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A note on a regularity criterion for the Navier–Stokes equations

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Abstract. We show that if u is a solution of the Navier–Stokes equations in the whole three-dimensional space and $\partial_3 u \in L^p(0,T;L^q(\mathbb{R}^3))$, $T>0$, where $2/p+3/q=1+3/q$ and $q\in(3,10/3]$, then u is regular on $(0,T]$.

1. Introduction. We consider the Navier–Stokes equations in the whole three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.2}$$

$$u|_{t=0} = u_0, \tag{1.3}$$

where $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ and $p=p(x,t)$ denote the unknown velocity and pressure, $\nu>0$ is the kinematic viscosity and $u_0=u_0(x)$ is the initial velocity vector field. In what follows, we put, without loss of generality, $\nu=1$.

It is well known that for $u_0\in L^2_\sigma$ (solenoidal functions from L^2) the problem (1.1)–(1.3) has a global weak solution u satisfying the energy inequality $\|u(t)\|_2^2/2+\int_0^t\|\nabla u(\tau)\|_2^2\,d\tau\le\|u_0\|_2^2/2$ for every $t\ge0$ (see [L] or [S]). Such solutions are called *Leray solutions*. If $u_0\in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$) then the Leray solutions are regular on a time interval $(0,\delta]$, $\delta>0$. This means that $\nabla u\in L^\infty(0,\delta;L^2)$, which then implies that also $u\in L^\infty(\varepsilon,\delta;W^{k,2})$ for any $\varepsilon\in(0,\delta)$ and $k\ge0$ and $u\in C^\infty((0,\delta)\times\mathbb{R}^3)$ (see [S]). However, the following fundamental question still remains open: Are the Leray solutions regular on every time interval

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$(0,T]$, $T>0$? Although the solution of this problem seems to be beyond the scope of the present techniques, there exist plenty of results showing that the answer is affirmative if a Leray solution satisfies some additional conditions. Let us mention here the following classical regularity result known as the Prodi–Serrin conditions (see [P] and [S] for $q>3$ and [ESS] for $q=3$): a Leray solution u with initial condition $u_0\in W^{1,2}_\sigma$ is regular on $(0,T]$ if $u\in L^p(0,T;L^q)$, where $2/p+3/q=1$ and $q\in[3,\infty]$. It is well known that if u and p solve the system (1.1)–(1.3) then the same is true for the rescaled functions u_λ, p_λ, $\lambda>0$, defined as

$$u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t),\qquad p_\lambda(x,t)=\lambda^2 p(\lambda x,\lambda^2 t).$$

The spaces $L^p(0,T;L^q)$, where $2/p+3/q=1$, are called *critical* since their norms are invariant with regard to the scaling, i.e. $\|u\|_{L^p(0,T;L^q)}=\|u_\lambda\|_{L^p(0,T/\lambda^2;L^q)}$ for every $\lambda>0$. In this sense the Prodi–Serrin conditions are optimal.

An analogous situation occurs for ∇u: It was proved in [B] that a Leray solution u with $u_0\in W^{1,2}_\sigma$ is regular on $(0,T]$ if $\nabla u\in L^p(0,T;L^q)$, where $2/p+3/q=2$ and $q\in(3/2,\infty)$. This result is optimal in the sense that $\|\nabla u\|_{L^p(0,T;L^q)}=\|\nabla u_\lambda\|_{L^p(0,T/\lambda^2;L^q)}$.

In the present note we focus on criteria in terms of only one directional derivative of the velocity field. Here, the optimal result was proved for $q\in(3/2,3]$ (see [KZ], [C], [Z], [NS] and [Sk]). For $q>3$, the situation is much less satisfactory; the best result proved so far can be found in [PP] with the Prodi–Serrin level being $3/2$ for $q\in(3,\infty]$. So the present state of the art is the following: u is regular on $(0,T]$ if

$$\partial_3 u\in L^p(0,T;L^q),\quad \frac{2}{p}+\frac{3}{q}\le 2,\quad \frac{3}{2}<q\le 3, \tag{1.4}$$

or

$$\partial_3 u\in L^p(0,T;L^q),\quad \frac{2}{p}+\frac{3}{q}\le \frac{3}{2},\quad 3<q\le\infty \tag{1.5}$$

(see [KZ], [C], [Z], [NS] and [Sk] for (1.4) and [PP] for (1.5)). It seems to be very difficult at present to prove an optimal result for $q>3$. Nevertheless, in this short note we make a step in this direction and close the gap for $q\to 3_+$. More precisely, our main result reads:

THEOREM 1.1. *Let $u=(u_1,u_2,u_3)$ be a weak solution to (1.1)–(1.3) corresponding to the initial condition $u_0\in W^{1,2}_\sigma$ and satisfying the energy inequality. Then u is regular on $(0,T]$, $T>0$, if*

$$\partial_3 u\in L^p(0,T;L^q),\qquad \frac{2}{p}+\frac{3}{q}\le 1+\frac{3}{q},\qquad 3<q\le\frac{10}{3}. \tag{1.6}$$

REMARK. Notice that $\lim_{q\to 3_+}(1+3/q)=2$.

Remark. Evidently, the condition (1.6) can be written as

$$\partial_3 u \in L^2(0,T;L^q), \qquad 3 < q \le 10/3.$$

Throughout the text we write for simplicity $\int f$ instead of $\int_{\mathbb{R}^3} f(x)\,dx$, L^p instead of $L^p(\mathbb{R}^3)$ and C_0^∞ instead of $C_0^\infty(\mathbb{R}^3)$. We define the Bochner spaces $L^t(0,T;X)$, where X is a Banach space, in the standard way (see [RRS]). We use standard Sobolev spaces. We do not differentiate between scalar and tensor functions and write, for example, $u = (u_1, u_2, u_3) \in L^2$ instead of $u = (u_1, u_2, u_3) \in (L^2)^3$. We denote $u_h = (u_1, u_2)$ and $\nabla_h = (\partial_1, \partial_2)$.

The following lemmas will be useful in the proof of Theorem 1.1. For the proof of Lemma 1.3 see [Z]. Lemma 1.2 can be proved using the same technique (see also [T]).

Lemma 1.2 (Troisi inequality). *Suppose that $r, p_1, p_2, p_3 \in (1,\infty)$ and $1 + 3/r = \sum_{i=1}^3 1/p_i$. Then there exists a constant $c > 0$ such that for every $f \in C_0^\infty$,*

$$\|f\|_r \le c \prod_{i=1}^3 \|\partial_i f\|_{p_i}^{1/3}.$$

Lemma 1.3. *Let $q \in [1,\infty)$ and $\lambda \in (0,\infty)$. Then there exists a constant $c > 0$ such that for every $v \in C_0^\infty$,*

$$\|v\|_{(2\lambda+1)q} \le c\|\partial_i v\|_q^{\frac{1}{2\lambda+1}} \left\|\partial_j |v|^\lambda\right\|_2^{\frac{1}{2\lambda+1}} \left\|\partial_k |v|^\lambda\right\|_2^{\frac{1}{2\lambda+1}}, \tag{1.7}$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$.

2. Proof of Theorem 1.1

Proof. Let $T^* = \sup\{\tau > 0;\ u \text{ is regular on } (0,\tau)\}$. Since $u_0 \in W_\sigma^{1,2}$, u is regular on a positive time interval and T^* is either equal to infinity (in which case the proof is finished) or it is a positive number and u is regular on $(0,T^*)$, that is, $\nabla u \in L^\infty_{\mathrm{loc}}([0,T^*);L^2)$. It is sufficient to prove that $T^* > T$.

Suppose that $T^* \le T$. We take $\delta > 0$ sufficiently small (it will be specified later) and fix $T_1 \in (0,T^*)$ such that $\|\partial_3 u\|_{L^2(T_1,T^*;L^q)} < \delta$. Fix $T_2 \in (T_1,T^*)$. The proof will be finished if we show that $\|\nabla u(T_2)\|_2 \le c < \infty$, where c is independent of T_2. Indeed, the standard extension argument then shows that the regularity of u can be extended beyond T^*, contrary to the definition of T^*.

Suppose that $q \in (3, 10/3)$ and

$$\frac{11q}{28-4q} \le \lambda \le \min\left(\frac{11q^2+18q}{-4q^2+22q+36}, \frac{q}{q-2}\right). \tag{2.1}$$

This condition will be used in most of the following estimates without further mention.

We now define

$$\mathcal{J}(T_2) = \max_{t\in[T_1,T_2]} (\|\nabla u_h\|_2^2 + \|\partial_3 u\|_2^2) + \int_{T_1}^{T_2} (\|\Delta u_h\|_2^2 + \|\nabla\partial_3 u\|_2^2)\, dt, \tag{2.2}$$

$$\mathcal{L}(T_2) = \max_{t\in[T_1,T_2]} \||u_3|^\lambda\|_2^2 + \int_{T_1}^{T_2} \|\nabla(|u_3|^\lambda)\|_2^2\, dt, \qquad \lambda > 0. \tag{2.3}$$

To prove regularity of u on $(0, T]$ it is now sufficient to show that $\mathcal{J} = \mathcal{J}(T_2)$ is bounded above on (T_1, T^*). In the following estimates, c will denote a generic constant which may change from line to line and is independent of T_2.

The estimate of $\mathcal{J}$. For $i = 1, 2$ we multiply the ith equation of (1.1) by $-\Delta u_i$ and integrate over the whole space. Similarly, for $j = 1, 2, 3$ we multiply the jth equation of (1.1) by $-\partial_3\partial_3 u_j$ and integrate over the whole space. Summing now all the equations over $i = 1, 2$ and $j = 1, 2, 3$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u_h\|_2^2 + \|\partial_3 u\|_2^2) + \|\Delta u_h\|_2^2 + \|\nabla\partial_3 u\|_2^2 \\ = \int [(u\cdot\nabla)u_h]\cdot\Delta u_h + \int \nabla_h p\cdot\Delta u_h + \int [(u\cdot\nabla)u]\cdot\partial_3\partial_3 u \\ \le c\int |\partial_3 u|\,|\nabla u_h|^2 + c\int |u_3|\,|\partial_3 u|(|\partial_3\nabla u| + |\Delta u_h|) =: J_1 + J_2,\end{aligned} \tag{2.4}$$

where the last inequality was proved in [Z]. By the Hölder and interpolation inequalities,

$$J_1 \le c\|\partial_3 u\|_q \|\nabla u_h\|_2^{(2q-3)/q} \|\Delta u_h\|_2^{3/q}. \tag{2.5}$$

Further, by the Hölder inequality, $J_2 \le c\|u_3\|_{2\lambda}\|\partial_3 u\|_{\frac{2\lambda}{\lambda-1}}(\|\Delta u_h\|_2 + \|\partial_3\nabla u\|_2)$ and using the interpolation inequality we get

$$J_2 \le c\||u_3|^\lambda\|_2^{1/\lambda} \|\partial_3 u\|_q^{\frac{q(2\lambda-3)}{\lambda(6-q)}} (\|\Delta u_h\|_2 + \|\partial_3\nabla u\|_2)^{\frac{6\lambda-3\lambda q+3q}{\lambda(6-q)}+1}. \tag{2.6}$$

Integrating over time we obtain from (2.4)–(2.6)

$$\begin{aligned}\mathcal{J}(T_2)^2 &\le c + c\int_{T_1}^{T_2} \|\partial_3 u\|_q \|\nabla u_h\|_2^{(2q-3)/q} \|\Delta u_h\|_2^{3/q}\, dt \\ &\quad + c\int_{T_1}^{T_2} \||u_3|^\lambda\|_2^{1/\lambda} \|\partial_3 u\|_q^{\frac{q(2\lambda-3)}{\lambda(6-q)}} (\|\Delta u_h\|_2 + \|\partial_3\nabla u\|_2)^{\frac{12\lambda-4\lambda q+3q}{\lambda(6-q)}}\, dt \\ &\le c + c\delta\mathcal{J}(T_2)^2 + c\mathcal{L}(T_2)^{1/\lambda}\Big(\int_{T_1}^{T_2} \|\partial_3 u\|_q^2\, dt\Big)^{\frac{q(2\lambda-3)}{2\lambda(6-q)}} \\ &\quad \times \Big(\int_{T_1}^{T_2} (\|\Delta u_h\|_2^2 + \|\partial_3\nabla u\|_2^2)\, dt\Big)^{\frac{12\lambda-4\lambda q+3q}{2\lambda(6-q)}}.\end{aligned}$$

Consequently,

$$\mathcal{J}(T_2)^2 \le c + c\delta\mathcal{J}(T_2)^2 + c\delta^{\frac{q(2\lambda-3)}{\lambda(6-q)}}\mathcal{L}(T_2)^{1/\lambda}\mathcal{J}(T_2)^{\frac{12\lambda-4\lambda q+3q}{\lambda(6-q)}}$$

and

$$\mathcal{J}(T_2) \le c + c\delta^{\frac{q(2\lambda-3)}{\lambda(6-q)}}\mathcal{L}(T_2)^{\frac{6-q}{q(2\lambda-3)}}. \tag{2.7}$$

The estimate of $\mathcal{L}$. Multiplying the 3rd equation of (1.1) by $|u_3|^{2\lambda-2}u_3$ and integrating over the whole space we get

$$\frac{d}{dt}\big\||u_3|^\lambda\big\|_2^2 + \frac{4\lambda-2}{\lambda}\big\|\nabla|u_3|^\lambda\big\|_2^2 = (-2\lambda)\int\partial_3 p\,|u_3|^{2\lambda-2}u_3 =: L. \tag{2.8}$$

By the standard procedure one gets

$$-\Delta\partial_3 p = 2\sum_{i,j=1}^{3}\partial_i\partial_j(u_i\partial_3 u_j) = 2\sum_{i=1}^{2}\sum_{j=1}^{3}\partial_i\partial_j(u_i\partial_3 u_j) + 2\sum_{j=1}^{3}\partial_3\partial_j(u_3\partial_3 u_j)$$

and so we have the following decomposition of $\partial_3 p$:

$$\partial_3 p = 2\sum_{i=1}^{2}\sum_{j=1}^{3}R_iR_j(u_i\partial_3 u_j) + 2\sum_{j=1}^{3}R_3R_j(u_3\partial_3 u_j) =: p_1 + p_2,$$

where $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ is the Riesz transform. Thus,

$$\begin{aligned} L &= (-2\lambda)\int\partial_3 p\,|u_3|^{2\lambda-2}u_3 = (-2\lambda)\int(p_1+p_2)|u_3|^{2\lambda-2}u_3 \\ &\le c\|u_h\|_a\|\partial_3 u\|_q\|u_3\|_b^{2\lambda-1} + c\|u_3\|_{\frac{2\lambda q}{q-1}}\|\partial_3 u\|_q\|u_3\|_{\frac{2\lambda q}{q-1}}^{2\lambda-1}, \end{aligned}$$

where

$$1 \le a, b \le \infty, \qquad \frac{1}{a} + \frac{1}{q} + \frac{2\lambda-1}{b} = 1. \tag{2.9}$$

By the Gagliardo–Nirenberg and interpolation inequalities,

$$\begin{aligned} L &\le c\|u_h\|_{3q}^{1-\alpha_3}\|\Delta u_h\|_2^{\alpha_3}\|\partial_3 u\|_q\|u_3\|_{2\lambda}^{(2\lambda-1)(1-\alpha_4)}\|u_3\|_{(2\lambda+1)q}^{(2\lambda-1)\alpha_4} \\ &\quad + c\|\partial_3 u\|_q\big\||u_3|^\lambda\big\|_{\frac{2q}{q-1}}^2, \end{aligned}$$

where

$$\begin{aligned} &\frac{1}{a} = (1-\alpha_3)\frac{1}{3q} - \frac{\alpha_3}{6}, \quad \frac{1}{b} = \frac{1-\alpha_4}{2\lambda} + \frac{\alpha_4}{(2\lambda+1)q}, \\ &\alpha_3 \in [0, 2/(q+2)], \qquad \alpha_4 \in [0,1]. \end{aligned} \tag{2.10}$$

Using further Lemmas 1.2 and 1.3 we have

$$\begin{aligned} L &\le c\|\nabla u_h\|_2^{\frac{2(1-\alpha_3)}{3}}\|\partial_3 u\|_q^{\frac{1-\alpha_3}{3}+1+\frac{(2\lambda-1)\alpha_4}{2\lambda+1}}(\|\Delta u_h\| + \|\partial_3\nabla u\|)_2^{\alpha_3} \\ &\times \big\||u_3|^\lambda\big\|_2^{\frac{(2\lambda-1)(1-\alpha_4)}{\lambda}}\big\|\nabla|u_3|^\lambda\big\|_2^{\frac{2(2\lambda-1)\alpha_4}{2\lambda+1}} + c\|\partial_3 u\|_q\big\||u_3|^\lambda\big\|_2^{(2q-3)/q}\big\|\nabla|u_3|^\lambda\big\|_2^{3/q} \end{aligned} \tag{2.11}$$

and integrating over time it follows from (2.8) and (2.11) that

$$(2.12)\qquad \mathcal{L}(T_2)^2 \le c + c\delta\mathcal{L}(T_2)^2 + c\mathcal{J}(T_2)^{\frac{2(1-\alpha_3)}{3}}\mathcal{L}(T_2)^{\frac{(2\lambda-1)(1-\alpha_4)}{\lambda}}$$
$$\times \int_{T_1}^{T_2} \|\partial_3 u\|_q^{\frac{1-\alpha_3}{3}+1+\frac{(2\lambda-1)\alpha_4}{2\lambda+1}} (\|\Delta u_h\| + \|\partial_3\nabla u\|)_2^{\alpha_3} \big\|\nabla|u_3|^\lambda\big\|_2^{\frac{2(2\lambda-1)\alpha_4}{2\lambda+1}}\,dt$$
$$\le c + c\delta\mathcal{L}(T_2)^2 + c\delta^{\frac{1-\alpha_3}{3}+1+\frac{(2\lambda-1)\alpha_4}{2\lambda+1}}\mathcal{J}(T_2)^{\frac{2(1-\alpha_3)}{3}+\alpha_3}$$
$$\times \mathcal{L}(T_2)^{\frac{(2\lambda-1)(1-\alpha_4)}{\lambda}+\frac{2(2\lambda-1)\alpha_4}{2\lambda+1}}.$$

The condition for the Hölder inequality in the last estimate is

$$(2.13)\qquad \frac{1}{2}\left[\frac{1-\alpha_3}{3}+1+\frac{(2\lambda-1)\alpha_4}{2\lambda+1}\right]+\frac{\alpha_3}{2}+\frac{(2\lambda-1)\alpha_4}{2\lambda+1}=1$$

and it follows from (2.9) and (2.10) that

$$(2.14)\qquad (1-\alpha_3)\frac{1}{3q}-\frac{\alpha_3}{6}+\frac{1}{q}+(2\lambda-1)\left[\frac{1-\alpha_4}{2\lambda}+\frac{\alpha_4}{(2\lambda+1)q}\right]=1.$$

Solving the equations (2.13) and (2.14) we get the following solution:

$$(2.15)\qquad \alpha_3=\frac{4\lambda q-28\lambda+11q}{\lambda q-10\lambda+2q},\qquad \alpha_4=\frac{(2\lambda+1)(-2\lambda q-6q+12\lambda)}{3(2\lambda-1)(\lambda q-10\lambda+2q)}.$$

Using (2.1), one can verify that $\alpha_3 \in [0, 2/(q+2)]$ and $\alpha_4 \in [0,1]$.

Having now α_3 and α_4 one infers from (2.7) and (2.12) that

$$\mathcal{L}(T_2) \le c + c\tilde{\delta}\mathcal{L}(T_2)^{\frac{(6-q)(6\lambda q-48\lambda+15q)}{3q(2\lambda-3)(\lambda q-10\lambda+2q)}}\mathcal{L}(T_2)^{\frac{6\lambda q-60\lambda+11q+18}{3(\lambda q-10\lambda+2q)}},$$

where $\tilde{\delta}$ is a positive power of δ. Consequently,

$$(2.16)\qquad \mathcal{L}(T_2) \le c + c\tilde{\delta}\mathcal{L}(T_2)^{\frac{12\lambda^2q^2-120\lambda^2q-2\lambda q^2+300\lambda q-48q^2+36q-288\lambda}{3q(2\lambda-3)(\lambda q-10\lambda+2q)}}.$$

One can verify that the power of $\mathcal{L}(T_2)$ in (2.16) is ≤ 2. So choosing δ sufficiently small, it follows from (2.16) that $\mathcal{L}(T_2)$ is bounded independently of T_2 and consequently $\mathcal{J}(T_2)$ is bounded as follows from (2.7). This implies the regularity of u on $(0, T]$. ■

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