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On eigenvalues of statistical hypersurfaces

Mirjana Milijević (Rio de Janeiro)

Abstract. We examine statistical hypersurfaces with shape operators having one or two constant eigenvalues, and establish the relation between the eigenvalues and the constant holomorphic sectional curvature of ambient manifolds.

1. Introduction. We will examine statistical hypersurfaces in holomorphic statistical manifolds of constant holomorphic sectional curvature c. A new definition of sectional curvature given by Opozda [9] will be used. Holomorphic statistical manifolds are defined as Kähler manifolds on which there exists a certain affine connection [6]. Naturally, on statistical manifolds we have two dual connections, which implies that we have two shape operators A and A^*.

In order to study the geometry of hypersurfaces, the principal curvatures, i.e. the eigenvalues of the shape operator, are very important. For example, when the ambient manifold is a complex hyperbolic space, Berndt and Ramos [1] obtained a classification of real hypersurfaces with three distinct principal curvatures. A classical result of Tashiro and Tachibana [10] from 1963 shows that there are no umbilical hypersurfaces, i.e. hypersurfaces with the shape operator satisfying $AX = \alpha X$, in nonflat complex space forms. This result was generalized by Djorić and Okumura [3] in 2009 to CR submanifolds of maximal CR dimension.

In [8], we initiated the study of CR statistical submanifolds of maximal CR dimension. Our first result is a generalization of Tashiro and Tachibana's results to statistical hypersurfaces: we proved the nonexistence of hypersurfaces whose shape operators satisfy $AX = \alpha X$ and $A^*X = \beta X$. In this paper we examine relations between the eigenvalues of the shape operators of a statistical hypersurface and the holomorphic sectional curvature of the

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ambient holomorphic statistical manifold. In particular, we are interested in a generalization of the results obtained by Élie Cartan [2], who classified all connected hypersurfaces with constant principal curvatures in the real hyperbolic space, and proved that the number of distinct principal curvatures is at most two. Furthermore, in [2] it is proven that every connected real hypersurface M with constant principal curvatures in $\mathbb{R}H^n$ is an open part of a homogeneous hypersurface. This means that when M is complete, the constancy of principal curvatures is equivalent to the existence of a closed subgroup of the isometry group G of $\mathbb{R}H^n$ such that M is an orbit of G. To obtain our results, we will use the assumption that the principal curvatures are constant. In this paper, we initiate the study of classification of statistical hypersurfaces.

Since in the case of statistical hypersurfaces we have two shape operators (A and A^*), we examine how many constant eigenvalues they have. Moreover, we want to see if the number of eigenvalues in question is the same as or different from the number of eigenvalues of the shape operator of hypersurfaces in complex space forms.

The paper is organized as follows. First, we examine the case when the shape operator A has only one constant eigenvalue. Next, we consider the case when one shape operator has only one eigenvalue and the dual shape operator has two eigenvalues. We prove an analogous result to the result obtained by Djorić and Okumura [3], in the case where the ambient manifold is complex projective space. We also show that the Reeb vector field U is an eigenvector of A (or A^*) if and only if $c \neq 0$ (c denotes the constant holomorphic sectional curvature of the ambient manifold).

2. Preliminaries. We denote by $(\overline{M}, \overline{g}, J)$ a complex manifold with complex structure J. (M, g) will denote a submanifold of $\overline{M}$, where the metric g is induced from $\overline{g}$.

Let M be an n-dimensional CR submanifold of maximal CR dimension of an $\frac{n+p}{2}$-dimensional manifold $\overline{M}$. Then M is necessarily odd-dimensional and there exists a unit vector ξ_x normal to T_xM such that

$$JT_xM \subset T_xM \oplus \operatorname{span}\{\xi_x\}, \qquad x \in M. \tag{2.1}$$

Hence, for any $X \in \Gamma(TM)$, we may write

$$JX = PX + u(X)\xi, \tag{2.2}$$

where P is an endomorphism acting on TM and u is a one-form on M.

Lemma 2.1 ([3]). *The subbundle $T_1^{\perp}M = \{\eta \in T^{\perp}M \mid \overline{g}(\eta, \xi) = 0\}$ is J-invariant and we can choose a local orthonormal basis of $T^{\perp}M$ in the* *following way:*

$$\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*},$$

where $\xi_{a^*} = J\xi_a$, $a = 1, \dots, q$, *and* $q = (p-1)/2$.

Moreover, we have

$$J\xi = -U \in \Gamma(TM), \tag{2.3}$$

$$P^2X = -X + u(X)U, \tag{2.4}$$

$$u(PX) = 0, \quad PU = 0, \tag{2.5}$$

$$g(U,U) = 1, \quad g(U,X) = u(X). \tag{2.6}$$

For more details we refer to [3].

Let $\overline{M}$ be a C^∞ manifold of dimension $\overline{m} \geq 2$, $\overline{\nabla}$ an affine connection on $\overline{M}$, and $\overline{g}$ a Riemannian metric on $\overline{M}$.

Definition 2.2.

(1) $(\overline{M}, \overline{\nabla}, \overline{g})$ is called a *statistical manifold* if:

- $\overline{\nabla}$ is torsion-free,
- $(\overline{\nabla}_X \overline{g})(Y,Z) = (\overline{\nabla}_Y \overline{g})(X,Z)$ for $X, Y, Z \in \Gamma(T\overline{M})$.

(2) $\overline{\nabla}^*$ is called the *dual connection* of $\overline{\nabla}$ with respect to $\overline{g}$ if

$$X\overline{g}(Y,Z) = \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}^*_X Z), \qquad X, Y, Z \in \Gamma(T\overline{M}).$$

Definition 2.3 ([9], [5]). For a statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g})$, we define

$$S^{(\overline{\nabla}, \overline{g})}(X,Y)Z = \tfrac{1}{2}\{R^{\overline{\nabla}}(X,Y)Z + R^{\overline{\nabla}^*}(X,Y)Z\},$$

$$S^{(\overline{\nabla}, \overline{g})}(X,Y,Z,W) = \overline{g}(S^{(\overline{\nabla}, \overline{g})}(Z,W)Y, X), \qquad X, Y, Z, W \in \Gamma(T\overline{M}),$$

where $R^{\overline{\nabla}}$ and $R^{\overline{\nabla}^*}$ are the curvature tensors of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

Definition 2.4 ([5]). A statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g})$ has *constant sectional curvature* c if

$$S^{(\overline{\nabla}, \overline{g})}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \qquad X, Y, Z \in \Gamma(T\overline{M}).$$

Fundamental equations for statistical submanifolds are given by P. W. Vos [11].

Definition 2.5. Let $(\overline{M}, \overline{\nabla}, \overline{g})$ be a statistical manifold and M a submanifold of $\overline{M}$. We denote by $T_x^\perp M$ the normal space of M, i.e. $T_x^\perp M := \{v \in T_x\overline{M} \mid \overline{g}(v,w) = 0,\ w \in T_xM\}$. We define

$$\nabla, \nabla^* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

$$h, h^* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^\perp M),$$

$$A, A^* : \Gamma(T^\perp M) \times \Gamma(TM) \to \Gamma(TM),$$

and

$$D, D^* : \Gamma(TM) \times \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M)$$

by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X,Y), & \overline{\nabla}_X V &= -A_V X + D_X V, \\ \overline{\nabla}^*_X Y &= \nabla^*_X Y + h^*(X,Y), & \overline{\nabla}^*_X V &= -A^*_V X + D^*_X V, \end{aligned}$$

for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The *shape operators* A and A^* are defined by

$$g(A_V X, Y) := \overline{g}(h^*(X,Y), V), \quad g(A^*_V X, Y) := \overline{g}(h(X,Y), V),$$

for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where g denotes the induced metric on M. The second fundamental forms are denoted by h and h^*.

DEFINITION 2.6 ([4]). Let $(\overline{M}, J, \overline{g})$ be a Kähler manifold and $\overline{\nabla}$ an affine connection of $\overline{M}$. Then $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ is called a *holomorphic statistical manifold* if

- $(\overline{M}, \overline{\nabla}, \overline{g})$ is a statistical manifold,
- $\overline{\omega} := \overline{g}(*, J*)$ is a $\overline{\nabla}$-parallel 2 - form on $\overline{M}$.

LEMMA 2.7 ([4]). *Let $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ be a holomorphic statistical manifold. Then $\overline{\nabla}_X JY = J\overline{\nabla}^*_X Y$ for $X, Y \in \Gamma(T\overline{M})$.*

DEFINITION 2.8 ([5]). A holomorphic statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ is said to be *of constant holomorphic sectional curvature* $c \in \mathbb{R}$ if

$$\begin{aligned} S^{(\overline{\nabla}, \overline{g})}(X,Y)Z = \tfrac{1}{4}c\{&\overline{g}(Y,Z)X - \overline{g}(X,Z)Y \\ &+ \overline{g}(JY,Z)JX - \overline{g}(JX,Z)JY + 2\overline{g}(X,JY)JZ\} \end{aligned}$$

for $X, Y, Z \in \Gamma(T\overline{M})$.

3. Statistical hypersurfaces. Let $(\overline{M}, \overline{\nabla}, \overline{g}, J)$ be a $2m$-dimensional holomorphic statistical manifold and M a $(2m-1)$-dimensional submanifold of $\overline{M}$, i.e. a hypersurface of $\overline{M}$. Let ξ be a unit normal vector field of M. The Gauss and Weingarten equations are given by (see [8])

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X,Y), & \overline{\nabla}^*_X Y &= \nabla^*_X Y + h^*(X,Y), \\ \overline{\nabla}_X \xi &= -AX + s(X)\xi, & \overline{\nabla}^*_X \xi &= -A^*X + s^*(X)\xi, \end{aligned}$$

respectively.

For the 1-forms s and s^*, we have

$$s(X) = -s^*(X).$$

The structure vector U, also called the *Reeb vector field*, is defined by

$$U := -J\xi \in \Gamma(TM).$$

For $X \in \Gamma(TM)$, JX decomposes into the tangent and normal parts as

$$JX = PX + g(X,U)\xi. \tag{3.1}$$

When $\overline{M}$ is of constant holomorphic sectional curvature c, the Codazzi equation is

$$\begin{aligned} \tfrac{1}{2}c\{g(X,U)PY - g(Y,U)PX + 2g(X,PY)U\} \\ = (\nabla^*_X A^* + \nabla_X A)Y - (\nabla^*_Y A^* + \nabla_Y A)X \\ + s(X)(A^* - A)Y - s(Y)(A^* - A)X. \end{aligned} \tag{3.2}$$

For more details on how to obtain (3.2) we refer to [7]. We also proved [7, Proposition 18] that

$$\nabla_X U = PA^*X + s^*(X)U, \tag{3.3}$$

$$\nabla^*_X U = PAX + s(X)U. \tag{3.4}$$

LEMMA 3.1. *Let M be a real hypersurface of dimension ≥ 3 in a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c. Suppose the shape operators A and A^* satisfy*

$$AU = \alpha U, \qquad AV = \beta V, \qquad A^*X = \gamma X,$$

for all $X \in \Gamma(TM)$ and $V \in U^\perp = \{Y \in \Gamma(TM) \mid g(Y,U) = 0\}$ and $\alpha, \beta, \gamma \in \mathbb{R}$, with $\alpha \neq \beta$. Then one of the following holds:

(1) $c = 0,\ A^* = 0,\ D_X\xi = 0,\ \beta = 0,$
(2) $c = 0,\ A^* = 0,\ D_V\xi = 0$ *for* $V \in U^\perp$, $\beta = 0,$
(3) $c = 2\gamma(\beta - \alpha) \neq 0,\ \beta = \gamma,\ D_X\xi = 0,$
(4) $c = 2\gamma(\beta - \alpha) \neq 0,\ \gamma = \alpha,\ D_X\xi = 0,$
(5) $c = 2\gamma(\beta - \alpha) \neq 0,\ \beta = \gamma,\ D_V\xi = 0$ *for* $V \in U^\perp$.

Proof. Differentiating $AU = \alpha U$, we get

$$(\nabla_X A)U + A\nabla_X U = \alpha \nabla_X U. \tag{3.5}$$

From the Codazzi equation, we get

$$\begin{aligned} -\tfrac{1}{2}cg(U,U)PX = (\nabla^*_X A^*)U + (\nabla_X A)U - (\nabla^*_U A^*)X - (\nabla_U A)X \\ + s(X)A^*U - s(X)AU - s(U)A^*X + s(U)AX. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\begin{aligned} -\tfrac{1}{2}cPX = (\nabla^*_X A^*)U + \alpha\nabla_X U - A\nabla_X U - (\nabla^*_U A^*)X - (\nabla_U A)X \\ + s(X)A^*U - \alpha s(X)U - s(U)A^*X + s(U)AX. \end{aligned} \tag{3.7}$$

From (3.7), it follows that

$$
\begin{aligned}
(3.8)\qquad & -\tfrac{1}{2}cg(PX,Y)\\
&= g((\nabla^*_X A^*)U,Y) + \alpha g(\nabla_X U,Y) - g(A\nabla_X U,Y)\\
&\quad - g((\nabla^*_U A^*)X,Y) - g((\nabla_U A)X,Y) + s(X)g(A^*U,Y)\\
&\quad - \alpha s(X)g(U,Y) - s(U)g(A^*X,Y) + s(U)g(AX,Y), \quad Y \in \Gamma(TM).
\end{aligned}
$$

Next, we interchange X and Y in (3.8) and subtract (3.8) and the equation obtained. The resulting equation is

$$
\begin{aligned}
(3.9)\qquad & -cg(PX,Y) = g((\nabla^*_X A^*)U,Y) - g((\nabla^*_Y A^*)U,X)\\
&+ \alpha g(\nabla_X U,Y) - \alpha g(\nabla_Y U,X) - g(A\nabla_X U,Y) + g(A\nabla_Y U,X)\\
&- g((\nabla^*_U A^*)X,Y) + g((\nabla^*_U A^*)Y,X) - g((\nabla_U A)X,Y) + g((\nabla_U A)Y,X)\\
&+ s(X)g(A^*U,Y) - s(Y)g(A^*U,X) - \alpha s(X)g(U,Y) + \alpha s(Y)g(U,X).
\end{aligned}
$$

If we replace Y by U in (3.9), we get

$$
\begin{aligned}
(3.10)\qquad & \\
0 &= g((\nabla^*_X A^*)U,U) - \alpha g(\nabla_U U,X) + g(A\nabla_U U,X) - g((\nabla^*_U A^*)X,U)\\
&\quad - g(X,(\nabla^*_U A^*)U) + g((\nabla_U A)U,X)\\
&\quad + s(X)g(A^*U,U) - s(U)g(A^*U,X) - \alpha s(X) + \alpha s(U)g(U,X),
\end{aligned}
$$

that is,

$$
\begin{aligned}
(3.11)\qquad & \\
0 &= -g((\nabla^*_X A^*)U,U) + g((\nabla^*_U A^*)X,U) + \alpha g(X,\nabla^*_U U) - g(X,A\nabla^*_U U)\\
&\quad - s(X)g(A^*U,U) + s(U)g(A^*U,X) + \alpha s(X) - \alpha s(U)g(U,X).
\end{aligned}
$$

From (3.9) and (3.11), we get

$$
\begin{aligned}
(3.12)\qquad -cg(PX,Y) &= g((\nabla^*_X A^*)U,Y) - g((\nabla^*_Y A^*)U,X)\\
&- g((\nabla^*_X A^*)U,U)g(U,Y) + g((\nabla^*_U A^*)X,U)g(U,Y)\\
&+ \alpha g(X,\nabla^*_U U)g(U,Y) - g(X,A\nabla^*_U U)g(U,Y)\\
&- s(X)g(A^*U,U)g(U,Y) + s(U)g(A^*U,X)g(U,Y)\\
&+ g((\nabla^*_Y A^*)U,U)g(U,X) - g((\nabla^*_U A^*)Y,U)g(U,X)\\
&- \alpha g(Y,\nabla^*_U U)g(U,X) + g(Y,A\nabla^*_U U)g(U,X)\\
&+ s(Y)g(A^*U,U)g(U,X) - s(U)g(A^*U,Y)g(U,X)\\
&+ \alpha g(\nabla_X U,Y) - \alpha g(\nabla_Y U,X) - g(A\nabla_X U,Y) + g(A\nabla_Y U,X)\\
&- g((\nabla^*_U A^*)X,Y) + g((\nabla^*_U A^*)Y,X) - g((\nabla_U A)X,Y)\\
&+ g((\nabla_U A)Y,X) + s(X)g(A^*U,Y) - s(Y)g(A^*U,X).
\end{aligned}
$$

By (3.12), we have

$$-cg(P^2X,PY) = g((\nabla^*_{PX}A^*)U,PY) - g((\nabla^*_{PY}A^*)U,PX) + \alpha g(\nabla_{PX}U,PY) - \alpha g(\nabla_{PY}U,PX) - g(A\nabla_{PX}U,PY) + g(A\nabla_{PY}U,PX) - g((\nabla^*_U A^*)PX,PY) + g((\nabla^*_U A^*)PY,PX) - g((\nabla_U A)PX,PY) + g((\nabla_U A)PY,PX) + s(PX)g(A^*U,PY) - s(PY)g(A^*U,PX). \tag{3.13}$$

From (3.10) and (3.13), we deduce that

$$0 = \{\gamma(\alpha-\beta)+c/2\}PV + s(U)(\beta-\gamma)V + s(V)(\gamma-\alpha)U, \tag{3.14}$$

where V is a tangent vector orthogonal to U. Since U, V and PV are linearly independent for $V \neq 0$, we conclude that $\gamma(\alpha-\beta)+c/2 = 0$, $s(U)(\beta-\gamma) = 0$ and $s(V)(\gamma-\alpha) = 0$.

We have the following cases:

(I) $c = 0$. Then $\gamma = 0$, that is, $A^* = 0$.

(1) $s(U) = 0$ and $s(V) = 0$ for V orthogonal to U, that is, $s(X) = 0$ for every tangent vector X. Then from the Codazzi equation, we get

$$0 = (\nabla_X A)Y - (\nabla_Y A)X.$$

Taking the inner product with U, we obtain $\beta = 0$.

(2) $s(U) = 0$ and $\alpha = 0$. Again from the Codazzi equation, we conclude that $\beta = 0$. This is a contradiction.

(3) $\beta = 0$ and $s(V) = 0$ for V orthogonal to U.

(II) $c \neq 0$. Then $c = 2\gamma(\beta-\alpha)$.

(1) $s(X) = 0$ for any tangent vector X. From the Codazzi equation multiplied with U, we have the equation

$$\gamma(\beta-\alpha) = \beta^2 - \alpha\beta,$$

which has a solution $\beta = \gamma$.

(2) $s(U) = 0$, $\gamma = \alpha$. In this case from the Codazzi equation, we conclude that $s(X) = 0$ for any tangent vector X.

(3) $\beta = \gamma$ and $s(V) = 0$ for any vector V orthogonal to U. ∎

LEMMA 3.2. *Let M be a real hypersurface of dimension > 3 in a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature $c \neq 0$. If the shape operator A^* $[A]$ of M has only one eigenvalue and that its dual shape operator A $[A^*]$ has exactly two distinct eigenvalues, then U is an eigenvector of A.*

Proof. Let γ be an eigenvalue of A^*, i.e. $A^*X = \gamma X$, and let α and β be eigenvalues of A. We denote by T_α and T_β the eigenspaces corresponding to the eigenvalues α and β, respectively. Let us assume that

$$U = pX + qY, \tag{3.15}$$

where $p, q \neq 0$, $X \in T_\alpha$ and $Y \in T_\beta$. We denote by $S_\alpha = \{V \in T_\alpha \mid g(V, X) = 0\}$ and $S_\beta = \{W \in T_\beta \mid g(W, Y) = 0\}$. In the following we will use similar techniques to [3]. From the Codazzi equation, we get

$$\begin{aligned} cg(V, PQ)U = {} & [(V\alpha) + s(V)\gamma - s(V)\alpha]Q \\ & - [(Q\alpha) + s(Q)\gamma - s(Q)\alpha]V + (\alpha \operatorname{Id} - A)[V, Q], \end{aligned} \tag{3.16}$$

where V and Q are vector fields mutually orthogonal in S_α. Now, we take the inner product with Q. It follows that

$$(V\alpha) + s(V)\gamma - s(V)\alpha = 0. \tag{3.17}$$

From (3.16) and (3.17), we get

$$cg(V, PQ)U = (\alpha \operatorname{Id} - A)[V, Q]. \tag{3.18}$$

If $c \neq 0$, from the last equation we conclude that PQ is orthogonal to S_α for any $Q \in S_\alpha$.

The Codazzi equation for $V \in S_\alpha$ and $X \in T_\alpha$ (from the decomposition (3.15)) gives

$$\begin{aligned} \tfrac{1}{2}c\{pPV + 2g(X, PV)U\} = {} & (X\alpha)V - (V\alpha)X + (\alpha \operatorname{Id} - A)[X, V] \\ & + s(X)(\gamma - \alpha)V - s(V)(\gamma - \alpha)X. \end{aligned} \tag{3.19}$$

Taking the inner product with V, we obtain

$$X\alpha + s(X)(\gamma - \alpha) = 0. \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$\tfrac{1}{2}c\{pPV + 2g(X, PV)U\} = (\alpha \operatorname{Id} - A)[X, V]. \tag{3.21}$$

We take the inner product of (3.21) with a particular vector $X \in T_\alpha$. The result is

$$cpg(X, PV) = 0.$$

From this equation it follows that PV is orthogonal to X, that is, $PV \in T_\beta$. Now, we have $0 = g(PV, U) = qg(PV, Y)$, that is, PV is orthogonal to Y. This means that $PV \in S_\beta$, i.e. $P(S_\alpha) \subset S_\beta$.

From the decomposition (3.15), it follows that the vectors $V \in S_\alpha$ are orthogonal to U, which means that $JV = PV$ for any $V \in S_\alpha$, i.e. P is injective on S_α and thus $\dim S_\beta \geq \dim S_\alpha \geq 2$. Similarly, interchanging the roles of S_α and S_β, we find that $P(S_\beta) \subset S_\alpha$. This means that $\dim S_\alpha = \dim S_\beta$, i.e. M is even-dimensional, which is a contradiction. Therefore, $p = 0$ or $q = 0$. ∎

From Lemmas 3.1 and 3.2, we can easily deduce the following theorem.

THEOREM 3.3. *Let M be a real hypersurface of dimension > 3 in a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c. Suppose the shape operator A $[A^*]$ of M has two distinct constant nonzero eigenvalues α and β, and its dual shape operator A^* $[A]$ has only one constant nonzero eigenvalue γ different from α and β. Then U is an eigenvector of A $[A^*]$ if and only if $c \neq 0$.*

4. Three-dimensional statistical hypersurfaces. In this section, we will examine principal curvatures of three-dimensional statistical hypersurfaces. Our first result is the following.

THEOREM 4.1. *Let M be a three-dimensional hypersurface in a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c. Suppose the shape operator A $[A^*]$ of M has one constant eigenvalue α, and its dual shape operator A^* $[A]$ has two constant eigenvalues β and γ, where $\beta \neq \gamma$, and U is an eigenvector corresponding to β. Then $c = 0$.*

Proof. We assume that the distribution T_γ is generated by a tangent vector X and that T_β is generated by U and PX. First, we will replace Y by U in the Codazzi equation. The resulting equation is

$$\begin{aligned}(4.1)\quad -\tfrac{1}{2}cPX = {}& \nabla^*_X(\beta U) - A^*\nabla^*_X U + \nabla_X(\alpha U) \\ & - A\nabla_X U - \nabla^*_U(\gamma X) + A^*\nabla^*_U X - \nabla_U(\gamma X) \\ & + A\nabla_U X + \beta s(X)U - \alpha s(X)U - \gamma s(U)X + \alpha s(U)X.\end{aligned}$$

Taking the inner product with X, we obtain

$$(4.2)\qquad 0 = (\beta-\gamma)g(\nabla^*_X U, X) + (\alpha-\gamma)g(\nabla_X U, X) + (\alpha-\gamma)s(U).$$

This means $0 = (\alpha-\gamma)g(\nabla_U X, X) + (\alpha-\gamma)s(U)$, from which it follows that $\alpha = \gamma$ or $g(\nabla_U X, X) = -s(U)$.

(1) If $\alpha = \gamma$, from (4.1) it follows that

$$(4.3)\qquad -\tfrac{1}{2}cPX = -\alpha\nabla^*_U X + A^*\nabla^*_U X + \beta s(X)U - \alpha s(X)U,$$

where we have used

$$\nabla^*_X U = PAX + s(X)U = \alpha PX + s(X)U, \qquad A^*\nabla^*_X U = \alpha\beta PX + \beta s(X)U,$$

and similarly

$$A\nabla_X U = \alpha^2 PX - \alpha s(X)U.$$

Now taking the inner product of (4.3) with U, we obtain $0 = (\beta-\alpha)s(X)$ because $\nabla_U U = s^*(U)U$. Since $\beta \neq \alpha$, we conclude that $s(X) = 0$ for

$X \in \Gamma(T_\gamma)$. The equation (4.3) becomes

$$-\tfrac{1}{2}cPX = -\alpha\nabla^*_U X + A^*\nabla^*_U X. \tag{4.4}$$

Taking the inner product with PX, we obtain

$$-\tfrac{1}{2}cg(PX,PX) = (\beta-\alpha)g(\nabla^*_U X, PX), \tag{4.5}$$

from which we conclude that $\nabla^*_U X$ is not orthogonal to PX. Therefore, $\nabla^*_U X = tX + vPX$. In this case, (4.5) becomes

$$-\tfrac{1}{2}cg(PX,PX) = (\beta-\alpha)g(tX + vPX, PX),$$

that is, $-c/2 = (\beta-\alpha)v$. If we take the inner product of $\nabla_U X$ with X and U respectively, we conclude that $\nabla_U X = -tX + v_1PX$. Furthermore, we have $\nabla_X U = \alpha PX$ and $\nabla^*_X U = \alpha PX$. Therefore, if we calculate $[X,U] = \nabla_X U - \nabla_U X = (\alpha - v_1)PX + tX$ and $[X,U] = \nabla^*_X U - \nabla^*_U X = (\alpha - v)PX - tX$, we conclude that $v_1 = v$ and $t = 0$. This means $\nabla^*_U X = vPX$ and $\nabla_U X = vPX$. Next, we calculate $g(\nabla_U X, PX) = v$ and $g(\nabla_U X, PX) = -g(X, \nabla^*_U PX)$. It follows that $\nabla^*_U PX$ is not orthogonal to X. Similarly, we conclude that $\nabla^*_U PX$ is orthogonal to U. Therefore, we can write $\nabla^*_U PX = t_2X + v_2PX$. If we take the inner products $g(\nabla^*_U PX, X) = -g(PX, \nabla_U X) = -v$ and $g(\nabla^*_U PX, X) = g(t_2X + v_2PX, X) = t_2$, we conclude $t_2 = -v$, i.e. $\nabla^*_U PX = -vX + v_2PX$. Similarly, $\nabla^*_{PX} U = -\alpha X + s(PX)U$. On the other hand, we have $g(\nabla_U PX, X) = -g(PX, \nabla^*_U X) = -v$ and $g(\nabla_U PX, U) = 0$. Therefore, we can write $\nabla_U PX = -vX - v_2PX$ since $g(\nabla_U PX, PX) = -g(PX, \nabla^*_U PX) = -v_2$. Moreover, we know that $\nabla_{PX} U = -\beta X + s^*(PX)U$. Let us replace X by PX, and Y by U, in the Codazzi equation to obtain

$$\begin{aligned} -\tfrac{1}{2}cP^2X &= \beta s(PX)U + \alpha^2 X - \alpha\beta X + \alpha s^*(PX)U \\ &\quad + \beta vX - \alpha vX - \beta s(U)PX + \alpha s(U)PX. \end{aligned} \tag{4.6}$$

Taking the inner product with U, we get $0 = (\beta-\alpha)s(PX)$, which means that $s(PX) = 0$ for $X \in \Gamma(T_\gamma)$. Therefore, $\nabla_{PX} U = -\beta X$ and $\nabla^*_{PX} U = -\alpha X$. On the other hand, taking the inner product with PX, we find that $0 = -\beta s(U) + \alpha s(U)$, i.e. $s(U) = 0$. Thus, we have proved that $s(Z) = 0$ for $Z \in \Gamma(TM)$. From $[U,PX] = \nabla^*_U PX - \nabla^*_{PX} U = -vX + v_2PX + \alpha X$ and $[U,PX] = \nabla_U PX - \nabla_{PX} U = -vX - v_2PX + \beta X$, it follows that $\alpha = \beta$. This is a contradiction.

(2) Let us examine the case when $g(\nabla_U X, X) = -s(U)$. We have $\nabla_U X = -s(U)X + rPX$ since $\nabla_U X$ is orthogonal to U. The equation (4.1) then becomes

$$\begin{aligned} -\tfrac{1}{2}cPX &= -\gamma\nabla^*_U X + A^*\nabla^*_U X - \gamma\nabla_U X \\ &\quad + \alpha\nabla_U X + (\beta-\alpha)s(X)U - (\gamma-\alpha)s(U)X. \end{aligned} \tag{4.7}$$

Taking the inner product with U, we get $0 = (\beta-\alpha)s(X)$. We will examine two cases.

2.1. $\beta=\alpha$. In this case, the equation (4.7) becomes

$$-\tfrac{1}{2}cPX=\alpha\gamma PX-\alpha^2PX-\gamma\nabla^*_UX+A^*\nabla^*_UX+(\alpha-\gamma)\nabla_UX. \tag{4.8}$$

Taking the inner product with PX, we get $-c/2=(\alpha-\gamma)(g(\nabla^*_UX,PX)+r)$. Since we can also write $\nabla^*_UX=s(U)X+r_1PX$, we have $-c/2=(\alpha-\gamma)(r+r_1)$. Next, we calculate $[U,X]=\nabla_UX-\nabla_XU=-s(U)X+rPX-\gamma PX-s^*(X)U$ and $[U,X]=\nabla^*_UX-\nabla^*_XU=s(U)X+r_1PX-\alpha PX-s(X)U$. Identifying the right hand sides of the equations obtained, and then taking the inner product with PX, X and U, we obtain $r-\gamma=r_1-\alpha$, $s(U)=0$, and $s(X)=0$, respectively. This means that $\nabla_UX=rPX$, $\nabla^*_UX=r_1PX$. The equation (4.7) then becomes

$$-\tfrac{1}{2}c=-\gamma r_1+\alpha r_1-\gamma r+\alpha r. \tag{4.9}$$

Since $r_1+r=-\frac{c}{2(\alpha-\gamma)}$ and $r_1-r=\alpha-\gamma$, we get

$$r_1=-\frac{c}{4(\alpha-\gamma)}+\frac{\alpha-\gamma}{2}\quad\text{and}\quad r=-\frac{c}{4(\alpha-\gamma)}+\frac{\gamma-\alpha}{2}.$$

Next we calculate $g(\nabla_UPX,U)=-g(PX,\nabla^*_UU)=-g(PX,PAU+s(U)U)=0$, $g(\nabla_UPX,X)=-g(PX,\nabla^*_UX)=-g(PX,r_1PX)=-r_1$. Therefore, $\nabla_UPX=-r_1X+pPX$. Also, $\nabla_{PX}U=-\beta X+s^*(PX)U$. Similarly, we obtain $\nabla^*_UPX=-rX-pPX$ and $\nabla^*_{PX}U=-\alpha X+s(PX)U$.

If we replace X by PX and Y by U in the Codazzi equation, we get

$$-\tfrac{1}{2}cPX=\gamma\alpha X-\alpha^2X+\alpha rX+2\alpha pPX-r\gamma X. \tag{4.10}$$

Taking the inner product with X, we obtain $\alpha^2+(-\gamma-r)\alpha+r\gamma=0$ with the solution $\alpha=r$. Then $[U,PX]=\nabla_UPX-\nabla_{PX}U=-r_1X+pPX+\alpha X-s^*(PX)U$ and $[U,PX]=\nabla^*_UPX-\nabla^*_{PX}U=-pPX-s(PX)U$. We conclude that $r=r_1$, which is a contradiction.

2.2. $s(X)=0$ for $X\in\Gamma(T_\gamma)$. First, we will write $\nabla^*_UX=s(U)X+pPX$, $\nabla^*_XU=\alpha PX$, $\nabla_UX=-s(U)X+p_1PX$ and $\nabla_XU=\gamma PX$. If we calculate $[U,X]$ using ∇ and ∇^*, we conclude that $s(U)=0$ and $p-p_1=\alpha-\gamma$, respectively. Next, we calculate $\nabla_UPX=-pX+rPX$, $\nabla_{PX}U=-\beta X+s^*(PX)U$, $\nabla^*_{PX}U=-p_1X+r_1PX$ and $\nabla^*_{PX}U=-\alpha X+s(PX)U$. Using the same reasoning, we calculate $[U,PX]=-pX+rPX+\beta X-s^*(PX)U$ and $[U,PX]=-p_1X+r_1PX+\alpha X-s(PX)U$. Equating the right hand sides, we obtain $s(PX)=0$, $r=r_1$ and $p-p_1=\beta-\alpha$. We conclude that $\nabla^*_UX=pPX$, $\nabla^*_XU=\alpha PX$, $\nabla_UX=p_1PX$, $\nabla_XU=\gamma PX$, $\nabla_UPX=-pX+rPX$, $\nabla_{PX}U=-\beta X$, $\nabla^*_UPX=-p_1X+rPX$ and $\nabla^*_{PX}U=-\alpha X$. Finally, we replace X by PX, and Y by U in the Codazzi equation. The result is

$$-\tfrac{1}{2}cPX=\gamma\alpha X-\alpha\beta X+\beta p_1X-p_1\gamma X,$$

from which we conclude that $c=0$. ■

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Mirjana Milijević
Institute of Mathematics and Statistics
Universidade Federal Fluminense
Niteroi, Rio de Janeiro, Brazil
E-mail: mirjana.mmilijevic@gmail.com