STUDIA MATHEMATICA
Online First version

Semi-Fredholm Toeplitz operators on the space of real analytic functions

by

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Abstract. We investigate Toeplitz operators on the space of real analytic functions on the real line. We completely characterize semi-Fredholm Toeplitz operators in terms of vanishing of their symbol.

1. Introduction. The theory of continuous linear operators on locally convex spaces is much less developed than the Hilbert or Banach space counterpart. There are prominent exceptions such as differential operators or convolution operators. However, much less is known about both concrete continuous linear operators on concrete locally convex spaces and about abstract operators. This paper is part of a project to investigate operators on the space of real analytic functions on the real line.

Let us recall that a function $f : \mathbb{R} \to \mathbb{C}$ is called real analytic if it develops locally near each point $x \in \mathbb{R}$ into a convergent power series (we emphasize that a real analytic function can be complex valued). The space $\mathcal{A}(\mathbb{R})$ of all real analytic functions on the real line is not a Banach space, it is not even metrizable. Importantly, it does not have a Schauder basis, as shown by Domański and Vogt [14]. On the other hand, it carries a natural locally convex topology and such fundamental tools of functional analysis as the open mapping theorem/closed graph theorem or the uniform boundedness principle are available. We briefly recall basic properties of $\mathcal{A}(\mathbb{R})$ in Section 2.

Real analytic functions appear in many areas of mathematics. We just mention here such fundamental results as the Cauchy–Kovalevskaya theorem or Holmgren's uniqueness theorem [30]. A function which is real analytic on $\mathbb{R}$ is the restriction of a function holomorphic on some complex neighborhood

2010 *Mathematics Subject Classification*: Primary 47B35, 46E10; Secondary 47A53, 30H10, 30H50, 46A13.
Key words and phrases: Toeplitz operator, real analytic, Fredholm, semi-Fredholm, Cauchy transform.
Received 10 August 2017; revised 7 May 2019.
Published online 20 December 2019.

DOI: 10.4064/sm170810-23-5

of $\mathbb{R}$. In other words, a real analytic function is a germ on $\mathbb{R}$ of a holomorphic function. Somewhat imprecisely we may say that we investigate the space of holomorphic functions on which we impose only one condition: their domain must contain the real line. Thus the domain depends on the function. This is different from classical function spaces like the Hardy, Bergman or Fock spaces, where the domain is fixed and common for all functions in the space considered. We mention these particular spaces here since on them one can define Toeplitz operators, which are the main object of study in this paper.

We call a continuous linear operator $T\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ a *Toeplitz operator* if on monomials its matrix is a Toeplitz matrix

$$M_T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix}, \tag{1.1}$$

with $\ldots, a_{-1}, a_0, a_1, \ldots \in \mathbb{C}$. More precisely, a continuous linear operator $T\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Toeplitz operator if locally near zero

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots \tag{1.2}$$

for some complex numbers $a_n, n \in \mathbb{Z}$. We emphasize that one should be careful with the matrix interpretation of a Toeplitz operator since, as already mentioned, the monomials do not form a Schauder basis in $\mathcal{A}(\mathbb{R})$. We shall say that the matrix M_T is *associated* with the operator T. We give examples of Toeplitz operators on $\mathcal{A}(\mathbb{R})$, also coming from interpolation theory, in Section 3. Note that condition (1.2) determines the operator, since the polynomials are dense in $\mathcal{A}(\mathbb{R})$.

In [8] we completely described this class of operators. We proved that an operator $T\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Toeplitz operator if and only if it is of the form

$$T = T_F = \mathcal{C} M_F, \tag{1.3}$$

where $\mathcal{C}\colon \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is the Cauchy transform (for definition see Section 3) and M_F is the operator of multiplication by $F \in \mathcal{X}(\mathbb{R})$. The space $\mathcal{X}(\mathbb{R})$ is defined as the inductive limit of the spaces $H(U \setminus K)$, where U runs through complex neighborhoods of $\mathbb{R}$ and K runs through compact subsets of $\mathbb{R}$:

$$\mathcal{X}(\mathbb{R}) := \operatorname{lim\,ind} H(U \setminus K).$$

We give the details of this construction in Section 3. Any $\tilde{F} \in H(U \setminus K)$ defines an element of $\mathcal{X}(\mathbb{R})$. Two such functions $F_i \in H(U_i \setminus K_i)$, $i = 1, 2$, for possibly different open sets U_i and compact sets K_i, define the same element in $\mathcal{X}(\mathbb{R})$ if they coincide as germs of functions, that is, there is an open set $U \subset U_1 \cap U_2$ containing $\mathbb{R}$ and a compact set $K \supset K_1 \cup K_2$ contained in $\mathbb{R}$

such that $F_1|_{U\setminus K} = F_2|_{U\setminus K}$. In this case, F_1 and F_2 define the same Toeplitz operator. We will denote by $[\tilde{F}]_\sim$ the element of $\mathcal{X}(\mathbb{R})$ that corresponds to $\tilde{F} \in H(U \setminus K)$.

It is natural to think about $\mathcal{X}(\mathbb{R})$ as the symbol space and explain properties of the operators T_F in terms of their symbols F. We shall discuss the details of the definition of Toeplitz operators on $\mathcal{A}(\mathbb{R})$ and examples which motivate our study in Section 3.

Recall that a Toeplitz operator $\mathcal{T}\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ on the Hardy space $H^2(\mathbb{T})$ on the unit circle is necessarily of the form

$$\mathcal{T}_\phi := PM_\phi, \tag{1.4}$$

where $P\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the Riesz projection and $M_\phi\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is multiplication by $\phi \in L^\infty(\mathbb{T})$. This fundamental result of Brown and Halmos [6] marked the beginning of the theory of Toeplitz operators. We refer the reader to [5] or [28] for excellent expositions of the theory of Toeplitz operators on Hardy spaces.

The methods used for the space of real analytic functions are completely different from those for Hardy spaces (in particular, this cannot be a metric theory). However, there is a rather surprising similarity between formulas (1.3) and (1.4). The similarity goes even further. In [8] we provided a complete characterization of Fredholm Toeplitz operators on $\mathcal{A}(\mathbb{R})$. Recall that an operator $\mathcal{T}_\phi\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ with ϕ continuous is a Fredholm operator if and only if ϕ does not vanish on $\mathbb{T}$. This classical result can be traced back to the work of Gohberg [18], Krein [23], Noether [29] among others (see [27] for some recent developments). Our characterization of Fredholm Toeplitz operators is surprisingly similar:

Theorem 1.1. *A Toeplitz operator $T_F\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with $F \in \mathcal{X}(\mathbb{R})$ is a Fredholm operator if and only if F is the equivalence class in $\mathcal{X}(\mathbb{R}) =$ $\operatorname{lim\,ind} H(U \setminus K)$ of a function $\tilde{F} \in H(U \setminus K)$, where U is an open simply connected neighborhood of $\mathbb{R}$ and K a compact subset of $\mathbb{R}$, such that $\tilde{F}$ does not vanish in $U \setminus K$. In this case,*

$$\operatorname{index} T_F = -\operatorname{winding} \tilde{F}.$$

If $\tilde{F} \in H(U \setminus K)$ does not vanish in $U \setminus K$, as in Theorem 1.1, then $\operatorname{winding} \tilde{F}$ is defined as the index of $\tilde{F} \circ \gamma$ with respect to 0, i.e.

$$\operatorname{winding} \tilde{F} = \operatorname{Ind}_{\tilde{F}\circ\gamma}(0) = \frac{1}{2\pi i} \int_{\tilde{F}\circ\gamma} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta. \tag{1.5}$$

The curve γ in (1.5) is any C^∞ Jordan curve in $U \setminus K$ such that K is in the interior of γ. Recall that by Jordan's theorem the complement of a C^∞

Jordan curve γ in $\mathbb{C}$ consists of two open connected sets, $I(\gamma)$ and $E(\gamma)$, the latter unbounded, with the common boundary γ. The set $I(\gamma)$ is called the *interior* of γ and $E(\gamma)$ the *exterior*. We shall use this notation frequently. Since U is simply connected (in particular connected) one easily checks that the definition (1.5) is correct: it does not depend on γ or on the representative $\tilde{F}$ of F. In this situation, we shall say that γ *surrounds* K (see Theorem 3.2 below).

Recall that an operator $A\colon X \to X$, where X is a Banach space, is a *Fredholm operator* if it is both a Φ_+- and Φ_--operator. An operator $A\colon X \to X$ is a *Φ_+-operator* if $\dim\ker A < \infty$ and the range of A is closed; and it is a *Φ_--operator* if $\operatorname{codim}\operatorname{im} A < \infty$. We adopt the same definitions for operators on $\mathcal{A}(\mathbb{R})$. We remark that a Φ_--operator on a Banach space necessarily has closed range. The same is true for operators on $\mathcal{A}(\mathbb{R})$, by the open mapping theorem (see [8, Proposition 5.1]).

Consider now a function $\tilde{F}$ holomorphic in some set $U \setminus K$, where U is an open set containing $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. By the definition of $\mathcal{X}(\mathbb{R})$ the same function considered on $\tilde{U} \setminus \tilde{K}$, where $\mathbb{R} \subset \tilde{U} \subset U$, $K \subset \tilde{K} \subset\subset \mathbb{R}$, defines the same equivalence class in $\mathcal{X}(\mathbb{R})$ and hence the same Toeplitz operator (cf. Proposition 3.1). In view of Theorem 1.1 this implies that there are precisely two situations when a Toeplitz operator $T_F\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, where F is the equivalence class of some $\tilde{F} \in H(U \setminus K)$, fails to be a Fredholm operator: either there exists a sequence $(z_n) \subset \mathbb{C} \setminus \mathbb{R}$ which accumulates at some $z \in K$ such that $\tilde{F}(z_n) = 0$ for all n, or there exists a sequence $(x_n) \subset \mathbb{R}$ which accumulates at $\pm\infty$ such that $\tilde{F}(x_n) = 0$ for all n. It turns out that these conditions govern the $\Phi_\pm$-properties as well.

Let $F \in \mathcal{X}(\mathbb{R})$ and let $\tilde{F} \in H(U \setminus K)$ be its representative: $F = [\tilde{F}]_\sim$.

(i) We say that F *has real zeros going to infinity* if there are $x_n \in \mathbb{R}$ such that $\lim |x_n| = \infty$ and $\tilde{F}(x_n) = 0$ for all n.
(ii) We say that F *has non-real zeros accumulating at a real point* if $\tilde{F}$ has zeros $z_n \notin \mathbb{R}$ whose limit $\lim z_n$ exists and belongs to $\mathbb{R}$.

We observe that these two properties depend only on the germ $F \in \mathcal{X}(\mathbb{R})$ and not on the choice of K, U or $\tilde{F} \in H(U \setminus K)$. The set of zeros of $\tilde{F}$ is discrete in $U \setminus K$. It follows that, given a non-zero germ $F \in \mathcal{X}(\mathbb{R})$, one can find U, K and a representative $\tilde{F} \in H(U \setminus K)$ of F that has no zeros in $U \setminus K$ if and only if $F \in \mathcal{X}(\mathbb{R})$ has neither real zeros going to infinity nor non-real zeros accumulating at a real point.

We can now formulate our main results:

Main Theorem 1.2. *A Toeplitz operator $T_F\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, where $F \in \mathcal{X}(\mathbb{R})$, is a Φ_+-operator if and only if F has no non-real zeros accumulating at a real point.*

Main Theorem 1.3. *A Toeplitz operator $T_F\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, where $F \in \mathcal{X}(\mathbb{R})$, is a Φ_--operator if and only if F has no real zeros going to infinity.*

It is natural to ask about the meaning of the assumptions of Theorem 1.2 for $F \in \mathcal{X}(\mathbb{R})$, which is the equivalence class of $\tilde{F} \in H(U)$ for some open neighborhood U of $\mathbb{R}$. Such a function belongs to $H(U \setminus K)$ for any $K \subset\subset \mathbb{R}$ and cannot have non-real zeros accumulating at a real point. We therefore have the following result:

Corollary 1.4. *Assume that $F \in \mathcal{X}(\mathbb{R})$ is the equivalence class of a function $\tilde{F} \in H(U)$, for some open neighborhood U of $\mathbb{R}$, which does not vanish identically. Then T_F is a Φ_+-operator. It is a Φ_--operator if and only if there are no real zeros of F going to infinity, that is, $\tilde{F}$ has only finitely many real zeros.*

This corollary is hardly surprising and can be proved by a direct argument. Observe that T_F is now just multiplication by F. Let x_n be the real zeros of $\tilde{F} \in H(U)$. Then

$$\operatorname{im} T_F = \{f \in \mathcal{A}(\mathbb{R})\colon f(x_n) = \cdots = f^{(m_n-1)}(x_n) = 0,\, n \in \mathbb{N}\},$$

where $m_n \in \mathbb{N}$ is the multiplicity of x_n as a zero of $\tilde{F}$. Thus the image is closed as the intersection of the kernels of continuous functionals. This space is of finite codimension if and only if there are only a finite number of zeros x_n. Naturally, the operator $f \mapsto \tilde{F} f$ is injective.

The following statement can be seen as a dual result.

Corollary 1.5. *Assume that $F \in \mathcal{X}(\mathbb{R})$ is the equivalence class of a function $\tilde{F} \in H_0(\mathbb{C}_\infty \setminus K)$ for some compact set $K \subset \mathbb{R}$ which does not vanish identically. Then T_F is a Φ_--operator. It is a Φ_+-operator if and only if there are no non-real zeros of $\tilde{F}$ accumulating at a real point, that is, there exists an open set $V \supset K$ such that in $V \setminus K$ there are only a finite number of zeros of $\tilde{F}$ which are not real.*

Corollary 1.5 is a consequence of Theorem 1.3. Indeed, if $\tilde{F} \in H_0(\mathbb{C}_\infty \setminus K)$ for a compact set $K \subset \mathbb{R}$, then $\tilde{F}$ does not vanish outside some large disc. We explain the relation of $\mathcal{X}(\mathbb{R})$ to these two specific cases in Section 3.

Recall that Douglas and Sarason [16] (see also [5, Theorem 2.75]) proved that the operator $\mathcal{T}_\phi\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ with $\phi \in L^\infty$ unimodular is a Φ_+-operator if and only if

$$\operatorname{dist}_{L^\infty}(\phi, C + H^\infty) < 1, \tag{1.6}$$

and $\mathcal{T}_\phi$ is a Φ_--operator if and only if

$$\operatorname{dist}_{L^\infty}(\phi, C + \overline{H^\infty}) < 1. \tag{1.7}$$

It seems interesting that our characterization is completely different. This may suggest that there exists a different characterization of the classes $\Phi_\pm$ in the spirit of (1.6) and (1.7) also for operators on $\mathcal{A}(\mathbb{R})$.

Together with our previous results we obtain the following characterization of the Toeplitz operators T_F.

Main Theorem 1.6. *Assume that $F \in \mathcal{X}(\mathbb{R})$ does not vanish identically. Consider the Toeplitz operator T_F.*

(i) *If F has no real zeros going to infinity and has no non-real zeros accumulating at a real point, then the kernel of T_F is finite-dimensional, the cokernel is finite-dimensional and the image of T_F is closed.*
(ii) *If F has non-real zeros accumulating at a real point, but has no real zeros going to infinity, then T_F is surjective and the kernel of T_F is of infinite dimension.*
(iii) *If F has real zeros going to infinity, but has no non-real zeros accumulating at a real point, then T_F is injective and its range is a closed subspace of infinite codimension.*
(iv) *If F has both real zeros going to infinity and non-real zeros accumulating at a real point, then T_F is injective and its range is a dense subspace of infinite codimension.*

For any $F \in \mathcal{X}(\mathbb{R})$ one of the above four cases holds true.

In a series of papers [9]–[12] Domański and Langenbruch and Domański, Langenbruch and Vogt [13] built a surprisingly rich theory of linear operators on $\mathcal{A}(\mathbb{R})$ whose associated matrix is diagonal. Such operators are called Hadamard multipliers and their theory provides the correct language to analyze the Euler differential equation. This motivated us to consider linear operators on $\mathcal{A}(\mathbb{R})$ whose associated matrix is a Toeplitz matrix. We give another motivation and examples in Section 3. As already stated, we completely characterized such operators in [8]. We also described Fredholm operators therein. Building on these results we proved in [20] a Coburn–Simonenko type theorem for operators T_F and invertibility criteria.

In [8] we introduced a method of investigating Toeplitz operators on $\mathcal{A}(\mathbb{R})$ by means of Toeplitz operators on the Hardy spaces on smooth Jordan curves. We develop and use these arguments in this paper as well. We strive to make the paper self-contained and accessible not only to experts. That is why we repeat some information concerning real analytic functions and Toeplitz operators which can also be found in [8] or [20]. An important part of the proof of Theorem 1.2 concerns Toeplitz operators on $H(K)$, the space of germs of holomorphic functions on a compact connected set K, i.e. a compact interval. We investigate this case in Section 5.

The paper is divided into six sections. In the next one we recall basic information on $\mathcal{A}(\mathbb{R})$. In Section 3 we provide examples of Toeplitz operators which motivate our study. Section 4 concerns our fundamental tools, i.e. the Hardy spaces on curves and the Cauchy transform. In Section 5 we investigate Toeplitz operators on $H(K)$ where K is a compact interval. This will serve as a tool in the proof of Theorem 1.2. Section 6 is devoted to the proofs of our main results.

2. The space of real analytic functions. We recall those basic properties of the space of real analytic functions on the real line which are relevant for our results. We refer the reader to [26], [19] and [21] for the functional-analytic background. Projective and injective limits of locally convex spaces are also studied in detail in [17]. The article [3] gives an in-depth presentation of locally convex inductive limits. We also refer to [7] for an instructive introduction to real analytic functions. There is also an interesting book [22], but of less functional-analytic flavor. In our presentation we essentially follow [15]. The reader may also consult our previous paper [8].

A function $f\colon \mathbb{R} \to \mathbb{C}$ is (*real*) *analytic at a point* x_0 if there exists $\delta > 0$ such that $f(x)$ is equal for $x \in (x_0 - \delta, x_0 + \delta)$ to the sum of a convergent power series,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{2.1}$$

A function is *real analytic* on $\mathbb{R}$ if it is real analytic at each point of $\mathbb{R}$. We denote by $\mathcal{A}(\mathbb{R})$ the space of all real analytic functions on $\mathbb{R}$.

There are two ways of making $\mathcal{A}(\mathbb{R})$ a locally convex space; they are equivalent by a fundamental result of Martineau [25]. We shall describe them now.

The power series (2.1) converges for complex z such that $|z - x_0| < \delta$. This implies that $f \in \mathcal{A}(\mathbb{R})$ extends to a function holomorphic on some open complex neighborhood of $\mathbb{R}$. Two such extensions $F_1 \in H(U_1)$ and $F_2 \in H(U_2)$, U_1, U_2 open neighborhoods of $\mathbb{R}$, define the same real analytic function on $\mathbb{R}$ if and only if there exists an open set $U \supset \mathbb{R}$ such that $F_1|_U = F_2|_U$. Here $H(U)$ stands for the Fréchet space of all holomorphic functions on U with the topology of uniform convergence on compact sets. This essentially means that

$$\mathcal{A}(\mathbb{R}) \cong \operatorname{lim\,ind}\{H(U), r_{U,V}\}$$

as vector spaces, where for open neighborhoods $V \subset U$ of the real line the map $r_{U,V}\colon H(U) \to H(V)$ is restriction. We refer the reader to [17, §23] for the construction of the inductive limit of an inductive spectrum of locally convex spaces. The above algebraic isomorphism provides $\mathcal{A}(\mathbb{R})$ with the so-called *inductive topology*. This is the strongest locally convex topology such that all restrictions $r_U : H(U) \to \mathcal{A}(\mathbb{R})$, $r_U(F) = F|_{\mathbb{R}}$, are continuous. This topology is Hausdorff [26, Lemma 24.6].

We now discuss the projective topology of $\mathcal{A}(\mathbb{R})$. For this we need to recall some information concerning the space $H(K)$ of germs of holomorphic functions over a compact set $K \subset \mathbb{R}$. This will also play an important role in Section 5. Our presentation is based on [3, p. 64]. The natural topology on $H(K)$ is the inductive topology of the system $\{H(U), r_{U,V}\}$, where this time the U are open neighborhoods of K and for open $U \supset V$ which contain K, the maps $r_{U,V}$ are again restrictions. That is,

$$H(K) := \lim\operatorname{ind}\{H(U), r_{U,V}\}.$$

If F is a holomorphic function on some open neighborhood V of K, we denote by $[(F,V)]_{\sim_K}$ the germ in $H(K)$ which it determines. The inductive topology of $H(K)$ is the strongest locally convex topology which makes continuous all maps $\pi_{V,K} : H(V) \to H(K)$,

$$\pi_{V,K}(F) := [(F,V)]_{\sim_K}. \tag{2.2}$$

Every compact set $K \subset \mathbb{R}$ has a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods. Hence,

$$H(K) = \lim\operatorname{ind} H(U_n)$$

is a countable inductive limit.

Any $f \in \mathcal{A}(\mathbb{R})$ is the restriction of some $F \in H(U)$, where U is an open neighborhood of $\mathbb{R}$, i.e. $f = r_U(F)$. For any compact $K \subset \mathbb{R}$ the function F defines the germ $[(F,U)]_{\sim_K} \in H(K)$. Let $R_K : \mathcal{A}(\mathbb{R}) \to H(K)$ be the map

$$R_K : f \mapsto [(F,U)]_{\sim_K}. \tag{2.3}$$

One easily observes that it is well-defined. We equip $\mathcal{A}(\mathbb{R})$ with the projective topology of the system $\{R_K : \mathcal{A}(\mathbb{R}) \to H(K)\}_{K\subset\mathbb{R}}$, where K runs through all compact subsets of $\mathbb{R}$ [26, p. 278] (see also [17, §6]). This topology is called the *projective topology*. Equivalently, we use the fact that algebraically

$$\mathcal{A}(\mathbb{R}) \cong \lim\operatorname{proj}\{H(K), R_{K,L}\} \tag{2.4}$$

to furnish $\mathcal{A}(\mathbb{R})$ with the corresponding topology. If $[(F,U)]_{\sim_L}$ is a germ in $H(L)$ and $K \subset L$, then $R_{K,L} : H(L) \to H(K)$ is defined by restriction, i.e. $R_{K,L}([(F,U)]_{\sim_L}) = [(F,U)]_{\sim_K}$. We refer the reader to [17, §6] for the construction of the projective limit of a projective spectrum. The projective topology is the weakest topology (and not only the weakest locally convex topology, see [17, §6, Satz 1.1]) such that all maps $R_K : \mathcal{A}(\mathbb{R}) \to H(K)$ are continuous. It is easy to observe that instead of taking all compact subsets of $\mathbb{R}$ one can restrict attention to any compact countable exhaustion $\{K_n\}_{n\in\mathbb{N}}$ of the real line, for instance the intervals $\{[-n,n]\}_{n\in\mathbb{N}}$.

We have the following fundamental result:

THEOREM 2.1 (Martineau [25]). *The above two topologies on $\mathcal{A}(\mathbb{R})$ coincide: the projective topology is equal to the inductive topology.*

In [15, Theorem 1.2] the authors give a simplified proof of this result. In [7, Theorem 1.27] there is a discussion of other known proofs.

Naturally the image of $\mathcal{A}(\mathbb{R})$ under R_K is dense in $H(K)$ for any $K \subset\subset \mathbb{R}$. It follows therefore from [17, §26, Satz 1.6] that algebraically

$$\mathcal{A}(\mathbb{R})' \cong (\operatorname{lim\,proj} H(K_n))' \cong \operatorname{lim\,ind} H(K_n)' \tag{2.5}$$

where as before $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of the real line.

One can identify the dual space of $H(K)$ for a compact $K \subset \mathbb{R}$ as a space of functions. Namely, this dual, when equipped with the strong topology, is isomorphic to the Fréchet space of functions holomorphic in $\mathbb{C}_\infty \setminus K$ which vanish at infinity, i.e.

$$H(K)'_b \cong H_0(\mathbb{C}_\infty \setminus K). \tag{2.6}$$

Recall that in general, in contrast to the Banach space case, there is no distinguished topology on the dual space of a locally convex space. The index b in $H(K)'_b$ indicates that the space of all continuous linear functionals on $H(K)$ is equipped with the strong topology, that is, the locally convex topology of uniform convergence on bounded subsets of $H(K)$ (we refer the reader to [26, Chapter 23, especially p. 269] for a presentation of topologies of the dual space).

Although we do not need it, it is worth mentioning that the isomorphism (2.5) also holds topologically when the dual of $\mathcal{A}(\mathbb{R})$ is considered with its strong topology and $\operatorname{lim\,ind} H(K_n)'$ is equipped with the corresponding inductive topology (cf. [15, Proposition 1.7]).

We now present the duality (2.6) in detail. Since we work with connected compact subsets of $\mathbb{R}$, i.e. finite closed intervals, let K be such a set. Let $[(g,U)]_{\sim_K} \in H(K)$. Shrinking U if necessary we may assume that it is simply connected (recall that we assume that simply connected sets are connected). Let γ be a C^∞ Jordan curve in $U \setminus K$. Assume that K is contained in the interior $I(\gamma)$ of γ. With this notation the duality between $H_0(\mathbb{C}_\infty \setminus K)$ and $H(K)$ is given by

$$\begin{aligned} H_0(\mathbb{C}_\infty \setminus K) \times H(K) \ni (f, [(g,U)]_{\sim_K}) \\ \mapsto \langle f, [(g,U)]_{\sim_K} \rangle := \frac{1}{2\pi i} \int_\gamma f(z) g(z)\, dz. \end{aligned} \tag{2.7}$$

This important result, known as the Grothendieck–Köthe–Silva duality, is proved in [21, pp. 372–378] (see also [2, Theorem 1.3.5]). We remark that in the general case, that is, when K is not necessarily connected, one needs a finite set of Jordan curves in (2.7), since U may be disconnected.

We can summarize (2.5) and (2.6) in the following way: the dual of $\mathcal{A}(\mathbb{R})$, when equipped with the strong topology, is isomorphic to $\lim\operatorname{ind} H_0(\mathbb{C}_\infty \setminus K_n)$, where $\{K_n\}$ is any compact exhaustion of $\mathbb{R}$:

$$\mathcal{A}(\mathbb{R})'_b \cong \lim\operatorname{ind} H_0(\mathbb{C}_\infty \setminus K_n). \tag{2.8}$$

As already remarked, for a compact exhaustion of $\mathbb{R}$ one can simply take finite closed intervals. Hence

$$\mathcal{A}(\mathbb{R})'_b \cong \lim\operatorname{ind} H_0(\mathbb{C}_\infty \setminus [-n, n]). \tag{2.9}$$

The latter space is usually denoted by $H_0(\mathbb{C} \setminus \mathbb{R})$. Algebraically one may interpret it as the union

$$\bigcup_{n=1}^{\infty} H_0(\mathbb{C}_\infty \setminus [-n, n]).$$

The isomorphism (2.9) holds both algebraically and topologically. Since we will work with continuous linear functionals on $\mathcal{A}(\mathbb{R})$, it is important to write down the isomorphism (2.9) explicitly. Let ξ be a continuous linear functional on $\mathcal{A}(\mathbb{R})$. Then there exists an interval $[-n, n]$ and a function $f \in H_0(\mathbb{C}_\infty \setminus [-n, n])$ such that for any $g \in \mathcal{A}(\mathbb{R})$,

$$\xi(g) = \frac{1}{2\pi i} \int_\gamma f(z)G(z)\, dz. \tag{2.10}$$

Here G is holomorphic in some open simply connected neighborhood of $\mathbb{R}$ and its restriction to $\mathbb{R}$ is g. The C^∞ Jordan curve γ lies in the common domain of holomorphy of f and G, and surrounds $[-n, n]$, i.e. $[-n, n] \subset I(\gamma)$. One easily observes that the formula (2.10) is correct, i.e. it does not depend on γ or G, or on the representative of the class of f in $\lim\operatorname{ind} H_0(\mathbb{C}_\infty \setminus [-n, n])$.

3. Toeplitz operators on $\mathcal{A}(\mathbb{R})$. We introduce the symbol space and define Toeplitz operators on the space of real analytic functions on $\mathbb{R}$. Consider the inductive spectrum

$$\{H(U \setminus K), r_{(U,K),(V,L)}, \mathfrak{P}, \succ\}.$$

We refer the reader to [17, §23, (3)] for the definition and construction of the inductive limit, which will be used below. The symbol $\mathfrak{P}$ stands for the set of all pairs (U, K), where U is an open neighborhood of $\mathbb{R}$ in $\mathbb{C}$ and K is a compact subset of $\mathbb{R}$. For $(V, L), (U, K) \in \mathfrak{P}$, by definition, $(V, L) \succ (U, K)$ if and only if $V \subset U$ and $K \subset L$. The relation $\succ$ directs the set $\mathfrak{P}$. If $(V, L) \succ (U, K)$ then $r_{(U,K),(V,L)} \colon H(U \setminus K) \to H(V \setminus L)$ is the restriction map, $r_{(U,K),(V,L)}(f) = f|_{V \setminus L}$. We define the symbol space

$$\mathcal{X}(\mathbb{R}) := \lim\operatorname{ind}\{H(U \setminus K), r_{(U,K),(V,L)}, \mathfrak{P}, \succ\}. \tag{3.1}$$

Somewhat imprecisely one can think about elements of $\mathcal{X}(\mathbb{R})$ as elements of the union

$$\bigcup_{U,K} H(U \setminus K)$$

with the convention that $F_1 \in H(U_1 \setminus K_1)$ and $F_2 \in H(U_2 \setminus K_2)$ define the same element if and only if there is an open neighborhood V of $\mathbb{R}$ with $V \subset U_1 \cap U_2$ and a compact set $K \subset \mathbb{R}$ with $K \supset K_1 \cup K_2$ such that $F_1|_{V\setminus K} = F_2|_{V\setminus K}$. Although this will not be used in the paper we remark that $\mathcal{X}(\mathbb{R})$ is a locally convex space, which is Hausdorff.

To every symbol $F \in \mathcal{X}(\mathbb{R})$ we assign the continuous linear operator

$$T_F \colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}),$$

which we will call a Toeplitz operator for the reason which will become clear shortly. By the definition of $\mathcal{X}(\mathbb{R})$ the element F is the equivalence class in $(\coprod_{U\setminus K} H(U \setminus K))/\sim$ of a function $\tilde{F} \in H(U \setminus K)$, i.e. $F = [\tilde{F}]_\sim$ for some open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$. Let $f \in \mathcal{A}(\mathbb{R})$. There exists a holomorphic function $\tilde{f} \in H(V)$, where V is an open neighborhood of $\mathbb{R}$, such that $r_V(\tilde{f}) = f$. There exists a simply connected (in particular connected) neighborhood W of $\mathbb{R}$ contained in $U\cap V$. Indeed, since $U\cap V$ is open, for any $n \in \mathbb{N}$ there is $r_n > 0$ such that $R_n := (-n, n) \times (-r_n, r_n) \subset U \cap V$. We set

$$W := \bigcup_{n=1}^{\infty} R_n.$$

When dealing with $f \in \mathcal{A}(\mathbb{R})$, $f = r_V(\tilde{f})$, and $F \in \mathcal{X}(\mathbb{R})$, $F = [\tilde{F}]_\sim$, $\tilde{F} \in H(U \setminus K)$, we will always assume that $U \cap V$ is simply connected, in particular connected.

Let $z \in W$ with W a simply connected neighborhood of $\mathbb{R}$ contained in $U \cap V$. Choose a C^∞ Jordan curve in $W \setminus K$ such that $\mathrm{Ind}_\gamma(z) = 1$ and $\mathrm{Ind}_\gamma(\zeta) = 1$ for any $\zeta \in K$. This is equivalent to saying that both z and K are in the interior $I(\gamma)$ of γ. The assumption on γ means in particular that γ is homologous in $\mathbb{C} \setminus K$ to the cycle defined by the loop $\gamma_R(t) := Re^{it}$ for $R > 0$ large enough.

We set

$$(T_F f)(z) := \frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta) \cdot \tilde{f}(\zeta)}{\zeta - z} \, d\zeta, \tag{3.2}$$

where $f = r_V(\tilde{f})$, $\tilde{f} \in H(V)$, V an open neighborhood of $\mathbb{R}$, and $F = [\tilde{F}]_\sim$ with $\tilde{F} \in H(U\setminus K)$, U an open set containing $\mathbb{R}$ and K a compact subset of $\mathbb{R}$.

The function $T_F f$, which so far depends on the choice of $\tilde{f}$ and $\tilde{F}$, is holomorphic in $I(\gamma)$. It follows from Cauchy's theorem that for $z \in W$ fixed the value of $(T_F f)(z)$ does not depend on the choice of γ provided that γ is a C^∞ Jordan curve in $W \setminus K$ such that both z and K are contained in $I(\gamma)$. Indeed, if γ_1 is another such curve, then $\gamma - \gamma_1$ is homologous to zero in $W \setminus (K \cup \{z\})$, since W is simply connected. For any $z \in W$ we can find a C^∞ Jordan curve γ in $W \setminus K$ such that $z \in I(\gamma)$ and $K \subset I(\gamma)$. Hence $(T_F f)(z)$ is defined for any $z \in W$, since by Cauchy's theorem the value of $(T_F f)(z)$ does not depend on γ. Also the function $(T_F f)(\cdot)$ is holomorphic in W. The restriction of $T_F f$ to $\mathbb{R}$, formally $r_W(T_F f)$, is therefore an element of $\mathcal{A}(\mathbb{R})$. We claim that the value $r_W(T_F f)$ for z real, i.e. $(T_F f)(z)$, does not depend on the set W, the extension $\tilde{f}$ and the representative $\tilde{F}$ used in (3.2). Each of these statements is a consequence of Cauchy's theorem, since W is simply connected. We shall show that the definition does not depend on the choice of the extension and the equivalence class. The reader may prefer to skip the elementary proof of this fact.

PROPOSITION 3.1. *Let $f \in \mathcal{A}(\mathbb{R})$ and $F \in \mathcal{X}(\mathbb{R})$. For z real the value $(T_F f)(z)$ defined in* (3.2) *does not depend on the extension $\tilde{f} \in H(V)$ of $f \in \mathcal{A}(\mathbb{R})$ and the representative $\tilde{F} \in H(U \setminus K)$ of $F \in \mathcal{X}(\mathbb{R})$.*

Proof. Assume that $f = r_V(\tilde{f})$ for some $\tilde{f} \in H(V)$ and also $f = r_{V_1}(\tilde{g})$ for some $\tilde{g} \in H(V_1)$, where V, V_1 are open neighborhoods of $\mathbb{R}$. Furthermore, let $F = [\tilde{F}]_\sim = [\tilde{G}]_\sim$ for $\tilde{F} \in H(U \setminus K)$ and $\tilde{G} \in H(U_1 \setminus K_1)$, where U, U_1 are open neighborhoods of $\mathbb{R}$ and K, K_1 are compact subsets of $\mathbb{R}$. Let W and W_1 be simply connected neighborhoods of $\mathbb{R}$ contained in $U \cap V$ and $U_1 \cap V_1$, respectively. Fix $z \in \mathbb{R}$. Choose a C^∞ Jordan curve γ in $W \setminus K$ such that both z and K are in $I(\gamma)$. Let also γ_1 be a C^∞ Jordan curve in $W_1 \setminus K_1$ with $z \in I(\gamma_1)$ and $K_1 \subset I(\gamma_1)$. We need to show that

$$\frac{1}{2\pi i} \int_\gamma \frac{(\tilde{F} \cdot \tilde{f})(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\gamma_1} \frac{(\tilde{G} \cdot \tilde{g})(\zeta)}{\zeta - z}\, d\zeta.$$

Since $r_V(\tilde{f}) = r_{V_1}(\tilde{g})$, we have $\tilde{f}|_{\mathbb{R}} = \tilde{g}|_{\mathbb{R}}$. Hence there is an open set $V_2 \subset V \cap V_1$ which contains $\mathbb{R}$ such that $\tilde{f}|_{V_2} = \tilde{g}|_{V_2}$. Also, if $[\tilde{F}]_\sim = [\tilde{G}]_\sim$ in $\operatorname{lim\,ind} H(U) = (\coprod_{U,K} H(U \setminus K))/\sim$, there exist $U_2 \subset U \cap U_1$ with $U_2 \supset \mathbb{R}$ and $K_2 \supset K \cup K_1$ with $K_2 \subset\subset \mathbb{R}$ such that $\tilde{F} = \tilde{G}$ in $U_2 \setminus K_2$. Choose a C^∞ Jordan curve γ_2 in $(W \cap W_1 \cap U_2 \cap V_2) \setminus K_2$ such that both z and K_2 are in $I(\gamma_2)$. The cycle $\gamma - \gamma_2$ is homologous to zero in $W \setminus (K \cup \{z\})$, and similarly $\gamma_1 - \gamma_2$ is homologous to zero in $W_1 \setminus (K_1 \cup \{z\})$. Hence by Cauchy's theorem,

$$\begin{aligned}\frac{1}{2\pi i} \int_\gamma \frac{(\tilde{F} \cdot \tilde{f})(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{(\tilde{F} \cdot \tilde{f})(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\gamma_2} \frac{(\tilde{G} \cdot \tilde{g})(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\gamma_1} \frac{(\tilde{G} \cdot \tilde{g})(\zeta)}{\zeta - z}\, d\zeta,\end{aligned}$$

since $\tilde{F} \cdot \tilde{f} = \tilde{G} \cdot \tilde{g}$ on γ_2. ∎

We have thus defined a linear map $f \mapsto (T_F f)|_{\mathbb{R}}$ on $\mathcal{A}(\mathbb{R})$. For a given $F \in \mathcal{X}(\mathbb{R})$ this map will be denoted simply by T_F.

It is elementary to check that T_F is a continuous operator on $\mathcal{A}(\mathbb{R})$ (see [8, p. 14] for the details). By Proposition 3.1, we can assume that any $F \in \mathcal{X}(\mathbb{R})$ is represented by $\tilde{F} \in H(U \setminus K)$ with U simply connected and K connected.

Observe also that T_F is the Cauchy transform of $\tilde{F} \cdot \tilde{f}$. The Cauchy transform, which will be denoted by $\mathcal{C}$, plays a fundamental role in this paper. It may be defined on various spaces. In our setting we have

$$\mathcal{C}\colon \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}).$$

We briefly recall the definition. Let $F \in \mathcal{X}(\mathbb{R})$ be the equivalence class of $\tilde{F} \in H(U \setminus K)$; we may assume that U is simply connected. Let $z \in U$ and choose a C^∞ Jordan curve in $U \setminus K$ such that both z and K are in $I(\gamma)$. We put

$$(\mathcal{C}F)(z) := \frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta)}{\zeta - z}\, d\zeta. \tag{3.3}$$

Naturally, for any $z \in U$ we can find such that z and K are in $I(\gamma)$. By Cauchy's theorem, $(\mathcal{C}F)(z)$ does not depend on γ. Hence (3.3) defines a function holomorphic in U. Its equivalence class $[\mathcal{C}F]_\sim$ in $\mathcal{X}(\mathbb{R})$ is by definition the Cauchy transform of F. We use the same symbol, that is, $\mathcal{C}F$, to denote this object. One easily shows that $\mathcal{C}F \in \mathcal{X}(\mathbb{R})$ does not depend on the choice of $\tilde{F}$. As already stated, $(\mathcal{C}F)(z)$ in (3.3) is holomorphic on some neighborhood of $\mathbb{R}$. Hence its restriction to the real line belongs to $\mathcal{A}(\mathbb{R})$.

By Cauchy's integral formula for any $F \in \mathcal{A}(\mathbb{R})$ we have $\mathcal{C}F = F$. That is, $\mathcal{C}^2 = \mathcal{C}$ and $\mathcal{C}$ is a continuous linear projection onto $\mathcal{A}(\mathbb{R})$. The existence of a continuous projection onto $\mathcal{A}(\mathbb{R})$ readily implies that $\mathcal{A}(\mathbb{R})$ is a closed subspace of $\mathcal{X}(\mathbb{R})$. Also, $I - \mathcal{C}$ is a projection onto $H_0(\mathbb{C} \setminus \mathbb{R})$. Hence this space is also a closed subspace of $\mathcal{X}(\mathbb{R})$. Furthermore, it is essentially a consequence of Cauchy's integral formula that

$$\mathcal{X}(\mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus H_0(\mathbb{C} \setminus \mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R})'.$$

We refer the reader to [8, Theorem 3.3] for the details.

In [8] we proved the following theorem,

THEOREM 3.2. *The following assertions are equivalent:*

(i) $T\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ *is a Toeplitz operator, i.e. a continuous linear operator such that locally near zero*

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots \tag{3.4}$$

for some complex numbers a_n, $n \in \mathbb{Z}$.

(ii) *There exists* $F \in \mathcal{X}(\mathbb{R})$ *such that*

$$T = T_F = \mathcal{C}M_F, \tag{3.5}$$

where $M_F\colon \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ *is multiplication by* F *and* $\mathcal{C}\colon \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \subset \mathcal{X}(\mathbb{R})$ *is the Cauchy projection. Then* (3.4) *holds with*

$$a_n = \frac{1}{2\pi i}\int_\gamma \tilde{F}(\zeta)\zeta^{-n-1}\,d\zeta, \tag{3.6}$$

where $F = [\tilde{F}]_\sim$ *with* $\tilde{F} \in H(U \setminus K)$, $U \supset \mathbb{R}$ *open simply connected,* $K \subset \mathbb{R}$ *compact and* γ *a* C^∞ *Jordan curve in* $U \setminus K$ *surrounding the origin and* K (*i.e.* $\{0\} \cup K \subset I(\gamma)$).

(iii) *There exist* $G \in \mathcal{A}(\mathbb{R})$ *and* $\Phi \in \mathcal{A}(\mathbb{R})'$ *such that*

$$(Tf)(z) = G(z)f(z) + \left\langle \frac{f(z) - f(\cdot)}{z - \cdot}, \Phi \right\rangle.$$

Then close to 0,

$$G(z) = \sum_{n=0}^{\infty} c_n z^n$$

and (3.4) *holds with* $a_n = c_n$, $n \in \mathbb{N}_0$, *and* a_{-n}, $n \in \mathbb{N}$, *the sequence of moments of* Φ, *i.e.*

$$a_{-n-1} = \langle z^n, \Phi\rangle, \qquad n = 0, 1, 2, \ldots.$$

Let $T\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ be a continuous linear operator. Since for each $n \in \mathbb{N}$ the function $T(x^n)$ belongs to $\mathcal{A}(\mathbb{R})$, it develops locally near zero in a power series

$$T(x^n)(\xi) = \sum_{m=0}^{\infty} a_{mn}\xi^m.$$

We say that the matrix $(a_{mn})_{m,n\in\mathbb{N}}$ is *associated* with the operator T. It is natural to call T a *Toeplitz operator* if this matrix is a Toeplitz matrix. According to Theorem 3.2 a continuous linear operator on $\mathcal{A}(\mathbb{R})$ is a Toeplitz operator if and only if $T = T_F$ for some $F \in \mathcal{X}(\mathbb{R})$.

Since

$$\mathcal{X}(\mathbb{R}) \cong \mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R})', \tag{3.7}$$

it is easy to give concrete examples of Toeplitz operators on $\mathcal{A}(\mathbb{R})$. Any $F \in \mathcal{A}(\mathbb{R})$ induces the Toeplitz operator T_F which is simply the multiplication operator M_F. The matrix of such an operator is lower-triangular and the coefficients a_n, $n \in \mathbb{N}_0$, are the Taylor coefficients of F at 0.

It follows from the description of $\mathcal{A}(\mathbb{R})'$ in Section 2 that any function $G \in H_0(\mathbb{C}_\infty \setminus K)$, where $K \subset \mathbb{R}$ is compact, determines the Toeplitz operator T_G. The associated matrix is upper-triangular and the coefficients a_{-n}, $n \in \mathbb{N}$, are the Taylor coefficients of G at ∞,

$$G(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}.$$

It follows from Theorem 3.2 and the decomposition (3.7) that any Toeplitz operator on $\mathcal{A}(\mathbb{R})$ is a sum of operators of this type, i.e. the sum of a multiplication operator M_F with $F \in \mathcal{A}(\mathbb{R})$ and the operator T_G, where $G \in H_0(\mathbb{C}_\infty \setminus K)$ for some $K \subset\subset \mathbb{R}$.

Our interest in Toeplitz operators on $\mathcal{A}(\mathbb{R})$ is partly motivated by what we call divided difference equations. We will briefly present this theory since it is a source of examples of Toeplitz operators. In our presentation we follow [24]. The reader acquainted with our previous research [20] may wish to skip this part of the paper.

The first order divided difference of $f \in \mathcal{A}(\mathbb{R})$ with respect to the points $x_1, x_2 \in \mathbb{R}$ is the quantity

$$\Delta^{(1)}[f; x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then the kth order divided difference of $f(z)$ with respect to the $k+1$ points $x_1, \ldots, x_{k+1} \in \mathbb{R}$ is defined inductively by the formula

$$\Delta^{(k)}[f; x_1, \ldots, x_{k+1}] = \frac{\Delta^{(k-1)}[f; x_2, \ldots, x_{k+1}] - \Delta^{(k)}[f; x_1, \ldots, x_k]}{x_{k+1} - x_1}.$$

It is an easy matter to show

$$\Delta^{(k)}[f; x_1, \ldots, x_{k+1}] = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(\zeta)}{\omega_{k+1}(\zeta)}\, d\zeta, \tag{3.8}$$

where $\omega_{k+1}(z) = (z - x_1) \ldots (z - x_{k+1})$ and γ is a C^∞ Jordan curve which surrounds the points $x_1, \ldots, x_{k+1}$ and is contained in (simply connected) V, where $f = r_V(\tilde{f})$ for some $\tilde{f} \in H(V)$. We may consider $\Delta^{(k)}[f; x_1, \ldots, x_{k+1}]$ as a function of one the points, for instance $z = x_1$. We have

$$\Delta^{(k)}[f; z, x_1, \ldots, x_k] = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(\zeta)}{(\zeta - x_1) \ldots (\zeta - x_k)} \frac{d\zeta}{\zeta - z}.$$

Let now $x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^N, \ldots, x_{k_N}^N \in \mathbb{R}$, $N \in \mathbb{N}$ and $f_1, \ldots, f_N \in \mathcal{A}(\mathbb{R})$. Consider the equation

$$\sum_{j=1}^N \Delta^{(k_j)}[f_j \cdot f; z, x_1^j, \ldots, x_{k_j}^j] = g(z), \tag{3.9}$$

where $g \in \mathcal{A}(\mathbb{R})$, to be solved for $f \in \mathcal{A}(\mathbb{R})$. It follows from (3.8) that (3.9) is of the form

$$T_F f = g,$$

where $F \in \mathcal{X}(\mathbb{R})$ and T_F is our Toeplitz operator.

Divided differences appear in interpolation theory. For instance, Newton's interpolation series can be written in the form

$$\sum_{k=0}^{\infty} \Delta^{(k)}[f; x_1, \ldots, x_{k+1}]\omega_k(z).$$

We refer again to [24] for an explanation. In general, consider the problem of finding a polynomial P which interpolates $f \in \mathcal{A}(\mathbb{R})$ at interpolation points $x_1, \ldots, x_m \in \mathbb{R}$. More specifically, let $\alpha_1, \ldots, \alpha_m$ be positive integers such that $\sum \alpha_i = n$. We look for a polynomial P satisfying

$$P(x_k) = f(x_k), \ldots, P^{(\alpha_k - 1)}(x_k) = f^{(\alpha_k - 1)}(x_k), \quad k = 1, \ldots, m. \tag{3.10}$$

The solution to this problem gives another example of Toeplitz operators. Namely one shows [24, Theorem 2.8, p. 68] that the polynomial P of degree less than n which solves (3.10) is given by

$$P(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{f}(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} \, d\zeta,$$

where $\tilde{f} \in H(V)$ is an extension of f, $\gamma \subset V$ is a C^∞ Jordan curve such that $I(\gamma)$ contains the interpolation points and z, and $\omega(z) = (z - x_1)^{\alpha_1} \ldots (z - x_m)^{\alpha_m}$.

We can write

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} \, d\zeta \\ &\quad = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{\omega(z)}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega(\zeta)} \frac{d\zeta}{\zeta - z} = (I - T_\omega T_{1/\omega}) f(z). \end{aligned} \tag{3.11}$$

Consider now the remainder term

$$\begin{aligned} R(z) = f(z) - P(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} \, d\zeta \\ &= \frac{\omega(z)}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega(\zeta)} \frac{d\zeta}{\zeta - z} = T_\omega T_{1/\omega} f(z). \end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that not only the solution to the interpolation problem but also the convergence properties of the solution are governed by Toeplitz operators of the form T_F for some $F \in \mathcal{X}(\mathbb{R})$.

4. Hardy spaces on curves and the Cauchy transform. Our method of analyzing Toeplitz operators on $\mathcal{A}(\mathbb{R})$ is based on investigating the properties of Toeplitz operators on Hardy spaces on curves. The main tool is the Cauchy transform. Our presentation is based on the beautiful exposition [1].

Let Ω be a finitely connected domain in the complex plane. We assume that the boundary of Ω consists of C^∞ Jordan curves $\gamma_0, \dots, \gamma_n \colon [0,1] \to \mathbb{C}$. We assume they have standard orientation, i.e. when moving along γ_i in the direction of the parameter increase one has the domain Ω on the left.

We denote by $C^k(\overline{\Omega})$ the space of all complex valued functions on $\overline{\Omega}$ whose partial derivatives up to and including order k exist and are continuous on Ω and extend continuously to $\overline{\Omega}$. The space $C^\infty(\overline{\Omega})$ is the set of functions in $C^k(\overline{\Omega})$ for all k. Let also $A^\infty(\overline{\Omega})$ denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$. We say that a function g defined on the boundary of Ω is C^∞ on $b\Omega$ if for each i the function $g(\gamma_i(t))$ is smooth on $[0,1]$ and all of its derivatives agree at the endpoints 0 and 1. We denote the set of all such functions by $C^\infty(b\Omega)$.

If u belongs to $C^\infty(b\Omega)$, then the *Cauchy transform* of u is the holomorphic function $\mathcal{C}_{b\Omega}u$ on Ω given by

$$(\mathcal{C}_{b\Omega}u)(z) = \frac{1}{2\pi i} \int\limits_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta. \tag{4.1}$$

Here is the first fundamental property of the Cauchy transform.

THEOREM 4.1 ([1, Theorem 3.1]). *The Cauchy transform maps $C^\infty(b\Omega)$ into $A^\infty(\overline{\Omega})$.*

Thus the Cauchy transform of a function in $C^\infty(b\Omega)$ extends smoothly up to the boundary. This means that $\mathcal{C}_{b\Omega}$ may be treated as a map on $C^\infty(b\Omega)$. However, the extension to $b\Omega$ is not given by (4.1). It is therefore crucial to give an explicit formula for $\mathcal{C}_{b\Omega}$ on $b\Omega$.

THEOREM 4.2 ([1, Theorem 3.4]). *Suppose $u \in C^\infty(b\Omega)$. If m is a positive integer, there is a function $\Psi \in C^\infty(\overline{\Omega})$ which vanishes to order m on the boundary such that the boundary values of $\mathcal{C}_{b\Omega}u$ are expressed via*

$$(\mathcal{C}_{b\Omega}u)(z) = u(z) - \frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}, \qquad z \in b\Omega.$$

Let $L^2(b\Omega)$ denote the space of all classes of functions on $b\Omega$ which are square integrable with respect to the arc length measure on $b\Omega$. In other words, $L^2(b\Omega)$ is the space of all functions u on $b\Omega$ such that $u(\gamma_i(t))$ is a measurable function on $[0,1]$ for each i and

$$\|u\|^2_{L^2(b\Omega)} := \sum_{i=0}^{n} \int_0^1 |u(\gamma_i(t))|^2 |\gamma_i'(t)|\, dt < \infty.$$

Let $A^\infty(b\Omega)$ denote the space of functions on $b\Omega$ which are boundary values of functions in $A^\infty(\overline{\Omega})$. The *Hardy space* $H^2(b\Omega)$ is defined to be the closure of $A^\infty(b\Omega)$ in $L^2(b\Omega)$. We can now formulate the next fundamental property of the Cauchy transform.

THEOREM 4.3 ([1, Theorem 4.1]). *The Cauchy transform extends to a bounded operator from $L^2(b\Omega)$ onto $H^2(b\Omega)$.*

If γ is a C^∞ Jordan curve then by Jordan's theorem it divides the plane into domains $I(\gamma)$ and $E(\gamma)$, the latter unbounded. If $\Omega = I(\gamma)$ for such a curve, we shall write $C^\infty(\gamma), L^2(\gamma), H^2(\gamma)$ and $\mathcal{C}_\gamma$ to denote the corresponding spaces and the Cauchy transform.

Let now $\phi \in L^\infty(b\Omega)$. We define the Toeplitz operator $T_{b\Omega,\phi}\colon H^2(b\Omega) \to H^2(b\Omega)$ in the following way:

$$T_{b\Omega,\phi} f := \mathcal{C}_{b\Omega}(\phi \cdot f).$$

We now intend to relate the operators $T_{b\Omega,\phi}$ to the previously defined operator $T_F\colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$. By the definition, the function $f \in H^2(b\Omega)$ and also $T_{b\Omega,\phi} f$ are defined on $b\Omega$ only. There is however another definition of the Hardy space: classically, the Hardy space is defined as the space of all holomorphic functions H on Ω such that

$$\sup_{\epsilon>0} \Big(\sum_{i=0}^{n} \int |H(\gamma_{i,\epsilon}(t))|^2 |\gamma'_{i,\epsilon}(t)|\, dt \Big)^{1/2} < \infty. \tag{4.2}$$

The C^∞ Jordan curves $\gamma_{i,\epsilon}$ are appropriately constructed dilations of the curves γ_i. We refer to reader to [1, p. 17] for the details. It is a standard fact that the two definitions are equivalent and the equivalence is realized by the Cauchy transform [1, Theorems 6.1, 6.2]. We may therefore think about functions in $H^2(b\Omega)$ as either functions on $b\Omega$ or holomorphic functions in Ω which satisfy the growth condition (4.2).

We will tacitly use the following fact which relates the operators $T_{b\Omega,\phi}$ and T_F.

PROPOSITION 4.4 ([8, Proposition 5.3]). *Let γ be a C^∞ Jordan curve in the complex plane. For any $\phi \in C(\gamma)$, $f \in H^2(\gamma)$ and $z \in I(\gamma)$,*

$$\frac{1}{2\pi i} \int_\gamma \frac{\mathcal{C}_\gamma(\phi \cdot f)(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{(\phi \cdot f)(\zeta)}{\zeta - z}\, d\zeta. \tag{4.3}$$

We will use the Coburn–Simonenko theorem for the operators $T_{\gamma,\phi}$, the Toeplitz operators on $H^2(\gamma)$ defined by means of the Cauchy transform $\mathcal{C}_\gamma$.

THEOREM 4.5 ([4, Theorem 6.17]). *Assume that γ is a C^∞ Jordan curve and let $\phi \in L^\infty(\gamma)$. Then either $T_{\gamma,\phi}$ or its adjoint is injective.*

There is a well-known characterization of Fredholm Toeplitz operators on $H^2(\gamma)$ with continuous symbols.

THEOREM 4.6. *Assume that γ is a C^∞ Jordan curve and let $\phi \in C(\gamma)$. If $\phi \neq 0$ on γ then $T_{\gamma,\phi}$ is a Fredholm operator and*

$$\operatorname{index} T_{\gamma,\phi} = -\operatorname{Ind}_{\phi\circ\gamma}(0).$$

A much more general result is proved in [27, Proposition 4.1.6] (see also [8, Theorem 5.4]).

5. Toeplitz operators on $H(K)$. We need also consider operators of the form T_F on the spaces $H(K)$, K a compact interval which contains the origin. This is precisely what is needed in the proof of Theorem 1.2. Recall that

$$\mathcal{A}(\mathbb{R}) = \operatorname{lim\,proj} H(K_n),$$

where K_n, $n \in \mathbb{N}$, is any compact exhaustion of $\mathbb{R}$.

We will now define operators $T_F\colon H(K) \to H(K)$ and establish their basic properties.

Let K be a finite closed interval. We define the symbol space

$$\mathcal{X}(K) := \operatorname{lim\,ind}\{H(U \setminus K), r_{(U,K),(V,K)}, \mathfrak{F}, \succ\},$$

where $\mathfrak{F}$ is the set of pairs (U, K), where U is an open neighborhood of the fixed compact set K directed by the relation $(V, K) \succ (U, K)$ if and only if $V \subset U$, and $r_{(U,K),(V,K)}\colon H(U \setminus K) \to H(V \setminus K)$ is the restriction map. As in Section 3, $\mathcal{X}(K)$ can be thought of as the union

$$\bigcup_{U \supset K} H(U \setminus K)$$

with $F_1 \in H(U_1 \setminus K)$ and $F_2 \in H(U_2 \setminus K)$ defining the same element if and only if there is an open set $U \subset U_1 \cap U_2$ with $U \supset K$ such that $F_1|_{U\setminus K} = F_2|_{U\setminus K}$. We again refer the reader to [17, §23] for the construction of the inductive limit. We will now assign to any $F \in \mathcal{X}(K)$ the Toeplitz operator $T_F\colon H(K) \to H(K)$.

Assume that $F \in \mathcal{X}(K) = (\coprod_{U\supset K} H(U \setminus K))/\sim$ is given and $F = [\tilde{F}]_\sim$ for some $\tilde{F} \in H(U \setminus K)$, where U is an open neighborhood of K. Let also $[(f, V)]_{\sim_K} \in H(K)$. Let $W \subset U \cap V$ be an open simply connected set which contains K. Let γ be a C^∞ Jordan curve contained in W such that $K \subset I(\gamma)$. For $z \in I(\gamma)$ define

$$(5.1) \qquad (T_{F,K}([(f, V)]_{\sim_K}))(z) := \frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta) f(\zeta)}{\zeta - z}\, d\zeta.$$

The function $T_{F,K}([(f, V)]_{\sim_K})$ is holomorphic in $I(\gamma)$. Since $K \subset I(\gamma)$, this function defines the germ $\pi_{I(\gamma),K}(T_{F,K}([(f, V)]_{\sim_K}))$ on K (see (2.2) for the definition of $\pi_{V,K}$).

Proposition 5.1. *The map*

$$[(f, V)]_{\sim_K} \mapsto \pi_{I(\gamma),K}(T_{F,K}([(f, V)]_{\sim_K}))$$

defines a continuous linear operator on $H(K)$ denoted by $T_{F,K}$.

Proof. It follows easily from Cauchy's theorem that the map is well-defined. The argument follows the proof of Proposition 3.1. It is elementary that $T_{F,K}$ is a continuous operator on $H(K)$. Indeed, by [26, Proposition 24.7], $T_{F,K}$ is continuous if and only if for any open neighborhood V of K, the composition $T_{F,K} \circ \pi_{V,K} \colon H(V) \to H(K)$ is continuous. However, this composition factors through the map $H(V) \to H(I(\gamma))$ with γ as above, which is continuous by elementary estimates of the Cauchy integral. Also, $\pi_{I(\gamma),K}$ is continuous by the definition of the inductive topology of $H(K)$. ∎

Let $F \in \mathcal{X}(K)$ and let $F = [\tilde{F}]_\sim$ for some $\tilde{F} \in H(U \setminus K)$, where U is an open neighborhood of K. The set U may be assumed to be simply connected. There exists $r > 0$ such that for $z \in B(0,r)$ and any $n \in \mathbb{N}_0$,

$$\frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta)\zeta^n}{\zeta - z}\, d\zeta = F_{-n} + F_{-n+1}z + \cdots + F_{-1}z^{n-1} + F_0 z^n + \cdots \tag{5.2}$$

where

$$F_n = \frac{1}{2\pi i} \int_\gamma \tilde{F}(\zeta)\zeta^{-n-1}\, d\zeta$$

and γ is a C^∞ Jordan curve in $U \setminus K$ which surrounds K. The proof of this fact is elementary and follows [8, proof of Proposition 4.1]. The first step is the decomposition

$$\tilde{F} = \tilde{F}_+ + \tilde{F}_-,$$

where $\tilde{F}_+ \in H(U)$ and $\tilde{F}_- \in H_0(\mathbb{C}_\infty \setminus K)$, which is an immediate consequence of the Cauchy integral formula in $U \setminus K$. Formula (5.2) means that it is legitimate to call the operators $T_{F,K} \colon H(K) \to H(K)$ defined by (5.1) also Toeplitz operators.

Although we do not need it for our main results, it seems of interest that there is a characterization of Toeplitz operators on $H(K)$ very similar to the characterization of such operators on $\mathcal{A}(\mathbb{R})$.

THEOREM 5.2. *Let K be a compact interval which contains the origin. The following assertions are equivalent:*

(i) *$T \colon H(K) \to H(K)$ is a Toeplitz operator, i.e. a continuous linear operator such that locally near zero,*

$$T(\zeta^n)(z) = a_{-n} + a_{-n+1}z + a_{-n+2}z^2 + \cdots \tag{5.3}$$

for some complex numbers a_n, $n \in \mathbb{Z}$.

(ii) *There exists $F \in \mathcal{X}(K)$ such that $T = T_{F,K}$. Then (5.3) holds with*

$$a_n = \frac{1}{2\pi i} \int_\gamma \tilde{F}(\zeta)\zeta^{-n-1}\, d\zeta,$$

where $F = [\tilde{F}]_\sim$ with $\tilde{F} \in H(U \setminus K)$, U is an open simply connected neighborhood of K, and γ is a C^∞ Jordan curve in $U \setminus K$ such that $K \subset I(\gamma)$.

Some comment concerning the notation in Theorem 5.2 is in order. For a continuous linear operator $T\colon H(K) \to H(K)$ we denoted by $T(\zeta^n)$ the value of T on the germ on K defined by the monomial $\zeta \mapsto \zeta^n$, formally $T([(\zeta^n, \mathbb{C})]_{\sim_K})$. Furthermore, $T(\zeta^n)$ is by definition a germ on K, so it is defined by a function holomorphic on some open neighborhood of K, which by assumption contains the origin. Formally, $T(\zeta^n) = [(f, V)]_{\sim_K}$, where V is an open neighborhood of K and $f \in H(V)$. This neighborhood contains a small disc around zero, so it is meaningful to consider the Taylor expansion of the representative f of $T(\zeta^n)$ around zero. This expansion is well-defined, that is, it does not depend on the holomorphic representative of $T(\zeta^n)$ chosen. This is the meaning of (i) in Theorem 5.2.

The proof of Theorem 5.2 is essentially the same as the proof of Theorem 3.2 ([8, Theorem 1]). We repeat it for the convenience of the reader.

Proof of Theorem 5.2. We show that (i) implies (ii); the comments before the formulation of the theorem are essentially the proof of (ii)⇒(i).

Let $T1 = [(F_+, U)]_{\sim_K}$, that is, $F_+ \in H(U)$ defines the germ $T([(1, \mathbb{C})]_{\sim_K})$. In a sufficiently small disc centered at zero,

$$F_+(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Define $\varphi \in H(K)'$ by $\varphi([(f, V)]_{\sim_K}) := T([(f, V)]_{\sim_K})(0)$. Since $0 \in K$, φ is well-defined. Also, φ is continuous, since T is continuous by assumption and the evaluation functionals at points of K are continuous on $H(K)$ by [26, Proposition 24.7]. It follows from (5.3) that $\varphi(\zeta^n) = a_{-n}$ for $n \in \mathbb{N}_0$. Since $\varphi \in H(K)'$, there is $G_- \in H_0(\mathbb{C}_\infty \setminus K)$ such that

$$\varphi([(f, V)]_{\sim_K}) = \langle f, G_- \rangle = \frac{1}{2\pi i} \int_\gamma f(\zeta) G_-(\zeta)\, d\zeta$$

for $[(f, V)]_{\sim_K} \in H(K)$, where γ is a C^∞ Jordan curve in $V \setminus K$ such that $K \subset I(\gamma)$. As usual we assume that V is simply connected. For sufficiently large $|z|$, there is an expansion

$$G_-(z) = \sum_{m=1}^{\infty} \frac{G_{-m}}{z^m}$$

and we obtain

$$a_{-n} = \varphi(\zeta^n) = \langle \zeta^n, G_- \rangle = G_{-(n+1)}, \qquad n \in \mathbb{N}_0.$$

We set

$$\tilde{F} = F_+ + zG_- - a_0.$$

Then $\tilde{F} \in H(U \setminus K)$ and the Taylor expansions of the representatives of $T(\zeta^n)$ and $T_{F,K}(\zeta^n)$, where $F = [\tilde{F}]_\sim$, are equal in sufficiently small discs around zero for any $n \in \mathbb{N}_0$. Thus the germs $T(\zeta^n)$ and $T_{F,K}(\zeta^n)$ are equal for $n \in \mathbb{N}_0$. Since polynomials are dense in $H(K)$, we infer that $T = T_{F,K}$. ■

What is really needed in order to prove our main results is a characterization of Fredholm Toeplitz operators on $H(K)$. The proof is similar to, actually essentially easier than, the case of Toeplitz operators on $\mathcal{A}(\mathbb{R})$, investigated in [8]. We consider it safer to provide the details, even if it means repeating some arguments.

THEOREM 5.3. *Let $F \in \mathcal{X}(K)$, where K is a compact interval. The operator $T_{F,K}\colon H(K) \to H(K)$ is a Fredholm operator if and only if there exists an open set $U \supset K$ and a function $\tilde{F} \in H(U \setminus K)$ such that $\tilde{F}$ does not vanish in $U \setminus K$ and $F = [\tilde{F}]_\sim$.*

In other words, $T_{F,K}$ is Fredholm if and only if the zeros of no representative $\tilde{F}$ of F accumulate at K.

Proof of Theorem 5.3. Assume that $F = [\tilde{F}]_\sim$ for some $\tilde{F} \in H(U \setminus K)$ such that $\tilde{F} \neq 0$ in $U \setminus K$, where U is an open simply connected neighborhood of K.

Let γ_1 be a C^∞ Jordan curve in $U \setminus K$ such that $K \subset I(\gamma_1)$. By Theorem 4.6, $T_{\tilde{F},\gamma_1}\colon H^2(\gamma_1) \to H^2(\gamma_1)$ is a Fredholm operator with index $T_{\tilde{F},\gamma_1} = -\operatorname{Ind}_{\tilde{F}\circ\gamma_1}(0)$, since $\tilde{F}$ does not vanish on γ_1. Since $\tilde{F} \neq 0$ in $U \setminus K$, it follows from Cauchy's theorem and the formula

$$\operatorname{Ind}_{\tilde{F}\circ\gamma_1}(0) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta$$

that the index of $T_{\tilde{F},\gamma_1}\colon H^2(\gamma_1) \to H^2(\gamma_1)$ does not depend on γ_1. That is, there is a natural number n such that for any C^∞ Jordan curve $\gamma \subset U \setminus K$ with $K \subset I(\gamma)$,

$$\operatorname{index}\big(T_{\tilde{F},\gamma}\colon H^2(\gamma) \to H^2(\gamma)\big) = n. \tag{5.4}$$

Since U is simply connected, for any two such curves γ_1, γ_2 the cycle $\gamma_1 - \gamma_2$ is homologous to zero in $U \setminus K$.

Since $T_{\tilde{F},\gamma_1}$ is a Fredholm operator, there exist linearly independent functions $f_1^1, \ldots, f_{N_1}^1 \in H^2(\gamma_1)$ which span $\ker(T_{\tilde{F},\gamma_1}\colon H^2(\gamma_1) \to H^2(\gamma_1))$. We recall the following crucial fact:

LEMMA 5.4 ([8, Lemma 5.8]). *Assume that $\tilde{F} \in H(U \setminus K)$ and $\tilde{F}(z) \neq 0$ for $z \in U \setminus K$, where $U \supset \mathbb{R}$ is open and simply connected and $K \subset \mathbb{R}$*

is a finite closed interval. Let $\gamma \subset U \setminus K$ be a C^∞ Jordan curve such that $K \subset I(\gamma)$ and let $f \in H^2(\gamma)$. If $T_{\tilde{F},\gamma} f = 0$, then $f \in H(U)$.

In [8] we assumed that $U \subset \mathbb{R}$ but the situation when U is only a complex neighborhood of K can be treated in the same way.

Lemma 5.4 shows that $f_1^1, \ldots, f_{N_1}^1 \in H(U)$. Let now $\gamma_2 \subset U \setminus K$ be another C^∞ Jordan curve in $U \setminus K$ with $K \subset I(\gamma_2)$. By Lemma 5.4, both kernels $\ker(T_{\tilde{F},\gamma_j} \colon H^2(\gamma_j) \to H^2(\gamma_j))$, $j = 1, 2$, can be viewed as finite-dimensional subspaces of $H(U)$. Moreover, by Cauchy's theorem these two subsets of $H(U)$ coincide. Hence there exists $N \in \mathbb{N}$ and linearly independent functions $f_1, \ldots, f_N$ such that for any C^∞ Jordan curve $\gamma \subset U \setminus K$ with $K \subset I(\gamma)$,

$$\ker\bigl(T_{\tilde{F},\gamma} \colon H^2(\gamma) \to H^2(\gamma)\bigr) = \operatorname{Span}_{H^2(\gamma)}\{f_1, \ldots, f_N\}. \tag{5.5}$$

We claim that

$$\ker\bigl(T_{F,K} \colon H(K) \to H(K)\bigr) = \operatorname{Span}_{H(K)}\{[(f_1, U)]_{\sim K}, \ldots, [(f_N, U)]_{\sim K}\}.$$

Indeed, assume that $[(f, V)]_{\sim K} \in H(K)$ and $T_{F,K}([(f, V)]_{\sim K}) = 0$. This means that

$$\frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta) f(\zeta)}{\zeta - z}\, d\zeta = 0, \qquad z \in I(\gamma),$$

where γ is a C^∞ Jordan curve in an open simply connected set $W \subset U \cap V$ such that $K \subset I(\gamma)$. Hence $T_{\tilde{F},\gamma} f = 0$, where $T_{\tilde{F},\gamma}$ is the Toeplitz operator on $H^2(\gamma)$ defined by $\tilde{F}$. By (5.5) we have $f \in \operatorname{Span}\{f_1, \ldots, f_N\}$ in $H^2(\gamma)$, so also

$$[(f, V)]_{\sim K} \in \operatorname{Span}_{H(K)}\{[(f_1, U)]_{\sim K}, \ldots, [(f_N, U)]_{\sim K}\}.$$

Since $f_1, \ldots, f_N$ are linearly independent in $H(U)$, they form a basis of

$$\ker\bigl(T_{F,K} \colon H(K) \to H(K)\bigr).$$

It follows from (5.4) and (5.5) that there exists a natural number M such that for any C^∞ Jordan curve $\gamma \subset U \setminus K$ with $K \subset I(\gamma)$,

$$\dim \operatorname{coker}\bigl(T_{\tilde{F},\gamma} \colon H^2(\gamma) \to H^2(\gamma)\bigr) = M. \tag{5.6}$$

We shall show that the dimension of $H(K)/\operatorname{im} T_{F,K}$ is also M.

Let $\gamma_1 \subset U \setminus K$ be a C^∞ Jordan curve with $K \subset I(\gamma_1)$. There exist functions $f_1^1, \ldots, f_M^1 \in H^2(\gamma_1)$ such that the classes

$$f_1^1 + \operatorname{im} T_{\tilde{F},\gamma_1}, \ldots, f_M^1 + \operatorname{im} T_{\tilde{F},\gamma_1}$$

are a basis of $H^2(\gamma_1)/\operatorname{im} T_{\tilde{F},\gamma_1}$. We need a generalization of Lemma 5.4, also proved in [8].

Lemma 5.5 ([8, Lemma 5.9]). *Assume that $\tilde{F} \in H(U \setminus K)$, where U is an open neighborhood of a connected set $K \subset\subset \mathbb{R}$, and $\tilde{F}(z) \neq 0$ for $z \in U \setminus K$.*

Assume that $V \supset K$ is an open set such that $U \cap V$ is simply connected. Let $\gamma \subset (U \cap V) \setminus K$ be a C^∞ Jordan curve with $K \subset I(\gamma)$. Assume that

(i) $f \in H(V)$,
(ii) $h \in H^2(\gamma)$,
(iii) $f(z) = \mathcal{C}_\gamma(\tilde{F}h)(z)$ *for all* $z \in I(\gamma)$.

Then $h \in H(U \cap V)$. Furthermore, if $V = I(\tilde{\gamma})$ for a C^∞ Jordan curve $\tilde{\gamma} \subset U \setminus K$ with $K \subset I(\tilde{\gamma})$ and $f \in H^2(\tilde{\gamma})$, then also $h \in H^2(\tilde{\gamma})$.

Let $\gamma_2 \subset U \setminus K$ be a C^∞ Jordan curve contained in $I(\gamma_1)$ such that $K \subset I(\gamma_2)$. We shall show that the classes

$$f_1^1 + \operatorname{im} T_{\tilde{F},\gamma_2}, \ldots, f_M^1 + \operatorname{im} T_{\tilde{F},\gamma_2}$$

form a basis of $H^2(\gamma_2)/\operatorname{im} T_{\tilde{F},\gamma_2}$. It follows from (5.6) that it suffices to show that these classes are linearly independent in $H^2(\gamma_2)/\operatorname{im} T_{\tilde{F},\gamma_2}$. Assume that for some scalars

$$\alpha_1 f_1^1 + \cdots + \alpha_M f_M^1 = T_{\tilde{F},\gamma_2} f$$

for some $f \in H^2(\gamma_2)$. It follows from Lemma 5.5 that $f \in H^2(\gamma_1)$. Thus, by Cauchy's theorem and a standard limit argument in the Hardy space,

$$\alpha_1 f_1^1 + \cdots + \alpha_M f_M^1 = T_{\tilde{F},\gamma_1} f.$$

This implies that $\alpha_1 = \cdots = \alpha_M = 0$, since $f_1^1 + \operatorname{im} T_{\tilde{F},\gamma_1}, \ldots, f_M^1 + \operatorname{im} T_{\tilde{F},\gamma_1}$ are linearly independent in $H^2(\gamma_1)/\operatorname{im} T_{\tilde{F},\gamma_1}$.

Assume that $[(g,V)]_{\sim_K} \in H(K)$. We shall show that there exist scalars $\alpha_1, \ldots, \alpha_M$ and a germ $[(f,W)]_{\sim_K} \in H(K)$ such that

(5.7)
$$[(g,V)]_{\sim_K} - \alpha_1[(f_1^1, I(\gamma_1))]_{\sim_K} - \cdots - \alpha_M[(f_M^1, I(\gamma_1))]_{\sim_K} = T_{F,K}[(f,W)]_{\sim_K}.$$

There exists a C^∞ Jordan curve $\gamma \subset U \setminus K$ with $K \subset I(\gamma)$ such that $g \in H^2(\gamma)$. We may assume that $\gamma \subset I(\gamma_1)$. This implies that $g - \alpha_1 f_1^1 - \cdots - \alpha_M f_M^1 = T_{\tilde{F},\gamma} f$ for some $f \in H^2(\gamma)$, proving (5.7) with $W = I(\gamma)$.

Assume that $F \in \mathcal{X}(K)$ and that there does not exist an open set $U \supset K$ and $\tilde{F} \in H(U \setminus K)$ with $\tilde{F} \neq 0$ in $U \setminus K$ such that $F = [\tilde{F}]_\sim$. We shall show that $T_{F,K}\colon H(K) \to H(K)$ is not a Fredholm operator. The assumption on F implies that if $F = [\tilde{F}]_\sim$ for $\tilde{F} \in H(U \setminus K)$ and U an open neighborhood of K, then there exists a sequence $(z_n) \subset U \setminus K$ with $z_n \to z \in K$ such that $\tilde{F}(z_n) = 0$ for all n. Assume that such an $\tilde{F} \in H(U \setminus K)$ has been chosen, where U is a simply connected neighborhood of K.

Let γ_{outer} be a C^∞ Jordan curve in $U \setminus K$ with $K \subset I(\gamma_{\text{outer}})$ such all points z_n are in $I(\gamma_{\text{outer}})$, which may be assumed without loss of generality. For each $n \in \mathbb{N}$ choose a C^∞ Jordan curve γ_n in $U \setminus K$ with $K \subset I(\gamma_n)$ which does not pass through any zero of $\tilde{F}$ and such that $\gamma_n \subset I(\gamma_{\text{outer}})$. We also assume that $z_1, \dots, z_n \in I(\gamma_{\mathrm{outer}}) \setminus \overline{I(\gamma_n)}$. Then

$$n \leq \frac{1}{2\pi i} \int\limits_{\gamma_{\mathrm{outer}}} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta$$

by the argument principle. This means that

$$\frac{1}{2\pi i} \int\limits_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta \to -\infty$$

as $n \to \infty$. Therefore,

$$\operatorname{index}\big(T_{\tilde{F},\gamma_n} \colon H^2(\gamma_n) \to H^2(\gamma_n)\big) = -\operatorname{Ind}_{\tilde{F}\circ\gamma_n}(0) \to \infty,$$

so also $\dim\ker T_{\tilde{F},\gamma_n} \to \infty$ as $n \to \infty$. This, however, immediately implies that

$$\dim\ker\big(T_{F,K} \colon H(K) \to H(K)\big) = \infty,$$

finishing the proof of Theorem 5.3. ∎

We also have the following interesting property of Toeplitz operators on $H(K)$.

THEOREM 5.6. *Let $T_{F,K} \colon H(K) \to H(K)$ be a Toeplitz operator, where K is a compact interval in $\mathbb{R}$ and $F \in \mathcal{X}(K)$. Then either the range of $T_{F,K}$ is of finite codimension, in which case it is closed, or this range is dense in $H(K)$.*

Proof. Let $F \in \mathcal{X}(K)$. Either there is $\tilde{F} \in H(U \setminus K)$, with U an open simply connected neighborhood of K, which defines the class F, i.e. $F = [\tilde{F}]_\sim$, such that $\tilde{F} \neq 0$ in $U \setminus K$, or for any $\tilde{F}$ with $F = [\tilde{F}]_\sim$ there is a sequence (z_n) which accumulates at a point of K such that $\tilde{F}(z_n) = 0$ for all n. In the first case, $T_{F,K}$ is a Fredholm operator by the previous theorem and its range is a subspace of finite codimension. Such a subspace is necessarily closed in $H(K)$. This can be shown as in the case of $\mathcal{A}(\mathbb{R})$ [8, Proposition 5.1]. The proof relies on the open mapping theorem, since $H(K)$ is both ultrabornological and has a web.

Assume now that $\tilde{F}(z_n) = 0$ for a sequence $(z_n) \subset U \setminus K$ with $z_n \to z \in K$. Then, as shown in the proof of Theorem 5.3, there exist C^∞ Jordan curves γ_n in $U \setminus K$ such that $K \subset I(\gamma_n)$ for all n and

$$\dim\ker\big(T_{\tilde{F},\gamma_n} \colon H^2(\gamma_n) \to H^2(\gamma_n)\big) = \infty \tag{5.8}$$

for each n. We may choose γ_n to satisfy the following conditions:

(i) $\gamma_{n+1} \subset I(\gamma_n)$,
(ii) $\bigcap_n I(\gamma_n) = K$.

It follows from (5.8) that there is $N \in \mathbb{N}$ such that $T_{\tilde{F},\gamma_n} \colon H^2(\gamma_n) \to H^2(\gamma_n)$ is not injective for $n > N$. By the Coburn–Simonenko theorem (Theorem 4.5), $T_{\tilde{F},\gamma_n}$ has dense image in $H^2(\gamma_n)$. Let $[(f,V)]_{\sim_K} \in H(K)$. Then $f \in H^2(\gamma_n)$ for n large enough. There is therefore a sequence $(g_\nu) \subset H^2(\gamma_n)$ such that $T_{\tilde{F},\gamma_n} g_\nu \to f$ in $H^2(\gamma_n)$ as $\nu \to \infty$. This implies that $T_{\tilde{F},\gamma_n} g_\nu \to f$ in $H(I(\gamma_n))$. By the definition of the inductive topology we also have $T_{F,K}([(g_\nu, I(\gamma_n))]_{\sim_K}) \to [(f,V)]_{\sim_K}$ in $H(K)$. ■

6. Proofs of main results. The hypotheses of both Theorems 1.2 and 1.3, and consequently also of Theorem 1.6, concern the zeros of the representatives of the symbols $F \in \mathcal{X}(\mathbb{R})$. It is important to realize that they cover all the cases. Indeed, assume that $F \in \mathcal{X}(\mathbb{R})$, $F \neq 0$, is the equivalence class of a function $\tilde{F} \in H(U \setminus K)$, U an open neighborhood of $\mathbb{R}$, K a compact subset of $\mathbb{R}$, and there is a sequence $(z_n) \subset U \setminus K$ such that $\tilde{F}(z_n) = 0$. Naturally (z_n) cannot accumulate at a point in $\mathbb{R} \setminus K$, unless $\tilde{F}$ vanishes identically. It follows from Proposition 3.1 that we can shrink U and enlarge K without affecting the equivalence class $[\tilde{F}]_\sim$ and, as a result, also the operator T_F. Hence we can assume that either there are no real zeros of $\tilde{F}$ or they accumulate at $\pm\infty$, since the real zeros accumulating at a point of K can be swallowed up by a larger compact set K. Similarly, by shrinking U, we can assume that either there are no zeros in $U \setminus \mathbb{R}$ or they accumulate at points which belong to K. This is essentially [8, Proposition 5.14]. This means that one of the four cases listed in Theorem 1.6 holds true.

We now establish some results which are of independent interest and will be used in the proof of Theorem 1.2. In order to prove the next theorem we need some information concerning the adjoint operator of the Toeplitz operator $T_F \colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ (see [20, Proposition 4.1] for the details).

The dual space to $\mathcal{A}(\mathbb{R})$ is isomorphic as a linear space to

$$\operatorname{lim\,ind} H_0(\mathbb{C}_\infty \setminus [-n,n]).$$

The duality is given by

$$\langle [g]_\sim, f \rangle = \frac{1}{2\pi i} \int_\delta g(z) \tilde{f}(z)\, dz, \tag{6.1}$$

where $g \in H_0(\mathbb{C}_\infty \setminus [-n,n])$ for some $n \in \mathbb{N}$ and $[g]_\sim$ is the equivalence class of g in $\operatorname{lim\,ind} H_0(\mathbb{C}_\infty \setminus [-n,n])$. Also, $f = r_U(\tilde{f})$, where $\tilde{f} \in H(U)$ with U a simply connected open neighborhood of $\mathbb{R}$. The C^∞ Jordan curve $\delta \subset U \setminus K$ satisfies $K \subset I(\delta)$.

In the sense of the duality (6.1) the adjoint operator T'_F is defined by

$$T'_F([g]_\sim)(z) := \frac{1}{2\pi i} \int_{-\varDelta} \frac{\tilde{F}(\zeta) g(\zeta)}{\zeta - z}\, d\zeta. \tag{6.2}$$

Strictly speaking, $T'_F([g]_\sim)$ is the equivalence class of the function (6.2) in $\operatorname{lim\,ind} H_0(\mathbb{C}_\infty \setminus [-n,n])$. We explain the notation used in (6.2). Namely, $F =$

$[\tilde{F}] \in \mathcal{X}(\mathbb{R})$ for some $\tilde{F} \in H(U \setminus K)$, where U is an open simply connected neighborhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. Also, $g \in H_0(\mathbb{C}_\infty \setminus L)$ for some $L = [-n, n]$. The C^∞ Jordan curve Δ is contained in $U \setminus (K \cup L)$ and satisfies $K \cup L \subset I(\Delta)$. The point z belongs to the exterior $E(\Delta)$ of Δ. Let K_1 be any connected compact subset of $\mathbb{R}$ such that $K \cup L \subset K_1$. Naturally, for any $z \in \mathbb{C} \setminus K_1$ we can find a C^∞ Jordan curve $\Delta \subset U \setminus K_1$ such that $K_1 \subset I(\Delta)$ and $z \in E(\Delta)$. Hence, (6.2) defines a holomorphic function in $\mathbb{C}_\infty \setminus K_1$ which vanishes at ∞. The arguments similar to those which follow the definition of T_F in Section 3 show that the definition is correct.

We now consider symbols $F \in \mathcal{X}(\mathbb{R})$ which have non-real zeros accumulating at a real point.

THEOREM 6.1. *Assume that $F \in \mathcal{X}(\mathbb{R})$, $F = [\tilde{F}]_\sim$ for $\tilde{F} \in H(U \setminus K)$, $\tilde{F}$ does not vanish identically, where U is an open simply connected neighborhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. Furthermore assume there is a sequence $(z_n) \subset U \setminus \mathbb{R}$ such that $z_n \to z \in K$ and $\tilde{F}(z_n) = 0$. Then the image of $T_F \colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is dense in $\mathcal{A}(\mathbb{R})$.*

Proof. We shall show that T'_F is injective. This readily implies that the range of T_F is dense.

Assume that $T'_F([\tilde{g}]_\sim) = 0$, where $\tilde{g} \in H_0(\mathbb{C}_\infty \setminus L)$ with $L = [-n, n]$ for some $n \in \mathbb{R}$. We shall show that $\tilde{g} \equiv 0$.

Let K_1 be a compact connected subset of $\mathbb{R}$ which contains both K and L. Choose C^∞ Jordan curves γ_{outer} and γ_{inner} in $U \setminus K_1$ such that $K_1 \subset I(\gamma_{\text{inner}})$ and $\gamma_{\text{inner}} \subset I(\gamma_{\text{outer}})$. Let Ω be the domain whose boundary consists of the curves γ_{outer} and γ_{inner}.

By the choice of $\gamma_{\text{inner}}, \gamma_{\text{outer}}$ the product $\tilde{F}\tilde{g}$ is defined on some neighborhood of the closure of Ω. The Cauchy transform $\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g})$ is defined for $z \in \Omega$ by

$$(\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g}))(z) = \frac{1}{2\pi i} \int_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{\text{inner}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta. \tag{6.3}$$

As recalled above, $T'_F([\tilde{g}]_\sim)$ is the equivalence class of the function

$$\frac{1}{2\pi i} \int_{-\gamma_{\text{inner}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta,$$

since $z \in E(\gamma_{\text{inner}})$ and $K \cup L \subset I(\gamma_{\text{inner}})$ by assumption. Since $T'_F([\tilde{g}]_\sim) = 0$, we have

$$(\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g}))(z) = \frac{1}{2\pi i} \int_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in \Omega$. This function belongs to $H(I(\gamma_{\text{outer}}))$, hence it is smooth on γ_{inner}. In particular, by Theorem 4.1 for $z \in \gamma_{\text{inner}}$ we also have

$$(\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g}))(z) = \frac{1}{2\pi i} \int\limits_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta,$$

since $\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g})$ is a smooth function on the boundary of Ω.

On the other hand, by Theorem 4.2 there exists $\Psi \in C^\infty(\overline{\Omega})$ which vanishes to a positive order on $b\Omega$ such that

$$\big(\mathcal{C}_{b\Omega}(\tilde{F}\tilde{g})\big)(z) = (\tilde{F}\cdot\tilde{g})(z) - \frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for all $z \in b\Omega$, in particular for $z \in \gamma_{\text{inner}}$. We conclude that for $z \in \gamma_{\text{inner}}$,

$$\frac{1}{2\pi i} \int\limits_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta = (\tilde{F}\cdot\tilde{g})(z) - \frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}. \tag{6.4}$$

Both sides of the equality are holomorphic in $I(\gamma_{\text{inner}}) \setminus K_1$ and belong to $C^1(I(\gamma_{\text{outer}}) \setminus K_1)$. Since they are equal on γ_{inner}, they must be equal for $z \in I(\gamma_{\text{inner}}) \setminus K_1$.

The sequence (z_n) accumulates at a point $z \in K \subset K_1 \subset I(\gamma_{\text{inner}})$, so we may assume that $z_n \in I(\gamma_{\text{inner}})$ for all n. Since $\tilde{F}(z_n) = 0$ for all n, we have

$$\frac{1}{2\pi i} \int\limits_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z_n}\, d\zeta = -\frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\Psi(\zeta)}{\zeta - z_n}\, d\zeta \wedge d\bar{\zeta}.$$

Both sides are holomorphic in $I(\gamma_{\text{inner}})$. Since they are equal on a set with an accumulation point in $I(\gamma_{\text{inner}})$, they must be equal in $I(\gamma_{\text{inner}})$. Since, as we have argued,

$$\frac{1}{2\pi i} \int\limits_{\gamma_{\text{outer}}} \frac{\tilde{F}(\zeta)\tilde{g}(\zeta)}{\zeta - z}\, d\zeta = (\tilde{F}\cdot\tilde{g})(z) - \frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for all $z \in I(\gamma_{\text{inner}}) \setminus K_1$, we conclude that $\tilde{F}\cdot\tilde{g} \equiv 0$ in $I(\gamma_{\text{inner}}) \setminus K_1$ and hence $\tilde{g} = 0$. ∎

Now we investigate symbols $F \in \mathcal{X}(\mathbb{R})$ which have real zeros going to infinity. The next theorem is essentially [8, Theorem 5.17]. Since some of the original arguments were not precise enough, we repeat the proof.

THEOREM 6.2. *Assume that $F \in \mathcal{X}(\mathbb{R})$ with $F = [\tilde{F}]_\sim$, where $\tilde{F} \in H(U \setminus K)$, U is an open simply connected neighborhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. Furthermore, assume that there is a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ such that $\tilde{F}(x_n) = 0$ for all n. Then if $g \in \mathcal{A}(\mathbb{R})$ belongs to the range of T_F, then*

$$\lim_{n\to\infty} g(x_n) = 0.$$

Hence T_F is not surjective.

Proof. Let $g \in \mathcal{A}(\mathbb{R})$ be such that $g = T_F f$ for some $f \in \mathcal{A}(\mathbb{R})$. Thus there is an open set $V \supset \mathbb{R}$ and a function $\tilde{f} \in H(V)$ such that $f = r_V(\tilde{f})$.

Also, $g = r_W(\tilde{g})$, where W is an open simply connected neighborhood of $\mathbb{R}$ and $\tilde{g} \in H(W)$. As usual we assume that $U \cap V$ is simply connected. Let γ be a C^∞ Jordan curve in $(U \cap V) \setminus K$ such that $K \subset I(\gamma)$. For any $z \in U \cap V$ we can find such a curve with $z \in I(\gamma)$. The Cauchy integral

$$\frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta)\tilde{f}(\zeta)}{\zeta - z}\, d\zeta = (\mathcal{C}_\gamma(\tilde{F}\tilde{f}))(z)$$

then defines a holomorphic function in $U \cap V$ (recall that $U \cap V$ is simply connected, so the definition is correct). By the definition of T_F, the restriction of this function to $\mathbb{R}$ is equal to g. Furthermore, the equality $g = T_F f$ means that there exists an open set $\tilde{W} \subset U \cap V \cap W$ containing $\mathbb{R}$, which may be assumed to be simply connected, such that for $z \in \tilde{W}$,

$$\tilde{g}(z) = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{F}(\zeta)\tilde{f}(\zeta)}{\zeta - z}\, d\zeta \tag{6.5}$$

where $\gamma \subset (U \cap V) \setminus K$ is a C^∞ Jordan curve with $K \subset I(\gamma)$ and $z \in I(\gamma)$. There exists an interval $\mathcal{I} := [a, b]$, $a < b$, contained in $I(\gamma)$. The equality (6.5) holds for all $z \in \mathcal{I}$. Choose a C^∞ Jordan curve $\gamma_1 \subset \tilde{W} \setminus (K \cup \mathcal{I})$ with $K \subset I(\gamma_1)$ and $\mathcal{I} \subset I(\gamma_1)$. Since $\gamma - \gamma_1$ is homologous to zero in $U \cap V \setminus (K \cup \mathcal{I})$, by Cauchy's theorem we have

$$\tilde{g}(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{\tilde{F}(\zeta)\tilde{f}(\zeta)}{\zeta - z}\, d\zeta = (\mathcal{C}_{\gamma_1}(\tilde{F}\tilde{f}))(z) \tag{6.6}$$

for all $z \in \mathcal{I}$. Hence (6.6) holds for all $z \in I(\gamma_1)$.

Since according to Theorem 4.1 the function $\mathcal{C}_{\gamma_1}(\tilde{F} \cdot \tilde{f})$ extends smoothly to γ_1 and $\tilde{g} \in C^\infty(\gamma_1)$, we also have $\tilde{g}(z) = \mathcal{C}_{\gamma_1}(\tilde{F}\tilde{f})(z)$ for all $z \in \gamma_1$. Theorem 4.2 yields a function $\Psi \in C^\infty(\overline{I(\gamma_1)})$ vanishing to a positive order on γ_1 such that for all $z \in \gamma_1$,

$$\tilde{g}(z) = \big(\mathcal{C}_{\gamma_1}(\tilde{F}\tilde{f})\big)(z) = (\tilde{F} \cdot \tilde{f})(z) - \frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}. \tag{6.7}$$

Both sides of (6.7) are holomorphic in $\tilde{W} \setminus \overline{I(\gamma_1)}$ and belong to $C^1(\tilde{W} \setminus I(\gamma_1))$. Since the equality holds for all $z \in \gamma_1$, it must hold for all $z \in \tilde{W} \setminus I(\gamma_1)$. Since $\tilde{F}(x_n) = 0$, for sufficiently large n we have

$$\tilde{g}(x_n) = -\frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - x_n}\, d\zeta \wedge d\bar{\zeta}.$$

The function

$$G(z) := \frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

is holomorphic in $\mathbb{C}_\infty \setminus \overline{I(\gamma_1)}$ and vanishes at ∞. This implies that

$$\lim_{n\to\infty} \tilde{g}(x_n) = 0.$$

We have shown that if $\tilde{F}$ vanishes on a sequence (x_n) of real numbers accumulating at ∞ and g belongs to the range of T_F and is represented by $\tilde{g}$, i.e. $g = r_W(\tilde{g})$ for some open neighborhood W of $\mathbb{R}$, then $\tilde{g}(x_n) \to 0$ as $n \to \infty$. Hence also $\lim g(x_n) = 0$. Since the range of T_F does not contain any polynomial, the operator T_F is not surjective. ∎

COROLLARY 6.3. *Assume that* $F \in \mathcal{X}(\mathbb{R})$, $F = [\tilde{F}]_\sim$, *where* $\tilde{F} \in H(U\setminus K)$, *$U$ is an open simply connected neighborhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. Furthermore, assume that there exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ and $\tilde{F}(x_n) = 0$ for all n. If a function $g \in \mathcal{A}(\mathbb{R})$ belongs to the range of the Toeplitz operator T_F, then g vanishes either on almost every point x_n, or only at a finite number of these points.*

Proof. We use the notation from the proof of Theorem 6.2. We have

$$\tilde{g}(z) = \tilde{F} \cdot \tilde{f}(z) - \frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for all $z \in \tilde{W} \setminus I(\gamma_1)$. Since $\tilde{F}(x_n) = 0$, we obtain

$$\tilde{g}(x_n) = -\frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - x_n}\, d\zeta \wedge d\bar{\zeta}$$

for sufficiently large n. If the function

$$G(z) := \frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

vanishes identically, then $\tilde{g}(x_n) = 0$ for almost every x_n. Otherwise, G is holomorphic in $\mathbb{C}_\infty \setminus \overline{I(\gamma_1)}$ and so cannot vanish outside some disc $|z| > R$ with R large enough. Hence, $\tilde{g}(x_n) = -G(x_n)$ does not vanish for n large enough. ∎

COROLLARY 6.4. *Assume that* $F \in \mathcal{X}(\mathbb{R})$, $F = [\tilde{F}]_\sim$, *where* $\tilde{F} \in H(U\setminus K)$, *$U$ is an open simply connected neighborhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. Furthermore, assume that there exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ and $\tilde{F}(x_n) = 0$ for all n. If in the range of T_F there exists a function g such that $g(x_n) = 0$ for almost every n, then $\tilde{F}$ is a meromorphic function with a finite number of real poles.*

Proof. Again we use the notation from the proof of Theorem 6.2. We have shown that

$$\tilde{g}(x_n) = -\frac{1}{2\pi i} \iint_{I(\gamma_1)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = -G(x_n),$$

where Ψ is C^∞ smooth in $\overline{I(\gamma_1)}$ and vanishes to a positive order on γ_1. Since G is holomorphic in $\mathbb{C}_\infty \setminus \overline{I(\gamma_1)}$ and it vanishes on (x_n), which tends to ∞, it must vanish identically. Thus, in view of (6.7),

$$\tilde{g}(z) = \tilde{f}(z) \cdot \tilde{F}(z)$$

for all $z \in \tilde{W} \setminus I(\gamma_1)$. Since $\tilde{g}$ is holomorphic in $\tilde{W}$, we infer that $\tilde{f} \cdot \tilde{F}$ extends holomorphically onto $\tilde{W}$. Thus $\tilde{F}$ is holomorphic in $\tilde{W}$ except at zeros of $\tilde{f}$. Only a finite number of them can be in $\mathbb{R} \cap I(\gamma_1)$. This implies that $\tilde{F}$ is holomorphic on a possibly smaller neighborhood of $\mathbb{R}$ except for a finite number of real poles. ∎

Proof of Theorem 1.2. Let $F \in \mathcal{X}(\mathbb{R})$ satisfy the hypothesis of the theorem. One of the following two possibilities holds true:

- (i) The symbol F is represented by a function $\tilde{F} \in H(U \setminus K)$ with U an open neighborhood of $\mathbb{R}$ and K a compact subset of $\mathbb{R}$ such that $\tilde{F}$ does not vanish in $U \setminus K$.
- (ii) There exists an open set U which contains $\mathbb{R}$, a compact set K contained in $\mathbb{R}$, a sequence $(x_n) \subset \mathbb{R}$ which accumulates at $\pm\infty$ and a function $\tilde{F} \in H(U \setminus K)$ which represents F such that $\tilde{F}$ vanishes in $U \setminus K$ at the points x_n, $n \in \mathbb{N}$, only.

In the former case, T_F is a Fredholm operator according to Theorem 1.1.

Now assume that (ii) holds true. Let V be an open neighborhood of K such that $\tilde{F}$ does not vanish in $V \setminus K$. As usual, we assume that U is simply connected. Similarly, V is simply connected. In [8] we showed:

THEOREM 6.5 ([8, Theorem 5.16]). *Assume that $\tilde{F} \in H(U \setminus K)$ and $\tilde{F}$ does not vanish identically. If there exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ and $\tilde{F}(x_n) = 0$ for all n, then T_F is injective, where $F = [\tilde{F}]_\sim$.*

Thus in order to prove the theorem it suffices to show that the image of

$$T_F \colon \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$$

is closed. Assume that for a net $f_\alpha \in \mathcal{A}(\mathbb{R})$, we have $T_F f_\alpha \to g$ in $\mathcal{A}(\mathbb{R})$ for some $g \in \mathcal{A}(\mathbb{R})$. We need to show that there exists $f \in \mathcal{A}(\mathbb{R})$ such that $T_F f = g$.

We have $|x_n| \to \infty$. We can assume that

$$|x_1| \le |x_2| \le \cdots \tag{6.8}$$

and that the x_n are the only zeros of $\tilde{F}$ in $U \setminus K$. Set $\mathcal{K}_n := [-|x_n|, |x_n|]$. Then $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots$ and $\bigcup_{n=1}^\infty \mathcal{K}_n = \mathbb{R}$. It may however happen that $\mathcal{K}_n = \mathcal{K}_{n+1}$. We now redefine the sets $\mathcal{K}_n$. We proceed by induction. Let K_1 be the first among the sets $\mathcal{K}_m$, $m \in \mathbb{N}$, such that $K = K_0 \subset \mathcal{K}_m$. If the sets $K_1 \subsetneq \cdots \subsetneq K_n$ have already been chosen, let K_{n+1} be the first set $\mathcal{K}_m$ such that $K_n \subsetneq \mathcal{K}_m$.

With this choice of the sets K_n we have

(i) $K_0 \subsetneq K_1 \subsetneq \cdots$;
(ii) $\bigcup_{n=0}^{\infty} K_n = \mathbb{R}$;
(iii) each K_n contains at least n zeros of $\tilde{F}$.

Let γ_n, $n \in \mathbb{N}_0$, be a C^∞ Jordan curve in $U \setminus K_n$ which is contained in $\operatorname{int}(K_{n+1} \times [-\epsilon_{n+1}, \epsilon_{n+1}])$ and contains K_n, i.e. $K_n \subset I(\gamma_n)$. The number $\epsilon_{n+1} > 0$ is chosen in such a way that $K_{n+1} \times [-\epsilon_{n+1}, \epsilon_{n+1}] \subset U$. Note that the choice of K_n guarantees that $\tilde{F}$ does not vanish on γ_n.

For $m \in \mathbb{N}$ let γ_{0m} let be a small C^∞ Jordan curve such that $\gamma_{0m} \subset (V \setminus K) \cap I(\gamma_m)$, $K \subset I(\gamma_{0m})$ and all points x_n, $n \in \mathbb{N}$, belong to the exterior of γ_{0m}. It follows from the construction that

$$n \leq \frac{1}{2\pi i} \int_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{0n}} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta.$$

Now since $\tilde{F}$ does not vanish in a neighborhood of K, the value

$$\frac{1}{2\pi i} \int_{\gamma_{0n}} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta$$

is constant for each $n \in \mathbb{N}$ (as V is simply connected, for any $m, n \in \mathbb{N}$ the curves γ_{0m} and γ_{0n} are homologous in $V \setminus K$). Thus

$$\operatorname{Ind}_{\tilde{F} \circ \gamma_n}(0) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta \to \infty.$$

Observe that $\tilde{F} \neq 0$ on γ_n, since $\tilde{F}$ vanishes on the x_n only. Hence

$$T_{\tilde{F},\gamma_n} \colon H^2(\gamma_n) \to H^2(\gamma_n)$$

is a Fredholm operator and we have

$$\operatorname{index}\big(T_{F,\gamma_n} \colon H^2(\gamma_n) \to H^2(\gamma_n)\big) = -\operatorname{Ind}_{\tilde{F} \circ \gamma_n}(0) \to -\infty.$$

We conclude that there exists n_0 such that for all $n \geq n_0$,

$$\operatorname{index} T_{\tilde{F},\gamma_n} < 0.$$

This implies that for $n \geq n_0$, we must have

$$\dim H^2(\gamma_n)/\operatorname{im} T_{\tilde{F},\gamma_n} > 0. \tag{6.9}$$

By the Coburn–Simonenko theorem either $T_{\tilde{F},\gamma_n}$ is injective or it has dense image. Since $T_{\tilde{F},\gamma_n}$ is a Fredholm operator, the image of $T_{\tilde{F},\gamma_n}$ is closed. By (6.9) it is of positive codimension. Hence for $n \geq n_0$ the operator $T_{\tilde{F},\gamma_n}$ is injective.

To sum up, there exists n_0 such that for any $n \geq n_0$ and any C^∞ Jordan curve $\gamma_n \subset \operatorname{int}(K_{n+1} \times [-\epsilon_{n+1}, \epsilon_{n+1}]) \setminus K_n$ which surrounds K_n, the operator $T_{\tilde{F},\gamma_n}$ is injective.

Consider now the spaces $H(K_n)$ and the operators $T_{F,K_n}\colon H(K_n) \to H(K_n)$ for $n \geq n_0$ studied in Section 5. Since $K \subset K_n$, we may consider F to belong also to $\mathcal{X}(K_n)$ for all n. The space $\mathcal{A}(\mathbb{R})$ carries the projective topology of the system $(R_K\colon \mathcal{A}(\mathbb{R}) \to H(K))_{K\subset\subset\mathbb{R}}$. Since $T_F f_\alpha \to g$ in $\mathcal{A}(\mathbb{R})$, we also have $R_{K_n}T_F f_\alpha \to R_{K_n} g$ in $H(K_n)$ for any n (the restriction operator R_K was defined in (2.3)). By definition of R_{K_n} and T_F, $R_{K_n}T_F f_\alpha$ is the germ on K_n of the function

$$\int\limits_{\Gamma_\alpha} \frac{(\tilde{F}\cdot\tilde{f}_\alpha)(\zeta)}{\zeta - z}\, d\zeta,$$

where $f_\alpha = r_{V_\alpha}(\tilde{f}_\alpha)$ for some $\tilde{f}_\alpha \in H(V_\alpha)$, V_α is an open neighborhood of $\mathbb{R}$ and Γ_α is a C^∞ Jordan curve in $(U \cap V_\alpha) \setminus K_n$ such that $K_n \subset I(\Gamma_\alpha)$. As usual, we assume that $U \cap V_\alpha$ is simply connected. Thus $R_{K_n}T_F f_\alpha = T_{F,K_n}([(\tilde{f}_\alpha, V_\alpha)]_{\sim K_n})$.

By Theorem 5.3 the operators $T_{F,K_n}\colon H(K_n) \to H(K_n)$ are Fredholm operators, since $\tilde{F}$ does not vanish in a small vicinity of K_n. This implies that the image of T_{F,K_n} is closed in $H(K_n)$: the proof is virtually the same as the proof of the fact that a Fredholm operator on $\mathcal{A}(\mathbb{R})$ has closed image, which relies on the fact that $\mathcal{A}(\mathbb{R})$ is ultrabornological and has a web (see [26, Chapter 24] for explanation); for such spaces the open mapping theorem holds. This is the so called de Wilde theory [26, Theorem 24.30]. The space $H(K)$ has the same properties. We conclude that the image of T_{F,K_n} is closed (see [8, Proposition 5.1]) for the details).

Hence for each $n \geq n_0$ there exists a germ $[(\tilde{f}^n, V_n)]_{\sim K_n} \in H(K_n)$ such that

$$T_{F,K_n}([(\tilde{f}^n, V_n)]_{\sim K_n}) = R_{K_n} g$$

in $H(K_n)$. The sets V_n are open neighborhoods of K_n, which may be assumed to be simply connected and such that $U \cap V_n$ is simply connected. The function $g \in \mathcal{A}(\mathbb{R})$ is the restriction to $\mathbb{R}$ of a function $\tilde{g} \in H(W)$, where W is an open neighborhood of $\mathbb{R}$. We claim that the functions $\tilde{f}^n \in H(V_n)$ define an element of $\mathcal{A}(\mathbb{R})$. Indeed, there exists a C^∞ Jordan curve $\tilde{\gamma}_n$ contained in $W \cap V_{n+1} \cap \operatorname{int}(K_{n+1} \times [-\epsilon_{n+1}, \epsilon_{n+1}]) \setminus K_n$ with $K_n \subset I(\tilde{\gamma}_n)$ such that

$$T_{\tilde{F},\tilde{\gamma}_n}(\tilde{f}^{n+1}) = T_{\tilde{F},\tilde{\gamma}_n}(\tilde{f}^n) = \tilde{g}.$$

Naturally, this equality is a consequence of Cauchy's theorem. Thus

$$T_{\tilde{F},\tilde{\gamma}_n}(\tilde{f}^{n+1} - \tilde{f}^n) = 0,$$

and since $T_{\tilde{F},\tilde{\gamma}_n}$ is injective, $\tilde{f}^{n+1} = \tilde{f}^n$ in $H^2(\tilde{\gamma}_n)$. This implies that the Cauchy transforms of $\tilde{f}^n$ and $\tilde{f}^{n+1}$ are equal in $I(\tilde{\gamma}_n)$. This in particular means that $\tilde{f}^n = \tilde{f}^{n+1}$ on K_n. For $z \in \mathbb{R}$, set $f(z) := \tilde{f}^n(z)$ if $z \in K_n$. The definition is correct and f is real analytic on $\mathbb{R}$, since for any $z_0 \in \mathbb{R}$ there is $\delta > 0$ such that for some $n \in \mathbb{N}$ we have $f(z) = \tilde{f}^n(z)$ forall real z with $|z - z_0| < \delta$. It follows from Cauchy's theorem that $T_F f = g$. We have shown that if there is an open set $V \supset K$ such that $F(z) \neq 0$ for all $z \in V \setminus K$, then T_F has closed range.

Assume now that for any $\tilde{F} \in H(U \setminus K)$ with $F = [\tilde{F}]_\sim$, U an open neighborhood of $\mathbb{R}$ and K a compact subset of $\mathbb{R}$, there does not exist an open set $V \supset K$ such that $\tilde{F}$ does not vanish in $V \setminus K$. This means that for any such $\tilde{F}$ there is a sequence $(z_n) \subset \mathbb{C} \setminus K$ which accumulates at points of K and $\tilde{F}(z_n) = 0$ for all n. After shrinking U and enlarging K, which does not affect $[\tilde{F}]_\sim$, we may assume that one of the following cases holds true:

(i) there exists a sequence $(z_n) \subset U \setminus \mathbb{R}$ which accumulates at points of K such that $\tilde{F}$ vanishes in $U \setminus K$ on the z_n only,
(ii) there exists a sequence $(z_n) \subset U \setminus \mathbb{R}$ which accumulates at points of K and there exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \pm\infty$ such that $\tilde{F}$ vanishes in $U \setminus K$ on the z_n and x_n only.

In [8] we proved the following result:

Theorem 6.6 ([8, Theorem 5.15]). *Assume that $\tilde{F} \in H(U \setminus K)$ and there exists a sequence $(z_n) \subset U \setminus \mathbb{R}$ with $z_n \to z \in K$, $\tilde{F}(z_n) = 0$ for all n and $\tilde{F}(z) \neq 0$ in $\mathbb{R} \setminus K$. Then*

$$\dim \ker T_F = \infty,$$

where $F = [\tilde{F}]_\sim$. In particular, $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is not a Fredholm operator.

It follows from Theorem 6.6 that if (i) holds, then T_F is not a Φ_+-operator. We assume therefore that apart from the existence of (z_n) there also exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ such that $\tilde{F}(x_n) = 0$ for all n. By Theorem 6.1 the image of T_F is dense in $\mathcal{A}(\mathbb{R})$, and Theorem 6.2 shows that T_F is not surjective. Hence the image of T_F is not closed and T_F is not a Φ_+-operator. ∎

Observe that we have proved the following fact.

Corollary 6.7. *Assume that $\tilde{F} \in H(U \setminus K)$ satisfies*

(i) *there is an open set $V \supset K$ such that $\tilde{F} \neq 0$ in $V \setminus K$,*
(ii) *there is a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \pm\infty$ such that $\tilde{F}(x_n) = 0$ for all n.*

Then the image of T_F is closed in $\mathcal{A}(\mathbb{R})$, where $F = [\tilde{F}]_\sim$.

Proof of Theorem 1.3. Assume that there is a compact set $L \subset \mathbb{R}$ such that there are no zeros of $\tilde{F}$ in $\mathbb{R} \setminus (K \cup L)$. If there is also an open set $V \supset K$ such that $\tilde{F}(z) \neq 0$ in $V \setminus K$, then T_F is a Fredholm operator (cf. Theorem 1.1), hence also a Φ_--operator. Indeed, we can then choose a smaller $U \supset \mathbb{R}$ such that the zeros of $\tilde{F}$ are all in L. In view of Proposition 3.1 such zeros are negligible. We assume therefore that such a set V does not exist, i.e. there is a sequence $(z_n) \subset \mathbb{C} \setminus \mathbb{R}$ with $z_n \to z \in K$ such that $\tilde{F}(z_n) = 0$ for all n (we remark that the zeros which accumulate at a point of K may be assumed not to be real since otherwise they are negligible according to Proposition 3.1) and the z_n are the only zeros of $\tilde{F}$ in $U \setminus (K \cup L)$. As usual, U is assumed to be simply connected. We show that the operator T_F is then surjective, hence also a Φ_--operator. Indeed, let γ_{outer} be a C^∞ Jordan curve in $U \setminus (K \cup L)$ which does not pass through any zero of $\tilde{F}$ and $K \cup L \subset I(\gamma_{\text{outer}})$. We may assume that the points z_n are all in $I(\gamma_{\text{outer}})$. Choose also a C^∞ Jordan curve $\gamma_n \subset I(\gamma_{\text{outer}}) \setminus (K \cup L)$ with $K \cup L \subset I(\gamma_n)$ such that $z_1, \ldots, z_n \in E(\gamma_n)$ and $z_{n+1}, \ldots \in I(\gamma_n)$. It follows that

$$n \le \frac{1}{2\pi i} \int_{\gamma_{\text{outer}}} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta.$$

This implies that

$$\int_{\gamma_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)}\, d\zeta \to -\infty.$$

Since γ_n contains no zero of $\tilde{F}$, the operator $T_{\tilde{F},\gamma_n} : H^2(\gamma_n) \to H^2(\gamma_n)$ is a Fredholm operator and

$$\operatorname{index} T_{\tilde{F},\gamma_n} = -\operatorname{Ind}_{\tilde{F}\circ\gamma_n}(0) \to \infty.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$\dim\ker T_{\tilde{F},\gamma_n} > 0$$

for any $n \ge n_0$. By the Coburn–Simonenko theorem (Theorem 4.5), the operator $T_{\tilde{F},\gamma_n}$ is surjective, since as a Fredholm operator it has closed range. Let us sum up the argument. We have shown that there exists $n_0 \in \mathbb{N}$ such that for any C^∞ Jordan curve γ with $\gamma \subset I(\gamma_{\text{outer}}) \setminus (K \cup L)$ and $K \cup L \subset I(\gamma)$ which does not pass through any zero of $\tilde{F}$, if $z_1, \ldots, z_n \in E(\gamma)$ for $n \ge n_0$, then the operator $T_{\tilde{F},\gamma} : H^2(\gamma) \to H^2(\gamma)$ is surjective.

Let $g \in \mathcal{A}(\mathbb{R})$. There exists $\tilde{g} \in H(W)$, where W is a simply connected open neighborhood of $\mathbb{R}$, such that $g = r_W(\tilde{g})$. Choose a C^∞ Jordan curve γ with $\gamma \subset W$, $\gamma \subset I(\gamma_{\text{outer}}) \setminus (K \cup L)$, $K \cup L \subset I(\gamma)$, and $z_1, \ldots, z_n \in E(\gamma)$ for $n \ge n_0$. The curve γ is assumed not to contain any zero of $\tilde{F}$.

Naturally, $\tilde{g} \in H^2(\gamma)$, since it is holomorphic on an open set which contains γ. Since $T_{\tilde{F},\gamma} : H^2(\gamma) \to H^2(\gamma)$ is surjective, there is $\tilde{f} \in H^2(\gamma)$ such that $\tilde{g} = T_{\tilde{F},\gamma}\tilde{f}$. Since $K \cup L \subset I(\gamma)$, there are no real zeros of $\tilde{F}$ outside γ. We will show that $\tilde{f}$ can be continued holomorphically on some neighborhood of $\mathbb{R}$. The argument is similar to the proof of Lemma 5.5. Since we cannot apply Lemma 5.5 directly, we repeat the necessary arguments.

Let $\tau(\gamma(t))$ denote the unit tangent vector to the curve γ at the point $\gamma(t)$ pointing in the direction of the orientation of the curve. For a sufficiently small $\epsilon > 0$ the curve $\gamma_\epsilon(t) := \gamma(t) + i\epsilon\tau(\gamma(t))$ is a C^∞ Jordan curve such that $K \cup L \subset I(\gamma_\epsilon)$. Clearly $\gamma_\epsilon \subset I(\gamma)$ and we may assume that no zero of $\tilde{F}$ lies on γ_ϵ. By a standard limit argument in Hardy spaces [1, Theorem 6.3] and Cauchy's theorem,

$$\tilde{g}(z) = \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{(\tilde{F} \cdot \tilde{f})(\zeta)}{\zeta - z}\, d\zeta = \mathcal{C}_{\gamma_\epsilon}(\tilde{F} \cdot \tilde{f})(z) \tag{6.10}$$

for all $z \in I(\gamma_\epsilon)$. Recall that we identify $\tilde{f} \in H^2(\gamma)$ with its Cauchy transform, which is a function holomorphic in $I(\gamma)$ and satisfies the growth estimate (4.2). In (6.10) we also use Proposition 4.4. Both $\tilde{g}$ and $\mathcal{C}_{\gamma_\epsilon}(\tilde{F}\cdot\tilde{f})$ belong to $C^\infty(\overline{I(\gamma_\epsilon)})$, the latter by Theorem 4.1. Since they are equal in $I(\gamma_\epsilon)$, they must be equal on γ_ϵ. Theorem 4.2 yields $\Psi \in C^\infty(\overline{I(\gamma_\epsilon)})$ vanishing to a positive order on γ_ϵ such that for all $z \in \gamma_\epsilon$,

$$\tilde{g}(z) = (\tilde{F} \cdot \tilde{f})(z) - \frac{1}{2\pi i} \iint_{I(\gamma_\epsilon)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

For $z \in \gamma_\epsilon$ we therefore have

$$\tilde{f}(z) = \frac{1}{\tilde{F}(z)}\Big(\tilde{g}(z) + \frac{1}{2\pi i} \iint_{I(\gamma_\epsilon)} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\Big) =: H(z). \tag{6.11}$$

We use here the fact that $\tilde{F}$ does not vanish on γ_ϵ. Since only a finite number of the zeros z_n belong to $E(\gamma_\epsilon)$ and they are not real, we may shrink U and assume that there is no zero of $\tilde{F}$ in $U \cap E(\gamma_\epsilon)$. Thus the right hand side of (6.11) is holomorphic in $E(\gamma_\epsilon) \cap U \cap W$ and belongs to $C^1(\overline{E(\gamma_\epsilon)} \cap U \cap W)$. Shrinking $U \cap W$ if necessary we may assume that it is simply connected. The function $\tilde{f}$ is holomorphic in $I(\gamma)$ and so belongs to $C^1(\overline{E(\gamma_\epsilon)} \cap I(\gamma))$. Since (6.11) holds on γ_ϵ, it must also hold in the open set $E(\gamma_\epsilon) \cap I(\gamma)$. This means that H is the holomorphic extension of $\tilde{f}$ onto some open neighborhood of $\mathbb{R}$. We denote the extended function also by $\tilde{f}$. By Cauchy's theorem $g = T_F f$, where f is the restriction to $\mathbb{R}$ of $\tilde{f}$. This completes the proof of surjectivity of T_F.

Assume that for any $\tilde{F} \in H(U \setminus K)$ which represents F, there does not exist a compact set $L \subset \mathbb{R}$ such that $\tilde{F}$ does not vanish in $\mathbb{R} \setminus (K \cup L)$. This means that for any $\tilde{F}$ such that $F = [\tilde{F}]_\sim$, there exists a sequence $(x_n) \subset \mathbb{R}$ accumulating at $\pm\infty$ such that $\tilde{F}(x_n) = 0$ for all n. By Theorem 6.2, if there is a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \infty$ and $\tilde{F}(x_n) = 0$ for all n, then T_F with $F = [\tilde{F}]_\sim$ is not a Φ_--operator, since the range does not contain any polynomial. ■

Now it suffices to observe that we have also proved Theorem 1.6.

Proof of Theorem 1.6. Part (i) is just Theorem 1.1. Theorem 6.1 says that if there is a sequence $(z_n) \subset \mathbb{C} \setminus \mathbb{R}$ with $z_n \to z \in K$ and $\tilde{F}(z_n) = 0$ for all n, then the image of T_F is dense. If additionally there are no real zeros of $\tilde{F}$ accumulating at $\pm\infty$, then according to Theorem 6.6 the kernel of T_F is of infinite dimension. Also under these assumptions the proof of Theorem 1.3 shows that T_F is surjective. This establishes (ii).

It follows from Theorem 6.5 that if there exists a sequence $(x_n) \subset \mathbb{R}$ with $x_n \to \pm\infty$ and $\tilde{F}(x_n) = 0$ for all n, then T_F is injective. If additionally there are no zeros of $\tilde{F}$ around K, then according to Corollary 6.7 the image of T_F is closed. The fact that the range is of infinite codimension is a consequence of Theorem 6.2. Altogether this proves (iii).

Part (iv) is Theorem 6.5, Theorem 6.1 and Theorem 6.2. Also, as we argued at the beginning of Section 6 we can assume that one of the conditions listed in the theorem holds true. ∎

Acknowledgements. The author thanks the referee for recommending various improvements in exposition.

The research was supported by National Center of Science (Poland), grant no. UMO-2013/10/A/ST1/00091.

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