, T_{r-1}$  are  $\sigma$ -invariant. Then from  $T \mathcal{G} P[n-1] = \sigma P[n-1]$ , we find that  $d_1 T_r$  is  $\sigma$ -invariant, and by Lemma 3.3,  $T_r$  is  $\sigma$ -invariant.

The same construction in Lie brackets of length  $n$  gives a  $\sigma$ -invariant element  $S$  such that

$$(3.3) \quad P[n-1]_n + S - d_1 T_n = \sigma P[n-1]_n.$$

By successive applications of  $\sigma$ , we deduce the system of equations

$$\begin{aligned} P[n-1]_n + S - d_1 T_n &= \sigma P[n-1]_n, \\ \sigma P[n-1]_n + S - d_1 \sigma T_n &= \sigma^2 P[n-1]_n, \\ &\dots \\ \sigma^{k-1} P[n-1]_n + S - d_1 \sigma^{k-1} T_n &= P[n-1]_n. \end{aligned}$$

Summing these equations gives

$$(3.4) \quad kS = d_1 (T_n + \sigma T_n + \dots + \sigma^{k-1} T_n).$$

We define  $T'$  and  $P'$  by

$$T'_i = \begin{cases} T_i & \text{for } i < n, \\ \frac{k-1}{k}T_n + \frac{k-2}{k}\sigma T_n + \cdots + \frac{1}{k}\sigma^{k-2}T_n & \text{for } i = n, \\ 0 & \text{for } i > n, \end{cases}$$

and

$$P' = T' \mathcal{G}P[n-1].$$

By construction,  $P'_{<n} = P[n-1]_{<n}$ . On the other hand,

$$(3.5) \quad P'_n = P[n-1]_n + S' - \sum_{i=0}^{k-2} \frac{k-i-1}{k} d_1 \sigma^i T_n,$$

with  $S'$  invariant by  $\sigma$ . We apply  $\sigma$  to this equality and obtain

$$\sigma P'_n = \sigma P[n-1]_n + S' - \sum_{i=0}^{k-2} \frac{k-i-1}{k} d_1 \sigma^{i+1} T_n.$$

Using (3.3) and extracting the value of  $\sigma^{k-1}T_n$  of (3.4), we have

$$\begin{aligned} \sigma P'_n &= P[n-1]_n + S - d_1 T_n + S' - \sum_{i=0}^{k-3} \frac{k-i-1}{k} d_1 \sigma^{i+1} T_n \\ &\quad - \frac{1}{k} \left( kS - \sum_{i=0}^{k-2} d_1 \sigma^i T_n \right). \end{aligned}$$

After reordering the indices, this formula becomes

$$\sigma P'_n = P[n-1]_n + S' - \sum_{j=0}^{k-2} \frac{k-j-1}{k} d_1 \sigma^j T_n = P'_n.$$

We write  $P[n] = P'$  and the inductive step is closed. Now by (a), the sequence  $P[n]$  converges to a MC-element  $P$  with  $\sigma P = P$ .

We now use  $P$  to construct a MC-element  $\Omega$  invariant by  $\sigma$  and  $\tau$ . Since  $\sigma\tau = \tau\sigma^{k-1}$ , we deduce that  $\tau P$  is also  $\sigma$ -invariant. On the other hand, since  $P$  and  $\mathbf{v}_1$  are gauge equivalent, there is an element  $\mathbf{w}$  such that  $\mathbf{w}\mathcal{G}P = \mathbf{v}_1$  and we set

$$\alpha = (-\tau\mathbf{w}) * \mathbf{w},$$

i.e.,  $\alpha$  can be represented as a composition of paths,

$$\begin{array}{ccc} P & \xleftarrow{\alpha} & \tau P \\ & \swarrow \mathbf{w} & \nearrow \tau\mathbf{w} \\ & \mathbf{v}_1 = \tau\mathbf{v}_1 & \end{array}$$



From (2.8) and from property (iii) of the BCH-product, recalled in Section 2, we get

$$\alpha \mathcal{G}P = \tau P, \quad \tau \alpha = (-\mathfrak{w}) * \tau \mathfrak{w} = -((-\tau \mathfrak{w}) * \mathfrak{w}) = -\alpha.$$

We define a MC-element  $\Omega$  by

$$(3.6) \quad \Omega = \frac{\alpha}{2} \mathcal{G}P$$

and deduce that

$$\tau \Omega = \left( -\frac{\alpha}{2} \right) \mathcal{G} \tau P = \frac{\alpha}{2} \mathcal{G}(-\alpha) \mathcal{G} \tau P = \frac{\alpha}{2} \mathcal{G}P = \Omega.$$

Thus the MC-element  $\Omega$  is  $\tau$ -invariant. We now prove that  $\alpha$  is  $\sigma$ -invariant, which implies the  $\sigma$ -invariance of  $\Omega$ .

The linear part of  $\alpha \mathcal{G}P = \tau P$  is  $P_1 - d_1 \alpha_1 = \tau P_1$ . This gives the  $\sigma$ -invariance of  $d_1 \alpha_1$  and by Lemma 3.2(i) the  $\sigma$ -invariance of  $\alpha_1$ . Suppose that the components  $\alpha_j$  of  $\alpha$  in Lie brackets of length  $j$ , are  $\sigma$ -invariant for any  $j \leq n$ . The component of the equality  $\alpha \mathcal{G}P = \tau P$  in Lie brackets of length  $n+1$  gives

$$P_{n+1} + S - d_1 \alpha_{n+1} = (\tau P)_{n+1},$$

where  $S$  is  $\sigma$ -invariant by induction. This implies the  $\sigma$ -invariance of  $d_1 \alpha_{n+1}$  and therefore of  $\alpha_{n+1}$  as required. ■

**4. Symmetric Lie model of a triangle.** Let  $\Delta^2$  be the simplicial complex of the triangle, with three vertices and three edges. In the spirit of the model of the interval [6], we are looking for a model of the triangle as a cdgl whose generators are the simplices of  $\Delta^2$ , whose linear part of the differential is the differential of the associated simplicial complex and whose sub-cdgl's corresponding to edges are LS-intervals. We call it a *Lie model of  $\Delta^2$* . The next result proves existence and unicity of such a model.

PROPOSITION 4.1. *Let  $\Delta^2$  be the simplicial complex of the triangle.*

(1) *A Lie model of  $\Delta^2$  is the cdgl*

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{e}), d),$$

*with  $|\mathfrak{v}_i| = -1$ ,  $|\mathfrak{x}_i| = 0$  and  $|\mathfrak{e}| = 1$ . The elements  $\mathfrak{v}_i$  are MC-elements, the intervals  $[\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{x}_1]$ ,  $[\mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{x}_2]$  and  $[\mathfrak{v}_3, \mathfrak{v}_1, \mathfrak{x}_3]$  are LS-intervals, and*

$$d\mathfrak{e} = [\mathfrak{e}, \mathfrak{v}_1] + \mathfrak{x}_1 * \mathfrak{x}_2 * \mathfrak{x}_3.$$

(2) *Let  $\mathfrak{L}_2$  be as before and  $(\widehat{\mathbb{L}}(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{e}), d')$  be a cdgl such that*

- (i) *the differentials  $d$  and  $d'$  coincide on the generators  $\mathfrak{v}_i$  and  $\mathfrak{x}_i$ ,*
- (ii) *the linear parts  $d_1 \mathfrak{e}$  and  $d'_1 \mathfrak{e}$  coincide,*
- (iii)  *$d'_{\mathfrak{v}_1}(\mathfrak{e}) \in \widehat{\mathbb{L}}(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)$ .*

*Then  $d = d'$ .*

*Proof.* (1) The proof is reduced to: “ $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$  is a  $d_{v_1}$ -cycle”. Let  $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a, b, x), d)$  be a LS-interval. The composition of morphisms as in [6, Theorem 2] gives a cdgl map

$$\varphi: \mathfrak{L}_1 \rightarrow (\widehat{\mathbb{L}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), d)$$

with  $\varphi(a) = \varphi(b) = \mathbf{v}_1$  and  $\varphi(x) = \mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ . We deduce

$$\begin{aligned} d(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) &= d\varphi(x) = \varphi(dx) = \varphi\left(\text{ad}_x b + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}(b - a)\right) \\ &= \varphi([x, b]) = -\text{ad}_{v_1}(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3). \end{aligned}$$

This implies  $d_{v_1}(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) = 0$  as expected.

(2) We decompose  $d_{v_1}$  and  $d'_{v_1}$  according to bracket size as  $d_{v_1} = \sum_{j \geq 1} d_j$  and  $d'_{v_1} = \sum_{j \geq 1} d'_j$ . By hypothesis, we have  $d_1 = d'_1$ . Suppose, by induction, that  $d_j = d'_j$  for any  $j < n$ . From  $d^2 = d'^2 = 0$ , we deduce

$$d_1 d'_n = - \sum_{j=2}^n d_j d'_{n+1-j} = - \sum_{j=2}^n d_j d_{n+1-j} = d_1 d_n.$$

Therefore, the element  $d_n \mathbf{e} - d'_n \mathbf{e}$  is a  $d_1$ -cycle of degree 0 in  $\widehat{\mathbb{L}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ . From the first assertion of Lemma 3.2, we deduce  $d_n \mathbf{e} = d'_n \mathbf{e}$ . ■

Denote by  $\Gamma$  the boundary of the triangle and by  $\mathfrak{L}(\Gamma)$  its Lie model. As the symmetric group  $\Sigma_3$  coincides with the dihedral group  $D_6$ , we may consider the action of  $\Sigma_3$  on  $\mathfrak{L}(\Gamma)$  described at the beginning of Section 3 and apply Theorem 3.4.

**THEOREM 4.2.** *Let  $\Omega$  be a  $\Sigma_3$ -invariant MC-element in  $\mathfrak{L}(\Gamma)$ . Then the cdgl  $\mathfrak{L}_2$  is a  $\Sigma_3$ -cdgl isomorphic to the cdgl  $(\mathfrak{L}(\Gamma) \widehat{\Pi} \mathbb{L}(\mathbf{e}'), d)$  satisfying the following properties:*

(i) *for some  $\beta$  of degree 0, we have*

$$d\mathbf{e}' = -[\Omega, \mathbf{e}'] + \beta * \mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3 * (-\beta),$$

(ii)  *$\sigma(\mathbf{e}') = \mathbf{e}'$  and  $\tau(\mathbf{e}') = -\mathbf{e}'$ .*

*Proof.* Let  $\beta$  be a path from  $\Omega$  to  $\mathbf{v}_1$  and set  $\mathbf{e}' = e^{\text{ad}_\beta}(\mathbf{e})$ . From Proposition 2.2, we get

$$d_\Omega \mathbf{e}' = d_\Omega e^{\text{ad}_\beta}(\mathbf{e}) = e^{\text{ad}_\beta}(d_{v_1} \mathbf{e}) = \beta * \mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3 * (-\beta).$$

Observe that  $\sigma(\beta) * (-\mathbf{x}_1)$  and  $\beta$  are two paths from  $\Omega$  to  $\mathbf{v}_1$ . From Lemma 3.2(ii) applied to  $(-\beta) * \sigma(\beta) * (-\mathbf{x}_1)$ , there exists  $\lambda \in \mathbb{Q}$  such that

$$\sigma(\beta) = \beta * \lambda(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * \mathbf{x}_1.$$

Replacing  $\sigma(\beta)$  by its value in  $\sigma(d_\Omega \mathbf{e}')$ , we get

$$\begin{aligned} \sigma d_\Omega(\mathbf{e}') &= \sigma(\beta) * \mathbf{x}_2 * \mathbf{x}_3 * \mathbf{x}_1 * (-\sigma(\beta)) \\ &= \beta * \lambda(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * \mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3 * \mathbf{x}_1 \\ &\quad * (-\mathbf{x}_1) * (-\lambda)(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * (-\beta) \\ &= \beta * \mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3 * (-\beta) = d_\Omega(\mathbf{e}'). \end{aligned}$$

On the other hand, recall that  $\tau(\mathbf{x}_2) = -\mathbf{x}_2$ ,  $\tau(\mathbf{x}_1) = -\mathbf{x}_3$ ,  $\tau(\mathbf{v}_1) = \mathbf{v}_1$  and  $\tau(\mathbf{v}_2) = \mathbf{v}_3$ . Applying Lemma 3.2(ii) to  $(-\beta) * \tau(\beta)$ , we obtain  $\mu \in \mathbb{Q}$  such that

$$\tau(\beta) = \beta * \mu(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3).$$

Once again, a similar computation gives

$$\begin{aligned} \tau(d_\Omega(\mathbf{e}')) &= \beta * \mu(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * (-\mathbf{x}_3) * (-\mathbf{x}_2) * (-\mathbf{x}_1) \\ &\quad * (-\mu)(\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * (-\beta) \\ &= \beta * (-\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3) * (-\beta) = -d_\Omega(\mathbf{e}'). \quad \blacksquare \end{aligned}$$

**PROBLEM.** In Theorem 3.4, we prove that the Lie model of a graph consisting of a circuit contains a MC-element invariant under the group of automorphisms of the graph. Is this result true in general: does the Lie model of any finite graph have a MC-element invariant under the automorphisms of the graph?

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Urtzi Buijs, Aniceto Murillo  
Departamento de Álgebra, Geometría y Topología  
Universidad de Málaga  
Ap. 59  
29080 Málaga, Spain  
E-mail: ubuijs@uma.es  
aniceto@uma.es

Yves Félix  
Institut de Mathématiques et Physique  
Université Catholique de Louvain-la-Neuve  
Louvain-la-Neuve, Belgium  
E-mail: Yves.Felix@uclouvain.be

Daniel Tanré  
Département de Mathématiques, UMR 8524  
Université de Lille 1  
59655 Villeneuve d'Ascq Cedex, France  
E-mail: Daniel.Tanre@univ-lille1.fr