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On surjections between Banach spaces of continuous functions on separable nonmetrizable compact lines

by

Artur Michalak (Poznań)

Abstract. For a compact subset K of $[0,1]$ and a subset A of K, we denote by K_A the modification of the two-arrows space with base K and duplicated set A. We study necessary conditions for the existence of continuous linear surjections between the Banach spaces $C(K_A)$ of all real continuous functions on K_A spaces. We show that if there exists a continuous linear surjection from $C(K_A)$ onto $C(L_B)$ and A is a member of the additive Borel class Σ_α for some ordinal number $1 \le \alpha \le \omega_1$, then $B \in \Sigma_{\max\{3,1+\alpha\}}$.

1. Introduction. For a compact subset K of $[0,1]$ and a subset A of K, we denote by K_A the space

$$K_A = \{(t,0) : t \in K\} \cup \{(t,1) : t \in A\}$$

equipped with the order topology where $(t,p) < (u,r)$ if either $t < u$ or $t = u$ and $p < r$. Each K_A is a compact, separable, sequentially compact, hereditarily separable and hereditarily Lindelöf space. All these properties are well known for $[0,1]_{(0,1)}$, called the two-arrows or double-arrow space (see [2, p. 270]). For other K_A spaces the proofs of these properties are similar. The family of K_A spaces coincides with the class of all separable compact lines. Ostaszewski [9] showed that a linearly ordered set L equipped with the order topology is compact and separable if and only if L is homeomorphic to some K_A. We focus on the Banach spaces $C(K_A)$ of all real continuous functions on nonmetrizable K_A spaces. A space K_A is metrizable if and only if A is countable. The Banach spaces $C(L)$ on nonmetrizable compact lines L have been studied by many authors (see [1], [4], [6], [7], [8]).

Our investigations are closely related to the problems of topological classification of K_A spaces and of isomorphic classification of $C(K_A)$ spaces

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(see [3]). The best partial results concerning both problems may be found in Marciszewski's paper [6]. It is shown in [6] for instance that there exist $2^{\mathfrak{c}}$ pairwise nonisomorphic $C(K_A)$ spaces and that for Borel subsets A of K and B of L of exact multiplicative Borel classes Π_α and Π_β, respectively, with $\beta > \alpha \geq \omega$ the spaces $C(K_A)$ and $C(L_B)$ are not isomorphic.

Our first goal is to find necessary conditions for the existence a continuous linear surjection between $C(K_A)$ and $C(L_B)$. We show that such a surjection is generated by a sequence of some specific right continuous regular functions defined on subsets of B with values in A. This allows us to show that

(1) if $A \in \Sigma_\alpha$ for some ordinal number $1 \leq \alpha < \omega_1$ and there exists a continuous linear surjection from $C(K_A)$ onto $C(L_B)$, then $B \in \Sigma_{\max\{3,1+\alpha\}}$,
(2) if $A \in \Pi_\alpha$ for some ordinal number $1 \leq \alpha < \omega_1$ and there exists a continuous linear surjection from $C(K_A)$ onto $C(L_B)$, then $B \in \Sigma_{2+\alpha}$,
(3) there does not exist a continuous linear surjection from $C([0,1]_{[0,1]})$ onto $C([0,1]_{[0,1]\setminus\mathbb{Q}})$.

The last result is the answer to Marciszewski's question [6, p. 261]. Our considerations also provide some information on topological classification of K_A spaces.

The paper is divided into four sections. The fundamental properties of $C(K_A)$ spaces are gathered in the second section. Continuous linear surjections between $C(K_A)$ spaces are studied in the third section. Our results concerning the topological classification of K_A spaces are included in the fourth section.

2. Preliminaries. The closure of a subset B of a topological space is denoted by $\overline{B}$. The Banach space of all real continuous functions f on a compact Hausdorff space L equipped with the norm

$$\|f\| = \sup_{t\in L} |f(t)|$$

is denoted by $C(L)$. The closed linear hull in the norm topology of a subset B of a Banach space X is denoted by $\overline{\text{lin}}(B)$. The topological dual of a Banach space X is denoted by X^*. We write χ_B for the characteristic function of a set B and $f|_B$ for the restriction of a function f to B.

Let B be a nonempty subset of $[0,2]$. We set

$$B_l = \{s \in B : (s-\varepsilon, s) \cap B \neq \emptyset \text{ for every } \varepsilon > 0\},$$
$$B_r = \{s \in B : (s, s+\varepsilon) \cap B \neq \emptyset \text{ for every } \varepsilon > 0\}.$$

It is easy to see that for every subset B of $[0,2]$ the set $B \setminus (B_l \cap B_r)$ is countable. We define the *variation* of a function $f : B \to \mathbb{R}$ by

$$\operatorname{var}(f,B) = \sup\Big\{|f(t_1)|+\sum_{k=1}^{n-1}|f(t_{k+1})-f(t_k)| : t_1,\dots,t_n \in B,\ t_1<\dots<t_n\Big\}.$$

The Banach space of all functions $f : B \to \mathbb{R}$ such that $\operatorname{var}(f,B) < \infty$ is denoted by $\mathrm{BV}(B)$. Let X be a Banach space. We define the *weak variation* of a function $f : B \to X$ by

$$\operatorname{Var}(f,B) = \sup\{\operatorname{var}(x^* \circ f, B) : x^* \in X^*,\ \|x^*\| \le 1\}.$$

Let K be a compact subset of $[0,1]$ and let A be a subset of K. Every nonempty closed subset of K_A has the minimum and maximum which are elements of the set. For any $s,t,u \in K_A$ with $s < t$ and $(\min(K),0) < u < \max(K_A)$, the sets $(s,t) = \{v \in K_A : s < v < t\}$, $[(\min(K),0),u) = \{v \in K_A : v < u\}$, $(u,\max(K_A)] = \{v \in K_A : v > u\}$ are called *open intervals.*

The case most important for us is when K is a compact subset of $[0,1]$ and A is a dense subset of K. We then define the set A_* in the following way: if $\overline{A} = [\min(\overline{A}),\max(\overline{A})]$, we put $A_* = A$, if $[\min(\overline{A}),\max(\overline{A})] \setminus \overline{A} = \bigcup_{n=1}^{\infty}(a_n,b_n)$, we put $A_* = A \cup \{(a_n+b_n)/2 : n \in \mathbb{N}\}$. For every $t \in A_*$, we define $p_t : \overline{A}_A \to \mathbb{R}$ by

$$p_t = \begin{cases} \chi_{[(\min(\overline{A}),0),(t,0)]} & \text{if } t \in A,\\ \chi_{[(\min(\overline{A}),0),\max\{(u,r)\in\overline{A}_A : u\le t\}]} & \text{if } t \in A_* \setminus A,\\ 1 & \text{if } t = 2. \end{cases}$$

It is easy to see that the sets

$$\overline{A}_A \setminus [(\min(\overline{A}),0),(t,0)] = [(t,1),\max(\overline{A}_A)] = ((t,0),\max(\overline{A}_A)] \quad \text{and}$$
$$\overline{A}_A \setminus [(\min(\overline{A}),0),\max\{(u,r) \in \overline{A}_A : u \le s\}]$$
$$= [(\min\{u\in\overline{A} : u\ge s\},0),\max(\overline{A}_A)] = (\max\{(u,r)\in\overline{A}_A : u\le s\},\max(\overline{A}_A)]$$

are closed and open in $\overline{A}_A$ for all $t \in A$ and $s \in A_* \setminus A$. Consequently, the function p_t is continuous on $\overline{A}_A$ for every $t \in A_* \cup \{2\}$.

PROPOSITION 2.1. *Let K be a compact subset of $[0,1]$, let A be a dense subset of K and let X be a Banach space.*

(a) *The linear hull of $\{p_t : t \in A_* \cup \{2\}\}$ in $C(K_A)$ is a dense subset of $C(K_A)$.*
(b) *For every continuous linear operator $S : C(K_A) \to X$, the function $f_S : A_* \cup \{2\} \to X$ given by*

$$f_S(t) = S(p_t) \quad \textit{for every } t \in A_* \cup \{2\}$$

satisfies $\operatorname{Var}(f_S, A_* \cup \{2\}) \le \|S\|$.

Proof. (a) It is easy to check that for all $t,u \in A_*$ with $t < u$,

$$p_t p_u = p_{\min\{t,u\}}.$$

If $t \in A$, then

$$p_t(t,0) = 1 \neq 0 = p_t(t,1).$$

If $(s,p),(u,r) \in K_A$ and $s<u$ and there exists $t \in A$ such that $s<t<u$, then

$$p_t(s,p) = 1 \neq 0 = p_t(u,r).$$

If $(s,p),(u,r) \in K_A$ and $s<u$ and $(s,u)\cap K = \emptyset$, then $t=(s+u)/2 \in A_*$ and

$$p_t(s,p) = 1 \neq 0 = p_t(u,r).$$

Thus we have shown that the linear hull of $\{p_t : t \in A_* \cup \{2\}\}$ is an algebra of functions that separates the points of K_A and contains 1. An appeal to the Stone–Weierstrass theorem completes the proof of (a).

(b) For any $t_1,\dots,t_n \in A_* \cup \{2\}$ and $x^* \in X^*$ with $t_1 < \cdots < t_n$ we have

$$\begin{aligned}
&|x^*(f_S(t_1))| + \sum_{j=1}^{n-1} |x^*(f_S(t_{j+1})) - x^*(f_S(t_j))| \\
&\quad \le \max\Big\{\Big|S^*(x^*)\Big(\epsilon_1 p_{t_1} + \sum_{j=1}^{n-1}\epsilon_{j+1}(p_{t_{j+1}} - p_{t_j})\Big)\Big| : |\epsilon_j| = 1,\ 1 \le j \le n\Big\} \\
&\quad \le \|S^*\|\,\|x^*\| \max\{|\epsilon_j| : |\epsilon_j| = 1,\ 1\le j\le n\} \le \|S^*\|\,\|x^*\|.
\end{aligned}$$

This shows that $\mathrm{Var}(f_S, A_* \cup \{2\}) \le \|S\|$. ∎

COROLLARY 2.2. *Let K be a compact subset of $[0,1]$, let A be a dense subset of K and let X be a Banach space. For every continuous linear operator $S : C(K_A) \to X$ and $x^* \in X^*$, the limits*

$$\lim_{A_* \ni s \to t-} x^* \circ f_S(s) \quad \text{and} \quad \lim_{A_* \ni s \to u+} x^* \circ f_S(s)$$

exist for each point $t \in K_l$ and $u \in K_r$.

PROPOSITION 2.3. *Let K be a compact subset of $[0,1]$ and let A be a nonempty subset of K.*

(a) *There exists a continuous linear surjection from $C(K_A)$ onto $C(\overline{A}_A)$.*
(b) *There exists a continuous surjection from K_K onto K_A.*
(c) *$C(K_A)$ is isometrically isomorphic to a subspace of $C([0,1]_{[0,1]})$.*

Moreover, if A is dense in K, then $C(K_A)$ is isometrically isomorphic to the subspace $\overline{\mathrm{lin}}(\{p_t : t \in A_* \cup \{2\}\})$ of $C([0,1]_{[0,1]})$.

Proof. (a) It is easy to see that $\overline{A}_A$ is a closed subset of K_A. By the Tietze theorem the map $f \mapsto f|_{\overline{A}_A}$ is a continuous linear surjection from $C(K_A)$ onto $C(\overline{A}_A)$.

(b) Let $\psi_{K,A} : K_K \to K_A$ be given by

$$\psi_{K,A}(t,r) = \begin{cases} (t,r) & \text{if } t \in A, \\ (t,0) & \text{if } t \notin A, \end{cases}$$

for every $(t,r) \in K_K$. It is easy to check that for every $(t,r) \in K_A$ the sets $\{u \in K_K : \psi_{K,A}(u) > (t,r)\}$ and $\{u \in K_K : \psi_{K,A}(u) < (t,r)\}$ are open intervals in K_K. Consequently, $\psi_{K,A}$ is continuous.

(c) Assume first that A is dense in K. Let $I : \{p_t : t \in A_* \cup \{2\}\} \to \{p_t : t \in [0,1] \cup \{2\}\}$ be given by

$$I(p_t) = p_t$$

for every $t \in A_* \cup \{2\}$. Let $t_1, \dots, t_n \in A_* \cup \{2\}$ be such that $t_1 < \dots < t_n$ and let $c_1, \dots, c_n \in \mathbb{R}$. It is easy to check that

$$\begin{aligned}\Big\| I\Big(\sum_{j=1}^{n} c_j p_{t_j}\Big)\Big\| &= \sup\Big\{\Big|\sum_{j=1}^{n} c_j p_{t_j}(u,r)\Big| : 0 \le u \le 1,\, r \in \{0,1\}\Big\} \\ &= \max\Big\{\Big|\sum_{j=k}^{n} c_j\Big| : 1 \le k \le n\Big\} \\ &\ge \sup\Big\{\Big|\sum_{j=1}^{n} c_j p_{t_j}(u,r)\Big| : (u,r) \in K_A\Big\} = \Big\|\sum_{j=1}^{n} c_j p_{t_j}\Big\|.\end{aligned}$$

For every $t_k \in \{t_1, \dots, t_n\} \cap A$ we have

$$\Big|\sum_{j=1}^{n} c_j p_{t_j}(t_k, 0)\Big| = \Big|\sum_{j=k}^{n} c_j\Big|.$$

Moreover

$$\Big|\sum_{j=1}^{n} c_j p_{t_j}(\min(K), 0)\Big| = \Big|\sum_{j=1}^{n} c_j\Big|.$$

Let $t_k \in \{t_1, \dots, t_n\} \cap (A_* \setminus A)$. If $\max\{u \in K : u \le t_k\}$ is in A, then

$$\Big|\sum_{j=1}^{n} c_j p_{t_j}(\max\{u \in K : u \le t_k\}, 1)\Big| = \Big|\sum_{j=k}^{n} c_j\Big|.$$

If $\max\{u \in K : u \le t_k\}$ is not in A, then

$$\Big|\sum_{j=1}^{n} c_j p_{t_j}(\max\{u \in K : u \le t_k\}, 0)\Big| = \Big|\sum_{j=k}^{n} c_j\Big|.$$

Thus we have shown that

$$\Big\|\sum_{j=1}^{n} c_j p_{t_j}\Big\| = \max\Big\{\Big|\sum_{j=k}^{n} c_j\Big| : 1 \le k \le n\Big\}.$$

Hence I has an extension to a linear isometry from $\overline{\mathrm{lin}}(\{p_t : t \in A_* \cup \{2\}\})$ in $C(K_A)$ into $C([0,1]_{[0,1]})$. By Proposition 2.1(a) the closed linear hull coincides with $C(K_A)$.

Suppose now that A is not dense in K. Since $\psi_{K,A}$ is a continuous surjection, $C(K_A)$ is isometric to a subspace of $C(K_K)$. From the first part of the proof it follows that $C(K_K)$ is isometrically isomorphic to a subspace of $C([0,1]_{[0,1]})$. ■

We say that a real function f on a compact subset K of $[0,1]$ is *regular* if the limits

$$\lim_{K\ni s\to t-} f(s) = f(t-) \quad \text{and} \quad \lim_{K\ni s\to u+} f(s) = f(u+)$$

exist for all $t \in K_l$ and $u \in K_r$. We will need the following well known fact.

PROPOSITION 2.4. *Let K be a compact subset of $[0,1]$. If $f : K \to [0,1]$ is a right continuous regular function, then for every open subset U of $[0,1]$ the set $f^{-1}(U)$ is an F_σ subset of K.*

Proof. It is easy to see that if $\mathcal{F}$ is a family of left-closed intervals in $\mathbb{R}$ (i.e. intervals of the form $[a,b)$ for some $a, b \in \mathbb{R}$, $a < b$), then there exists an open subset V of $\mathbb{R}$ and a countable subset N of $\mathbb{R}$ which does not contain any infinite strictly decreasing sequence together with its limit such that

$$\bigcup_{M\in\mathcal{F}} M = V \cup N.$$

Consequently, $\bigcup_{M\in\mathcal{F}} M$ is an F_σ subset of $\mathbb{R}$.

For $a \in [0,1)$, $b \in (0,1]$, $t \in \{f > a\}$ and $u \in \{f < b\}$ there exist $\delta_1, \delta_2 > 0$ such that $[t, t+\delta_1) \cap K \subset \{f > a\}$ and $[u, u+\delta_2) \cap K \subset \{f < b\}$. Therefore for every open $U \subset [0,1]$ there exists a family $\mathcal{F}$ of left-closed intervals in $\mathbb{R}$ such that $f^{-1}(U) = K \cap \bigcup_{M\in\mathcal{F}} M$. Consequently, $f^{-1}(U)$ is an F_σ subset of K. ■

3. On continuous linear surjections between $C(K_A)$ spaces. Let K be a compact subset of $[0,1]$ and let A be a dense subset of K. We assign to each continuous linear operator $S : C(K_A) \to C([0,1]_{[0,1]})$ a function $h_S : [0,1]_{[0,1]} \to \mathrm{BV}(A_* \cup \{2\})$ given by

$$h_S((t,r))(u) = S(p_u)(t,r)$$

for all $u \in A_* \cup \{2\}$ and $(t,r) \in [0,1]_{[0,1]}$. The next proposition gathers together the elementary properties of the functions h_S.

PROPOSITION 3.1. *Let K be a compact subset of $[0,1]$, let A be a dense subset of K and let E be a nonempty subset of $[0,1]$. Let $S : C(K_A) \to C([0,1]_{[0,1]})$ be a continuous linear operator.*

(a) *The map* $h_S : [0,1]_{[0,1]} \to (\mathrm{BV}(A_* \cup \{2\}), \tau_p)$ *is continuous where* τ_p *is the topology of pointwise convergence on* $A_* \cup \{2\}$. *Consequently,*

$$\lim_{s \to t-} h_S(s,r)(v) = h_S(t,0)(v) \quad \text{and} \quad \lim_{s \to u+} h_S(s,r)(v) = h_S(u,1)(v)$$

for all $r \in \{0,1\}$, $v \in A_* \cup \{2\}$, $t \in (0,1]$, $u \in [0,1)$.

(b) *If* $S(C(K_A)) \subset \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$, *then* $h_S(u,1) = h_S(u,0)$ *for every* $u \in [0,1] \setminus E$.

(c) *We have*

$$\{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\} = \{t \in [0,1] : h_S(t,1) \neq h_S(t,0)\}$$

where $\delta_{(t,r)}$ *is the Dirac measure at* $(t,r) \in [0,1]_{[0,1]}$.

(d) *If* $S(C(K_A)) = \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$, *then*

$$\mathrm{var}(h_S(t,1) - h_S(t,0), A_* \cup \{2\}) \geq 1/c$$

for every $t \in E$ *where* $c = \inf\{a > 0 : \{y \in \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\}) : \|y\| < 1\} \subset S(\{x \in C(K_A) : \|x\| < a\})\}$.

(e) *The sets*

$$E_1^+ = \{t \in [0,1] : \text{there is } u \in A_r \text{ with } h_S(t,1)(u+) \neq h_S(t,0)(u+)\},$$
$$E_1^- = \{t \in [0,1] : \text{there is } u \in A_l \text{ with } h_S(t,1)(u-) \neq h_S(t,0)(u-)\}$$

are countable.

(f) *The set*

$$E_2 = \{t \in [0,1] : \text{there is } u \in A_r \cap A_l \text{ with } h_S(t,1)(u) \neq h_S(t,0)(u) \text{ and } h_S(t,1)(u+) = h_S(t,0)(u+) = h_S(t,1)(u-) = h_S(t,0)(u-)\}$$

is countable.

Proof. (a) The Dirac measure $\delta_{(t,r)}$ at $(t,r) \in [0,1]_{[0,1]}$ is a continuous linear functional on $C([0,1]_{[0,1]})$. The function

$$(t,r) \mapsto h_S(t,r)(u) = S(p_u)(t,r) = \delta_{(t,r)}(S(p_u))$$

is in $C([0,1]_{[0,1]})$ for every $u \in A_* \cup \{2\}$. By Proposition 2.1 the function

$$t \mapsto \delta_{(s,r)}(S(p_t))$$

is in $\mathrm{BV}(A_* \cup \{2\})$ for every $(s,r) \in [0,1]_{[0,1]}$.

(b) It is clear that $p_t(u,1) = p_t(u,0)$ for all $t \in E$ and $u \in [0,1] \setminus E$. Therefore $f(u,1) = f(u,0)$ for all $f \in \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$ and $u \in [0,1] \setminus E$. Hence

$$h_S(u,1)(t) - h_S(u,0)(t) = S(p_t)(u,1) - S(p_t)(u,0) = 0$$

for all $u \in [0,1] \setminus E$ and $t \in A_* \cup \{2\}$.

(c) Suppose $t \in \{s \in [0,1] : S^*(\delta_{(s,1)} - \delta_{(s,0)}) \neq 0\}$ and $h_S(t,1) = h_S(t,0)$. Then there exists $f \in C(K_A)$ such that $(\delta_{(t,1)} - \delta_{(t,0)})(S(f)) \neq 0$. We pick $\varepsilon > 0$ such that $|(\delta_{(t,1)} - \delta_{(t,0)})(S(f))| > 3\|S\|\varepsilon$. In view of Proposition 2.1

there exist $u_1,\dots,u_n \in A_* \cup \{2\}$ and $c_1,\dots,c_n \in \mathbb{R}$ such a $\|f-\sum_{j=1}^n c_j p_{u_j}\| < \varepsilon$. Consequently,

$$2\|S\|\varepsilon \ge 2\Big\|S\Big(f-\sum_{j=1}^n c_j p_{u_j}\Big)\Big\| \ge \Big|(\delta_{(t,1)}-\delta_{(t,0)})\Big(S\Big(f-\sum_{j=1}^n c_j p_{u_j}\Big)\Big)\Big|$$
$$= \Big|(\delta_{(t,1)}-\delta_{(t,0)})(S(f)) - \sum_{j=1}^n c_j\big(h_S(t,1)(u_j)-h_S(t,0)(u_j)\big)\Big| > 3\|S\|\varepsilon.$$

This contradiction shows that $\{t\in[0,1] : S^*(\delta_{(t,1)}-\delta_{(t,0)}) \ne 0\}$ is contained in $\{t\in[0,1] : h_S(t,1)\ne h_S(t,0)\}$.

If $t \in \{s\in[0,1] : h_S(s,1)\ne h_S(s,0)\}$, then there exists $u\in A_*\cup\{2\}$ such that $h_S(t,1)(u)\ne h_S(t,0)(u)$, so

$$S^*(\delta_{(t,1)}-\delta_{(t,0)})(p_u) = h_S(t,1)(u)-h_S(t,0)(u) \ne 0.$$

Thus $\{t\in[0,1] : h_S(t,1)\ne h_S(t,0)\} \subset \{t\in[0,1] : S^*(\delta_{(t,1)}-\delta_{(t,0)}) \ne 0\}$.

(d) Let $t\in E$. According to Proposition 2.1 for every $\varepsilon>0$ we can find $u_1,\dots,u_n\in A_*\cup\{2\}$ and $a_1,\dots,a_n\in\mathbb{R}$ such that $u_1<\dots<u_n$ and

$$\Big\|\sum_{j=1}^n a_j p_{u_j}\Big\| \le c \quad\text{and}\quad \Big\|S\Big(\sum_{j=1}^n a_j p_{u_j}\Big)-p_t\Big\| < \varepsilon.$$

It is easy to see that

$$\Big\|\sum_{j=1}^n a_j p_{u_j}\Big\| = \Big\|\Big(\sum_{k=1}^n a_k\Big)p_{u_1} + \sum_{j=1}^{n-1}\Big(\sum_{k=j+1}^n a_k\Big)(p_{u_{j+1}}-p_{u_j})\Big\|$$
$$= \max\Big\{\Big|\sum_{j=k}^n a_j\Big| : 1\le k\le n\Big\}.$$

Therefore for every $\varepsilon>0$ we can find $u_1,\dots,u_n\in A_*\cup\{2\}$ and $c_1,\dots,c_n\in\mathbb{R}$ such that $u_1<\dots<u_n$, $\max\{|c_j| : 1\le j\le n\}\le c$ and

$$2\varepsilon > 2\Big\|S\Big(c_1p_{u_1}+\sum_{j=1}^{n-1}c_{j+1}(p_{u_{j+1}}-p_{u_j})\Big)-p_t\Big\|$$
$$\ge \Big|(\delta_{(t,1)}-\delta_{(t,0)})\Big(p_t - S\Big(c_1p_{u_1}+\sum_{j=1}^{n-1}c_{j+1}(p_{u_{j+1}}-p_{u_j})\Big)\Big)\Big|$$
$$= \Big|-1-\Big(c_1\big(h_S(t,1)(u_1)-h_S(t,0)(u_1)\big)$$
$$+\sum_{j=1}^{n-1}c_{j+1}\big(\big(h_S(t,1)(u_{j+1})-h_S(t,0)(u_{j+1})\big)$$
$$-\big(h_S(t,1)(u_j)-h_S(t,0)(u_j)\big)\big)\Big)\Big|$$
$$\ge 1-\max\{|c_j| : 1\le j\le n\}\operatorname{var}\big(h_S(t,1)-h_S(t,0)\big).$$

Hence

$$\operatorname{var}(h_S(t,1) - h_S(t,0)), A_* \cup \{2\}) \geq 1/c.$$

(e) Let $\{u_n : n \in \mathbb{N}\}$ be a dense subset of $A_* \cup \{2\}$. Let $t \in E_1^+$. By Corollary 2.2 there exist $s \in A_r$ and $\delta, \varepsilon > 0$ such that $|h_S(t,1)(u) - h_S(t,0)(u)| > \varepsilon$ for every $u \in (s, s+\delta) \cap A_*$. Consequently, there exists n such that $h_S(t,1)(u_n) \neq h_S(t,0)(u_n)$. Hence

$$\begin{aligned} E_1^+ &\subset \bigcup_{n=1}^{\infty} \{t \in [0,1] : h_S(t,1)(u_n) \neq h_S(t,0)(u_n)\} \\ &= \bigcup_{n=1}^{\infty} \{t \in [0,1] : S(p_{u_n})(t,1) \neq S(p_{u_n})(t,0)\}. \end{aligned}$$

Since for every $f \in C([0,1]_{[0,1]})$ and $\varepsilon > 0$ the set $\{t \in [0,1] : |f(t,1) - f(t,0)| > \varepsilon\}$ is finite, the set E_1^+ is countable. Similarly we show that E_1^- is also countable.

(f) Suppose that E_2 is uncountable. It is clear that for every $u \in E_2$ we can find $s_u \in A_r \cap A_l$ such that $h_S(u,1)(s_u) \neq h_S(u,0)(s_u)$ and $h_S(u,1)(s_u+) = h_S(u,0)(s_u+) = h_S(u,1)(s_u-) = h_S(u,0)(s_u-)$. Obviously, at least one of the sets

$$\begin{aligned} B &= \{u \in E_2 : h_S(u,1)(s_u) \neq h_S(u,1)(s_u+)\}, \\ C &= \{u \in E_2 : h_S(u,0)(s_u) \neq h_S(u,0)(s_u+)\}, \end{aligned}$$

say B, is uncountable. It is easy to see that

$$\begin{aligned} &\{(x,y) \in \mathbb{R}^2 : x \neq y\} \\ &= \bigcup_{\{(a,b) \in \mathbb{Q}^2 : \, a \neq b\}} \{(x,y) \in \mathbb{R}^2 : |x-a| < |b-a|/4, \ |y-b| < |b-a|/4\}. \end{aligned}$$

Since $\mathbb{Q}^2$ is countable, we can find $a, b \in \mathbb{R}$ such that $a \neq b$ and the set

$$E_{2,1} = \{u \in E_2 : |h_S(u,1)(s_u) - a| < |b-a|/4, \ |h_S(u,1)(s_u+) - b| < |b-a|/4\}$$

is uncountable. According to Corollary 2.2 for every $u \in E_{2,1}$ there exists $\delta_u > 0$ such that

$$|h_S(u,1)(t) - h_S(u,1)(s_u+)| < |b-a|/4$$

for every $t \in (s_u - \delta_u, s_u + \delta_u) \setminus \{s_u\}$. We choose $\delta > 0$ such that the set $E_{2,2} = \{u \in E_{2,1} : \delta_u > \delta\}$ is uncountable. Since $[0,1]$ can be covered by finitely many intervals of length $\frac{1}{2}\delta$, there exist $0 \leq c < d \leq 1$ with $d - c < \delta$ such that the set $E_{2,3} = \{u \in E_{2,2} : s_u \in [c,d]\}$ is uncountable. Let $\psi : \mathbb{R} \to [0,1]$ be a continuous function such that $\psi(\{x \in \mathbb{R} : |x-a| \leq |b-a|/4\}) = \{1\}$ and $\psi(\{x \in \mathbb{R} : |x-b| \leq |b-a|/2\}) = \{0\}$. The space of all functions from $[c,d] \cap A_*$ to $\mathbb{R}$ equipped with the pointwise convergence topology is denoted by $\mathbb{R}^{[c,d]\cap A_*}$. According to (a) the function $\Psi : [0,1]_{[0,1]} \to \mathbb{R}^{[c,d]\cap A_*}$ given by

$$\Psi(t,r) = \psi \circ h_S(t,r)|_{[c,d]\cap A_*}$$

for every $(t,r) \in [0,1]_{[0,1]}$ is continuous. It is easy to see that for each $u \in E_{2,3}$ the function $\Psi(u,1)$ is the characteristic function of the singleton $\{s_u\}$ in $[c,d] \cap A_*$. Since

$$E_{2,3} \subset \bigcup_{s\in\{s_u :\, u\in E_{2,3}\}} \{t \in [0,1] : h_S(t,1)(s) \neq h_S(t,0)(s)\}$$

and $E_{2,3}$ is uncountable, the set $\{s_u : u \in E_{2,3}\}$ is also uncountable. Therefore $\{\Psi(u,1) : u \in E_{2,3}\}$ is an uncountable subset of $\mathbb{R}^{[c,d]\cap A_*}$ which is discrete in its subspace topology. The space $[0,1]_{[0,1]}$ is hereditarily separable (see [2, p. 270]) and so $\Psi([0,1]_{[0,1]})$ is also hereditarily separable. We have arrived at a contradiction. ■

The set $\{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\}$ contains information about which functions $p_t \in C([0,1]_{[0,1]})$ are in the range of the operator $S : C(K_A) \to C([0,1]_{[0,1]})$. If $p_t \in S(C(K_A))$ for some $t \in [0,1]$, then there exists $f \in C(K_A)$ such that $S(f) = p_t$ and $S^*(\delta_{(t,1)} - \delta_{(t,0)})(f) = (\delta_{(t,1)} - \delta_{(t,0)})(p_t) = -1$. According to Proposition 3.1(b, c), if $S(C(K_A)) = \overline{\text{lin}}(\{p_t : t \in E\})$, then

$$\{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\} = E.$$

In the following, for brevity, we write "either A_x or B_x for all x" (where A_x and B_x are some statements) to mean that "either A_x for all x, or B_x for all x".

Theorem 3.2. *Let K be an uncountable compact subset of $[0,1]$, let A be a dense subset of K and let E be a nonempty subset of $[0,1]$. Let $S : C(K_A) \to C([0,1]_{[0,1]})$ be a continuous linear operator such that $S(C(K_A)) \subset \overline{\text{lin}}(\{p_t : t \in E \cup \{2\}\})$ and the set $E \setminus \{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\}$ is countable. Let*

$$\begin{aligned} E_3 = \{t \in [0,1] : {} & h_S(t,1)(u+) = h_S(t,0)(u+),\ h_S(t,1)(u-) = h_S(t,0)(u-) \\ & \text{for every } u \in A_l \cap A_r \text{ and there exists } s \in A_l \cap A_r \text{ with} \\ & h_S(t,1)(s+) \neq h_S(t,1)(s-) \text{ and } h_S(t,1)(s) \neq h_S(t,0)(s)\}. \end{aligned}$$

Then the set $E \setminus E_3$ is countable and there exist a sequence (D_n) of subsets of E_3 and a sequence (g_n) of right continuous regular functions $g_n : \overline{D_n} \to [0,1]$ with the following properties:

1. *$E_3 = \bigcup_{n=1}^\infty D_n$, $g_n(D_n) \subset A$ and g_n is continuous at each point of D_n,*
2. *if $t \in (\overline{D_n})_l$ and $g_n(t-) \in A$, then there exists $\delta > 0$ such that either $g_n(u) \leq g_n(t-)$ or $g_n(u) \geq g_n(t-)$ for all $u \in (t-\delta, t) \cap \overline{D_n}$,*
3. *if $t \in (\overline{D_n})_r$ and $g_n(t) \in A$, then there exists $\delta > 0$ such that either $g_n(u) \leq g_n(t)$ or $g_n(u) \geq g_n(t)$ for all $u \in (t, t+\delta) \cap \overline{D_n}$,*

(4) *if* $t \in (D_n)_l \cap (D_n)_r$, *then there exists* $\delta > 0$ *such that either* $g_n(u_1) \leq g_n(t) \leq g_n(u_2)$ *or* $g_n(u_1) \geq g_n(t) \geq g_n(u_2)$ *for all* $u_1 \in (t-\delta, t) \cap \overline{D_n}$ *and* $u_2 \in (t, t+\delta) \cap \overline{D_n}$,

(5) *if* $t \in (\overline{D_n})_l \cap (\overline{D_n})_r$ *and* $g_n(t-) = g_n(t) \in A$ *and there exist sequences* $(s_n), (u_n) \subset \overline{D_n}$ *such that* (s_n) *is strictly increasing and* (u_n) *is strictly decreasing and* $\lim_{k\to\infty} s_k = t = \lim_{k\to\infty} u_k$, *and either* $g_n(s_k) < g_n(t) < g_n(u_k)$ *or* $g_n(s_k) > g_n(t) > g_n(u_k)$ *for all* k, *then* $t \in E$,

(6) *if* $t \in (\overline{D_n})_l \cap (\overline{D_n})_r$ *and* $g_n(t-) \neq g_n(t)$, *then* $t \in E$,

(7) *if* $t \in ((\overline{D_n})_l \cap (\overline{D_n})_r) \setminus E$ *and either* $g_n(t-) \in A$ *or* $g_n(t) \in A$, *then* $g_n(t-) = g_n(t)$ *and there exists* $\delta > 0$ *such that either* $g_n(u) \leq g_n(t)$ *or* $g_n(u) \geq g_n(t)$ *for all* $u \in (t-\delta, t+\delta) \cap \overline{D_n}$.

Proof. We use the notations from Proposition 3.1. Let

$$E_4 = \{t \in [0,1] : \text{there exists } u \in (A_* \cup \{2\}) \setminus (A_l \cap A_r) \text{ with } h_S(t,1)(u) \neq h_S(t,0)(u)\}.$$

By Proposition 3.1(c),

$$\{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\} = E_1^- \cup E_1^+ \cup E_2 \cup E_3 \cup E_4.$$

Since the set $(A_* \cup \{2\}) \setminus (A_l \cap A_r)$ is countable and

$$E_4 \subset \bigcup_{u \in (A_* \cup \{2\}) \setminus (A_l \cap A_r)} \{t \in [0,1] : h_S(t,1)(u) \neq h_S(t,0)(u)\},$$

E_4 is countable. By Proposition 3.1 the sets E_1^-, E_1^+ and E_2 are also countable. This shows that $E \setminus E_3$ is countable.

For any $a, b, c, d \in \mathbb{Q}$ with $a < b$ and $0 \leq c < d \leq 1$, let

$$\begin{aligned} E_{3,a,b,c,d,+} = \{t \in E_3 : {} & \text{there is } s \in (c,d) \cap A_r \cap A_l \text{ with } h_S(t,1)(u_1) \leq a, \\ & b \leq h_S(t,1)(u_2) \text{ for all } u_1 \in [c,s) \cap A, u_2 \in (s,d] \cap A, \\ & \text{and } h_S(t,1)(s) \neq h_S(t,0)(s)\}, \\ E_{3,a,b,c,d,-} = \{t \in E_3 : {} & \text{there is } s \in (c,d) \cap A_r \cap A_l \text{ with } h_S(t,1)(u_1) \geq b, \\ & a \geq h_S(t,1)(u_2) \text{ for all } u_1 \in [c,s) \cap A, u_2 \in (s,d] \cap A, \\ & \text{and } h_S(t,1)(s) \neq h_S(t,0)(s)\}. \end{aligned}$$

It follows from Corollary 2.2 that

$$E_3 = \bigcup_{a,b \in \mathbb{Q},\, a<b} \; \bigcup_{c,d \in [0,1] \cap \mathbb{Q},\, c<d} E_{3,a,b,c,d,-} \cup E_{3,a,b,c,d,+}.$$

Let $\{D_n : n \in \mathbb{N}\}$ be an enumeration of all sets in the family

$$\{E_{3,a,b,c,d,-} : a,b,c,d \in \mathbb{Q},\, a<b,\, 0 \leq c < d \leq 1\} \cup \{E_{3,a,b,c,d,+} : a,b,c,d \in \mathbb{Q},\, a<b,\, 0 \leq c < d \leq 1\}.$$

Suppose that $D_n = E_{3,a,b,c,d,+}$ for some $a, b, c, d \in \mathbb{Q}$ with $a < b$ and $0 \le c < d \le 1$. For every $t \in D_n$ there exists a unique $\tilde{g}_n(t) \in [c, d] \cap A$ such that

$$h_S(t, r)(\tilde{g}_n(t)-) \le a < b \le h_S(t, r)(\tilde{g}_n(t)+)$$

for each $r \in \{0, 1\}$.

Suppose first that D_n is finite. Then $\overline{D_n} = D_n$ and $(D_n)_l = (D_n)_r = \emptyset$. We put $g_n(t) = \tilde{g}_n(t)$ for every $t \in D_n$. It is clear that in this case properties (1)–(7) hold.

Suppose now that D_n is infinite. If $(t_k) \subset D_n$ is a strictly monotonic sequence with $\lim_{k\to\infty} t_k = t$ and $\liminf_{k\to\infty} \tilde{g}_n(t_k) \ne \limsup_{k\to\infty} \tilde{g}_n(t_k)$, then $\liminf_{k\to\infty} \tilde{g}_n(t_k)$ and $\limsup_{k\to\infty} \tilde{g}_n(t_k)$ are in K and there is $u \in A_*$ such that

$$u \in \Big(\liminf_{k\to\infty} \tilde{g}_n(t_k), \limsup_{k\to\infty} \tilde{g}_n(t_k)\Big).$$

Then the sets $\{k : \tilde{g}_n(t_k) > u\}$ and $\{k : \tilde{g}_n(t_k) < u\}$ are infinite and hence

$$\liminf_{k\to\infty} h_S(t_k, 0)(u) \le a < b \le \limsup_{k\to\infty} h_S(t_k, 0)(u).$$

This contradicts Proposition 3.1(a). Thus $\lim_{k\to\infty} \tilde{g}_n(t_k)$ exists and for any $u_1, u_2 \in [c, d] \cap A$ with $u_1 < \lim_{k\to\infty} \tilde{g}_n(t_k) < u_2$ we have

$$(\alpha)\qquad \begin{aligned} h_S(t, 0)(u_1) &= \lim_{k\to\infty} h_S(t_k, 0)(u_1) \le a < b \le \lim_{k\to\infty} h_S(t_k, 0)(u_2) \\ &= h_S(t, 0)(u_2). \end{aligned}$$

Thus we have shown that the function $g_n : \overline{D_n} \to [c, d]$ given by

$$g_n(t) = \begin{cases} \lim_{D_n \ni u\to t+} \tilde{g}_n(u) & \text{if } t \in (\overline{D_n})_r, \\ \tilde{g}_n(t) & \text{if } t \in D_n, \\ \lim_{D_n \ni u\to t-} \tilde{g}_n(u) & \text{if } t \in (\overline{D_n})_l \setminus (\overline{D_n})_r, \end{cases}$$

is well defined. It is clear that g_n is a right continuous regular function on $\overline{D_n}$ which is continuous at each point of D_n. If $(t_k) \subset D_n$ is an increasing sequence such that $t = \lim_{k\to\infty} t_k$, $g_n(t-) \in A$ and $g_n(t_k) = \tilde{g}_n(t_k) > g_n(t-)$ for every k, then

$$(\beta)\qquad h_S(t, 0)(g_n(t-)) = \lim_{k\to\infty} h_S(t_k, 0)(g_n(t-)) \le a.$$

If $(t_n) \subset D_n$ is an increasing sequence such that $t = \lim_{n\to\infty} t_n$, $g_n(t-) \in A$ and $g_n(t_k) = \tilde{g}_n(t_k) < g_n(t-)$ for every k, then

$$(\gamma)\qquad h_S(t, 0)(g_n(t-)) = \lim_{k\to\infty} h_S(t_k, 0)(g_n(t-)) \ge b.$$

This shows that g_n has property (2) for every n. If we replace in the above argument increasing sequences by decreasing sequences and $g_n(t-)$ by $g_n(t)$, we obtain the proof of (3). Combining (2), (3), (β) and (γ) we obtain (4) and (5). If $g_n(t-) \ne g_n(t)$ for some $t \in (\overline{D_n})_l \cap (\overline{D_n})_r$, then $g_n(t-), g_n(t) \in K$

and we can find $u \in A_*$ such that

$$u \in \bigl(\min\{g_n(t-), g_n(t)\}, \max\{g_n(t-), g_n(t)\}\bigr).$$

Applying (α) we obtain $h_S(t,0)(u) \neq h_S(t,1)(u)$ and $t \in E$. This shows that (6) holds. Combining (β), (γ) and (6) we obtain (7). ∎

COROLLARY 3.3. *If $E \subset [0,1]$ and there exists an open subset V of $[0,1]$ such that $E \cap V$ is of the second category in $[0,1]$ and $V \setminus E$ is dense in V, then there is no continuous linear operator $S : C([0,1]_{[0,1]}) \to C([0,1]_{[0,1]})$ with $S(C([0,1]_{[0,1]})) = \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$.*

Proof. We use the notations from Theorem 3.2. Suppose that such an operator S exists. Since $A = [0,1]$, all conditions of the form $g_n(t-) \in A$ and $g_n(t) \in A$ in Theorem 3.2 are fulfilled for every n and $t \in \overline{D_n}$. Since $E \setminus \bigcup_{n=1}^{\infty} D_n$ is countable, there exists $j \in \mathbb{N}$ such that $D_j \cap V$ is of the second category in $[0,1]$. Since $D_j \setminus ((D_j)_l \cap (D_j)_r)$ is countable, the set $(D_j)_l \cap (D_j)_r \cap V$ is of the second category in $[0,1]$. We pick an open nonempty subset W of V such that for every open nonempty interval W_1 in W the set $(D_j)_l \cap (D_j)_r \cap W_1$ is of the second category in $[0,1]$. For every $\eta > 0$, let

$$\begin{aligned} B_\eta = \{t \in \overline{D_j} : g_j(u_1) \le g_j(t) \le g_j(u_2) \text{ for all } u_1 \in (t-\eta, t) \cap \overline{D_j} \\ \text{and } u_2 \in (t, t+\eta) \cap \overline{D_j}\}, \\ C_\eta = \{t \in \overline{D_j} : g_j(u_1) \ge g_j(t) \ge g_j(u_2) \text{ for all } u_1 \in (t-\eta, t) \cap \overline{D_j} \\ \text{and } u_2 \in (t, t+\eta) \cap \overline{D_j}\}. \end{aligned}$$

According to Theorem 3.2(4),

$$(D_j)_l \cap (D_j)_r \cap W \subset V \cap \bigcup_{n=1}^{\infty} B_{1/n} \cup C_{1/n}.$$

Therefore at least one set in the family

$$\{(D_j)_l \cap (D_j)_r \cap W \cap B_{1/n} : n \in \mathbb{N}\} \cup \{(D_j)_l \cap (D_j)_r \cap W \cap C_{1/n} : n \in \mathbb{N}\},$$

say $(D_j)_l \cap (D_j)_r \cap W \cap B_{1/k}$, is of the second category in $[0,1]$. Then the closed set $\overline{(D_j)_l \cap (D_j)_r \cap W \cap B_{1/k}}$ contains a nonempty open interval U which is a subset of W. Since $U \cap \overline{D_j} \cap B_{1/k}$ is dense in U, the function $g_j|_{U \cap \overline{D_j} \cap B_{1/k}}$ is nondecreasing; as g_j is right continuous on U, it is nondecreasing on the whole of U. Let $t \in U \setminus E$. In view of Theorem 3.2(7) there exists $\delta > 0$ such that $g_j|_{[t,t+\delta) \cap \overline{D_j}}$ is constant. The choice of W guarantees that the set $(t, t+\delta) \cap D_j \cap U$ is of the second category in $[0,1]$, hence it is uncountable. But the set

$$(t, t+\delta) \cap D_j \subset \{u \in [0,1] : h_S(u,1)(g_j(t)) \neq h_S(u,0)(g_j(t))\}$$

is countable, a contradiction. ∎

Corollary 3.4.

(a) *If* $E \subset [0,1]$ *and there exists a continuous linear operator* $S : C([0,1]_{[0,1]}) \to C([0,1]_{[0,1]})$ *with* $S(C([0,1]_{[0,1]})) = \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$, *then there exist an open subset* U *of* $[0,1]$ *and a set* M *of the first category in* $[0,1]$ *such that* $E = U \cup M$.

(b) *There does not exist a continuous linear surjection from* $C([0,1]_{[0,1]})$ *onto* $C([0,1]_{[0,1]\setminus\mathbb{Q}})$.

Proof. (a) Let $\{V_n : n \in \mathbb{N}\}$ be a base of the topology of $[0,1]$. Let $N = \{n \in \mathbb{N} : V_n \cap E$ is of the first category in $[0,1]\}$. Let $F = [0,1] \setminus \bigcup_{n\in N} V_n$. If $F \cap E$ is of the first category, then so is E. Suppose that $F \cap E$ is of the second category. Let $W = \mathrm{int}\, F$ and $G = \mathrm{int}\, \overline{W \setminus E}$. If $G \neq \emptyset$, then $G \cap E$ is of the second category and $G \setminus E$ is dense in G. This contradicts Corollary 3.3. Thus $G = \emptyset$. Hence $W \setminus \overline{W \setminus E} \subset E$ and the set

$$E \setminus (W \setminus \overline{W \setminus E}) \subset (E \setminus F) \cup (F \setminus W) \cup (\overline{W \setminus E})$$

is of the first category in $[0,1]$.

(b) By Propositions 2.1 and 2.3 the subspace $\overline{\mathrm{lin}}(\{p_t : t \in ([0,1] \setminus \mathbb{Q}) \cup \{2\}\})$ of $C([0,1]_{[0,1]})$ is isomorphic to $C([0,1]_{[0,1]\setminus\mathbb{Q}})$. It is clear that $[0,1] \setminus \mathbb{Q}$ satisfies the assumption of Corollary 3.3. An appeal to Corollary 3.3 completes the proof. ∎

We use the standard Borel hierarchy of subsets of $[0,1]$ (see [5]). We denote by $D(0,1)$ the space of all real right continuous regular functions on $[0,1]$.

Theorem 3.5. *Let* K *be an uncountable compact subset of* $[0,1]$, *let* A *be a dense subset of* K *and let* E *be a nonempty subset of* $[0,1]$. *Let* $1 \le \alpha < \omega_1$ *be an ordinal number.*

If there exists a continuous linear operator $S : C(K_A) \to C([0,1]_{[0,1]})$ *such that* $S(C(K_A)) \subset \overline{\mathrm{lin}}(\{p_t : t \in E \cup \{2\}\})$ *and the set* $E \setminus \{t \in [0,1] : S^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\}$ *is countable, then*

(a) $E = (\bigcup_{n=1}^{\infty} (f_n^{-1}(A) \cap F_n)) \cup N$ *for some sequences* $(f_n) \subset D(0,1)$ *and* $(F_n) \subset \Sigma_3$ *and a countable subset* N *of* $[0,1]$,

(b) $E \in \Sigma_{\max\{3,1+\alpha\}}$ *whenever* $A \in \Sigma_\alpha$,

(c) $E \in \Sigma_{2+\alpha}$ *whenever* $A \in \Pi_\alpha$.

Proof. We use the notations from Theorem 3.2. For every n and $\eta > 0$, let

$$B_{n,\eta} = \{t \in \overline{D_n} : g_n(u_1) \le g_n(t) \le g_n(u_2) \text{ for all } u_1 \in (t-\eta, t) \cap \overline{D_n} \text{ and } u_2 \in (t, t+\eta) \cap \overline{D_n}\},$$

$$C_{n,\eta} = \{t \in \overline{D_n} : g_n(u_1) \ge g_n(t) \ge g_n(u_2) \text{ for all } u_1 \in (t-\eta, t) \cap \overline{D_n} \text{ and } u_2 \in (t, t+\eta) \cap \overline{D_n}\}.$$

For every n, let

$$L_{n,1} = (\overline{D_n})_l \cap (\overline{D_n})_r, \qquad L_{n,2} = \bigcup_{k=1}^{\infty} (B_{n,1/k} \cup C_{n,1/k}),$$
$$L_{n,3} = \{t \in \overline{D_n} : \text{there exists } \delta > 0 \text{ and } a \in \mathbb{R} \text{ with } g_n|_{(t-\delta,t]\cap\overline{D_n}} = a\},$$
$$L_{n,4} = \{t \in \overline{D_n} : \text{there exists } \delta > 0 \text{ and } a \in \mathbb{R} \text{ with } g_n|_{[t,t+\delta)\cap\overline{D_n}} = a\}.$$

The set $\overline{D_n}$ is an F_σ and a G_δ subset of $[0,1]$ for every n. The set $\overline{D_n} \setminus ((\overline{D_n})_l \cap (\overline{D_n})_r)$ is countable. Hence $L_{n,1}$ is a G_δ subset of $[0,1]$ for every n. Let $(t_k) \subset B_{n,\eta}$ be a strictly decreasing sequence with $\lim_{k\to\infty} t_k = t$ for some n and $\eta > 0$. Then for any $u_1 \in (t-\eta, t) \cap \overline{D_n}$ and $u_2 \in (t, t+\eta) \cap \overline{D_n}$ we can find N such that for every $k > N$, $u_1 \in (t_k - \eta, t_k)$ and $u_2 \in (t_k, t_k + \eta)$. Hence $g_n(u_1) \le g_n(t_k) \le g_n(u_2)$ for every $k > N$. Since g_n is right continuous at t, the point t is a member of $B_{k,\eta}$. This shows that for every $s \in \overline{D_n} \setminus B_{n,\eta}$ there exists $\delta > 0$ such that $[s, s+\delta) \cap \overline{D_n} \subset \overline{D_n} \setminus B_{n,\eta}$. Therefore there exists a family $\mathcal{F}$ of left-closed intervals in $[0,2]$ such that $\overline{D_n} \setminus B_{n,\eta} = \overline{D_n} \cap \bigcup_{M\in\mathcal{F}} M$. Similar arguments to the proof of Proposition 2.4 show that $\overline{D_n} \setminus B_{n,\eta}$ is an F_σ subset of $\overline{D_n}$ for every n and η. Therefore $B_{n,\eta}$ is an G_δ subset of $[0,1]$ for every n and $\eta > 0$. Similarly $C_{n,\eta}$ is a G_δ subset of $[0,1]$ for every n and $\eta > 0$. Consequently, $L_{n,2}$ is a $G_{\delta,\sigma}$ subset of $[0,1]$ for every n. For every $t \in L_{n,3}$ there exists $-1 < u < t$ such that $(u,t] \subset L_{n,3}$. Consequently, there exists a family $\mathcal{G}$ of right-closed intervals in $[-1,1]$ such that $L_{n,3} = \overline{D_n} \cap \bigcup_{M\in\mathcal{G}} M$. Similar arguments to the proof of Proposition 2.4 show that $L_{n,3}$ is an F_σ subset of $[0,1]$ for every n. Therefore $[0,1] \setminus L_{n,3}$ is a G_δ subset of $[0,1]$ for every n. Similar considerations show that $[0,1] \setminus L_{n,4}$ is a G_δ subset of $[0,1]$ for every n. Thus we have shown that $L_n = L_{n,1} \cap L_{n,2} \cap \big([0,1] \setminus L_{n,3}\big) \cap ([0,1] \setminus L_{n,4})$ is in Σ_3 for every n.

According to Theorem 3.2(4),

$$(D_n)_l \cap (D_n)_r \subset L_{n,1} \cap L_{n,2} \cap g_n^{-1}(A)$$

for every n. If $t \in L_n \cap g_n^{-1}(A)$ and $g_n(t) \ne g_n(t-)$, then $t \in E$ in view of Theorem 3.2(6). Suppose that $t \in L_n \cap g_n^{-1}(A)$ and $g_n(t) = g_n(t-)$. Then there exists k such that $t \in B_{n,1/k} \cup C_{n,1/k}$. Without loss of generality we may assume that $t \in B_{n,1/k}$. Since $t \notin L_{n,3} \cup L_{n,4}$, for every $\varepsilon > 0$ there exist $u_1 \in (t-\varepsilon, t) \cap \overline{D_n}$ and $u_2 \in (t, t+\varepsilon) \cap \overline{D_n}$ such that $g_n(u_1) < g_n(t) < g_n(u_2)$. By Theorem 3.2(5), $t \in E$. Thus we have shown that

$$L_n \cap g_n^{-1}(A) \subset E \quad \text{for every } n.$$

For each n, there exists a countable family $\{U_{n,k} : k \in \mathbb{N}\}$ of pairwise disjoint nonempty intervals in $[0,1]$ and a countable set $\{a_{n,k} : k \in \mathbb{N}\} \subset A$ such that for every $t \in L_{n,1} \cap L_{n,2} \cap L_{n,3} \cap g_n^{-1}(A)$ there exist $j, l \in \mathbb{N}$ such that $t \in U_{n,j}$ and $g_n|_{U_{n,j}\cap\overline{D_n}} = a_{n,l}$. Therefore

$$L_{n,1} \cap L_{n,2} \cap L_{n,3} \cap g_n^{-1}(A) \cap D_n \subset \bigcup_{k=1}^{\infty} \{u \in [0,1] : h_S(u,1)(a_{n,k}) \neq h_S(u,0)(a_{n,k})\}.$$

This shows that $L_{n,1} \cap L_{n,2} \cap L_{n,3} \cap g_n^{-1}(A) \cap D_n$ is countable. Similarly $L_{n,1} \cap L_{n,2} \cap L_{n,4} \cap g_n^{-1}(A) \cap D_n$ is countable. Hence

$$\bigcup_{n=1}^{\infty} L_n \cap g_n^{-1}(A) \subset E,$$

and the set

$$E \setminus \bigcup_{n=1}^{\infty} (L_n \cap g_n^{-1}(A)) \subset (E \setminus E_3) \cup \Big(E_3 \setminus \bigcup_{n=1}^{\infty} D_n\Big) \cup \bigcup_{n=1}^{\infty} [(D_n \setminus ((D_n)_l \cap (D_n)_r)) \cup (L_{n,1} \cap L_{n,2} \cap (L_{n,3} \cup L_{n,4}) \cap g_n^{-1}(A) \cap D_n)]$$

is countable.

For every compact subset L of $[0,1]$ and every right continuous regular function $f : L \to \mathbb{R}$, the function $\tilde{f} : [0,1] \to \mathbb{R}$ given by

$$\tilde{f}(t) = \begin{cases} f(\max\{s \in L : s \leq t\}) & \text{if } t \geq \min(L), \\ f(\min(L)) & \text{if } t < \min(L), \end{cases}$$

for every $t \in [0,1]$ is in $D(0,1)$. It is easy to see that $f^{-1}(M) = L \cap \tilde{f}^{-1}(M)$ for every $M \subset \mathbb{R}$. Thus we have shown (a).

By Proposition 2.4 for every n and any open subset U of $[0,1]$ the set $g_n^{-1}(U)$ is an F_σ subset of $\overline{D_n}$. Consequently, $g_n^{-1}(U)$ and $g_n^{-1}(W)$ are F_σ and G_δ subsets of $[0,1]$ respectively for every open $U \subset [0,1]$ and closed $W \subset [0,1]$ and for every n. It is easy to check that for every n and any ordinal number $\beta < \omega_1$,

$$\{g_n^{-1}(M) : M \subset \Sigma_\beta\} \subset \Sigma_{1+\beta} \quad \text{and} \quad \{g_n^{-1}(M) : M \subset \Pi_\beta\} \subset \Pi_{1+\beta}.$$

Gathering together all the facts above we obtain (b) and (c). ■

Corollary 3.6. *Let K, L be compact subsets of $[0,1]$ and let A, B be subsets of K, L, respectively. Let $1 \leq \alpha < \omega_1$ be an ordinal number.*

(a) *If $A \in \Sigma_\alpha$ and there exists a continuous linear surjection $S : C(K_A) \to C(L_B)$, then $B \in \Sigma_{\max\{3,1+\alpha\}}$.*
(b) *If $A \in \Pi_\alpha$ and there exists a continuous linear surjection $S : C(K_A) \to C(L_B)$, then $B \in \Sigma_{2+\alpha}$.*

Proof. Without loss of generality we may assume that $B \neq \emptyset$. It is easy to see that $B \in \Sigma_\alpha$ if $B_* \in \Sigma_\alpha$ and $\alpha \geq 2$. By Proposition 2.3(a) there exists a continuous linear surjection $S_1 : C(K_A) \to C(\overline{B}_B)$. By Proposition 2.3(c) there exists a continuous linear operator $S_2 : C(K_A) \to C([0,1]_{[0,1]})$ such that $S_2(C(K_A)) = \overline{\operatorname{lin}}(\{p_t : t \in B_* \cup \{2\}\})$.

If $K = \overline{A}$, an appeal to Theorem 3.5 completes the proof in this case.

Suppose now that $K \neq \overline{A}$ and $A \neq \emptyset$. The set $\overline{A}_A$ is a closed subset of K_A. By [4, Lemma 3.4] there exists a continuous linear extension operator $R : C(\overline{A}_A) \to C(K_A)$ such that $R(f)|_{\overline{A}_A} = f$ for every $f \in C(\overline{A}_A)$. Let $P : C(K_A) \to C(K_A)$ be the linear operator given by $P(f) = R(f|_{\overline{A}_A})$ for every $f \in C(K_A)$. Then P is a continuous linear projection and $C(K_A) = P(C(K_A)) + \ker P$. The space $P(C(K_A))$ is isomorphic to $C(\overline{A}_A)$. It is easy to see that $\ker P$ is isomorphic to

$$\{f \in C(K) : f|_{\overline{A}} = 0\}.$$

As $C(K)$ is separable, $\ker P$ is a separable subspace of $C(K_A)$. Hence $S_2(\ker P)$ is a separable subspace of $\overline{\operatorname{lin}}(\{p_t : t \in B_* \cup \{2\}\})$ in $C([0,1]_{[0,1]})$ and there exists a countable subset B_1 of $B_* \cup \{2\}$ such that $S_2(\ker P) \subset \overline{\operatorname{lin}}(\{p_t : t \in B_1\})$. For every $t \in B_*$, we pick $f_t \in C(K_A)$ such that $S_2(f_t) = p_t$. Let $u \in B_* \setminus B_1$. Then

$$\begin{aligned}\big((S_2|_{P(C(K_A))})^*(\delta_{(u,1)} - \delta_{(u,0)})\big)(P(f_u)) &= (\delta_{(u,1)} - \delta_{(u,0)})(S_2(P(f_u))) \\ &= (\delta_{(u,1)} - \delta_{(u,0)})\big(S_2(f_u - (f_u - P(f_u)))\big) \\ &= (\delta_{(u,1)} - \delta_{(u,0)})(p_u) - 0 = -1 \neq 0.\end{aligned}$$

Thus we have shown that the set

$$B_* \setminus \{t \in [0,1] : (S_2|_{P(C(K_A))})^*(\delta_{(t,1)} - \delta_{(t,0)}) \neq 0\} \subset B_1$$

is countable. An appeal to Theorem 3.5 completes the proof in this case.

Suppose now that $A = \emptyset$. Then $S_2(C(K))$ is a separable subspace of $\overline{\operatorname{lin}}(\{p_t : t \in B_* \cup \{2\}\})$ in $C([0,1]_{[0,1]})$ and there exists a countable subset B_2 of $B_* \cup \{2\}$ such that $S_2(C(K)) \subset \overline{\operatorname{lin}}(\{p_t : t \in B_2\})$. Consequently, B is countable. ∎

Corollary 3.7. *Let $1 < \alpha < \omega_1$ and $1 < \beta < \omega_1$ be ordinal numbers. Let K, L be compact subsets of $[0,1]$ and let A, B be Borel subsets of K, L respectively which are members of exact classes Σ_α and Σ_β, respectively. If the spaces $C(K_A)$ and $C(L_B)$ are isomorphic, then $\alpha \leq 1+\beta$ and $\beta \leq 1+\alpha$.*

Corollary 3.4 and Theorem 3.5 give only an approximate description of the class of all subsets E of $[0,1]$ such that there exists a continuous linear operator $S : C([0,1]_{[0,1]}) \to C([0,1]_{[0,1]})$ with $S(C([0,1]_{[0,1]})) = \overline{\operatorname{lin}}(\{p_t : t \in E \cup \{2\}\})$. It is not clear whether this class consists only of F_σ subsets of $[0,1]$.

4. On injections between K_A spaces. Corollary 3.6 provides some information on topological classification of K_A spaces. In this section we present another way of showing such results.

PROPOSITION 4.1. *Let K, L be compact subsets of $[0,1]$ and let A, B be subsets of K, L, respectively. Let $\psi_L : L_B \to L$ be given by $\psi_L(t,r) = t$ for every $(t,r) \in L_B$.*

If $f : K_A \to L_B$ is an injective map, then $g : K \to L$ given by

$$g(t) = \psi_L(f(t,0))$$

for every $t \in K$ is a left continuous and regular function on K with the following properties:

(1) *for every $t \in K$ and $u \in A \cap K_r$ we have $f(t,0) = (g(t),p)$ and $f(u,1) = (g(u+),r)$ for some $p, r \in \{0,1\}$, and $g^{-1}(\{s\})$ has at most two elements for every $s \in L$,*
(2) *g is continuous on $K \setminus A$, the set $(K \setminus A) \cap g^{-1}(B)$ is countable and $g|_{K\setminus(A\cup g^{-1}(B))}$ is injective,*
(3) *if $t \in K_l$ and $g(t) \in B$, then there exists $\delta > 0$ such that either $g(u) > g(t)$ or $g(u) < g(t)$ for all $u \in (t-\delta, t) \cap K$,*
(4) *if $t \in K_r$ and $g(t+) \in B$, then there exists $\delta > 0$ such that either $g(u) < g(t+)$ or $g(u) > g(t+)$ for all $u \in (t, t+\delta) \cap K$,*
(5) *if $t \in A \cap K_r$ and $g(t) = g(t+)$, then $g(t) \in B$,*
(6) *if $t \in A \cap K_l \cap K_r$ and $g(t) = g(t+)$, then there exists $\delta > 0$ such that either $g(u_1) < g(t) < g(u_2)$ or $g(u_1) > g(t) > g(u_2)$ for all $u_1 \in (t-\delta, t) \cap K$ and $u_2 \in (t, t+\delta) \cap K$,*
(7) *if $t \in (K \setminus A) \cap K_l \cap K_r$ and $g(t) \in B$, then there exists $\delta > 0$ such that either $g(u) < g(t)$ or $g(u) > g(t)$ for all $u \in ((t-\delta, t) \cup (t, t+\delta)) \cap K$.*

Proof. It is clear that for every $t \in K$ we have $(g(t),p) = f(t,0)$ for some $p \in \{0,1\}$. It is easy to see that the function ψ_L is continuous. Hence $\psi_L \circ f$ is a continuous function on K_A. To show the regularity of g we will need the following property of the topology of K_A: for a strictly monotonic sequence $(t_n) \subset K$ and any sequence $(r_n) \subset \{0,1\}$,

$$\lim_{n\to\infty}(t_n, r_n) = \begin{cases} (\lim_{n\to\infty} t_n, 0) & \text{if } (t_n) \text{ is increasing,} \\ (\lim_{n\to\infty} t_n, 1) & \text{if } (t_n) \text{ is decreasing and } \lim_{n\to\infty} t_n \in A, \\ (\lim_{n\to\infty} t_n, 0) & \text{if } (t_n) \text{ is decreasing and } \lim_{n\to\infty} t_n \notin A. \end{cases}$$

If $(t_n) \subset K$ is increasing, then

$$\lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} \psi_L(f(t_n,0)) = \psi_L\Big(f\Big(\lim_{n\to\infty} t_n, 0\Big)\Big) = g\Big(\lim_{n\to\infty} t_n\Big).$$

This shows that g is left continuous on K. If $(t_n) \subset K$ is strictly decreasing, then

$$\lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} \psi_L(f(t_n,0)) = \begin{cases} \psi_L(f(\lim_{n\to\infty} t_n, 1)) & \text{if } \lim_{n\to\infty} t_n \in A, \\ \psi_L(f(\lim_{n\to\infty} t_n, 0)) & \text{if } \lim_{n\to\infty} t_n \notin A. \end{cases}$$

This shows that g is a regular function on K which is continuous on $K \setminus A$. Moreover, for every $t \in A \cap K_r$ we have

$$(g(t+), p) = f(t,1) \quad \text{for some } p \in \{0,1\}.$$

Since f is injective, the set $g^{-1}(\{s\})$ has at most two elements for every $s \in L$. Thus we have shown (5). If $g^{-1}(\{g(t)\})$ has two elements for some $t \in K$, then $g(t) \in B$. Therefore $g|_{K\setminus(A\cup g^{-1}(B))}$ is injective.

If $(t_n) \subset K$ is increasing with $t = \lim_{n\to\infty} t_n$ and $g(t) \in B$, then

$$(\mathfrak{a}) \quad p_{g(t)}(f(t,0)) = \lim_{n\to\infty} p_{g(t)}(f(t_n,0)) = \begin{cases} 0 & \text{if } g(t_n) > g(t) \text{ for every } n, \\ 1 & \text{if } g(t_n) < g(t) \text{ for every } n, \end{cases}$$

where p_s denotes the continuous function $\chi_{[(\min(L),0),(s,0)]}$ on L_B for every $s \in B$. This and (1) show that g has property (3). If $(t_n) \subset K$ is decreasing with $t = \lim_{n\to\infty} t_n \in K \setminus A$ and $g(t) \in B$, then

$$(\mathfrak{b}) \quad p_{g(t)}(f(t,0)) = \lim_{n\to\infty} p_{g(t)}(f(t_n,0)) = \begin{cases} 0 & \text{if } g(t_n) > g(t) \text{ for every } n, \\ 1 & \text{if } g(t_n) < g(t) \text{ for every } n. \end{cases}$$

This and (1) show that g has property (7). If $(t_n) \subset K$ is decreasing with $t = \lim_{n\to\infty} t_n \in A$ and $g(t+) \in B$, then

$(\mathfrak{c})$

$$p_{g(t+)}(f(t,1)) = \lim_{n\to\infty} p_{g(t+)}(f(t_n,0)) = \begin{cases} 0 & \text{if } g(t_n) > g(t+) \text{ for every } n, \\ 1 & \text{if } g(t_n) < g(t+) \text{ for every } n. \end{cases}$$

Combining (1), $(\mathfrak{b})$, $(\mathfrak{c})$ we obtain (4). Combining, $(\mathfrak{a})$, $(\mathfrak{c})$ and the fact that $f(t,0) \neq f(t,1)$ for every $t \in A$ we obtain (6).

It is easy to see that for every $\delta > 0$ the sets

$$\{t \in K : g(u) < g(t) \text{ for every } u \in ((t-\delta,t) \cup (t,t+\delta)) \cap K\},$$
$$\{t \in K : g(u) > g(t) \text{ for every } u \in ((t-\delta,t) \cup (t,t+\delta)) \cap K\}$$

have at most $1/\delta$ elements each. In view of (7) the set

$$(K \setminus A) \cap g^{-1}(B) \subset \big((K \setminus A) \cap K_l \cap K_r \cap g^{-1}(B)\big) \cup (K \setminus (K_l \cap K_r))$$

is countable. ∎

Corollary 4.2. *Let K, L be compact subsets of $[0,1]$ and let A, B be subsets of K, L, respectively. Let $1 \le \alpha < \omega_1$ be an ordinal number. If there exists an injective map from K_A into L_B, then*

(a) $A = (f^{-1}(B) \setminus N_1) \cup N_2$ *for some left continuous regular function* f *on* K *and countable subsets* N_1 *and* N_2 *of* K,
(b) $A \in \Sigma_{\max\{3,1+\alpha\}}$ *whenever* $B \in \Sigma_\alpha$,
(c) $A \in \Pi_{\max\{3,1+\alpha\}}$ *whenever* $B \in \Pi_\alpha$.

Proof. We use the notations from Proposition 4.1. Let

$$M = \{t \in K_r : g \text{ is continuous at } t\}.$$

It is easy to see that $K \setminus M$ is countable. By Proposition 4.1(2, 5),

$$A \cap M \subset g^{-1}(B) \subset A \cup \big((K \setminus A) \cap g^{-1}(B)\big).$$

Consequently, $A = (g^{-1}(B) \setminus N_1) \cup N_2$ for some countable subsets N_1 and N_2 of K. ■

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Artur Michalak
Faculty of Mathematics and Computer Science
A. Mickiewicz University, Poznań
Umultowska 87
61-614 Poznań, Poland
E-mail: michalak@amu.edu.pl