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Finite-dimensional global attractor for globally modified Navier–Stokes equations with fractional dissipation

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Abstract. The paper deals with the globally modified Navier–Stokes equations with fractional dissipation in a periodic box. The existence, uniqueness and large time behavior of weak solutions are investigated. The existence and some sharp regularity results for the global attractor are established, and an upper bound for its fractal dimension is provided.

1. Introduction. The Navier–Stokes equations are one of the most essential models in fluid dynamics. Since Leray's work in the 1930s [28], Navier–Stokes equations have been widely studied by many mathematicians; see for example the monographs [12, 36, 37] and a large number of references therein. However, the global existence and uniqueness of solutions for the 3D Navier–Stokes equation still remains unsolved, and it is considered to be one of the most challenging problems of our time [14].

In the past years, various kinds of modified Navier–Stokes equations were introduced in different contexts, such as the Navier–Stokes-α models introduced by Chen, Titi et al. [5, 20], the Leray-α, Clark-α and simplified Bardina models introduced by Cao, Titi et al. [6, 1, 2], and some other modified Navier–Stokes equations introduced and studied by Ladyzhenskaya, Lions, Constantin, Sohr, Flandoli et al. [27, 29, 11, 18, 35].

In [4], Caraballo, Kloeden and Real introduced the modified Navier–Stokes equations

$$(1.1)\qquad \begin{cases}\partial_t u-\nu\Delta u+F_N(\|\nabla u\|)(u\cdot\nabla)u+\nabla P=f,\\ \nabla\cdot u=0,\end{cases}$$

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where $F_N(r)$ (for some $N \in \mathbb{R}^+$) is defined by

$$F_N(r) = \min\{1, N/r\}, \quad r \in \mathbb{R}^+.$$

As the modifying factor $F_N(\|\nabla u\|)$ depends on the L^2-norm of ∇u, which in turn depends on ∇u all over the domain and not just at or near one point x, they call (1.1) the *globally modified Navier–Stokes equations*. They proved that problem (1.1) with Dirichlet boundary conditions admits a unique global strong solution and the solution semigroup has a global attractor in $H^1(\Omega)$. Then in [26] (see also [3, 24] and references therein), the existence of a pullback attractor in $H^1(\Omega)$ for the nonautonomous globally modified Navier–Stokes equations was proved. More recently, Dong and Song [16] studied globally modified Navier–Stokes equations with fractional dissipation in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + \nu \Lambda^{2\alpha} u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = 0, \\ \nabla \cdot u = 0, \end{cases}$$

where the fractional operator $\Lambda^{2\alpha} = (-\Delta)^\alpha$ is defined by the Fourier transform (see Section 2). The authors proved the existence of weak solutions and the existence of a unique global smooth solution for $\alpha > 3/4$. Some time decay results and asymptotic stability results for weak solutions were also derived.

In this paper we consider globally modified Navier–Stokes equations with fractional dissipation in a periodic box $\Omega = [0, 2\pi]^3$:

$$\begin{cases} \partial_t u + \nu \Lambda^{2\alpha} u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = f, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{1.2}$$

with periodic boundary conditions, where $\nu > 0$ and $0 < \alpha \leq 1$. Without the modifying factor $F(\|\nabla u\|)$, problem (1.2) reduces to the Navier–Stokes equations with fractional dissipation, which has been extensively studied by many authors [17, 15, 21, 39].

Our consideration of model (1.2) is highly motivated by those works and [4, 16]. Irrefutably the above model may violate the basic laws of mechanics even for $\alpha = 1$ [4, 3, 26, 24]. Also, it may not bring new insights into solving the well-known Millennium Problem concerning the classical Navier–Stokes equations. However, from the viewpoint of mathematics, it may be of some interest, as it provides a well defined system of equations, just like the modified versions of the Navier–Stokes equations of Leray and others with other mollifications of the nonlinear term [11]. Recall that the fractional dissipated Navier–Stokes equations in $\mathbb{R}^3$ admit a unique global smooth solution when $\alpha \geq 5/4$ [39] (see also [7]). However, we shall see that the above modified model has a unique global smooth solution even for

$3/4 < \alpha \le 1$. For $0 \le \alpha \le 3/4$, the operator $\Lambda^{2\alpha}u$ is not strong enough to control the nonlinear term $F_N(\|\nabla u\|)(u\cdot\nabla)u$ and provide higher regularity for u when we perform a priori estimates, which seems necessary to prove the convergence of the modifying term $F_N(\|\nabla u_n\|)(u_n\cdot\nabla)u_n$ in the approximation process (see e.g. (3.16), where θ_2 needs to be positive). Therefore, we cannot prove the existence of strong solutions or improved regularity results (see Theorem 1.1 for the details).

Assume that the initial data and the forcing term are mean-free functions, i.e.,

$$\int_\Omega u_0\,dx = \int_\Omega f\,dx = 0,$$

so that the solution is also a mean-free function. Then the Poincaré inequality holds:

$$\lambda_1^\alpha\|u\|_{L^2} \le \|\Lambda^\alpha u\|_{L^2},$$

where λ_1 is the first eigenvalue of the operator Λ (see e.g. [13, 22, 19]). For $3/4 < \alpha \le 1$ and $u_0, f \in \mathbb{H}$, we prove that system (1.2) has a weak solution. Furthermore, if $u_0 \in \mathbb{H}^s$, $f \in \mathbb{H}^{s-\alpha}$ and $s \ge 1$, we show that the system has a unique global solution (see (2.1) for the definition of the spaces $\mathbb{H}^\gamma$, $\gamma \in \mathbb{R}$). For the large time behavior, with proper energy estimates, we prove that the solution semigroup generated by (1.2) admits a global attractor $\mathcal{A}$ in $\mathbb{H}^s$. In particular, using some decomposition technique and some delicate analysis, we prove that the attractor is actually bounded in $\mathbb{H}^{s-1+2\alpha}$ and has a finite fractal dimension in $\mathbb{H}^{s-1+2\alpha}$.

Our results may be viewed as a periodic counterpart of certain results in [4], where globally modified Navier–Stokes equations were first introduced. The authors proved the existence and uniqueness of global strong solutions for problem (1.1) with homogeneous Dirichlet boundary conditions and then they showed that the semigroup admits a global attractor, using Condition (C) [30] or the flattening property [25]. The existence results herein reveal that the modifying factor decreases the singularity of the quadratic term $u\cdot\nabla u$ so much that one can even control it with fractional dissipations.

This paper is organized as follows. In Section 2 we provide some preliminaries about function spaces and several useful lemmas. Then in Section 3 we prove the existence and uniqueness of the solution, while in Sections 4 and 5 we discuss the existence of a global attractor and an upper bound of its fractal dimension.

2. Notations and preliminaries. Let $\Omega = [0, 2\pi]^3$. The fractional operator $\Lambda^{2\alpha} = (-\Delta)^\alpha$ for any $\alpha \in \mathbb{R}$ can be defined as

$$\Lambda^{2\alpha} f(x) = \sum_{\xi \in \mathbb{Z}^3} |\xi|^{2\alpha} \widehat{f}(\xi) e^{i\xi \cdot x}$$

for any tempered distribution f, where $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$. In particular, $\Lambda = (-\Delta)^{1/2}$. Let $\dot{C}_p^\infty(\Omega)$ be the space of restrictions to Ω of infinitely differentiable functions that are 2π-periodic in each direction and with zero mean in Ω. For $s \in \mathbb{R}$, we denote by $H^s(\Omega)$ the closure of $\dot{C}_p^\infty(\Omega)$ under the norm

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2} = \Big(\sum_{\xi \in \mathbb{Z}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2 \Big)^{1/2},$$

that is the space of periodic functions with zero mean such that $\|f\|_{H^s} < \infty$. It is obvious that $H^{s_1}(\Omega) \hookrightarrow\hookrightarrow H^{s_2}(\Omega)$(compact imbedding) for any $s_1 > s_2$. Moreover, for $p \in [1, \infty]$, we denote by $H^{s,p}(\Omega)$ the space of periodic mean-free $L^p(\Omega)$ functions φ, which can be written as $\varphi = \Lambda^{-s}\psi$, with $\psi \in L^p$. This is normed by $\|\varphi\|_{H^{s,p}} = \|\Lambda^s \varphi\|_{L^p}$. For $s \in \mathbb{R}$, we denote

$$\begin{aligned} \mathbb{H}^s &= \{u \in [H^s(\Omega)]^3 : \operatorname{div} u = 0\}, \\ \mathbb{H}^{s,p} &= \{u \in [H^{s,p}(\Omega)]^3 : \operatorname{div} u = 0\}. \end{aligned} \tag{2.1}$$

In particular, when $s = 0$ we denote $\mathbb{H}^0$ by $\mathbb{H}$.

Throughout the paper, for any Banach space X, we denote its norm as $\|\cdot\|_X$, in particular, $\|\cdot\|_{L^2}$ will be abbreviated as $\|\cdot\|$. We use C to denote a positive constant which may be different in different places. To stress the dependence of C on some quantities or parameters such as ν, α, we denote it by $C(\nu, \alpha)$.

Now let us recall the definitions of global attractor and fractal dimension [31, 33, 37].

Definition 2.1. Let $\{S(t)\}_{t \geq 0}$ be a semigroup on a Banach space X. A subset $\mathscr{A} \subset X$ is called a *global attractor* for the semigroup if $\mathscr{A}$ enjoys the following properties:

(i) $\mathscr{A}$ is compact in X,
(ii) $\mathscr{A}$ is invariant, i.e., $S(t)\mathscr{A} = \mathscr{A}$ for any $t \geq 0$,
(iii) $\mathscr{A}$ attracts every bounded subset of X, i.e., for every $B \subset X$ bounded,

$$\lim_{t \to \infty} \operatorname{dist}(S(t)B, \mathscr{A}) = 0,$$

where dist is the Hausdorff semidistance between sets in X, defined as

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X, \qquad \forall A, B \subset X.$$

Definition 2.2. The *fractal dimension* of a compact set K in a Banach space X is defined as

$$\dim_{\mathrm{f}} K = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon}$$

where $N_\epsilon(K)$ is the minimal number of balls of radius ϵ in X needed to cover K.

The following product estimates play an essential role in our analysis (see e.g. [23, 13, 22, 19]).

LEMMA 2.1. *Suppose that $f, g \in \mathcal{S}$, the Schwartz class. Then for $s > 0$ and $1 < p < \infty$, there exists a positive constant C (depending only on s, p, p_i, q_i) such that*

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{2.2}$$

with $q_1, q_2 \in (1, \infty)$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

REMARK 2.1. We remark that although (2.2) is stated for functions in the Schwartz class, it is also valid for those functions belonging to certain Sobolev spaces for which the right-hand side of (2.2) is finite [13, 22, 19].

The following lemma is derived in [34], where the uniqueness of weak solutions to the globally modified Navier–Stokes equations (1.1) was obtained.

LEMMA 2.2. *For every $u, v \in \mathbb{H}^1$ and each $N > 0$, we have*

$$0 \le F_N(\|\nabla u\|)\|\nabla u\| \le N,$$
$$|F_N(\|\nabla u\|) - F_N(\|\nabla v\|)| \le \frac{F_N(\|\nabla u\|)F_N(\|\nabla v\|)}{N}\|\nabla u - \nabla v\|.$$

3. Existence and uniqueness of solutions. We now give the definition of weak solutions to system (1.2).

DEFINITION 3.1. Let $u_0, f \in \mathbb{H}$ and $\alpha \in (0,1]$. A function u is called a *weak solution* to system (1.2) if

$$u \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{H}^\alpha), \qquad \partial_t u \in L^2(0,T;\mathbb{H}^{-\alpha}), \qquad \forall T > 0,$$

and for any $\varphi \in \mathbb{H}^\alpha$,

$$\int_\Omega u(T)\varphi\,dx - \int_0^T\int_\Omega F_N(\|\nabla u\|)u \otimes u : \nabla\varphi\,dx\,ds + \nu\int_0^T\int_\Omega \Lambda^\alpha u \Lambda^\alpha\varphi\,dxds = \int_0^T\int_\Omega f\varphi\,dx\,ds + \int_\Omega u_0\varphi\,dx.$$

REMARK 3.1. Thanks to [29], if $u(t)$ is a weak solution of system (1.2), then $u \in C([0,T];\mathbb{H})$ for all $T > 0$.

Theorem 3.1. *Assume that* $u_0, f \in \mathbb{H}$ *and* $3/4 < \alpha \le 1$. *Then there exists a weak solution* $u(t)$ *to problem* (1.2) *with*

$$u \in L^\infty(\epsilon, T; \mathbb{H}^\alpha) \cap L^2(\epsilon, T; \mathbb{H}^{2\alpha}) \quad \textit{for all } T > \epsilon > 0. \tag{3.1}$$

If $u_0 \in \mathbb{H}^\alpha$, *then* $u(t)$ *is a strong solution in the sense that*

$$u \in L^\infty(0, T; \mathbb{H}^\alpha) \cap L^2(0, T; \mathbb{H}^{2\alpha}) \quad \textit{for all } T > 0. \tag{3.2}$$

Furthermore, if $u_0 \in \mathbb{H}^s$, $f \in \mathbb{H}^{s-\alpha}$ *and* $s \ge 1$, *then problem* (1.2) *has a unique solution* $u \in L^\infty(0, T; \mathbb{H}^s) \cap L^2(0, T; \mathbb{H}^{s+\alpha})$ *with*

$$\begin{aligned} \|\Lambda^s u(t)\|^2 + \nu \int_\tau^t \|\Lambda^{s+\alpha} u\|^2 \, d\tau &\le \|\Lambda^s u(\tau)\|^2 \\ &+ 2C(N, \alpha, \nu)(t-\tau)\left(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\right) + \frac{2(t-\tau)}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} \end{aligned} \tag{3.3}$$

for any $t \ge \tau \ge 0$. *Moreover, the map* $S(t) : \mathbb{H}^s \to \mathbb{H}^s$ *defined as* $S(t)u_0 = u(t)$ *is continuous in* $\mathbb{H}^s$ *and* $u \in C([0, T]; \mathbb{H}^s)$ *for any* $T > 0$.

Remark 3.2. When $\alpha = 1$ Theorem 3.1 recovers in some sense the results of [4, Theorem 7]. When $\alpha \in (3/4, 1)$, the smoothing effect of the operator is reduced compared to (1.1), and so is the regularity of the strong solution. Hence we are not able to prove the uniqueness of weak solutions and even strong solutions. When the initial data and forcing terms are more regular, they compensate for the reduced smoothing effect of the operator $\Lambda^{2\alpha}$, and the solution becomes more regular. We can then obtain the uniqueness of the solution.

Proof of Theorem 3.1. We implement the Galerkin approximation method to prove the existence of weak solutions. In the following, we will always assume that $3/4 < \alpha \le 1$. Let $\{\phi_j\}_{j=1}^\infty$ be an orthonormal basis of $\mathbb{H}$ consisting of eigenfunctions of the operator Λ. Consider the ordinary differential system

$$\begin{aligned} &\frac{du_m}{dt} + \nu\Lambda^{2\alpha} u_m + P_m F_N(\|\nabla u_m\|)(u_m \cdot \nabla) u_m = P_m f, \\ &u_m(0) = P_m u_0, \end{aligned} \tag{3.4}$$

where $u_m = \sum_{j=1}^m c_{jm}(t)\phi_j$, $\Lambda^{2\alpha} u_m = \sum_{j=1}^m \lambda_j^{2\alpha} c_{jm}(t)\phi_j$ and P_m is the orthogonal projection from $\mathbb{H}$ onto the space spanned by $\{\phi_1, \ldots, \phi_m\}$. By the standard existence theorem for ordinary differential equations, for each m there exists a local solution u_m to system (3.4) in the interval $[0, T_m)$.

Multiplying (3.4) by $u_m(t)$, we can deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu\|\Lambda^\alpha u_m\|^2 \le \|f\|\,\|u_m\|.$$

Using the Poincaré inequality

$$\lambda_1^{2\alpha}\|u_m\|^2 \le \|\Lambda^\alpha u_m\|^2,$$

we obtain

$$\frac{d}{dt}\|u_m\|^2 + \nu\|\Lambda^\alpha u_m\|^2 \le \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}. \tag{3.5}$$

Hence we have

$$\|u_m(t)\|^2 + \nu\int_0^t \|\Lambda^\alpha u_m(s)\|^2\,ds \le \frac{t}{\nu\lambda_1^{2\alpha}}\|f\|^2 + \|u_m(0)\|^2, \qquad \forall t \ge 0, \tag{3.6}$$

and also

$$\frac{d}{dt}\|u_m\|^2 + \nu\lambda_1^{2\alpha}\|u_m\|^2 \le \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}},$$

which in turn implies that

$$\begin{aligned}\|u_m(t)\|^2 &\le \|u_m(0)\|^2 e^{-\nu\lambda_1^{2\alpha}t} + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}(1-e^{-\nu\lambda_1^{2\alpha}t}) \\ &\le \|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}.\end{aligned} \tag{3.7}$$

Now we multiply (3.4) by $\Lambda^{2\alpha}u_m$ and integrate. Using Hölder's inequality, Young's inequality and the Gagliardo–Nirenberg inequality, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u_m\|^2 + \nu\|\Lambda^{2\alpha}u_m\|^2 \\ &\quad\le \|f\|\,\|\Lambda^{2\alpha}u_m\| + F_N(\|\nabla u_m\|)\Big|\int_\Omega (u_m\cdot\nabla u_m)\Lambda^{2\alpha}u_m\,dx\Big| \\ &\quad\le \|f\|\,\|\Lambda^{2\alpha}u_m\| + F_N(\|\nabla u_m\|)\|u_m\|_{L^6}\|\nabla u_m\|_{L^3}\|\Lambda^{2\alpha}u_m\| \\ &\quad\le \frac{1}{\nu}\|f\|^2 + \frac{\nu}{4}\|\Lambda^{2\alpha}u_m\|^2 + CN\|u_m\|^\theta\|\Lambda^{2\alpha}u_m\|^{2-\theta} \\ &\quad\le \frac{1}{\nu}\|f\|^2 + C(N,\nu,\alpha)\|u_m\|^2 + \frac{\nu}{2}\|\Lambda^{2\alpha}u_m\|^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}\|\Lambda^\alpha u_m\|^2 + \nu\|\Lambda^{2\alpha}u_m\|^2 \le \frac{2}{\nu}\|f\|^2 + C(N,\nu,\alpha)\|u_m\|^2. \tag{3.8}$$

Integrating (3.5) between t and $t+\tau$ and using (3.7), we get

$$\nu\int_t^{t+\tau}\|\Lambda^\alpha u_m\|^2\,ds \le \|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\Big(\tau + \frac{1}{\nu\lambda_1^{2\alpha}}\Big).$$

For given $t,\tau > 0$, define $a > 0$ by

$$a^2 = \frac{2}{\nu\tau}\Big\{\|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\Big(\tau + \frac{1}{\nu\lambda_1^{2\alpha}}\Big)\Big\}.$$

Set

$$D_m = \{s \in [t, t+\tau] : \|\Lambda^\alpha u_m\| \geq a\},$$

and denote by $|D_m|$ its Lebesgue measure. We have

$$\begin{aligned} a^2|D_m| &\leq \int_{D_m} \|\Lambda^\alpha u_m\|^2\, ds \leq \int_t^{t+\tau} \|\Lambda^\alpha u_m\|^2\, ds \\ &\leq \frac{1}{\nu}\bigg\{\|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\bigg(\tau + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg)\bigg\} = \frac{\tau a^2}{2}. \end{aligned}$$

Hence $|D_m| \leq \tau/2$. Therefore for any given $\epsilon > 0$ and $t \geq \epsilon$ there exists a $t_0 \in (t-\epsilon, t)$ such that

$$\|\Lambda^\alpha u_m(t_0)\|^2 \leq \frac{2}{\nu\epsilon}\bigg\{\|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\bigg(\epsilon + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg)\bigg\}. \tag{3.9}$$

Using (3.7), the Poincaré inequality and Gronwall's inequality on $[t_0, t]$, we deduce from (3.8) that for all $0 \leq t_0 \leq t$,

$$\|\Lambda^\alpha u_m(t)\|^2 \leq \|\Lambda^\alpha u_m(t_0)\|^2 + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg\{C(N,\nu,\alpha)\bigg(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg) + \frac{2\|f\|^2}{\nu}\bigg\}.$$

Combining this with (3.9), we obtain

$$\begin{aligned} \|\Lambda^\alpha u_m(t)\|^2 &\leq \frac{2}{\nu\epsilon}\bigg\{\|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\bigg(\epsilon + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg)\bigg\} \\ &\quad + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg\{C(N,\nu,\alpha)\bigg(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg) + \frac{2\|f\|^2}{\nu}\bigg\} \end{aligned} \tag{3.10}$$

for all $t \geq \epsilon$. Integrating (3.8) from ϵ to T and taking (3.7), (3.10) into consideration, we deduce that

$$\begin{aligned} &\|\Lambda^\alpha u_m(t)\|^2 + \int_\epsilon^T \|\Lambda^{2\alpha} u_m(t)\|^2\, ds \\ &\leq \int_\epsilon^T \bigg(C(N,\nu,\alpha)\|u_m\|^2 + \frac{2\|f\|^2}{\nu}\bigg)\, ds + \|\Lambda^\alpha u_m(\epsilon)\|^2 \\ &\leq \bigg\{C(N,\nu,\alpha)\bigg(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg) + \frac{2\|f\|^2}{\nu}\bigg\}(T-\epsilon) \\ &\quad + \frac{2}{\nu\epsilon}\bigg\{\|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}\bigg(\epsilon + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg)\bigg\} \\ &\quad + \frac{1}{\nu\lambda_1^{2\alpha}}\bigg\{C(N,\nu,\alpha)\bigg(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg) + \frac{2\|f\|^2}{\nu}\bigg\} \end{aligned} \tag{3.11}$$

for all $T \geq t \geq \epsilon$. Combining (3.6), (3.7), (3.10) and (3.11), we conclude that $\{u_m\}$ is bounded in

$$L^2(0,T;\mathbb{H}^\alpha)\cap L^\infty(0,T;\mathbb{H})\cap L^2(\epsilon,T;\mathbb{H}^{2\alpha})\cap L^\infty(\epsilon,T;\mathbb{H}^\alpha) \tag{3.12}$$

for all $T>\epsilon>0$. Now let us perform the estimates for $\{\frac{\partial u_m}{\partial t}\}$. For any $\varphi\in\mathbb{H}^\alpha$, using Hölder's inequality, the product estimates (see Lemma 2.1), the Gagliardo–Nirenberg inequality and Young's inequality, we deduce that

$$\begin{aligned}\Big|F_N(\|\nabla u_m\|)\int_\Omega (u_m\cdot\nabla u_m)\varphi\,dx\Big| &\le F_N(\|\nabla u_m\|)\|\Lambda^{-\alpha}(u_m\cdot\nabla u_m)\|\,\|\Lambda^\alpha\varphi\|\\ &\le F_N(\|\nabla u_m\|)\|\Lambda^{1-\alpha}(u_mu_m)\|\,\|\Lambda^\alpha\varphi\|\\ &\le 2CF_N(\|\nabla u_m\|)\|u_m\|_{L^6}\|\Lambda^{1-\alpha}u_m\|_{L^3}\|\Lambda^\alpha\varphi\|\\ &\le 2CN(\|\Lambda^\alpha u_m\|+\|u_m\|)\|\Lambda^\alpha\varphi\|.\end{aligned}$$

Thanks to (3.12), $\{F_N(\|\nabla u_m\|)(u_m\cdot\nabla)u_m\}$ is bounded in $L^2(0,T;\mathbb{H}^{-\alpha})$. Obviously, $\{-\Lambda^{2\alpha}u_m\}$ and $\{P_mf\}$ are bounded in $L^2(0,T;\mathbb{H}^{-\alpha})$. Hence, we conclude from (3.4) that

$$\Big\{\frac{\partial u_m}{\partial t}\Big\}\ \text{is bounded in } L^2(0,T;\mathbb{H}^{-\alpha}).$$

By the standard Aubin–Simon type compactness results [29, 37], there exists an element

$$u\in L^2(0,T;\mathbb{H}^\alpha)\cap L^\infty(0,T;\mathbb{H})\cap L^2(\epsilon,T;\mathbb{H}^{2\alpha})\cap L^\infty(\epsilon,T;\mathbb{H}^\alpha),$$

for all $\epsilon>0$, $T>\epsilon$, such that up to subsequences,

$$\begin{aligned} u_m&\to u &&\text{strongly in } L^2(0,T;\mathbb{H}),\\ u_m&\to u &&\text{a.e. in } (0,T)\times\Omega,\\ u_m&\to u &&\text{weakly in } L^2(\epsilon,T;\mathbb{H}^{2\alpha}),\\ u_m&\to u &&\text{weakly}^* \text{ in } L^\infty(\epsilon,T;\mathbb{H}^\alpha),\\ \frac{\partial u_m}{\partial t}&\to \partial_t u &&\text{weakly}^* \text{ in } L^2(\epsilon,T;\mathbb{H}^{-\alpha}).\end{aligned} \tag{3.13}$$

Thanks to the Gagliardo–Nirenberg inequality and Hölder's inequality, we deduce that

$$\begin{aligned}\int_\epsilon^T\|\nabla(u_m-u)\|^2\,ds &\le \int_\epsilon^T \|u_m-u\|^{2\theta_1}\|\Lambda^{2\alpha}(u_m-u)\|^{2-2\theta_1}\,ds\\ &\le \Big(\int_\epsilon^T\|u_m-u\|^2\,ds\Big)^{\theta_1}\Big(\int_\epsilon^T\|\Lambda^{2\alpha}(u_m-u)\|^2\,ds\Big)^{1-\theta_1},\end{aligned} \tag{3.14}$$

where $\theta_1=\frac{2\alpha-1}{2\alpha}$. As a consequence of (3.12) and (3.13), we know that for any $\epsilon>0$,

$$u_m\to u\quad \text{strongly in } L^2(\epsilon,T;\mathbb{H}^1).$$

Thus up to subsequences,

$$\|\nabla u_m\| \to \|\nabla u\| \quad \text{a.e. in } (\epsilon, T) \text{ for any } T > \epsilon > 0.$$

By the standard diagonal process, we can extract a subsequence of $\{u_m\}$ (still labeled by $\{u_m\}$) such that

$$\|\nabla u_m\| \to \|\nabla u\| \quad \text{a.e. in } (0, T) \text{ for any } T > 0.$$

Hence

$$F_N(\|\nabla u_m\|) \to F_N(\|\nabla u\|) \quad \text{a.e. in } (0, T) \text{ for any } T > 0. \tag{3.15}$$

Thanks to (3.13) and (3.15), we can take a test function $\varphi \in \mathbb{H}^\alpha$ in (3.4) and pass to the limit to deduce that u is a weak solution to system (1.2). As the calculations are rather similar to those in [4], we omit the details. Furthermore, if $u_0 \in \mathbb{H}^\alpha$, we get (3.2) immediately by integrating (3.8) on $[0, t]$ and passing to the limit.

If $u_0 \in \mathbb{H}^s$ and $f \in \mathbb{H}^{s-\alpha}$, we multiply (3.4) by $\Lambda^{2s} u_m$ and integrate. Note that $3/4 < \alpha \le 1$. Using Hölder's inequality, the product estimates, the Gagliardo–Nirenberg inequality and Young's inequality, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s u_m\|^2 + \nu\|\Lambda^{s+\alpha} u_m\|^2 \\ &\le \|f\|_{\mathbb{H}^{s-\alpha}}\|\Lambda^{s+\alpha} u_m\| + F_N(\|\nabla u_m\|)\Big|\int_\Omega (u_m \cdot \nabla u_m)\Lambda^{2s} u_m\,dx\Big| \\ &\le \frac{1}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} + \frac{\nu}{4}\|\Lambda^{s+\alpha} u_m\|^2 + F_N(\|\nabla u_m\|)\|\Lambda^{s+1-\alpha}(u_m u_m)\|\,\|\Lambda^{s+\alpha} u_m\| \\ &\le \frac{1}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} + \frac{\nu}{4}\|\Lambda^{s+\alpha} u_m\|^2 + CN\|u_m\|^{\theta_2}\|\Lambda^{s+\alpha} u_m\|^{2-\theta_2} \\ &\le C_0(N,\nu,\alpha)\|u_m\|^2 + \frac{1}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} + \frac{\nu}{2}\|\Lambda^{s+\alpha} u_m\|^2, \end{aligned} \tag{3.16}$$

where $\theta_2 = \frac{2\alpha - 3/2}{s+\alpha}$ and

$$C_0(N,\nu,\alpha) = \frac{2\alpha - 3/2}{2s + 2\alpha}(CN)^{(2s+2\alpha)/(2\alpha-3/2)}\left(\frac{\nu(s+\alpha)}{4s+3}\right)^{-(2s+3/2)/(2\alpha-3/2)}.$$

Thus

$$\frac{d}{dt}\|\Lambda^s u_m\|^2 + \nu\|\Lambda^{s+\alpha} u_m\|^2 \le 2C_0(N,\nu,\alpha)\|u_m\|^2 + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}, \tag{3.17}$$

which implies that for all $t \ge \tau \ge 0$,

$$\begin{aligned} &\|\Lambda^s u_m(t)\|^2 + \nu\int_\tau^t \|\Lambda^{s+\alpha} u_m\|^2\,d\varsigma \\ &\le \|\Lambda^s u(\tau)\|^2 + 2C_0(N,\nu,\alpha)(t-\tau)\left(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\right) + \frac{2(t-\tau)}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}. \end{aligned} \tag{3.18}$$

Hence $\{u_m\}$ is bounded in $L^\infty(0,T;\mathbb{H}^s)\cap L^2(0,T;\mathbb{H}^{s+\alpha})$ for all $T>0$. Passing to the limit, we obtain (3.3) with $C(N,\nu,\alpha)=C_0(N,\nu,\alpha)$ immediately.

Next, let us prove the uniqueness result. Let u, v be two solutions to system (1.2) corresponding to the initial conditions u_0, v_0 respectively. Let $\mathcal{P}$ be the Helmholtz–Leray projection and set $w=u-v$. It is obvious that w satisfies

$$w_t+\nu\Lambda^{2\alpha}w+F_N(\|\nabla u\|)\mathcal{P}(u\cdot\nabla)w+F_N(\|\nabla u\|)\mathcal{P}(w\cdot\nabla)v + \{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}\mathcal{P}(v\cdot\nabla)v=0. \tag{3.19}$$

Taking the inner product of (3.19) with $\Lambda^{2s}w$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s w\|^2+\nu\|\Lambda^{s+\alpha}w\|^2 &\le F_N(\|\nabla u\|)\Big|\int_\Omega(u\cdot\nabla w)\Lambda^{2s}w\,dx\Big| \\ &\quad+F_N(\|\nabla u\|)\Big|\int_\Omega(w\cdot\nabla v)\Lambda^{2s}w\,dx\Big| \\ &\quad+\Big|\{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}\int_\Omega(v\cdot\nabla v)\Lambda^{2s}w\,dx\Big| \\ &=: I_1+I_2+I_3.\end{aligned} \tag{3.20}$$

Note that $3/4<\alpha\le 1$. For I_1, using Hölder's inequality, the product estimates, the Gagliardo–Nirenberg inequality and Young's inequality, we deduce that

$$\begin{aligned}I_1&\le F_N(\|\nabla u\|)\|\Lambda^{s+1-\alpha}(uw)\|\,\|\Lambda^{s+\alpha}w\| \\ &\le CF_N(\|\nabla u\|)(\|u\|_{L^6}\|\Lambda^{s+1-\alpha}w\|_{L^3}+\|w\|_{L^6}\|\Lambda^{s+1-\alpha}u\|_{L^3})\|\Lambda^{s+\alpha}w\| \\ &\le CN\|\Lambda^s w\|^{\theta_3}\|\Lambda^{s+\alpha}w\|^{2-\theta_3}+C\|\Lambda^s w\|\|\Lambda^{s+\alpha}u\|\|\Lambda^{s+\alpha}w\| \\ &\le\frac{\nu}{6}\|\Lambda^{s+\alpha}w\|^2+C(N,\alpha,\nu)(1+\|\Lambda^{s+\alpha}u\|^2)\|\Lambda^s w\|^2,\end{aligned} \tag{3.21}$$

where $\theta_3=\frac{4\alpha-3}{2\alpha}$. Similarly,

$$I_2\le\frac{\nu}{6}\|\Lambda^{s+\alpha}w\|^2+C(N,\alpha,\nu)(1+\|\Lambda^{s+\alpha}v\|^2)\|\Lambda^s w\|^2. \tag{3.22}$$

Finally, for I_3, we use Hölder's inequality, Lemma 2.2, the product estimates and the imbedding of fractional Sobolev spaces to deduce that

$$\begin{aligned}I_3&\le|F_N(\|\nabla u\|)-F_N(\|\nabla v\|)|\,\|\Lambda^{s+1-\alpha}(vv)\|\,\|\Lambda^{s+\alpha}w\| \\ &\le C\frac{F_N(\|\nabla u\|)F_N(\|\nabla v\|)}{N}\|\nabla w\|\,\|v\|_{L^6}\|\Lambda^{s+3/2-\alpha}v\|\,\|\Lambda^{s+\alpha}w\|\end{aligned} \tag{3.23}$$

$$\le C\|\Lambda^s w\|\,\|\Lambda^{s+\alpha}v\|\,\|\Lambda^{s+\alpha}w\|$$
$$\le \frac{\nu}{6}\|\Lambda^{s+\alpha}w\|^2 + \frac{3C^2}{\nu}\|\Lambda^{s+\alpha}v\|^2\|\Lambda^s w\|^2.$$

Combining (3.20)–(3.23), we obtain

$$\frac{d}{dt}\|\Lambda^s w\|^2 \le C(N,\alpha,\nu)(\|\Lambda^{s+\alpha}u\|^2 + \|\Lambda^{s+\alpha}v\|^2 + 1)\|\Lambda^s w\|^2. \tag{3.24}$$

Gronwall's inequality then implies that

$$\|\Lambda^s w\|^2 \le \exp\Big\{C\int_0^t (\|\Lambda^{s+\alpha}u\|^2 + \|\Lambda^{s+\alpha}v\|^2 + 1)\,d\tau\Big\}\|\Lambda^s w(0)\|^2. \tag{3.25}$$

Since u and v are bounded in $L^2(0,T;\mathbb{H}^{s+\alpha})$, we conclude that the map $S(t): u_0 \to u(t)$ is continuous in $\mathbb{H}^s$. The uniqueness of the solution follows immediately.

Finally, let us verify that $u \in C([0,T];\mathbb{H}^s)$. Note that (3.3) implies that

$$u \in L^2(0,T;\mathbb{H}^{s+\alpha}),\ \forall T>0, \quad \text{i.e.,} \quad \Lambda^s u \in L^2(0,T;\mathbb{H}^\alpha),\ \forall T>0. \tag{3.26}$$

According to the standard Sobolev embedding results [33, 37], we need only show that

$$\Lambda^s u_t \in L^2(0,T;\mathbb{H}^{-\alpha}).$$

Indeed, for any $\varphi \in \mathbb{H}^\alpha$, we have

$$\langle \Lambda^s u_t, \varphi\rangle = -F_N(\|\nabla u\|)\langle \Lambda^s(u\cdot\nabla u),\varphi\rangle - \langle \Lambda^{s+2\alpha}u,\varphi\rangle + \langle \Lambda^s f,\varphi\rangle.$$

Therefore

$$|\langle \Lambda^s u_t,\varphi\rangle| \le \{F_N(\|\nabla u\|)\|\Lambda^{s-\alpha}(u\cdot\nabla u)\| + \|\Lambda^{s+\alpha}u\| + \|\Lambda^{s-\alpha}f\|\}\|\Lambda^\alpha\varphi\|,$$

which implies that

$$\|\Lambda^s u_t\|_{\mathbb{H}^{-\alpha}} \le F_N(\|\nabla u\|)\|\Lambda^{s-\alpha}(u\cdot\nabla u)\| + \|\Lambda^{s+\alpha}u\| + \|\Lambda^{s-\alpha}f\|.$$

Note that

$$F_N(\|\nabla u\|)\|\Lambda^{s-\alpha}(u\cdot\nabla)u\| \le CF_N(\|\nabla u\|)\|u\|_{L^6}\|\Lambda^{s+1-\alpha}u\|_{L^3}$$
$$\le CN\|\Lambda^{s+\alpha}u\|^{1-\theta_4}\|\Lambda^\alpha u\|^{\theta_4} \le CN(\|\Lambda^{s+\alpha}u\| + \|\Lambda^\alpha u\|)$$

for some $\theta_4 \in (0,1)$. Therefore

$$\|\Lambda^s u_t\|_{\mathbb{H}^{-\alpha}} \le C(\|\Lambda^{s+\alpha}u\| + \|\Lambda^\alpha u\| + \|\Lambda^{s-\alpha}f\|). \tag{3.27}$$

Combining (3.26) and the assumption $f \in \mathbb{H}^{s-\alpha}$, we find that

$$\int_0^T \|\Lambda^s u_t\|^2_{\mathbb{H}^{-\alpha}}\,ds < \infty,$$

that is, $\Lambda^s u_t \in L^2(0,T;\mathbb{H}^{-\alpha})$. Hence $\Lambda^s u \in C(0,T;\mathbb{H})$, i.e., $u \in C(0,T;\mathbb{H}^s)$.

This completes the proof of Theorem 3.1. ∎

Consider the following stationary globally modified Navier–Stokes equations corresponding to (1.2) in $[0,2\pi]^3$ with periodic boundary conditions:

$$\begin{cases} \nu \Lambda^{2\alpha} v + F_N(\|\nabla v\|)(v\cdot\nabla)v + \nabla P = f, \\ \nabla \cdot v = 0. \end{cases} \tag{3.28}$$

PROPOSITION 3.1. *Assume that* $f \in \mathbb{H}$ *and* $3/4 < \alpha \le 1$. *Then system* (3.28) *has a weak solution* $v \in \mathbb{H}^{2\alpha}$. *Moreover, if* $f \in \mathbb{H}^r$ *then* $v \in \mathbb{H}^{r+2\alpha}$ *for any* $r > 0$.

Proof. Performing the Galerkin approximation and calculations similar to Section 3, we can easily prove the following existence result for system (3.28). To prove the regularity result, we multiply (3.28) by v and integrate to deduce that

$$\nu \|\Lambda^\alpha v\|^2 \le \frac{1}{2\nu\lambda_1^{2\alpha}}\|f\|^2 + \frac{\nu\lambda_1^{2\alpha}}{2}\|v\|^2,$$

which implies that

$$\|v\|^2 \le \frac{\|f\|^2}{\nu^2\lambda_1^{4\alpha}}.$$

Now multiplying (3.28) by $\Lambda^{2r+2\alpha} v$ and integrating, we deduce that

$$\begin{aligned} &\nu\|\Lambda^{r+2\alpha}v\|^2 \\ &\quad \le \frac{1}{\nu}\|\Lambda^r f\|^2 + \frac{\nu}{4}\|\Lambda^{r+2\alpha}v\|^2 + F_N(\|\nabla v\|)\Big|\int_\Omega (v\cdot\nabla)v\Lambda^{2r+2\alpha}v\,dx\Big|. \end{aligned} \tag{3.29}$$

Similar to the deduction of (3.16), using the Gagliardo–Nirenberg inequality, Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned} &F_N(\|\nabla v\|)\Big|\int_\Omega (v\cdot\nabla v)\Lambda^{2r+2\alpha}v\,dx\Big| \\ &\quad \le F_N(\|\nabla v\|)\|\Lambda^r(v\cdot\nabla v)\|\,\|\Lambda^{r+2\alpha}v\| \\ &\quad \le CF_N(\|\nabla v\|)\|v\|_{L^6}\|\Lambda^{r+1}v\|_{L^3}\|\Lambda^{r+2\alpha}v\| \\ &\quad \le CN\|\Lambda^{r+1}v\|_{L^3}\|\Lambda^{r+2\alpha}v\| \le CN\|v\|^{\theta_5}\|\Lambda^{r+2\alpha}v\|^{2-\theta_5} \\ &\quad \le C_1(N,\alpha,\nu)\|v\|^2 + \frac{\nu}{4}\|\Lambda^{r+2\alpha}v\|^2, \end{aligned} \tag{3.30}$$

where $C_1(N,\alpha,\nu) = \frac{4\alpha-3}{4r+8\alpha}(CN)^{\frac{4r+8\alpha}{4\alpha-3}}\big(\frac{(2\alpha+r)\nu}{4\alpha+4r+3}\big)^{\frac{-3-4r-4\alpha}{4\alpha-3}}$. Combining (3.29) and (3.30), we obtain

$$\begin{aligned} \|\Lambda^{r+2\alpha}v\|^2 &\le \frac{2}{\nu^2}\|\Lambda^r f\|^2 + \frac{2}{\nu}C_1(N,\alpha,\nu)\|v\|^2 \\ &\le \frac{2}{\nu^2}\|\Lambda^r f\|^2 + \frac{2}{\nu^3\lambda_1^{4\alpha}}C_1(N,\alpha,\nu)\|f\|^2, \end{aligned} \tag{3.31}$$

that is, $v \in \mathbb{H}^{r+2\alpha}$. In particular, taking $r = s_0 + 1 - 2\alpha$, we conclude that $v \in \mathbb{H}^{s_0+1}$. ∎

4. Existence of a global attractor. In this section, we prove the existence of a global attractor for the globally modified Navier–Stokes system (1.2).

THEOREM 4.1. *Assume that* $3/4 < \alpha \le 1$, $f \in \mathbb{H}^{s-\alpha}$, $u_0 \in \mathbb{H}^s$ *and* $s \ge 1$. *Then system* (1.2) *generates a continuous semigroup* $\{S(t)\}_{t\ge 0}$ *in* $\mathbb{H}^s$ *and the semigroup has a global attractor* $\mathcal{A}$, *which is compact, connected in* $\mathbb{H}^s$, *and attracts all the bounded subsets of* $\mathbb{H}^s$ *in the* $\mathbb{H}^s$*-norm. Moreover, the global attractor is bounded in* $\mathbb{H}^{s_0+\alpha}$ *where* $s_0 = s - 1 + \alpha$.

Proof. By the standard existence theorem for global attractors [33, 37], to obtain the existence of a global attractor for the semigroup $\{S(t)\}_{t\ge 0}$ in $\mathbb{H}^s$, we need to prove the existence of an absorbing set, and the continuity and compactness of the semigroup in $\mathbb{H}^s$. Thanks to Theorem 3.1, we know that the semigroup is continuous. Thus we only need to verify the other two conditions.

STEP 1: *Absorbing set in* $\mathbb{H}^s$. Now we prove that the semigroup $\{S(t)\}_{t\ge 0}$ admits an absorbing set in $\mathbb{H}^s$, i.e., there exists a bounded set B_0 in $\mathbb{H}^s$ such that for any bounded subset $B \subset \mathbb{H}^s$, there exists a time $T(B)$ such that $S(t)B \subset B_0$ for any $t > T(B)$.

Let $u(t)$ be the solution of system (1.2). Multiplying (1.2) by u and integrating, similar to (3.5), we obtain

$$\frac{d}{dt}\|u\|^2 + \nu\|\Lambda^\alpha u\|^2 \le \frac{\|f\|^2}{\nu\lambda_1^{2\alpha}}.$$

Similar to (3.7), we can use the Poincaré inequality and Gronwall's inequality to deduce that

$$\begin{aligned} \|u(t)\|^2 &\le \|u_0\|^2 e^{-\nu\lambda_1^{2\alpha}t} + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2} \quad \text{(for all } t \ge 0) \\ &\le 2\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2} \quad \text{(for all } t \ge t_0 = t(u_0)). \end{aligned} \tag{4.1}$$

Multiplying (1.2) by $\Lambda^{2s}u$ and integrating, we have

$$\frac{d}{dt}\|\Lambda^s u\|^2 + \nu\|\Lambda^{s+\alpha}u\|^2 \le \|f\|_{\mathbb{H}^{s-\alpha}}\|\Lambda^{s+\alpha}u\| + F_N(\|\nabla u\|)\Big|\int_\Omega (u\cdot\nabla u)\Lambda^{2s}u\,dx\Big|.$$

Similar to (3.16), we can deduce that

$$\frac{d}{dt}\|\Lambda^s u\|^2 + \nu\|\Lambda^{s+\alpha}u\|^2 \le 2C_0(N,\nu,\alpha)\|u\|^2 + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} \tag{4.2}$$

with $C_0(N,\nu,\alpha) = \frac{2\alpha-3/2}{2s+2\alpha}(CN)^{(2s+2\alpha)/(2\alpha-3/2)}(\frac{\nu(s+\alpha)}{4s+3})^{-(2s+3/2)/(2\alpha-3/2)}$, which implies that

$$\begin{aligned}(4.3)\qquad \|\Lambda^s u(t_0)\|^2 &\le \|\Lambda^s u_0\| + \int_0^{t_0}\Big(2C_0(N,\nu,\alpha)\|u\|^2 + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\Big)\,d\tau \\ &\le \|\Lambda^s u_0\| + 2C_0(N,\nu,\alpha)t_0\Big(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\Big) + \frac{2t_0}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} =: \mathcal{K}_0.\end{aligned}$$

Using the Poincaré inequality and (4.1), we infer from (4.2) that for $t \ge t_0$,

$$(4.4)\qquad \frac{d}{dt}\|\Lambda^s u\|^2 + \nu\lambda_1^{2\alpha}\|\Lambda^s u\|^2 \le C_0(N,\nu,\alpha)\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}.$$

Gronwall's inequality then implies that

$$\|\Lambda^s u(t)\|^2 \le \|\Lambda^s u(t_0)\|^2 e^{-\nu\lambda_1^{2\alpha}(t-t_0)} + \frac{1}{\nu\lambda_1^{2\alpha}}\Big(C_0(N,\nu,\alpha)\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\Big).$$

Thanks to (4.3), we know that if

$$t \ge \frac{1}{\nu\lambda_1^{2\alpha}}\ln\mathcal{K}_0 + t_0,$$

then

$$\|\Lambda^s u(t)\|^2 \le \frac{1}{\nu\lambda_1^{2\alpha}}\Big(C_0(N,\nu,\alpha)\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\Big) + 1 =: \rho_0^2.$$

Therefore, there is an absorbing set B_0 for the semigroup $\{S(t)\}_{t\ge 0}$ in $\mathbb{H}^s$. Actually, B_0 can be chosen to be the ball centered at zero with radius ρ_0 in $\mathbb{H}^s$.

STEP 2: *Compactness of the semigroup.* To prove the compactness of the semigroup, we need only prove that for any bounded sequence $\{u_0^n\}$ in $\mathbb{H}^s$, the sequence $\{S(t)u_0^n\} = \{u^n(t)\}$ has a convergent subsequence in $\mathbb{H}^s$, for $t > t_1 > 0$. Similar to (3.18) and (3.27), we can prove that

$$\{\Lambda^s u^n(t)\} \text{ is bounded in } L^2(0,1;\mathbb{H}^\alpha),$$
$$\{\Lambda^s u^n_t(t)\} \text{ is bounded in } L^2(0,1;\mathbb{H}^{-\alpha}).$$

By the Aubin–Simon type compactness results [37], there exists an element u with

$$\Lambda^s u \in L^2(0,1;\mathbb{H}^\alpha), \qquad \Lambda^s u_t \in L^2(0,1;\mathbb{H}^{-\alpha}),$$

such that up to subsequences,

$$\Lambda^s u^n(t) \to \Lambda^s u(t) \quad \text{strongly in } L^2(0,1;\mathbb{H}),$$

i.e.,

$$u^n(t) \to u(t) \quad \text{strongly in } L^2(0,1;\mathbb{H}^s).$$

In particular, there exists a $\tau \in (0,1)$ such that

$$u^n(\tau) \to u(\tau) \quad \text{in } \mathbb{H}^s.$$

Recalling that the map $S(t) : \mathbb{H}^s \to \mathbb{H}^s$ is continuous, we infer that

$$S(t)u_0^n = S(t-\tau)S(\tau)u_0^n = S(t-\tau)u^n(\tau) \to S(t-\tau)u(\tau) \quad \text{in } \mathbb{H}^s,$$

for all $t \geq 1$. Therefore the semigroup $S(t)$ is compact for any $t \geq 1$. Thanks to the above results, we obtain the existence of a global attractor $\mathcal{A}$ in $\mathbb{H}^s$ immediately by the standard theorem on the existence of global attractors [33, 37]. Moreover, as $\mathbb{H}^s$ is connected, the attractor $\mathcal{A}$ is connected.

Step 3: *Regularity of the attractor.* Next, we prove that $\mathcal{A}$ is bounded in $\mathbb{H}^{s_0+\alpha}$. Since the attractor $\mathcal{A}$ is invariant, for any $t > 0$ and any $\chi \in \mathcal{A}$ there exists a $u_0 \in \mathcal{A}$ such that $S(t)u_0 = u(t) = \chi$. Since $S(t)u_0$ lies in $\mathcal{A}$ for any $t > 0$ ($\mathcal{A}$ is invariant), there exists a time-independent positive constant $\mathcal{K}_1$ such that

$$\|u(t)\|_{\mathbb{H}^s} = \|S(t)u_0\|_{\mathbb{H}^s} \leq \mathcal{K}_1 \quad \text{for any } t \geq 0. \tag{4.5}$$

Thanks to (3.3) (see also (3.18)), we know that, for any $t \in (0,1)$,

$$\begin{aligned} \nu\int_0^1 \|\Lambda^{s+\alpha}u\|^2\,d\tau \\ \leq \|\Lambda^s u_0\|^2 + 2C_0(N,\nu,\alpha)\left(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\right) + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}, \end{aligned} \tag{4.6}$$

where $C_0(N,\nu,\alpha)$ is as in (4.2). Similar to (3.9), there must exist a $t_0 \in (0,1)$ such that

$$\begin{aligned} \|\Lambda^{s+\alpha}u(t_0)\|^2 \\ \leq \frac{1}{\nu}\|\Lambda^s u_0\|^2 + \frac{2C_0(N,\nu,\alpha)}{\nu}\left(\|u_0\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\right) + \frac{2}{\nu^2}\|f\|^2_{\mathbb{H}^{s-\alpha}}. \end{aligned} \tag{4.7}$$

Let v be a solution of system (3.28). Since v is a fixed point of the solution semigroup, it must belong to $\mathcal{A}$. Moreover, from (3.31) we know that $v \in \mathbb{H}^{s+\alpha}$ $(= \mathbb{H}^{s_0+1})$ with

$$\begin{aligned} \|v\|_{\mathbb{H}^{s+\alpha}} &\leq \frac{2}{\nu^2}\|\Lambda^{s-\alpha}f\|^2 + \frac{2}{\nu}C_1(N,\alpha,\nu)\|v\|^2 \\ &\leq \frac{2}{\nu^2}\|\Lambda^{s-\alpha}f\|^2 + \frac{2}{\nu^3\lambda_1^{4\alpha}}C_1(N,\alpha,\nu)\|f\|^2. \end{aligned} \tag{4.8}$$

Now set $w(x,t) = u(x,t) - v(x)$; it is obvious that w satisfies

$$\begin{aligned} w_t + \nu\Lambda^{2\alpha}w + F_N(\|\nabla u\|)\mathcal{P}(u\cdot\nabla)w + F_N(\|\nabla u\|)\mathcal{P}(w\cdot\nabla)v \\ + \{F_N(\|\nabla u\|) - F_N(\|\nabla v\|)\}\mathcal{P}(v\cdot\nabla)v = 0. \end{aligned} \tag{4.9}$$

Taking the inner product with $\Lambda^{2s_0+2\alpha}w$ yields

$$(4.10)\quad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^{s_0+\alpha}w\|^2+\nu\|\Lambda^{s_0+2\alpha}w\|^2 \\ &\le F_N(\|\nabla u\|)\Big|\int_\Omega (u\cdot\nabla w)\Lambda^{2s_0+2\alpha}w\,dx\Big| \\ &\quad + F_N(\|\nabla u\|)\Big|\int_\Omega (w\cdot\nabla v)\Lambda^{2s_0+2\alpha}w\,dx\Big| \\ &\quad + \Big|\{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}\int_\Omega (v\cdot\nabla v)\Lambda^{2s_0+2\alpha}w\,dx\Big| \\ &=: J_1+J_2+J_3. \end{aligned}$$

For J_1, using Hölder's inequality, the product estimates and the Gagliardo–Nirenberg inequality, we deduce that

$$(4.11)\quad \begin{aligned} J_1 &\le F_N(\|\nabla u\|)\|\Lambda^{s_0}(u\cdot\nabla w)\|\,\|\Lambda^{s_0+2\alpha}w\| \\ &\le CF_N(\|\nabla u\|)(\|u\|_{L^6}\|\Lambda^{s_0+1}w\|_{L^3}+\|\Lambda^{s_0}u\|_{L^p}\|\nabla w\|_{L^q})\|\Lambda^{s_0+2\alpha}w\| \\ &\le CN\|\Lambda^{s_0+1-\alpha}w\|^{\theta_6}\|\Lambda^{s_0+2\alpha}w\|^{2-\theta_6} \\ &\quad + C\|\Lambda^{s_0+1-\alpha}u\|^{1-\theta_7}\|\Lambda^{s_0+1}u\|^{\theta_7}\|\Lambda^{s_0+1-\alpha}w\|^{\theta_8}\|\Lambda^{s_0+2\alpha}w\|^{2-\theta_8}, \end{aligned}$$

where $\theta_6=\frac{2\alpha-3/2}{3\alpha-1}$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and

$$\Big(\frac{1}{2}-\frac{s_0+1-\alpha}{3}\Big)(1-\theta_7)+\Big(\frac{1}{2}-\frac{s_0+1}{3}\Big)\theta_7=\frac{1}{p}-\frac{s_0}{3},$$
$$\Big(\frac{1}{2}-\frac{s_0+1-\alpha}{3}\Big)\theta_8+\Big(\frac{1}{2}-\frac{s_0+2\alpha}{3}\Big)(1-\theta_8)=\frac{1}{q}-\frac{1}{3}.$$

Taking $p=12/5$, $q=12$, we get $\theta_7=1-\frac{9}{12\alpha}$, $\theta_8=\frac{2\alpha+s_0-9/4}{3\alpha-1}$. (Here we may assume that s_0 is not too large, say $s_0\le 2$, so that $\frac{2\alpha+s_0-9/4}{3\alpha-1}\le 1$. Otherwise $s\ge 2$ ($s_0=s-1+\alpha$), and we may choose $q=\infty$, so the proof would be much easier.) Note that $s_0=s-1+\alpha\ge\alpha$, $3/4<\alpha\le 1$, and it is easy to check that $\theta_7<\theta_8$. Hence, using Young's inequality, we deduce from (4.11) that

$$(4.12)\quad \begin{aligned} J_1 &\le C(1+\|\Lambda^{s_0+1-\alpha}u\|^{2(1-\theta_7)/\theta_8}\|\Lambda^{s_0+1}u\|^{2\theta_7/\theta_8})\|\Lambda^{s_0+1-\alpha}w\|^2 \\ &\quad + \frac{\nu}{6}\|\Lambda^{s_0+2\alpha}w\|^2. \end{aligned}$$

On the other hand, using Hölder's inequality, the product estimates and the Gagliardo–Nirenberg inequality and Young's inequality, we deduce that

$$(4.13)\quad \begin{aligned} J_2 &\le F_N(\|\nabla u\|)\|\Lambda^{s_0}(w\cdot\nabla v)\|\,\|\Lambda^{s_0+2\alpha}w\| \\ &\le CF_N(\|\nabla u\|)(\|v\|_{L^6}\|\Lambda^{s_0+1}w\|_{L^3}+\|\Lambda^{s_0+1}v\|\,\|w\|_{L^\infty})\|\Lambda^{s_0+2\alpha}w\| \end{aligned}$$

$$
\begin{aligned}
&\le C\|\Lambda v\|\,\|\Lambda^{s_0+1-\alpha}w\|^{\theta_9}\|\Lambda^{s_0+2\alpha}w\|^{2-\theta_9}\\
&\quad + C\|\Lambda^{s_0+1}v\|\,\|\Lambda^{s_0+1-\alpha}w\|^{\theta_{10}}\|\Lambda^{s_0+2\alpha}w\|^{2-\theta_{10}}\\
&\le C\|\Lambda v\|^{2/\theta_9}\|\Lambda^{s_0+1-\alpha}w\|^2 + \frac{\nu}{6}\|\Lambda^{s_0+2\alpha}w\|^2\\
&\quad + C\|\Lambda^{s_0+1}v\|^{2/\theta_{10}}\|\Lambda^{s_0+1-\alpha}w\|^2,
\end{aligned}
$$

and also

$$
\begin{aligned}
J_3 &\le |F_N(\|\nabla u\|) - F_N(\|\nabla v\|)|\|\Lambda^{s_0}(v\cdot\nabla v)\|\,\|\Lambda^{s_0+2\alpha}w\|\\
&\le C\frac{F_N(\|\nabla u\|)F_N(\|\nabla v\|)}{N}\|v\|_{L^\infty}\|\Lambda^{s_0+1}v\|\,\|\nabla w\|\,\|\Lambda^{s_0+2\alpha}w\|\\
&\le C\|\Lambda^{s_0+1}v\|^2\|\Lambda^{s_0+1-\alpha}w\|\,\|\Lambda^{s_0+2\alpha}w\|\\
&\le C\|\Lambda^{s_0+1}v\|^2\|\Lambda^{s_0+1-\alpha}w\|^2 + \frac{\nu}{6}\|\Lambda^{s_0+2\alpha}w\|^2.
\end{aligned}
\tag{4.14}
$$

Combining (4.10) and (4.12)–(4.14), we get

$$
\begin{aligned}
&\frac{d}{dt}\|\Lambda^{s_0+\alpha}w\|^2 + \nu\|\Lambda^{s_0+2\alpha}w\|^2\\
&\le C(1+\|\Lambda^{s_0+1-\alpha}u\|^{2(1-\theta_7)/\theta_8}\|\Lambda^{s_0+1}u\|^{2\theta_7/\theta_8})\|\Lambda^{s_0+1-\alpha}w\|^2\\
&\quad + C(\|\Lambda v\|^{2/\theta_9} + \|\Lambda^{s_0+1}v\|^{2/\theta_{10}} + C\|\Lambda^{s_0+1}v\|^2)\|\Lambda^{s_0+1-\alpha}w\|^2.
\end{aligned}
\tag{4.15}
$$

Since $u(t)$ lies in $\mathcal{A}\subseteq\mathbb{H}^s=\mathbb{H}^{s_0+1-\alpha}$ and $v\in\mathbb{H}^{s+\alpha}=\mathbb{H}^{s_0+1}$, taking into account (4.5) and (4.8) we deduce from (4.15) that

$$
\frac{d}{dt}\|\Lambda^{s_0+\alpha}w\|^2 + \nu\|\Lambda^{s_0+2\alpha}w\|^2 \le C + C\|\Lambda^{s_0+1}u\|^{2\theta_7/\theta_8}.
$$

Integrating this inequality between τ and t $(0\le t-1\le\tau<t)$ yields

$$
\|\Lambda^{s_0+\alpha}w(t)\|^2 \le C + \|\Lambda^{s_0+\alpha}w(\tau)\|^2 + C\int_\tau^t \|\,\Lambda^{s_0+1}u(\varsigma)\|^{2\theta_7/\theta_8}\,d\varsigma.
$$

Integrating this with respect to τ between $t-1$ and t, we have

$$
\|\Lambda^{s_0+\alpha}w(t)\|^2 \le C\Big\{1+\int_{t-1}^t \|\Lambda^{s_0+\alpha}w(\tau)\|^2\,d\tau + \int_{t-1}^t \|\Lambda^{s_0+1}u(\varsigma)\|^{2\theta_7/\theta_8}\,d\varsigma\Big\}.
\tag{4.16}
$$

Note that $\theta_7<\theta_8$ and $s_0+1=s+\alpha$. Thanks to (3.3), we deduce that

$$
\begin{aligned}
&\int_{t-1}^t \|\Lambda^{s_0+1}u(\varsigma)\|^{2\theta_7/\theta_8}\,d\varsigma \le 1+\int_{t-1}^t \|\Lambda^{s+\alpha}u(\varsigma)\|^2\,d\varsigma\\
&\le \frac{1}{\nu}\|\Lambda^s u(t-1)\|^2 + \frac{2C}{\nu}\bigg(\|u_0\|^2+\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg) + \frac{2}{\nu^2}\|f\|^2_{\mathbb{H}^{s-\alpha}} + 1 \le C.
\end{aligned}
\tag{4.17}
$$

On the other hand, as $s_0 \le s$, we have

$$\int\limits_{t-1}^{t} \|\Lambda^{s_0+\alpha} w(\tau)\|^2\, d\tau \le \int\limits_{t-1}^{t} \|\Lambda^{s_0+\alpha} u(\tau)\|^2\, d\tau + \|\Lambda^{s_0+\alpha} v\|^2 \le C \int\limits_{t-1}^{t} \|\Lambda^{s+\alpha} u(\tau)\|^2\, d\tau + \|\Lambda^{s_0+\alpha} v\|^2 \le C. \tag{4.18}$$

Inserting (4.17) and (4.18) into (4.16), we obtain

$$\|\Lambda^{s_0+\alpha}\chi\|^2 = \|\Lambda^{s_0+\alpha} u(t)\|^2 \le \|\Lambda^{s_0+\alpha} w(t)\|^2 + \|\Lambda^{s_0+\alpha} v\|^2 \le C_{\mathcal{A}}^2. \tag{4.19}$$

From the deductions above, we know that the bound of $\|\Lambda^{s_0+\alpha}\chi\|^2$ (or $\|\Lambda^{s_0+\alpha} u(t)\|^2$) may depend on $\|f\|, \|f\|_{\mathbb{H}}^{s-\alpha}, N, \alpha, \nu$ and the $\mathbb{H}^s$ bound of the attractor $\mathcal{A}$, but it is independent of χ. Hence we denote it by $C_{\mathcal{A}}^2$ and conclude that $\mathcal{A}$ is bounded in $\mathbb{H}^{s_0+\alpha}$. ∎

5. Finite-dimensionality of the attractor

THEOREM 5.1. *Assume that* $3/4 < \alpha \le 1$, $u_0 \in \mathbb{H}^s$, $f \in \mathbb{H}^{s-\alpha}$ *and* $s \ge 1$. *Then the fractal dimension of the attractor* $\mathcal{A}$ *derived in Theorem* 4.1 *is finite.*

To prove Theorem 5.1 we shall use the following abstract results derived in [8] (see also [10, 9]).

LEMMA 5.1. *Let* H_0 *be a separable Hilbert space and let* M *be a bounded closed set in* H_0. *Assume that there exists a mapping* $S_0 : M \to H_0$ *such that* $M \subseteq S_0 M$ *and*

(i) S_0 *is Lipschitz on* M, *i.e., there exists* $L > 0$ *such that*

$$\|S_0 v_1 - S_0 v_2\|_{H_0} \le L\|v_1 - v_2\|_{H_0}, \qquad v_1, v_2 \in M,$$

(ii) *there exist finite-dimensional orthoprojectors* P_1 *and* P_2 *on* H_0 *such that for* $v_1, v_2 \in M$,

$$\|S_0 v_1 - S_0 v_2\|_{H_0} \le \eta\|v_1 - v_2\|_{H_0} + K(\|P_1(v_1 - v_2)\|_{H_0} + \|P_2(v_1 - v_2)\|_{H_0})$$

for some constants $0 < \eta < 1$ *and* $K > 0$.

Then

$$\dim_{\mathrm{f}} M \le (\dim P_1 + \dim P_2) \ln\left\{1 + \frac{8\sqrt{2}(1+L)K}{1-\eta}\right\}\left(\ln\frac{2}{1+\eta}\right)^{-1}.$$

Denote $Z_m = \operatorname{span}\{(e_j - \frac{k_j k}{|k|^2})e^{ik\cdot x} : j = 1, 2, 3,\ |k| = \sqrt{\sum_{i=1}^3 |k_i|^2} \le m\}$, where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, $k \ne 0$, and e_1, e_2, e_3 represent the canonical basis of $\mathbb{R}^3$. Let $P_m : L^2(\Omega) \to Z_m$ be the projection operator. Similar to [38, Lemma 3.4] (see also [8, Lemma 2.12]), we have the following lemma.

Lemma 5.2. *Let $\eta \geq 0$ and $\beta > 0$. For any $\epsilon > 0$, there exists a positive integer $m(\epsilon)$ such that for $m \geq m(\epsilon)$, we have*

$$\|\varphi\|_{H^\eta} \leq \epsilon\|\varphi\|_{H^{\eta+\beta}} + \|P_m\varphi\|_{H^\eta}, \qquad \forall\varphi \in H^{\eta+\beta},$$

where $m(\epsilon) = [\epsilon^{-1/\beta}]$, the integer part of $\epsilon^{-1/\beta}$.

Lemma 5.3. *The projection operator $P_m : L^2(\Omega) \to Z_m$ has a finite-dimensional range with*

$$\dim P_m \leq 8(4m^3 + 6m^2 + 8m + 3).$$

Proof. Note that for any $k \neq k' \in \mathbb{Z}^3$, $e^{ik\cdot x}$ is orthogonal to $e^{ik'\cdot x}$ in L^2. Hence

$$\dim P_m \leq \sum_{|k_1|+|k_2|+|k_3|\leq m} \dim P_m\bigg\{\bigg(e_j - \frac{k_jk}{|k|^2}\bigg)e^{ik\cdot x} : j = 1,2,3\bigg\}. \tag{5.1}$$

By Euler's formula, we have

$$\begin{aligned}
&\bigg(e_j - \frac{k_jk}{|k|^2}\bigg)e^{ik\cdot x} \\
&\quad= \bigg(e_j - \frac{k_jk}{|k|^2}\bigg)\big(\cos(k_1x_1 + k_2x_2 + k_3x_3) + i\sin(k_1x_1 + k_2x_2 + k_3x_3)\big) \\
&\quad= \bigg(e_j - \frac{k_jk}{|k|^2}\bigg)(\cos k_1x_1 \cos k_2x_2 \cos k_3x_3 - \cos k_1x_1 \sin k_2x_2 \sin k_3x_3 \\
&\qquad - \sin k_1x_1 \sin k_2x_2 \cos k_3x_3 - \sin k_1x_1 \cos k_2x_2 \sin k_3x_3) \\
&\qquad + i\bigg(e_j - \frac{k_jk}{|k|^2}\bigg)(\sin k_1x_1 \cos k_2x_2 \cos k_3x_3 - \sin k_1x_1 \sin k_2x_2 \sin k_3x_3 \\
&\qquad + \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 - \cos k_1x_1 \sin k_2x_2 \cos k_3x_3),
\end{aligned}$$

which implies that

$$\dim P_m\bigg\{\bigg(e_j - \frac{k_jk}{|k|^2}\bigg)e^{ik\cdot x} : j = 1,2,3\bigg\} \leq 24, \qquad \forall k \in \mathbb{Z}^3. \tag{5.2}$$

Moreover,

$$\operatorname{Card}\bigg\{k = (k_1,k_2,k_3) \in \mathbb{Z}^3 : \sum_{i=1}^{3}|k_i| \leq m\bigg\} = \frac{4}{3}m^3 + 2m^2 + \frac{8}{3}m + 1. \tag{5.3}$$

From (5.1)–(5.3), it follows that

$$\dim P_m \leq 8(4m^3 + 6m^2 + 8m + 3). \ \blacksquare$$

Lemma 5.4. *Assume that $3/4 < \alpha \leq 1$, $u_0 \in \mathbb{H}^s$, $f \in \mathbb{H}^{s-\alpha}$ and $s \geq 1$. Let $\mathcal{A}$ be the global attractor of system (1.2) derived in Theorem 4.1. Let $u(t)$, $v(t)$ be two solutions of system (1.2) corresponding to the initial data $u_0, v_0 \in \mathcal{A}$ respectively. Let $w(t) = u(t)-v(t)$, and let ϑ be a positive constant* *such that* $3/2 < s-1+2\alpha-\vartheta$. *Then for any* $\beta \in [s-1+2\alpha-\vartheta, s-1+2\alpha]$, *we have*

$$\|\Lambda^\beta w(t)\|^2 \le \exp\Big\{C_0\int_0^t(\|\Lambda^{s+\alpha}u\|^2+\|\Lambda^{s+\alpha}v\|^2+1)\,d\tau\Big\}\|\Lambda^\beta w(0)\|^2 \tag{5.4}$$

for some positive constant C_0.

Proof. Similar to the proof of the uniqueness result (see (3.20)), we find that w satisfies, for any $s-1+2\alpha-\vartheta \le \beta \le s-1+2\alpha$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta w\|^2 &+ \nu\|\Lambda^{\beta+\alpha}w\|^2\\ &\le F_N(\|\nabla u\|)\Big|\int_\Omega (u\cdot\nabla w)\Lambda^{2\beta}w\,dx|\\ &\quad + F_N(\|\nabla u\|)\Big|\int_\Omega (w\cdot\nabla v)\Lambda^{2\beta}w\,dx\Big|\\ &\quad + \Big|\{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}\int_\Omega (v\cdot\nabla v)\Lambda^{2\beta}w\,dx\Big|\\ &=: I_1+I_2+I_3.\end{aligned} \tag{5.5}$$

Since $\mathcal{A}$ is bounded in $\mathbb{H}^{s_0+\alpha}$, we have

$$\|\Lambda^{s_0+\alpha}u(t)\| \le C_{\mathcal{A}}, \qquad \|\Lambda^{s_0+\alpha}v(t)\| \le C_{\mathcal{A}}, \qquad \forall t\ge 0.$$

For I_1, using Hölder's inequality, the product estimates, the Gagliardo–Nirenberg inequality and Young's inequality, we deduce that

$$\begin{aligned} I_1 &\le F_N(\|\nabla u\|)\|\Lambda^{\beta+1-\alpha}(uw)\|\,\|\Lambda^{\beta+\alpha}w\|\\ &\le C_1F_N(\|\nabla u\|)(\|u\|_{L^\infty}\|\Lambda^{\beta+1-\alpha}w\|+\|w\|_{L^\infty}\|\Lambda^{\beta+1-\alpha}u\|)\|\Lambda^{\beta+\alpha}w\|\\ &\le C_1\|\Lambda^{s_0+\alpha}u\|\,\|\Lambda^\beta w\|^{\theta_{11}}\|\Lambda^{\beta+\alpha}w\|^{2-\theta_{11}} + C_2\|\Lambda^{s+\alpha}u\|\,\|\Lambda^\beta w\|\,\|\Lambda^{\beta+\alpha}w\|\\ &\le C_1C_{\mathcal{A}}\|\Lambda^\beta w\|^{\theta_{11}}\|\Lambda^{\beta+\alpha}w\|^{2-\theta_{11}} + C_2\|\Lambda^{s+\alpha}u\|\,\|\Lambda^\beta w\|\,\|\Lambda^{\beta+\alpha}w\|\\ &\le \frac{\nu}{6}\|\Lambda^{\beta+\alpha}w\|^2 + C(\alpha,\nu)(1+\|\Lambda^{s+\alpha}u\|^2)\|\Lambda^\beta w\|^2,\end{aligned} \tag{5.6}$$

where $\theta_{11}=\frac{2\alpha-1}{\alpha}$, $C(\alpha,\nu)=\max\{\frac{2\alpha-1}{2\alpha}C_1C_{\mathcal{A}}(\frac{\alpha\nu}{6C_1C_{\mathcal{A}}})^{\frac{1}{1-2\alpha}}, \frac{3C_2^2}{\nu}\}$, and $C_{\mathcal{A}}$ is from (4.19). Similarly,

$$I_2 \le \frac{\nu}{6}\|\Lambda^{\beta+\alpha}w\|^2 + C(\alpha,\nu)(1+\|\Lambda^{s+\alpha}v\|^2)\|\Lambda^\beta w\|^2. \tag{5.7}$$

Finally, for I_3, we use Hölder's inequality, Lemma 2.2, the product estimates and imbeddings of fractional Sobolev spaces to deduce that

$$
\begin{aligned}
(5.8)\qquad I_3 &\le |F_N(\|\nabla u\|)-F_N(\|\nabla v\|)|\,\|\Lambda^{s+1-\alpha}(vv)\|\,\|\Lambda^{\beta+\alpha}w\|\\
&\le C_3\frac{F_N(\|\nabla u\|)F_N(\|\nabla v\|)}{N}\|\nabla w\|\,\|v\|_{L^\infty}\|\Lambda^{\beta+1-\alpha}v\|\,\|\Lambda^{\beta+\alpha}w\|\\
&\le \frac{C_3}{N}\|\Lambda^{\beta}w\|\,\|\Lambda^{s_0+\alpha}v\|\,\|\Lambda^{s+\alpha}v\|\,\|\Lambda^{\beta+\alpha}w\|\\
&\le \frac{C_3C_{\mathcal{A}}}{N}\|\Lambda^{\beta}w\|\,\|\Lambda^{s+\alpha}v\|\,\|\Lambda^{\beta+\alpha}w\|\\
&\le \frac{\nu}{6}\|\Lambda^{\beta+\alpha}w\|^2+\frac{3C_3^2C_{\mathcal{A}}^2}{2N^2\nu}\|\Lambda^{s+\alpha}v\|^2\|\Lambda^{\beta}w\|^2.
\end{aligned}
$$

Combining (5.5)–(5.8), we obtain

$$
\frac{d}{dt}\|\Lambda^{\beta}w\|^2 \le C_0(\|\Lambda^{s+\alpha}u\|^2+\|\Lambda^{s+\alpha}v\|^2+1)\|\Lambda^{\beta}w\|^2,
$$

where $C_0=\max\{\frac{2\alpha-1}{2\alpha}C_1C_{\mathcal{A}}(\frac{\alpha\nu}{6C_1C_{\mathcal{A}}})^{\frac{1}{1-2\alpha}},\frac{3C_2^2}{\nu},\frac{3C_3^2C_{\mathcal{A}}^2}{2N^2\nu}\}$. Gronwall's inequality then implies exactly (5.4):

$$
\begin{aligned}
(5.9)\qquad \|\Lambda^{\beta}w(t)\|^2 &\le \exp\Big\{C_0\int_0^t(\|\Lambda^{s+\alpha}u\|^2+\|\Lambda^{s+\alpha}v\|^2+1)\,d\tau\Big\}\|\Lambda^{\beta}w(0)\|^2\\
&=: \widetilde{C}(t)\|\Lambda^{\beta}w(0)\|^2.
\end{aligned}
$$

Obviously, $\widetilde{C}(t)$ is a monotone function with respect to t and it is finite for finite time t. ■

LEMMA 5.5. *Assume that* $3/4<\alpha\le 1$, $u_0\in\mathbb{H}^s$, $f\in\mathbb{H}^{s-\alpha}$ *and* $s\ge 1$. *Let* $\mathcal{A}$ *be the global attractor of system* (1.2). *Let* $u(t),v(t)$ *be two solutions of system* (1.2) *corresponding to the initial data* $u_0,v_0\in\mathcal{A}$ *respectively. Set* $w(t)=u(t)-v(t)$, $\gamma=s-1+2\alpha$ $(\ge 2\alpha)$. *Then there exists a positive constant* C *such that*

$$
\|w(t)\|_{\mathbb{H}^\gamma}\le Ce^{-\frac{\nu\lambda_1^{2\alpha}t}{2}}\|w(0)\|_{\mathbb{H}^\gamma}+\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2+2C_{\mathcal{A}})\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}},\ \forall t\ge 0,
$$

where $\epsilon_0=\frac{1}{2}\min\{\frac{2\alpha-1}{2\alpha+1},\gamma-1,2\vartheta\}$ *and* $\widetilde{C}(t)$ *is from* (5.9) (*or* (5.4)).

Proof. Similar to (3.19), we know that w satisfies

$$
\begin{aligned}
(5.10)\qquad w_t+\nu\Lambda^{2\alpha}w+F_N(\|\nabla u\|)\mathcal{P}(u\cdot\nabla)w+F_N(\|\nabla u\|)\mathcal{P}(w\cdot\nabla)v&\\
+\{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}\mathcal{P}(v\cdot\nabla)v&=0.
\end{aligned}
$$

By semigroup theory [32], the solution of (5.10) can be given by

$$
\begin{aligned}
w(t)=e^{-\nu(-\Delta)^\alpha t}w(0)-\int_0^t e^{-\nu(-\Delta)^\alpha(t-\tau)}\big\{&F_N(\|\nabla u\|)\mathcal{P}(u\cdot\nabla)w\\
&+F_N(\|\nabla u\|)\mathcal{P}(w\cdot\nabla)v+\big(F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\big)\mathcal{P}(v\cdot\nabla)v\big\}\,d\tau.
\end{aligned}
$$

Moreover,

$$\|e^{-(-\Delta)^\alpha t}\|_{\mathbb{H}^\beta,\mathbb{H}^{\beta'}} \le Ce^{-\lambda_1^{2\alpha}t/2}t^{-\frac{\beta'-\beta}{2\alpha}}, \qquad \beta' \ge \beta \ge 0,\ t>0.$$

Hence,

(5.11)

$$\begin{aligned}\|w(t)\|_{\mathbb{H}^\gamma} &\le \|e^{-\nu(-\Delta)^\alpha t}w(0)\|_{\mathbb{H}^\gamma} + \int_0^t \big\|e^{-\nu(-\Delta)^\alpha (t-\tau)}\{F_N(\|\nabla u\|)u\cdot\nabla w \\ &\quad + F_N(\|\nabla u\|)w\cdot\nabla v + \big(F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\big)v\cdot\nabla v\}\big\|_{\mathbb{H}^\gamma}\,d\tau \\ &\le Ce^{-\nu\lambda_1^{2\alpha}t/2}\|w(0)\|_{\mathbb{H}^\gamma} + C\int_0^t e^{-\nu\lambda_1^{2\alpha}(t-\tau)/2}(t-\tau)^{-1+\epsilon}\Psi(\tau)\,d\tau,\end{aligned}$$

where

$$\begin{aligned}\Psi(\tau) = \big\|F_N(\|\nabla u\|)u\cdot\nabla w + F_N(\|\nabla u\|)w\cdot\nabla v \\ + \{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}v\cdot\nabla v\big\|_{\mathbb{H}^{\gamma-2(1-\epsilon)\alpha}},\end{aligned}$$

and $0<\epsilon<1$ will be determined later. Setting $\sigma=\gamma-2(1-\epsilon)\alpha$, we have

$$\begin{aligned}\big\|F_N(\|\nabla u\|)u\cdot\nabla w &+ F_N(\|\nabla u\|)w\cdot\nabla v \\ &+ \{F_N(\|\nabla u\|)-F_N(\|\nabla v\|)\}v\cdot\nabla v\big\|_{\mathbb{H}^\sigma} \\ &\le F_N(\|\nabla u\|)\|u\cdot\nabla w\|_{\mathbb{H}^\sigma} \\ &\quad + F_N(\|\nabla u\|)\|w\cdot\nabla v\|_{\mathbb{H}^\sigma} \\ &\quad + |F_N(\|\nabla u\|)-F_N(\|\nabla v\|)|\cdot\|v\cdot\nabla v\|_{\mathbb{H}^\sigma} \\ &=: S_1+S_2+S_3.\end{aligned} \tag{5.12}$$

Next, we estimate S_1,S_2,S_3 one by one. By the product estimates, we have

$$S_1 \le CF_N(\|\nabla u\|)(\|u\|_{L^{p_1}}\|\nabla w\|_{\mathbb{H}^{\sigma,q_1}} + \|u\|_{\mathbb{H}^{\sigma,p_2}}\|\nabla w\|_{L^{q_2}}) \tag{5.13}$$

for any positive integers p_i, q_i satisfying

$$\frac{1}{p_i}+\frac{1}{q_i}=\frac{1}{2}, \qquad i=1,2.$$

Set $\delta_0=\min\big\{\frac{2\alpha-1}{2\alpha+1},\gamma-1\big\}$ and let $\epsilon\in(0,\delta_0)$. Since $3/4<\alpha\le 1$ and $\gamma\ge 2\alpha$, we have

$$\frac{2\alpha-1}{2\alpha+1}\le\frac{2\gamma+4\alpha-5}{4\alpha+2}.$$

Therefore

$$\epsilon<\frac{2\gamma+4\alpha-5}{4\alpha+2}.$$

Hence, it is easy to check that there exists a pair (p_1,q_1) of positive integers such that $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{2}$, and

$$\begin{aligned}&\frac{1}{p_1}\geq\frac{1}{2}-\frac{\gamma}{3},\\&\frac{1}{q_1}\geq\frac{1}{2}-\frac{\gamma-\epsilon-1-\sigma}{3},\\&\sigma+1\leq\gamma-\epsilon.\end{aligned}$$

By the Sobolev inequality, we have

$$\|u\|_{L^{p_1}}\leq C\|u\|_{\mathbb{H}^\gamma},\qquad\|\nabla w\|_{\mathbb{H}^{\sigma,q_1}}\leq C\|w\|_{\mathbb{H}^{\gamma-\epsilon}},$$

and thus

$$\|u\|_{L^{p_1}}\|\nabla w\|_{\mathbb{H}^{\sigma,q_1}}\leq C\|u\|_{\mathbb{H}^\gamma}\|w\|_{\mathbb{H}^{\gamma-\epsilon}}. \tag{5.14}$$

On the other hand, since

$$\epsilon<\frac{2\gamma+4\alpha-5}{4\alpha+2},\qquad\epsilon<\gamma-1,$$

there exists a pair (p_2,q_2) of positive integers such that $\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{2}$, and

$$\begin{aligned}&\frac{1}{p_2}\geq\frac{1}{2}-\frac{\gamma-\sigma}{3},\\&\frac{1}{q_2}\geq\frac{1}{2}-\frac{\gamma-\epsilon-1}{3},\\&\gamma-\epsilon>1.\end{aligned}$$

By the Sobolev inequality, we have

$$\|u\|_{\mathbb{H}^{\sigma,p_2}}\leq C\|u\|_{\mathbb{H}^\gamma},\qquad\|\nabla w\|_{L^{q_2}}\leq C\|w\|_{\mathbb{H}^{\gamma-\epsilon}},$$

and hence

$$\|u\|_{\mathbb{H}^{\sigma,p_2}}\|\nabla w\|_{L^{q_2}}\leq C\|u\|_{\mathbb{H}^\gamma}\|w\|_{\mathbb{H}^{\gamma-\epsilon}}. \tag{5.15}$$

Combining (5.13)–(5.15), we obtain

$$S_1\leq C\|u\|_{\mathbb{H}^\gamma}\|w\|_{\mathbb{H}^{\gamma-\epsilon}} \tag{5.16}$$

when $\epsilon\in(0,\delta_0)$. Similarly, for S_2 we have

$$S_2\leq C\|v\|_{\mathbb{H}^\gamma}\|w\|_{\mathbb{H}^{\gamma-\epsilon}} \tag{5.17}$$

when $\epsilon\in(0,\delta_0)$. For S_3, using Lemma 2.2, the product estimate and the Sobolev inequality, it is easy to deduce that

(5.18)

$$S_3\leq C\frac{F_N(\|\nabla u\|)F_N(\|\nabla v\|)}{N}\|\nabla w\|\,\|v\|_{\mathbb{H}^\gamma}\|v\|_{\mathbb{H}^{\gamma-\epsilon}}\leq C\|v\|_{\mathbb{H}^\gamma}^2\|w\|_{\mathbb{H}^{\gamma-\epsilon}}$$

when $\epsilon \in (0, \delta_0)$. Now combining (4.19) and (5.16)–(5.18), we get

$$(S_1 + S_2 + S_3) \le C(\|u\|_{\mathbb{H}^\gamma} + \|v\|_{\mathbb{H}^\gamma} + \|v\|^2_{\mathbb{H}^\gamma})\|w\|_{\mathbb{H}^{\gamma-\epsilon}} \le C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\|w\|_{\mathbb{H}^{\gamma-\epsilon}}. \tag{5.19}$$

Now set $\epsilon_0 = \frac{1}{2}\min\{\frac{2\alpha-1}{2\alpha+1}, \gamma - 1, 2\vartheta\}$. Inserting (5.19) into (5.11) and using (4.19) and (5.4) (note that $s - 1 + 2\alpha - \vartheta \le \gamma - \epsilon_0 \le s - 1 + 2\alpha$), we obtain

$$\begin{aligned}\|w(t)\|_{\mathbb{H}^\gamma} &\le Ce^{-\nu\lambda_1^{2\alpha}t/2}\|w(0)\|_{\mathbb{H}^\gamma} \\ &\quad + C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\int_0^t e^{-\nu\lambda_1^{2\alpha}(t-\tau)/2}(t-\tau)^{-1+\epsilon_0}\|w(\tau)\|_{\mathbb{H}^{\gamma-\epsilon_0}}\,d\tau \\ &\le Ce^{-\nu\lambda_1^{2\alpha}t/2}\|w(0)\|_{\mathbb{H}^\gamma} \\ &\quad + C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}}\int_0^t e^{-\nu\lambda_1^{2\alpha}(t-\tau)/2}(t-\tau)^{-1+\epsilon_0}\,d\tau \\ &\le Ce^{-\nu\lambda_1^{2\alpha}t/2}\|w(0)\|_{\mathbb{H}^\gamma} + \Gamma(\epsilon_0)\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}},\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. This completes the proof of Lemma 5.5. ∎

Proof of Theorem 5.1. Let $u(t), v(t)$ be two solutions of system (1.2) with initial conditions $u_0, v_0 \in \mathcal{A}$ respectively and set $w = u - v$. It follows from Lemmas 5.2 and 5.5 that

$$\begin{aligned}\|w(t)\|_{\mathbb{H}^\gamma} &\le Ce^{-\nu\lambda_1^{2\alpha}t/2}\|w(0)\|_{\mathbb{H}^\gamma} + C\nu^{-\epsilon_0}(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}}, \\ &\le (Ce^{-\nu\lambda_1^{2\alpha}t/2} + \varepsilon\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t))\|w(0)\|_{\mathbb{H}^\gamma} \\ &\quad + \nu^{-\epsilon_0}C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t)\|P_m w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}}\end{aligned} \tag{5.20}$$

where $m = m(\varepsilon) = [\varepsilon^{-1/\epsilon_0}]$ and $\epsilon_0 = \frac{1}{2}\min\{\frac{2\alpha-1}{2\alpha+1}, \gamma - 1, 2\vartheta\}$.

Now set

$$Ce^{-\nu\lambda_1^{2\alpha}t_0/2} = \tfrac{1}{4}, \quad \text{i.e.,} \quad t_0 = \frac{2\ln 4C}{\nu\lambda_1^{2\alpha}},$$

and

$$\varepsilon\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})\widetilde{C}(t_0) = \tfrac{1}{4},$$

i.e.,

$$\varepsilon = \tfrac{1}{4}\nu^{\epsilon_0}C^{-1}(C_{\mathcal{A}}^2 + 2C_{\mathcal{A}})^{-1}\widetilde{C}(t_0))^{-1}.$$

Here

$$\begin{aligned}\widetilde{C}(t_0) = &\exp\Big\{C_0\int_0^{t_0}(\|\Lambda^{s+\alpha}u\|^2+\|\Lambda^{s+\alpha}v\|^2+1)\,d\tau\Big\}\\ \le &\exp\bigg\{2C_0\|\Lambda^s u_0\|^2+4C(N,\nu,\alpha)C_0t_0\bigg(\|u_0\|^2+\frac{\|f\|^2}{(\nu\lambda_1^{2\alpha})^2}\bigg)\\ &+\frac{4C_0t_0}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}+C_0t_0\bigg\},\end{aligned}$$

with $C(N,\nu,\alpha)=\frac{2s+2\alpha}{2\alpha-3/2}CN\big(\frac{\nu(s+\alpha)}{2CN(2s+3/2)}\big)^{-(2s+1/2)/(2\alpha-3/2)}$ (see (3.3) and (5.9)). Thanks to (5.20), we obtain

$$\|w(t_0)\|_{\mathbb{H}^\gamma}\le\tfrac{1}{2}\|w_0\|_{\mathbb{H}^\gamma}+\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2+2C_{\mathcal{A}})\widetilde{C}(t_0)\|P_m w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}}.$$

Setting

$$\begin{aligned}&M=\mathcal{A},\quad H_0=\mathbb{H}^\gamma=\mathbb{H}^{s-1+2\alpha},\\ &S_0=S(t_0):w(0)\mapsto w(t_0)\quad\text{with}\quad t_0=\frac{2\ln 4C}{\nu\lambda_1^{2\alpha}},\\ &\eta=1/2,\qquad K=\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2+2C_{\mathcal{A}})\widetilde{C}(t_0),\\ &m=[C\nu^{-1}(C_{\mathcal{A}}^2+2C_{\mathcal{A}})^{1/\epsilon_0}\widetilde{C}(t_0)^{1/\epsilon_0}],\\ &\dim P_2=0,\quad P_1=P_m\quad\text{with}\quad\dim P_m\le 8(4m^3+6m^2+8m+3),\end{aligned}$$

it is easy to check that S_0 satisfies condition (ii) of Lemma 5.1. Furthermore, thanks to (5.4), we have

$$\|w(t_0)\|_{\mathbb{H}^\gamma}\le\widetilde{C}(t_0)\|w(0)\|_{\mathbb{H}^\gamma}.$$

Hence setting $L=\widetilde{C}(t_0)$, we find that S_0 satisfies condition (i) of Lemma 5.1. Thanks to Lemma 5.1, we then obtain

$$\begin{aligned}\dim_{\mathrm{f}}\mathcal{A}\le\,&8(4m^3+6m^2+8m+3)\\ &\times\ln\{1+16\sqrt{2}(1+\widetilde{C}(t_0))\nu^{-\epsilon_0}C(C_{\mathcal{A}}^2+2C_{\mathcal{A}})\widetilde{C}(t_0)\}\big(\ln\tfrac{4}{3}\big)^{-1}.\end{aligned}$$

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