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On the Guedj–Rashkovskii conjecture

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In memory of Professor Józef Siciak

Abstract. We prove some cases when the Guedj–Rashkovskii conjecture holds.

1. Introduction. Let Δ^n be the unit polydisc in $\mathbb{C}^n$ and let $\mathcal{E}$ be the function class defined in Section 2. The complex Monge–Ampère operator, $(dd^c\cdot)^n$, is well-defined on functions from $\mathcal{E}$. Also, let $\nu_u(z)$ denote the Lelong number of the plurisubharmonic function u at z. In this note we are interested in the so-called *residual mass conjecture*. This conjecture is named after Vincent Guedj and Alexander Rashkovskii.

THE GUEDJ–RASHKOVSKII CONJECTURE. *Let $u \in \mathcal{E}(\Delta^n)$. If $\nu_u(0) = 0$, then $\int_{\{0\}}(dd^c u)^n = 0$.*

Wiklund [10, Proposition 5.4] proved that this conjecture is true when u is toric and $u^{-1}(-\infty) = \{0\}$. Traditionally, a toric function is known as poly-radial. Recently, Kim and Rashkovskii [7, Theorem 1.2] removed the assumption that $u^{-1}(-\infty) = \{0\}$. In Lemma 2.2 we give an alternative proof of their result, and by using Lemma 2.2 we make improvements in Theorems 2.5 and 2.6. We refer the reader to [7] for the history of this conjecture and its importance, and to [2] for an example that shows that this conjecture is non-trivial. See also [10].

2. The results. Following the notation introduced and studied by the second-named author in [3, 4, 5], for a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$

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we define

$$\mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F}(\Omega) = \Big\{\varphi \in \mathcal{PSH}(\Omega) : \exists\{u_j\} \subset \mathcal{E}_0(\Omega),\ \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E}(\Omega) = \{\varphi \in \mathcal{PSH}(\Omega) : \forall\omega \Subset \Omega\ \exists\varphi_\omega \in \mathcal{F}(\Omega) \text{ such that } \varphi_\omega = \varphi \text{ on } \omega\}.$$

We refer to [6, 9] for further information on the above function classes.

Let $u \in \mathcal{PSH}(\Omega)$, $u \le 0$. Set

$$u^j = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \le u \text{ on } \Delta(0, 1/j)^n, \text{ and } \varphi \le 0 \text{ on } \Omega\},$$

where $\Delta(0, \frac{1}{j}) \subset \mathbb{C}$ is the open disc centered at the origin with radius $1/j$. Then we define

$$\tilde{u} = \Big(\lim_{j\to\infty} u^j\Big)^*.$$

We shall need the following lemma in the proof of Lemma 2.2.

Lemma 2.1. *If $u \in \mathcal{E}(\Delta^n)$, then $\tilde{u} \in \mathcal{F}(\Delta^n)$ and*

$$(dd^c\tilde{u})^n = \Big(\int_{\{0\}} (dd^c u)^n\Big)\delta_{\{0\}}$$

where $\delta_{\{0\}}$ is Dirac measure at 0.

Proof. Since $u^j \nearrow \tilde{u}$ a.e. in Δ^n, we deduce that $(dd^c u^j)^n \to (dd^c\tilde{u})^n$ weakly as $j \to \infty$. On the other hand, $(dd^c u^j)^n = 0$ on $\Delta^n \setminus \overline{\Delta(0, 1/j)^n}$ and $(dd^c u^j)^n = (dd^c u)^n$ on $\Delta(0, 1/j)^n$. Therefore,

$$(dd^c\tilde{u})^n = c\delta_{\{0\}},$$

where $c \ge \int_{\{0\}} (dd^c u)^n$. By [1, Lemma 4.1] we obtain the asserted formula. ∎

Lemma 2.2 is due to Kim and Rashkovskii [7, Theorem 1.2]. Here we present an alternative proof. We shall make use of Lemmas 2.2 and 2.4 in Theorem 2.5.

Lemma 2.2. *Let $u \in \mathcal{E}(\Delta^n)$ be such that $u(z_1, \dots, z_n) = u(|z_1|, \dots, |z_n|)$. If $\nu_u(0) = 0$, then*

$$\int_{\{0\}} (dd^c u)^n = 0.$$

Proof. Without loss of generality we can assume that $u \in \mathcal{F}(\Delta^n)$. From $u(z_1, \dots, z_n) = u(|z_1|, \dots, |z_n|)$ and the maximum principle we get

$$u(z_1, \dots, z_n) \ge \frac{u(r, \dots, r)}{\log r} \log r \ge \frac{u(r, \dots, r)}{\log r} (\log|z_1| + \dots + \log|z_n|),$$

where $r = \min\{|z_j| : 1 \le j \le n\}$. Furthermore, since $u(r,\dots,r)/\log r$ decreases to 0 when r decreases to 0, we have

$$u(z_1,\dots,z_n) \ge \frac{u(r,\dots,r)}{\log r}(\log|z_1| + \dots + \log|z_n|)$$

for all $z \in \Delta(0,r)^n$. By the definition of $\tilde{u}$ we have

$$\tilde{u}(z) \ge \frac{u(r,\dots,r)}{\log r}(\log|z_1| + \dots + \log|z_n|)$$

for all $z \in \Delta^n$, and all $0 < r < 1$. By letting $r \to 0^+$ we arrive at $\tilde{u} \equiv 0$. Lemma 2.1 now yields the assertion. ∎

Let $u \in \mathcal{PSH}(\Delta^n)$, $u \le 0$, and set

$$\begin{aligned} u_s(z_1,\dots,z_n) &= \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)\,d\theta_1\cdots d\theta_n \\ &= \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u(e^{i\theta_1}|z_1|,\dots,e^{i\theta_n}|z_n|)\,d\theta_1\cdots d\theta_n. \end{aligned}$$

This construction yields

(1) $\nu_{u_s}(0) = \nu_u(0)$;
(2) if $u \in \mathcal{F}(\Delta^n)$, then $u_s \in \mathcal{F}(\Delta^n)$ and $\int_{\Delta^n}(dd^c u_s)^n \le \int_{\Delta^n}(dd^c u)^n$.

Indeed, (1) follows from [8, pp. 176–177], while (2) is proved below.

LEMMA 2.3. *Let $u \in \mathcal{F}(\Delta^n)$. Then $u_s \in \mathcal{F}(\Delta^n)$ and*

$$\int_{\Delta^n}(dd^c u_s)^n \le \int_{\Delta^n}(dd^c u)^n.$$

Proof. First, we consider the case when there is a $c > 0$ such that

$$u \ge c\max\{\ln|z_1|,\dots,\ln|z_n|,-1\}.$$

Obviously, since $u_s \ge c\max\{\ln|z_1|,\dots,\ln|z_n|,-1\}$ we have $u_s \in \mathcal{E}_0(\Delta^n)$. Set

$$h_t = \max\{t\ln|z_1|,\dots,t\ln|z_n|,-1\} \in \mathcal{E}_0(\Delta^n).$$

Then $h_t \searrow -1$ on Δ^n as $t \nearrow \infty$. By integration by parts [4, p. 166 and Corollary 5.6], and since $dd^c h_t \wedge (dd^c u_s)^{n-1}$ is $(\mathbb{S}^1)^n$-invariant, we get

$$\begin{aligned} \int_{\Delta^n}(dd^c u_s)^n &= \lim_{t\to\infty}\int_{\Delta^n} -h_t(dd^c u_s)^n = \lim_{t\to\infty}\int_{\Delta^n} -u_s\,dd^c h_t \wedge (dd^c u_s)^{n-1} \\ &= \lim_{t\to\infty}\int_{\Delta^n} -u\,dd^c h_t \wedge (dd^c u_s)^{n-1} = \lim_{t\to\infty}\int_{\Delta^n} -h_t\,dd^c u \wedge (dd^c u_s)^{n-1} \\ &= \int_{\Delta^n} dd^c u \wedge (dd^c u_s)^{n-1} \le \Big(\int_{\Delta^n}(dd^c u)^n\Big)^{1/n}\Big(\int_{\Delta^n}(dd^c u_s)^n\Big)^{(n-1)/n}, \end{aligned}$$

and the conclusion follows.

In the general case, we set

$$u_j = \max(u, j\max\{\ln|z_1|, \ldots, \ln|z_n|, -1\}).$$

Then $u_j \searrow u$, and $(u_j)_s \searrow u_s$, as $j \to \infty$. By the conclusion in the first case, we have

$$\sup_{j\geq 1} \int_{\Delta^n} (dd^c(u_j)_s)^n \leq \sup_{j\geq 1} \int_{\Delta^n} (dd^c u_j)^n = \int_{\Delta^n} (dd^c u)^n < \infty.$$

This implies that $u_s \in \mathcal{F}(\Delta^n)$. [4, Proposition 5.1] now yields

$$\begin{aligned}\int_{\Delta^n} (dd^c u_s)^n &= \lim_{j\to\infty} \int_{\Delta^n} (dd^c(u_j)_s)^n \\ &\leq \lim_{j\to\infty} \int_{\Delta^n} (dd^c u_j)^n = \int_{\Delta^n} (dd^c u)^n. \ \blacksquare\end{aligned}$$

LEMMA 2.4. *Let $u, v \in \mathcal{E}(\Delta^n)$ be such that $v(z_1, \ldots, z_n) = v(|z_1|, \ldots, |z_n|)$. Then*

$$\int_{\{0\}} dd^c u \wedge (dd^c v)^{n-1} = \int_{\{0\}} dd^c u_s \wedge (dd^c v)^{n-1}.$$

Proof. Without loss of generality we can assume that $u, v \in \mathcal{F}(\Delta^n)$. Set

$$h_t = \max\{t\ln|z_1|, \ldots, t\ln|z_n|, -1\} \in \mathcal{E}_0(\Delta^n).$$

Then

$$h_t \nearrow \begin{cases} 0 & \text{on } \Delta^n \setminus \{0\}, \\ -1 & \text{on } \{0\} \end{cases}$$

as $t \searrow 0$. By integration by parts [4, p. 166], and because $(dd^c h_t)\wedge(dd^c v)^{n-1}$ is $(\mathbb{S}^1)^n$-invariant, we get

$$\begin{aligned}\int_{\{0\}} dd^c u \wedge (dd^c v)^{n-1} &= \lim_{t\to 0^+} \int_{\Delta^n} -h_t\, dd^c u \wedge (dd^c v)^{n-1} \\ &= \lim_{t\to 0^+} \int_{\Delta^n} -u\, dd^c h_t \wedge (dd^c v)^{n-1} = \lim_{t\to 0^+} \int_{\Delta^n} -u_s\, dd^c h_t \wedge (dd^c v)^{n-1} \\ &= \lim_{t\to 0^+} \int_{\Delta^n} -h_t\, dd^c u_s \wedge (dd^c v)^{n-1} = \int_{\{0\}} dd^c u_s \wedge (dd^c v)^{n-1}. \ \blacksquare\end{aligned}$$

In Theorems 2.5 and 2.6 we prove two cases when the Guedj–Rashkovskii conjecture holds.

THEOREM 2.5. *Let $u, v \in \mathcal{E}(\Delta^n)$ satisfy $v(z_1, \ldots, z_n) = v(|z_1|, \ldots, |z_n|)$ and $u \geq v$. If $\nu_u(0) = 0$, then $\int_{\{0\}} (dd^c u)^n = 0$.*

Proof. From Lemma 2.2 and the fact that $\nu_{u_s}(0) = \nu_u(0) = 0$ we have

$$\int_{\{0\}} (dd^c u_s)^n = 0.$$

By [1, Lemmas 4.1 and 4.4] together with Lemma 2.4 we deduce that

$$\int_{\{0\}} (dd^c u)^n \leq \int_{\{0\}} dd^c u \wedge (dd^c v)^{n-1} = \int_{\{0\}} dd^c u_s \wedge (dd^c v)^{n-1}$$
$$\leq \Big(\int_{\{0\}} (dd^c u_s)^n \Big)^{1/n} \Big(\int_{\{0\}} (dd^c v)^n \Big)^{(n-1)/n} = 0. \blacksquare$$

THEOREM 2.6. *Let $u, v \in \mathcal{E}(\Delta^n)$ and $w \in \mathcal{PSH}(\Delta^n)$, $w \leq 0$, be such that $v(z_1, \ldots, z_n) = v(|z_1|, \ldots, |z_n|)$ and*

$$u \geq -|v|^t |w|^{1-t}$$

for some $t \in (0, 1]$. If $\nu_u(0) = 0$, then $\int_{\{0\}} (dd^c u)^n = 0$.

Proof. The case $t = 1$ follows from Theorem 2.5. Assume therefore that $t \in (0, 1)$. By Young's inequality we have, for all $\varepsilon > 0$,

$$u \geq -|v|^t |w|^{1-t} = -\left|\frac{v}{\epsilon}\right|^t |\epsilon^{t/(1-t)} w|^{1-t} \geq \frac{1}{\varepsilon} tv + \varepsilon^{t/(1-t)}(1-t)w.$$

Hence,

$$u \geq \frac{t}{\varepsilon} \max\{u, v\} + \varepsilon^{t/(1-t)}(1-t)w \quad \text{ for all } 0 < \varepsilon < t.$$

This implies that

$$\tilde{u} \geq \frac{t}{\varepsilon} \widetilde{\max\{u, v\}} + \varepsilon^{t/(1-t)}(1-t)\tilde{w}.$$

By Lemma 2.1 and Theorem 2.5 we now have $\widetilde{\max\{u, v\}} = 0$, and therefore

$$\tilde{u} \geq \varepsilon^{t/(1-t)}(1-t)\tilde{w} \quad \text{ for all } 0 < \varepsilon < t.$$

Let $\varepsilon \to 0^+$ to conclude that $\tilde{u} = 0$. $\blacksquare$

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