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Sharpening a theorem of Forelli

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In memory of Professor Józef Siciak

Abstract. We prove that a complex-valued function defined on the unit ball in $\mathbb{C}^n$ is holomorphic if and only if it is holomorphic along every complex vector line and satisfies a weak differentiability condition on some real vector planes. We thereby sharpen a classical theorem by Forelli.

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the unit ball in $\mathbb{C}^n$. A theorem due to Forelli [5] (see also [9, Theorem 4.4.5], and [6, 7, 8] for recent developments) asserts that a function $f: \mathbb{B}_n \to \mathbb{C}$ is holomorphic if the following two conditions hold:

(i) for every complex vector line $\Lambda \subset \mathbb{C}^n$ the restriction $f|_{\mathbb{B}_n \cap \Lambda}$ is a holomorphic function;
(ii) f is of class $\mathcal{C}^\infty$ at the origin 0.

The notion that appears in (ii) is recalled below; it is weaker than $\mathcal{C}^\infty$ in a neighborhood of 0. The purpose of this note is to offer a simple proof of a sharper result, in which the condition (ii) is significantly weakened.

Let $\mathbb{F}$ stand for $\mathbb{R}$ or $\mathbb{C}$, and let X be a finite-dimensional $\mathbb{F}$-vector space. A complex-valued function g, defined in a neighborhood of the origin $0 \in X$, is said to be of *class* $\mathcal{C}^\infty$ *at* 0 or of *class* $\mathcal{C}^\infty(X, 0)$ if for every positive integer k there is an open neighborhood of 0 in which g is of class $\mathcal{C}^k$. Of course, when $\mathcal{C}^k$ functions are involved, X is always regarded as a *real* vector space, with the scalars restricted to $\mathbb{R}$ if $\mathbb{F} = \mathbb{C}$.

Let $\mathbb{G}_p(\mathbb{F}^n)$ denote the Grassmannian of p-dimensional vector subspaces of $\mathbb{F}^n$. As in [1], we consider $\mathbb{G}_p(\mathbb{F}^n)$ endowed with the structure of a real algebraic variety.

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For any integer $m \geq 2$, we set

$$\mathcal{T}(m) = \big\{Q \in \mathbb{G}_2(\mathbb{R}^m) \colon e_j \in Q \text{ for some } j \in \{1, \ldots, m\}\big\},$$

where $e_1, \ldots, e_m$ is the standard vector basis for $\mathbb{R}^m$. If $m \geq 3$, then $\mathcal{T}(m)$ is the union of m algebraic subsets of $\mathbb{G}_2(\mathbb{R}^m)$, each isomorphic to real projective space $\mathbb{RP}^{m-2}$; in particular, $\mathcal{T}(m)$ represents a "small" portion of $\mathbb{G}_2(\mathbb{R}^m)$.

When convenient we henceforth identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the map

$$(z_1, \ldots, z_n) \mapsto (x_1, y_1, \ldots, x_n, y_n),$$

where $z_j = x_j + iy_j$ for $j = 1, \ldots, n$. This identification means precisely that $\mathbb{C}^n$ is regarded as a real vector space.

Theorem 1. *Let $f \colon \mathbb{B}_n \to \mathbb{C}$ be a function such that for every complex line $\Lambda \in \mathbb{G}_1(\mathbb{C}^n)$ the restriction $f|_{\mathbb{B}_n \cap \Lambda}$ is holomorphic. Then the following conditions are equivalent:*

(a) *f is holomorphic on $\mathbb{B}_n$.*
(b) *f is of class $\mathcal{C}^\infty(\mathbb{C}^n, 0)$.*
(c) *for every real plane $Q \in \mathbb{G}_2(\mathbb{R}^{2n})$ the restriction $f|_{\mathbb{B}_n \cap Q}$ is of class $\mathcal{C}^\infty(Q, 0)$.*
(d) *for every real plane $Q \in \mathcal{T}(2n)$ the restriction $f|_{\mathbb{B}_n \cap Q}$ is of class $\mathcal{C}^\infty(Q, 0)$.*

The differentiability condition in (d) is very weak: it is imposed only on the restrictions of f to the interesection of $\mathbb{B}_n$ with real vector planes forming a small subset of $\mathbb{G}_2(\mathbb{R}^{2n})$.

To prove Theorem 1 we need the following auxiliary result.

Proposition 2. *Let $f \colon B(r) \to \mathbb{C}$ be a function defined on the ball*

$$B(r) = \{x \in \mathbb{R}^m \colon \|x\| < r\}$$

for some $r > 0$. Assume that $m \geq 2$ and the following two conditions hold:

(1) *For every real plane $Q \in \mathcal{T}(m)$ the restriction $f|_{B(r) \cap Q}$ is of class $\mathcal{C}^\infty(Q, 0)$.*
(2) *For every real line $L \in \mathbb{G}_1(\mathbb{R}^m)$ the restriction $f|_{B(r) \cap L}$ is real analytic.*

Then f is a real analytic function in a neighborhood of 0.

Proof. For each vector $u \in \mathbb{R}^m$ the function $t \mapsto f(tu)$ is well defined and real analytic in a neighborhood of $0 \in \mathbb{R}$. Given a positive integer k, we can therefore define the function

$$\delta^k(f) \colon \mathbb{R}^m \to \mathbb{C}, \qquad u \mapsto \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(tu)\bigg|_{t=0}.$$

We set $\delta^0(f) = f(0)$.

For a fixed plane $P \in \mathcal{T}(m)$ and a positive integer k, the function $f|_{B(r)\cap P}$ is of class $\mathcal{C}^k$ in a neighborhood of $0 \in P$. Since $\delta^k(f)|_P = \delta^k(f|_{B(r)\cap P})$, it follows that the restriction $\delta^k(f)|_P$ is a real polynomial homogeneous function of degree k. If $l \subset \mathbb{R}^m$ is an affine line that is parallel to one of the coordinate axes, then $l \subset Q$ for some real plane $Q \in \mathcal{T}(m)$. Consequently, $\delta^k(f)|_l$ is a polynomial function, which in view of [2, Lemma 1] implies that $\delta^k(f)$ is a real polynomial function, necessarily homogeneous of degree k.

Consider the series of homogeneous polynomial functions

$$\sum_{k=0}^{\infty} P_k, \quad \text{where} \quad P_k = \frac{1}{k!}\delta^k(f).$$

By (2) and the definition of P_k, for every real line $L \in \mathbb{G}_1(\mathbb{R}^m)$ the series $\sum_{k=0}^{\infty} P_k(x)$ converges to $f(x)$ for all x in some neighborhood of 0 in L. Hence, according to [3, Lemma 3] (see also [11] for the first proof of the cited result), the series $\sum_{k=0}^{\infty} P_k$ converges to a real analytic function $\tilde{f}\colon B(\tilde{r}) \to \mathbb{C}$ defined on the ball $B(\tilde{r})$ for some $\tilde{r}$ with $0 < \tilde{r} \le r$. For every real line $L \in \mathbb{G}_1(\mathbb{R}^m)$, the functions f and $\tilde{f}$ coincide on a neighborhood of $0 \in L$. Since f and $\tilde{f}$ are real analytic on $B(\tilde{r}) \cap L$, we get $f = \tilde{f}$ on $B(\tilde{r}) \cap L$. Thus $f = \tilde{f}$ on $B(\tilde{r})$, which completes the proof. ∎

Proof of Theorem 1. The implications (a)⇒(b)⇒(c)⇒(d) are obvious, and hence it remains to prove (d)⇒(a).

Suppose that (d) holds. We first show that f is holomorphic in a neighborhood of $0 \in \mathbb{C}^n$. Since for every complex line $\Lambda \in \mathbb{G}_1(\mathbb{C}^n)$ the restriction $f|_{\mathbb{B}_n \cap \Lambda}$ is a holomorphic function, it follows that for every real line $L \in \mathbb{G}_1(\mathbb{R}^{2n})$ the restriction $f|_{\mathbb{B}_n \cap L}$ is a real analytic function. Hence, by Proposition 2, the function f is real analytic in a neighborhood of $0 \in \mathbb{C}^n = \mathbb{R}^{2n}$. Consequently, there exists a real constant ρ, satisfying $0 < \rho \le 1$, such that on the ball $\rho\mathbb{B}_n$ the function f is the sum of a uniformly convergent series $\sum_{k=0}^{\infty} F_k$, where each

$$F_k\colon \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{C}$$

is a homogeneous polynomial function of $2n$ real variables of degree k. In order to show holomorphicity of f on $\rho\mathbb{B}_n$ it suffices to demonstrate that every F_k is in fact a complex (holomorphic) polynomial function. This can be done as follows. Note that for a fixed vector $v \in \mathbb{C}^n$, we have

$$k!F_k(v) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(tv)\bigg|_{t=0}.$$

Since the function $\zeta \mapsto f(\zeta v)$ is holomorphic in a neighborhood of $0 \in \mathbb{C}$, we get

$$\mu^k F_k(v) = F_k(\mu v) \quad \text{ for all } \mu \in \mathbb{C}.$$

We now regard F_k as a polynomial in $z = (z_1, \ldots, z_n)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$, with complex coefficients. There is a unique representation

$$F_k = \sum_{p+q=k} F_k^{pq},$$

where F_k^{pq} is a polynomial in $(z, \bar{z})$ of bidegree (p, q). Since

$$\mu^k F_k(v) = F_k(\mu v) = \sum_{p+q=k} \mu^p \bar{\mu}^q F_k^{pq}(v)$$

for all $\mu \in \mathbb{C}$, we get $F_k^{pq}(v) = 0$ if $(p, q) \neq (k, 0)$. Consequently, F_k is a polynomial in z, and hence f is holomorphic on $\rho\mathbb{B}_n$.

It is now easy to see that f is holomorphic on $\mathbb{B}_n$. Indeed, the power series representation (in the complex sense) of f around $0 \in \mathbb{C}^n$ converges at each point $b \in \mathbb{B}_n \setminus \{0\}$ since the restriction $f|_{\mathbb{B}_n \cap \mathbb{C}b}$ is a holomorphic function. Hence holomorphicity of f on $\mathbb{B}_n$ follows from Abel's Lemma in several variables [4, (9.1.2)]. ∎

In conclusion, we present a version of Siciak's theorem [10] (see also [12] for a convenient formulation) in the context of this note.

Let X be a finite-dimensional complex vector space. A function $r \colon U \to \mathbb{C}$, defined on an open subset U of X, is said to be *rational* if there exist polynomial functions (in the complex sense) $\varphi, \psi \colon X \to \mathbb{C}$ such that ψ is not identically 0 on X and $\psi r = \varphi$ on U. If, in addition, $\psi^{-1}(0) \cap U = \emptyset$, then r is called a *regular* function. Note that if the function r is locally bounded and rational, then it is in fact regular since the polynomials φ, ψ in the definition of a rational function can be chosen relatively prime. Clearly, any regular function on U is holomorphic.

THEOREM 3. *Let $f \colon \mathbb{B}_n \to \mathbb{C}$ be a holomorphic function such that for every complex line $\Lambda \in \mathbb{G}_1(\mathbb{C}^n)$ the restriction $f|_{\mathbb{B}_n \cap \Lambda}$ is regular. Then f is regular on $\mathbb{B}_n$.*

Proof. We use induction on n. The case $n = 1$ is obvious. Suppose that $n \geq 2$. For any complex hyperplane $\Sigma \in \mathbb{G}_{n-1}(\mathbb{C}^n)$ the intersection $\mathbb{B}_n \cap \Sigma$ can be identified with the unit ball in $\Sigma \cong \mathbb{C}^{n-1}$, and hence $f|_{\mathbb{B}_n \cap \Sigma}$ is a regular function by the induction hypothesis. In view of [12, Theorem 2], f is therefore a rational function. Thus f is a regular function, being holomorphic and rational. ∎

Let us observe that the assumptions in Theorem 3 can be reformulated in accordance with Theorem 1.

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