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Classification of (k,μ)-contact manifolds with divergence free Cotton tensor and vanishing Bach tensor

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Abstract. We first prove that a (k,μ)-contact manifold of dimension $2n+1$ with divergence free Cotton tensor is flat in dimension 3, and in higher dimensions, locally isometric to $S^n(4)\times E^{n+1}$. Finally, we show that a Bach flat non-Sasakian (k,μ)-contact manifold is flat in dimension 3, and in each higher dimension, there is a unique (k,μ)-contact manifold locally isometric, up to a D-homothetic deformation, to the unit tangent sphere bundle of a space of constant curvature $\neq 1$. This result provides an example of a Bach flat metric that is neither Einstein nor conformally flat.

1. Introduction. The notion of Bach tensor was introduced by R. Bach [Ba21] to study conformal relativity. This is a symmetric traceless $(0,2)$ type tensor B on an m-dimensional Riemannian manifold (M,g), defined as

$$B(X,Y)=\frac{1}{m-3}\sum_{i,j=1}^{m}(\nabla_{e_i}\nabla_{e_j}W)(X,e_i,e_j,Y)+\frac{1}{m-2}\sum_{i,j=1}^{m}\mathrm{Ric}(e_i,e_j)W(X,e_i,e_j,Y)\tag{1.1}$$

where $(e_i)_{i=1}^m$ is a local orthonormal frame on (M,g), Ric is the Ricci tensor of type $(0,2)$ and W denotes the Weyl tensor of type $(0,4)$ defined by

$$W=R-\frac{2}{m-2}\mathrm{Ric}\odot g+\frac{r}{(m-1)(m-2)}g\odot g.\tag{1.2}$$

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Here $\odot$ is the well known Kulkarni–Nomizu product defined for two symmetric $(0,2)$-tensors s and t as

$$(s \odot t)(X,Y,Z,U) = \tfrac{1}{2}[t(X,U)s(Y,Z) + t(Y,Z)s(X,U) - t(X,Z)s(Y,U) - t(Y,U)s(X,Z)],$$

where X, Y, Z, U denote arbitrary vector fields on M. We will denote the Levi-Civita connection, curvature tensor and Ricci operator of g by ∇, R, and Q respectively, so that $g(QX,Y) = \mathrm{Ric}(X,Y)$. These conventions will be followed throughout this paper. We recall the *Cotton tensor* C which is a $(0,3)$-tensor defined by

$$C(X,Y,Z) = (\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z) - \frac{1}{2(m-1)}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \tag{1.3}$$

It is also well known (see Kuhnel and Rademacher [KR97]) that $\operatorname{div} W = \frac{m-3}{m-2}C$. In view of (1.1)–(1.3), the Bach tensor can be expressed as (Chen and He [CH13])

$$B(X,Y) = \frac{1}{m-2}\Big[\sum_{i=1}^{m}(\nabla_{e_i} C)(e_i,X)Y + \sum_{i,j=1}^{m} \mathrm{Ric}(e_i,e_j)W(X,e_i,e_j,Y)\Big]. \tag{1.4}$$

In dimension 3, the Weyl tensor W vanishes, and hence the Bach tensor expression reduces to

$$B(X,Y) = \sum_{i=1}^{3}(\nabla_{e_i} C)(e_i,X)Y. \tag{1.5}$$

A Riemannian manifold is said to be *Bach flat* if $B = 0$. In view of the definition (1.1), we see that Bach flatness is a natural generalization of Einstein and conformal flatness. We also note that Bach flat metrics in dimension 4 are the critical points of the Weyl functional $\mathcal{W}(g) = \int_M |W_g|^2 \,\mathrm{dvol}_g$. In this paper, we classify (k,μ)-contact manifolds (mentioned in Section 2) whose metrics have (i) divergence-free Cotton tensor, and (ii) vanishing Bach tensor. *Case (ii) provides a contact-theoretic example of a Bach flat metric in any odd dimension > 3 that is neither Einstein nor conformally flat.* We prove the following two results.

THEOREM 1.1. *If the Cotton tensor of a non-Sasakian (k,μ)-contact manifold M of dimension $2n+1$ is divergence free, then M is flat in dimension 3, and in higher dimensions, locally isometric to the trivial bundle $S^n(4) \times E^{n+1}$.*

COROLLARY 1.2. *If a non-Sasakian (k, μ)-contact manifold M of dimension $2n + 1$ has harmonic Weyl tensor, then M is flat in dimension 3, and in higher dimensions, locally isometric to $S^n(4) \times E^{n+1}$.*

The identity $\operatorname{div} W = \frac{m-3}{m-2} C$ shows that W harmonic is equivalent to $C = 0$. Spaces with $C = 0$ are also known as *C-spaces* in general relativity [S63].

THEOREM 1.3. *If a non-Sasakian (k, μ)-contact manifold M of dimension $2n + 1$ is Bach flat, then it is flat in dimension 3, and in each higher dimension, there exists a unique pair (k_n, μ_n) of values of (k, μ), depending on n as shown in Table 1 in the proof. Each (k_n, μ_n)-contact manifold is locally isometric, up to a D-homothetic deformation, to the unit tangent sphere bundle of some space of constant curvature $\neq 1$. In higher dimensions, the Ricci curvature is negative along the Reeb vector field.*

REMARK 1. The proof of Theorem 1.3 involves solving two nonlinear algebraic equations using computer solvers. As $n \to \infty$ (i.e. $\dim M \to \infty$), we observe that $k_n \to (3 - \sqrt{13})/2$, $\mu_n \to 2$, and this limiting value 2 of μ_n is fixed (invariant) under D-homothetic deformations, as explained in Section 2.

REMARK 2. Sasakian manifolds with purely transversal Bach tensor were recently studied by Ghosh and Sharma [GS17]. We note the following result of Tripathi and Kim [T04]: "A non-Sasakian Einstein (k, μ)-contact manifold is 3-dimensional and flat". On the other hand, we note the following result of Gouli-Andreou and Tsolakidou [GT04]: "A conformally flat contact metric manifold whose Reeb vector field is an eigenvector of the Ricci operator, is of constant curvature." As the Reeb vector field of a (k, μ)-contact manifold is an eigenvector of the Ricci operator, Theorems 1.1 and 1.3 provide a generalization of the two aforementioned results. Furthermore, though a (k_n, μ_n)-contact metric is Bach flat, it is neither conformally flat nor Einstein.

2. Preliminaries. A $(2n + 1)$-dimensional smooth manifold is said to be *contact* if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. Given a contact 1-form η there always exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarization of $d\eta$ on the contact subbundle D (defined by $\eta = 0$) yields a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \qquad \eta(X) = g(X, \xi), \qquad \varphi^2 = -I + \eta \otimes \xi.$$

g is called the *associated metric* of η and (φ, η, ξ, g) a *contact metric structure*. We recall the self-adjoint [Bl10] operators $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. The tensors h, $h\varphi$ are trace-free and $h\varphi = -\varphi h$. The following formula holds on a contact metric manifold [Bl10]:

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{2.1}$$

The contact metric structure on M is said to be *Sasakian* if the almost Kähler structure on the cone manifold $(M \times \mathbb{R}^+, r^2 g + dr^2)$ over M is Kähler (see Blair [Bl10] and Boyer–Galicki [BG08]). By a (k,μ)*-contact manifold* for some real numbers (k,μ) we mean a contact metric manifold $M(\varphi,\xi,\eta,g)$ of dimension $2n+1$ whose curvature tensor satisfies (Blair et al. [BKP95])

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}. \tag{2.2}$$

Interestingly, this class of manifolds is preserved under D-homothetic deformations (Tanno [T68]) $\bar\eta = a\eta$, $\bar\xi = \frac{1}{a}\xi$, $\bar\varphi = \varphi$, $\bar g = ag + a(a-1)\eta \otimes \eta$ for a positive constant a, and includes Sasakian manifolds (for which $k=1$ and $h=0$) and the trivial sphere bundle $S^n(4) \times E^{n+1}$ (for which $k=\mu=0$, a result of Blair [Bl77]). Under a D-homothetic deformation, a (k,μ)-contact metric structure changes to a $(\bar k,\bar\mu)$-contact metric structure according to $\bar k = (k + a^2 - 1)/a^2, \bar\mu = (\mu + 2a - 2)/a$. In particular, if $\mu = 2$, then $\bar\mu = 2$. Thus $\mu = 2$ is a fixed under D-homothetic deformations. For (k,μ)-contact manifolds, we know [BKP95] that

$$\mathrm{Ric}(X,\xi) = 2nkg(X,\xi), \tag{2.3}$$

$$h^2 = (1-k)(I - \eta \otimes \xi). \tag{2.4}$$

This shows that $k \le 1$, and equality holds when $k=1$, $h=0$, i.e. M is Sasakian. For the non-Sasakian case, i.e. $k < 1$, the (k,μ)-nullity condition (1.2) determines the curvature of M completely, and the curvature tensor is given by Boeckx [Bo00] as

$$\begin{aligned} g(R(X,Y)Z,U) &= (1-\mu/2)(g(Y,Z)g(X,U) - g(X,Z)g(Y,U)) \\ &+ g(Y,Z)g(hX,U) - g(X,Z)g(hY,U) - g(Y,U)g(hX,Z) \\ &+ g(X,U)g(hY,Z) + \frac{1-\mu/2}{1-k}\big(g(hY,Z)g(hX,U) - g(hX,Z)g(hY,U)\big) \\ &- \frac{\mu}{2}\big(g(\varphi Y,Z)g(\varphi X,U) - g(\varphi X,Z)g(\varphi Y,U)\big) \\ &+ \frac{k-\mu/2}{1-k}\big(g(\varphi hY,Z)g(\varphi hX,U) - g(\varphi hX,Z)g(\varphi hY,U)\big) \\ &+ \mu g(\varphi X,Y)g(\varphi Z,U) \\ &+ \eta(X)\eta(U)\big((k-1+\mu/2)g(Y,Z) + (\mu-1)g(hY,Z)\big) \\ &- \eta(X)\eta\big(Z\big)\big((k-1+\mu/2)g(Y,U) + (\mu-1)g(hY,U)\big) \\ &+ \eta(Y)\eta(Z)\big((k-1+\mu/2)g(X,U) + (\mu-1)g(hX,U)\big) \\ &- \eta(Y)\eta(U)\big((k-1+\mu/2)g(X,Z) + (\mu-1)g(hX,Z)\big). \end{aligned} \tag{2.5}$$

The following formulas are also valid for a (k,μ)-contact manifold [BKP95]:

$$\begin{aligned} QY &= [2(n-1)-n\mu]Y + [2(n-1)+\mu]hY \\ &\quad + [2(1-n)+n(2k+\mu)]\eta(Y)\xi, \end{aligned} \tag{2.6}$$

$$\begin{aligned} (\nabla_X h)Y &= ((1-k)g(X,\varphi Y) - g(X,\varphi hY))\xi \\ &\quad - \eta(Y)((1-k)\varphi X + \varphi hX) - \mu\eta(X)\varphi hY, \end{aligned} \tag{2.7}$$

$$(\nabla_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX). \tag{2.8}$$

Moreover, the scalar curvature equals

$$r = 2n(2(n-1)+k-n\mu) \tag{2.9}$$

which is constant. In [Bo00], Boeckx introduced a number

$$I = \frac{1-\mu/2}{\sqrt{1-k}},$$

which is invariant under D-homothetic deformations, and proved that a non-Sasakian (k,μ)-contact manifold is locally isometric, up to a D-homothetic deformation, to the unit tangent sphere bundle of some space of constant curvature $\neq 1$ if and only if $I > -1$.

3. Proofs of the results. We note that the last term of the Bach tensor in (1.4) can be written as

$$\sum_{i,j=1}^{2n+1} g(Qe_i,e_j)g(W(X,e_i)e_j,Y) = -\sum_{i=1}^{2n+1} g(QW(X,e_i)Y,e_i)$$

and hence (1.4) assumes the form

$$B(X,Y) = \frac{1}{2n-1}\sum_{i=1}^{2n+1}[(\nabla_{e_i}C)(e_i,X,Y) - g(QW(X,e_i)Y,e_i)]. \tag{3.1}$$

Now we prove the following lemma.

LEMMA 3.1. *For a (k,μ)-contact manifold M of dimension $2n+1$,*

$$\begin{aligned} \sum_{i=1}^{2n+1}(\nabla_{e_i}C)(e_i,Y,Z) &= (\mu k - 2k - \mu - n\mu)\{3g(hY,Z) - 3g(Y,Z) \\ &\quad + 3(2n+1)\eta(Y)\eta(Z)\} \\ &\quad + (\mu^2 + (n-3)\mu - 2nk)\{(1-\mu)g(hY,Z)\} \\ &\quad + (k-1)(g(Y,Z) - (2n+1)\eta(Y)\eta(Z))\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned}(3.3)\quad &\sum_{i=1}^{2n+1} g(QW(X,e_i)Y,e_i)\\ &= \frac{(2n-2+\mu)}{2n-1}\{2(1-n)(2n+\mu+k-n\mu)g(hX,Y)\\ &\quad + 2(k-1)(n-1)(1-\mu)g(X,Y)\\ &\quad + 2(k-1)(2n+1)(n-1)(\mu-1)\eta(X)\eta(Y)\}\\ &\quad + \frac{[2(n-1)-n(2k+\mu)]2(n-1)(\mu-1)}{2n-1}g(hX,Y).\end{aligned}$$

Proof. First, we derive the expression of the Cotton tensor. Differentiating (2.6) and using (2.1) gives

$$\begin{aligned}(3.4)\quad (\nabla_X Q)Y &= [2(n-1)+\mu](\nabla_X h)Y\\ &- [2nk+n\mu-2n+2]\{\eta(Y)(\varphi X+\varphi hX)+g(Y,\varphi X+\varphi hX)\xi\}.\end{aligned}$$

Next, using (2.7) gives

$$\begin{aligned}(3.5)\quad (\nabla_X h)Y-(\nabla_Y h)X &= 2(1-k)g(X,\varphi Y)\xi\\ &+ (\mu-1)\{\eta(Y)\varphi hX-\eta(X)\varphi hY\}+(k-1)\{\eta(Y)\varphi X-\eta(X)\varphi Y\}.\end{aligned}$$

Since the scalar curvature of a (k,μ)-contact manifold is constant, the Cotton tensor reduces to

$$C(X,Y,Z)=g((\nabla_X Q)Y-(\nabla_Y Q)X,Z).$$

Therefore, making use of (3.4) along with (3.5) in the foregoing equation we obtain

$$\begin{aligned}(3.6)\quad &C(X,Y,Z)\\ &= (\mu k-2k-\mu-n\mu)\{2g(\varphi X,Y)\eta(Z)+\eta(Y)g(\varphi X,Z)-\eta(X)g(\varphi Y,Z)\}\\ &\quad + (\mu^2+(n-3)\mu-2nk)\{\eta(Y)g(\varphi hX,Z)-\eta(X)g(\varphi hY,Z)\}.\end{aligned}$$

Differentiating it and using (1.1) we have

$$\begin{aligned}(3.7)\quad &(\nabla_U C)(X,Y,Z)\\ &= (\mu k-2k-\mu-n\mu)\{2g((\nabla_U\varphi)X,Y)\eta(Z)\\ &\quad - 2g(\varphi X,Y)g(Z,\varphi U+\varphi hU)+\eta(Y)g((\nabla_U\varphi)X,Z)\\ &\quad - g(\varphi X,Z)g(Y,\varphi U+\varphi hU)-\eta(X)g((\nabla_U\varphi)Y,Z)\}\\ &\quad + g(\varphi Y,Z)g(X,\varphi U+\varphi hU)\\ &\quad + (\mu^2+(n-3)\mu-2nk)\{\eta(Y)g((\nabla_U\varphi h)X,Z)\\ &\quad - g(\varphi hX,Z)g(Y,\varphi U+\varphi hU)\\ &\quad - \eta(X)g((\nabla_U\varphi h)Y,Z)+g(\varphi hY,Z)g(X,\varphi U+\varphi hU)\}.\end{aligned}$$

Now, for any contact metric manifold we know (see [BS90]) that (a) $(\operatorname{div}\varphi)(X) = -2n\eta(X)$ and (b) $(\operatorname{div}\varphi h)X = 2n\eta(X) - \operatorname{Ric}(X,\xi) = -2n(k-1)\eta(X)$, where we have used (2.3). Thus, setting $X = U = e_i$ in (3.7), summing over $i = 1,\dots,2n+1$ and using (a) and (b) we obtain (3.2).

As the Weyl tensor is trace free, the last term of (3.1) can be written as

$$\begin{aligned}(3.8)\qquad \sum_{i=1}^{2n+1} g(QW(X,e_i)Y,e_i) &= [2(n-1)-n\mu]g(W(X,e_i)Y,e_i)\\ &\quad + [2(n-1)+\mu]g(hW(X,e_i)Y,e_i)\\ &\quad + [2(1-n)+n(2k+\mu)]g(W(X,e_i)Y,\xi)g(e_i,\xi)\\ &= [2(n-1)+\mu]g(W(X,e_i)Y,he_i)\\ &\quad + [2(n-1)-n(2k+\mu)]g(W(X,\xi)\xi,Y).\end{aligned}$$

Recall the Weyl conformal curvature tensor

$$\begin{aligned}(3.9)\qquad & g(W(X,Y)Z,U)\\ &= g(R(X,Y)Z,U) - \frac{1}{2n-1}\{g(QY,Z)g(X,U)\\ &\quad - g(QX,Z)g(Y,U) + g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)\}\\ &\quad + \frac{r}{2n(2n-1)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}.\end{aligned}$$

The term $g(R(X,e_i)Y,he_i)$ is calculated using (2.5) and $\operatorname{Tr} h = \operatorname{Tr}(\varphi h) = \operatorname{Tr}(\varphi h^2) = 0$ as follows:

$$\begin{aligned}(3.10)\qquad \sum_{i=1}^{2n+1} g(R(X,e_i)Y,he_i) &= (2-2n+n\mu-2\mu-k)g(hX,Y)\\ &\quad + 2(n-1)(k-1)g(X,Y) + 2(k-1)(n\mu-n+1)\eta(X)\eta(Y).\end{aligned}$$

Using (3.10) in (3.9) and the fact that h is trace-free provides

$$\begin{aligned}(3.11)\qquad & \sum_{i=1}^{2n+1} g(W(X,e_i)Y,he_i)\\ &= (2-2n+n\mu-2\mu-k)g(hX,Y)\\ &\quad + 2(n-1)(k-1)g(X,Y) + 2(k-1)(n\mu-n+1)\eta(X)\eta(Y)\\ &\quad - \frac{1}{2n-1}\{g(QhX,Y) + g(hQX,Y) - \operatorname{Tr}(Qh)g(X,Y)\}\\ &\quad + \frac{r}{2n(2n-1)}g(hX,Y).\end{aligned}$$

On the other hand, by (2.4) and (2.6), we see that

$$\begin{aligned} Qh = hQ &= [2(n-1) - n\mu]h + [2(n-1)+\mu]h^2 \\ &= [2(n-1) - n\mu]h + (k-1)[2(n-1)+\mu]\varphi^2, \end{aligned}$$

and hence $\mathrm{Tr}(Qh) = -2n(k-1)[2(n-1)+\mu]$. Using these facts and (2.9) we find that (3.11) reduces to

$$\begin{aligned} \sum_{i=1}^{2n+1} g(W(X,e_i)Y, he_i) &= \frac{2(1-n)(2n+\mu+k-n\mu)}{2n-1} g(hX,Y) \\ &+ \frac{2(k-1)(n-1)(1-\mu)}{2n-1} g(X,Y) \\ &+ \frac{2(k-1)(2n+1)(n-1)(\mu-1)}{2n-1} \eta(X)\eta(Y). \end{aligned} \tag{3.12}$$

Further, using (2.2), (2.3) and (2.9) we get

$$g(W(X,\xi)\xi, Y) = \frac{2(n-1)(\mu-1)}{2n-1} g(hX,Y). \tag{3.13}$$

Thus, inserting (3.12) and (3.13) in (3.8) we obtain (3.3), completing the proof of the Lemma 3.1. ∎

Proof of Theorem 1.1. By hypothesis, the Cotton tensor is divergence free. Substituting ξ for Y and Z in (3.2) we find

$$3(\mu k - 2k - \mu - n\mu) - (k-1)(\mu^2 + (n-3)\mu - 2nk) = 0. \tag{3.14}$$

Next, replacing Y by hY in (3.2), contracting the resulting equation with Y and Z, and also using (2.4) we get

$$3(\mu k - 2k - \mu - n\mu) - (\mu-1)(\mu^2 + (n-3)\mu - 2nk) = 0.$$

Subtracting the above two equations we obtain

$$(k-\mu)(\mu^2 + (n-3)\mu - 2nk) = 0. \tag{3.15}$$

First, suppose that $k \neq \mu$. Then from (3.14) and (3.15) we find that either $k = \mu = 0$, or $k = \mu = n+3$, or $k = n - \frac{1}{n}, \mu = -2(n-1)$.

Next, suppose that $k = \mu$. Then from (3.14) it follows that either $k = \mu = 0$, or $k = n+3 = \mu$, or $k = \mu = 4$. Since $k < 1$, from the above solution sets of k and μ we conclude that the only possibility is $k = \mu = 0$. Thus, applying Blair's theorem stated in Section 1, we complete the proof. ∎

Proof of Theorem 1.3. Since the Bach tensor vanishes, (3.2) and (3.3) provide

$$\begin{aligned}(\mu k - 2k - \mu - n\mu)&\{3g(hX,Y) - 3g(X,Y)\\ &+ 3(2n+1)\eta(X)\eta(Y)\} + (\mu^2 + (n-3)\mu - 2nk)\{(1-\mu)g(hX,Y)\\ &+ (k-1)(g(X,Y) - (2n+1)\eta(X)\eta(Y))\}\\ &- \frac{2n-2+\mu}{2n-1}\{2(1-n)(2n+\mu+k-n\mu)g(hX,Y)\\ &+ 2(k-1)(n-1)(1-\mu)g(X,Y)\\ &+ 2(k-1)(2n+1)(n-1)(\mu-1)\eta(X)\eta(Y)\}\\ &- \frac{[2(n-1) - n(2k+\mu)]2(n-1)(\mu-1)}{2n-1}g(hX,Y) = 0.\end{aligned} \tag{3.16}$$

Substituting ξ for X and Y in the above equation we find

$$\begin{aligned}3(\mu k - 2k - \mu - n\mu) - (k-1)(\mu^2 + (n-3)\mu - 2nk)&\\ + \frac{2(k-1)(n-1)(1-\mu)(2n-2+\mu)}{2n-1} &= 0.\end{aligned} \tag{3.17}$$

Next, substituting hX for X in (3.16), using (2.4) and then contracting the resulting equation with X and Y gives

$$\begin{aligned}3(\mu k - 2k - \mu - n\mu) + (1-\mu)(\mu^2 + (n-3)\mu - 2nk)&\\ + \frac{2(n-1)}{2n-1}[(2n-2+\mu)(2n+\mu+k-n\mu)&\\ - (\mu-1)(2n-2-2nk-n\mu)] &= 0.\end{aligned} \tag{3.18}$$

Though we could not solve the nonlinear algebraic equations (3.17) and (3.18) analytically, the computer solver (such as MATLAB and SAGE) provided the solutions for each n as shown in Table 1.

Table 1

n	k	μ
1	0	0
2	−0.8020	0.8727
3	−0.7752	1.1073
4	−0.7091	1.2776
5	−0.6479	1.4024
10	−0.4842	1.6949
100000	−0.3028	2
1000000	−0.3028	2

To get the precise limiting values of k and μ, as $n \to \infty$, we first compute the value of k from (3.18) as $k = N/D$ where $N = -4 + 12n^2 - 8n^3 + (-6 + 18n - 8n^2 + 4n^3)\mu + (6 - 13n + 4n^2)\mu^2 + (2n - 1)\mu^3$ and $D = 10 - 14n - 4n^2 + (-5 + 2n + 8n^2)\mu$, and then substitute it in (3.17). This yields a lengthy equation in n and μ, which can be rearranged as $64n^8(\mu - 2)^2 =$ terms of degree ≤ 7 in n along with powers of $\mu \leq 6$. From this we infer that $\mu \to 2$ as $n \to \infty$. We can verify it also by dividing (3.18) by n^2 and letting $n \to \infty$. Substituting $\mu = 2$ in (3.17), dividing by n and letting $n \to \infty$, we obtain $k^2 - 3k - 1 = 0$, and as $k < 1$, the limiting value of k is $(3 - \sqrt{13})/2$. The significance of the limiting value 2 for μ is that it is fixed under D-homothetic deformations, as pointed out in Section 2.

Computing the Boeckx invariant I, as defined at the end of Section 2, for each (k_n, μ_n) we find that $I > -1$, and applying his result mentioned at the end of Section 2 we conclude that they are locally isometric, up to a D-homothetic deformation, to the unit tangent sphere bundle of some space of constant curvature $\neq 1$. Finally, from (2.3) we have $\mathrm{Ric}(\xi, \xi) = 2nk$, and for $n > 1$, as $k < 0$, it turns out that $\mathrm{Ric}(\xi, \xi) < 0$, completing the proof. ■

Concluding remarks. (i) For Bach flat metrics in dimensions > 3, we notice from Theorem 1.3 that $k < 0$. Hence the positive eigenvalue $\lambda = \sqrt{1 - k}$ of h is > 1. Thus, according to a result of Blair [Bl10, p. 220], there exists a vector field $V \perp \xi$ such that $\nabla_V \xi = -\sqrt{-k}\, V$ everywhere. The vector field V is said to be a *special direction*, and the Reeb vector field ξ falls backward in the direction of V itself.

(ii) The standard contact metric on the unit tangent bundle $T_1M(c)$ over an n-dimensional space of constant curvature c is known to be a (k, μ)-contact metric such that $k = c(2 - c)$, $\mu = -2c$. Hence $\mu^2 + 4(k + \mu) = 0$, which is incompatible with (3.17) and (3.18), as checked on MATLAB. Thus the Bach flat (k, μ)-contact manifold cannot be locally isometric to $T_1M(c)$. However, as explained at the end of the proof of Theorem 1.3, it is locally isometric, up to a D-homothetic deformation, to $T_1M(c)$ for $c \neq 1$.

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