FUNDAMENTA
MATHEMATICAE
Online First version

Lusternik–Schnirelmann category of relation matrices on finite spaces and simplicial complexes

by

Kohei Tanaka (Nagano)

Abstract. This paper focuses on matrices associated with maps on finite spaces (posets) and simplicial complexes. A map between finite spaces (simplicial complexes) and its homotopy (contiguous) class can be classified by the associated matrix. As an application, we propose a combinatorial method for calculating the Lusternik–Schnirelmann category of a map on finite spaces. Categorical (contractible) subspaces can be distinguished by a fundamental operation on components of the associated matrix.

1. Introduction. The Lusternik–Schnirelmann (LS) category $\mathrm{cat}(X)$ of a space X is a homotopy invariant simply defined by the minimal number (minus one) of categorical (contractible) open sets covering X [LS34]. This idea can be naturally extended to the LS category $\mathrm{cat}(f)$ of a continuous map $f\colon X \to Y$. Although the LS category was originally introduced as a lower bound of the number of critical points on a manifold, nowadays it is used in various areas of algebraic topology (see [CL$^+$03]).

A simplicial approach to LS category has recently been developed [AS13, FMV15, SS17]. For a simplicial map $f\colon K \to L$, the *simplicial LS category* $\mathrm{scat}(f)$ is defined in combinatorial terms using contiguity relation. We have the inequality $\mathrm{cat}(|f|) \leq \mathrm{scat}(f)$, where $|f|$ is the induced map on the geometric realizations $|K| \to |L|$. Although this inequality may be strict, the simplicial approximation theorem provides a better estimate. For the kth barycentric subdivision $\mathrm{sd}^k(K)$ of K and a simplicial approximation $\lambda_k\colon \mathrm{sd}^k(K) \to K$ to the identity on $|K|$, we show that the simplicial LS category $\mathrm{scat}(f \circ \lambda_k)$ becomes smaller and converges to $\mathrm{cat}(|f|)$ as $k \to \infty$ (Theorem 5.4). This fact states that for any continuous map f between finite

2010 *Mathematics Subject Classification*: Primary 55M30; Secondary 06A07, 05E45.
Key words and phrases: Lusternik–Schnirelmann category, Möbius function, finite space, simplicial complex.
Received 30 November 2018; revised 31 August 2019.
Published online 12 December 2019.

DOI: 10.4064/fm674-8-2019

regular cell complexes, the LS category $\mathrm{cat}(f)$ agrees with $\mathrm{scat}(g)$ for some simplicial map g. Moreover, finite simplicial complexes are closely related to finite spaces (posets) even in homotopy theory [Sto66, BM12], hence $\mathrm{cat}(f)$ agrees with $\mathrm{cat}(h)$ for some map h on finite spaces.

This paper proposes a combinatorial way to calculate the LS category of a map on a finite space using the associated relation matrix (filter/ideal relation). A finite space has only finitely many open sets and open covers. Hence, to calculate the LS category, it is important to distinguish categorical open sets. We provide a method for discriminating categorical sets of a map using the associated $\{0, 1\}$-matrix with the operation of switching $\{0, 1\}$-components. The proposed combinatorial approach can contribute to the design and implementation of algorithms for computing the LS category.

The remainder of this paper is organized as follows: Section 2 recalls basic properties and homotopy theory of finite spaces and simplicial complexes. Section 3 discusses a finite setting of Raptis's classification theorem on posets [Rap10]. A map between finite spaces and its homotopy class are completely classified by the associated relation matrix. Moreover, we consider a simplicial analog of Raptis's argument. Section 4 concerns a classical formula for Möbius numbers for relation matrices associated to maps [Bac77]. We provide an alternative proof of the formula by calculating the Möbius number of the associated relation matrix. Section 5 is devoted to the calculation of the LS category of maps on finite spaces and simplicial complexes. We demonstrate how to compute the LS category of a map on finite spaces with the associated relation matrix.

2. Preliminaries

2.1. Finite space. We first recall the homotopy theory of finite spaces established by Stong [Sto66]. In this paper, a *finite space* is a T_0 space consisting of finitely many points. Every point p in a finite space P has the minimal open neighborhood O_p defined as the intersection of all open neighborhoods U of p. A partial order $p \leq q$ can be defined on P by $O_p \supset O_q$.

Moreover, a finite poset P admits a topology, called the *Alexandroff topology*, and can be regarded as a finite space. An open set of P is a subset closed under the upper order (called *filter*). In this respect, finite spaces and finite posets are identified in this study. There is another option, namely, to choose *ideals* closed under the lower order as open sets of P. This space corresponds to the opposite poset P^{op} of P.

A map on finite spaces is continuous if and only if it preserves the order. We should note that an open set of a finite space P corresponds to a continuous map from P to the poset $0 < 1$. Indeed, an open set $Q \subset P$ gives a map $P \to \{0,1\}$ sending $p \in Q$ to 1 and $p \notin Q$ to 0. Conversely, a map $f\colon P \to \{0,1\}$ determines an open set $f^{-1}(1)$ of P.

A partial order can be defined on the set $\mathrm{Map}(P,Q)$ of maps between finite spaces P and Q. For two maps $f, g\colon P \to Q$, the order $f \le g$ implies $f(p) \le g(p)$ for every $p \in P$. Two maps f, g between finite spaces are homotopic if and only if there exists a finite sequence of maps $f = h_0, h_1, \dots, h_n = g$, and either $h_i \le h_{i+1}$ or $h_i \ge h_{i+1}$ holds for each i.

A special case of homotopic maps $f, g\colon P \to Q$ is considered as follows: There exists a point $p \in P$ such that $f(q) = g(q)$ for every $q \ne p$, and either $f(p) \ge g(p)$ or $f(p) \le g(p)$. This is denoted by $f \simeq_p g$. The relation $\simeq_p$ generates the homotopy relation on maps between finite spaces. This fact will be used in Section 5.

Lemma 2.1 ([BM12, Lemma 4.10]). *For homotopic maps $f, g\colon P \to Q$ on finite spaces, there exist maps $f = h_0, h_1, \dots, h_n = g$ and points $p_0, \dots, p_{n-1} \in P$ such that $h_i \simeq_{p_i} h_{i+1}$ for each i.*

2.2. Simplicial complex. We next recall some fundamental properties and homotopy theory of (abstract) simplicial complexes. A basic reference is Spanier's book [Spa66, Chapter 3]. A (finite) *simplicial complex K* consists of a finite set $V(K)$ of vertices and a set $\Sigma(K) \subset 2^{V(K)}$ of simplices satisfying the face relation as follows:

(1) $\{v\} \in \Sigma(K)$ for every $v \in V(K)$.
(2) For any simplex $\sigma \in \Sigma(K)$, every non-empty subset $\tau \subset \sigma$ belongs to $\Sigma(K)$.

We deal only with finite simplicial complexes in this study. An n-simplex $\sigma \in \Sigma(K)$ consists of $n+1$ vertices, and its dimension is $\dim(\sigma) = n$. A *simplicial map* $f\colon K \to L$ between simplicial complexes is a map $f\colon V(K) \to V(L)$ on vertices with $f(\sigma) \in \Sigma(L)$ for $\sigma \in \Sigma(K)$.

Definition 2.2. Two simplicial maps $f, g\colon K \to L$ are *contiguous*, denoted by $f \simeq g$, if $f(\sigma) \cup g(\sigma) \in \Sigma(L)$ for every $\sigma \in \Sigma(K)$.

For a simplicial complex K, the (geometric) *realization* $|K|$ is a space constructed by gluing geometric simplices

$$|\sigma| = \left\{ (x_0, \dots, x_{\dim(\sigma)}) \in \mathbb{R}^{\dim(\sigma)+1} \;\middle|\; 0 \le x_i \le 1, \sum_i x_i = 1 \right\}$$

for $\sigma \in \Sigma(K)$. A simplicial map $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$.

Definition 2.3. For a simplicial complex K, the *barycentric subdivision* $\mathrm{sd}(K)$ is a simplicial complex defined as follows: The set $V(\mathrm{sd}(K))$ of vertices consists of the barycenters b_σ of the realized simplices $|\sigma|$ for $\sigma \in \Sigma(K)$, and an n-simplex of $\mathrm{sd}(K)$ is formed by

$$\{b_{\sigma_0}, \ldots, b_{\sigma_n} \mid \sigma_0 \subset \cdots \subset \sigma_n\}.$$

Let $\mathrm{sd}^k(K)$ denote the kth iterated barycentric subdivision, $\mathrm{sd}^k(K) = \mathrm{sd}(\mathrm{sd}^{k-1}(K))$.

The natural inclusion induces a homeomorphism $|\mathrm{sd}(K)| \overset{\cong}{\hookrightarrow} |K|$ on the realizations, and $|\mathrm{sd}^k(K)|$ is identified with $|K|$ for any $k \geq 0$.

Definition 2.4. Let K be a simplicial complex and P be a finite space.

- The *face poset* $\mathcal{F}(K) = \Sigma(K)$ consists of simplices of K. In this study, the order on $\mathcal{F}(K)$ is defined by reverse inclusion, i.e., $\sigma \leq \tau$ in $\mathcal{F}(K)$ implies $\sigma \supset \tau$.
- The *order complex* $\mathcal{K}(P)$ is a simplicial complex with $V(\mathcal{K}(P)) = P$ and totally ordered subsets (chains) of P as simplices. It should be noted that $\mathcal{K}(P) = \mathcal{K}(P^{\mathrm{op}})$ for the opposite finite space P^{op}.

Remark 2.5. For a simplicial complex K, the barycentric subdivision $\mathrm{sd}(K)$ is isomorphic to $\mathcal{K}(\mathcal{F}(K))$.

Definition 2.6. For a finite space P, the *barycentric subdivision* $\mathrm{sd}(P)$ is the finite space defined as $\mathcal{F}(\mathcal{K}(P))$. It is equipped with a natural map $\tau_P\colon \mathrm{sd}(P) \to P$ sending $p_0 < \cdots < p_n$ to the initial element p_0. This map was introduced in [HV93] to study homotopical properties of barycentric subdivisions of finite spaces, and used in [BM08] to study simple homotopy types of finite spaces.

The simplicial approximation theorem plays a central role in the homotopy theory on simplicial complexes and their realizations.

Definition 2.7. Let $\varphi\colon |K| \to |L|$ be a continuous map between the realizations of simplicial complexes K and L. A simplicial map $f\colon K \to L$ is called a *simplicial approximation* to φ if $\varphi(\alpha) \in |\sigma|$ implies $|f|(\alpha) \in |\sigma|$ for every $\alpha \in |K|$ and $\sigma \in \Sigma(L)$.

Theorem 2.8 ([Spa66, Theorem 3.5.8]). *Let $\varphi\colon |K| \to |L|$ be a continuous map between the realizations of simplicial complexes K and L. For sufficiently large $k \geq 0$, there exists a simplicial approximation $f\colon \mathrm{sd}^k(K) \to L$ to φ.*

We refer the reader to [Spa66, Chapter 3] for details on simplicial approximations.

3. Classification theorem for finite spaces and simplicial complexes

3.1. Raptis's classification theorem for finite spaces. It is well known that the classifying space $\mathcal{B}G$ of a topological group G classifies principal G-bundles. The construction of classifying spaces can be applied not only to groups but also to posets, small categories, and more general topological categories. A natural question to ask is what these generalized classifying spaces classify. Several studies have been concerned with this question [Moe95, MW07, Seg78, Wei05], in particular, Raptis focused on the case of (not necessarily finite) posets and showed that the classifying space $\mathcal{B}P = |\mathcal{K}(P)|$ of a poset P classifies open covers well-indexed by P [Rap10, Section 5]. An open cover $\{U_p \mid p \in P\}$ of an arbitrary space X is *well-indexed* by P if it satisfies the following two conditions:

(1) $U_p \subset U_q$ for $p \geq q$.
(2) For each $x \in X$, the subset $\{p \in P \mid x \in U_p\}$ has a unique maximal element in P.

This definition is now considered in the finite setting using $\{0,1\}$-matrices.

Definition 3.1. Let X and P be finite spaces. A *filter relation* R of X and P^{op} is an open set of $X \times P^{\mathrm{op}}$. We should note that choosing an open set of a finite space Q is equivalent to determining an order-preserving map $Q \to \{0,1\}$, where $\{0,1\}$ is regarded as a poset with the order $0 < 1$. Hence, we can identify a filter relation R of X and P with a map

$$R\colon\ X \times P^{\mathrm{op}} \to \{0,1\}.$$

When we regard a filter relation as a map, we call it a *relation matrix.* A filter relation (relation matrix) R of X and P^{op} is called *principal* if $R(x) = \{p \in P \mid (x,p) \in R\}$ has a unique maximal element p_x in P for each $x \in X$.

Remark 3.2. A relation matrix $X \times P^{\mathrm{op}} \to \{0,1\}$ corresponds to a continuous map

$$P^{\mathrm{op}} \to \mathrm{Map}(X, \{0,1\})$$

by the exponential law. We have a natural homeomorphism from $\mathrm{Map}(X, \{0,1\})$ to the finite space $\mathcal{O}(X)$ of open sets of X with the inclusion order. In this respect, a relation matrix of X and P can be identified with a continuous map

$$P^{\mathrm{op}} \to \mathcal{O}(X).$$

A principal relation matrix of X and P^{op} ensures that $R(x) \neq \emptyset$ for every x, and the above associated map $P^{\mathrm{op}} \to \mathcal{O}(X)$, $p \mapsto U_p$, gives rise to an open cover $\{U_p \mid p \in P\}$ of X. Thus, our principal relation matrices agree with open coverings of X well-indexed by P. To combinatorially calculate the LS category in Section 5, we primarily employ descriptions based on matrices rather than open covers.

Definition 3.3. For a continuous map $f\colon X \to P$ between finite spaces, a relation matrix $f^*\colon X \times P^{\mathrm{op}} \to \{0,1\}$ is defined as follows:

$$f^*(x,p) = \begin{cases} 1 & \text{if } f(x) \geq p \text{ in } P, \\ 0 & \text{otherwise.} \end{cases}$$

This relation matrix is principal because $f^*(x) = \{p \in P \mid f(x) \geq p\}$ has the unique maximal element $p_x = f(x)$ in P. Moreover, for a principal relation matrix $R\colon X \times P^{\mathrm{op}} \to \{0,1\}$, a continuous map $R^*\colon X \to P$ is obtained by $R^*(x) = p_x$.

Let $\mathrm{RM}(X,P)$ denote the finite space of principal relation matrices of X and consider P^{op} as a subspace of $\mathrm{Map}(X \times P^{\mathrm{op}}, \{0,1\})$. Raptis showed that the above correspondences $f \mapsto f^*$ and $R \mapsto R^*$ determine bijections between $\mathrm{Map}(X,P)$ and $\mathrm{RM}(X,P)$. We immediately notice that these correspondences preserve orders on the two mapping spaces.

Proposition 3.4 ([Rap10, Proposition 5.2]). *For finite spaces X and P, there is a natural homeomorphism between $\mathrm{Map}(X,P)$ and $\mathrm{RM}(X,P)$ as finite spaces.*

Remark 3.5. A map $\varphi\colon X \to Y$ on finite spaces induces the pull-back

$$\varphi^*\colon \mathrm{RM}(Y,P) \to \mathrm{RM}(X,P)$$

by precomposing with the map $\varphi \times \mathrm{id}_{P^{\mathrm{op}}}$. The *universal relation matrix* of a finite space P is the adjacency matrix $\upsilon\colon P \times P^{\mathrm{op}} \to \{0,1\}$ given by $\upsilon(q,p) = 1$ if and only if $q \geq p$. This corresponds to the identity map on P. For a map $f\colon X \to P$ of finite spaces, the associated relation matrix f^* in Definition 3.3 agrees with the pull-back $f^*(\upsilon)$.

Now we consider homotopy classes of maps between finite spaces. Let us recall the notion of *concordance* between open covers of an arbitrary space X. Two coverings $\mu, \nu\colon P^{\mathrm{op}} \to \mathcal{O}(X)$ of X well-indexed by P are *concordant* if there exists a well-indexed open cover $\xi\colon P^{\mathrm{op}} \to \mathcal{O}(X \times [0,1])$ that restricts to μ and ν at $X \times \{0\}$ and $X \times \{1\}$, respectively.

Lemma 3.6. *For finite spaces X and P, two open coverings $P^{\mathrm{op}} \to \mathcal{O}(X)$ well-indexed by P are concordant if and only if they are homotopic as maps.*

Proof. Suppose that two well-indexed open coverings μ and ν are concordant by a well-indexed open covering $P^{\mathrm{op}} \to \mathcal{O}(X \times [0,1])$. This corresponds to a homotopy $H\colon X \times [0,1] \to P$ by [Rap10, Proposition 5.2]. By the homotopy theory of finite spaces, there are maps $H_0 = h_0, h_1, \ldots, h_n = H_1\colon X \to P$ such that for each i, either $h_i \leq h_{i+1}$ or $h_i \geq h_{i+1}$. Furthermore, either $h_i^* \leq h_{i+1}^*$ or $h_i^* \geq h_{i+1}^*$ as maps $P^{\mathrm{op}} \to \mathcal{O}(X)$. Since $\mu = h_0^*$ and $\nu = h_n^*$, these are homotopic.

Conversely, suppose that $\mu, \nu \colon P^{\mathrm{op}} \to \mathcal{O}(X)$ are homotopic as maps. Without loss of generality, we may assume that $\mu \leq \nu$. This implies that $\mu^* \leq \nu^*$ as maps $X \to P$. A continuous map $\iota\colon [0,1] \to \{0,1\}$ defined by $\iota(0) = 0$ and $\iota(0,1] = 1$ induces a homotopy connecting μ^* and ν^* as follows:

$$X \times [0,1] \xrightarrow{\mathrm{id}_X \times \iota} X \times \{0,1\} \xrightarrow{\mu^* \amalg \nu^*} P.$$

This determines a map $P^{\mathrm{op}} \to \mathcal{O}(X \times [0,1])$ and so μ and ν are concordant. ∎

By the above lemma, for finite spaces X and P, the set $\mathrm{RM}[X,P]$ of homotopy classes of maps in $\mathrm{RM}(X,P)$ and the set $[\mathrm{Cov}_P(X)]$ of concordance classes of $\mathrm{Cov}_P(X)$ can be identified. The proposition below follows immediately by using the maps in Definition 3.3.

Proposition 3.7 ([Rap10, Proposition 5.4]). *For finite spaces P and X, there is a natural bijection between $[X,P]$ and $\mathrm{RM}[X,P]$.*

Remark 3.8. McCord presented a weak homotopy equivalence $\psi\colon \mathcal{B}P \to P$ for a finite space P (see [McC66]). This is defined by $\psi(x) = p_n$ for a point $x \in \mathcal{B}P$ lying on the interior of the realization of a simplex $p_0 < \cdots < p_n$ in $\mathcal{K}(P)$. McCord's weak homotopy equivalence induces an isomorphism $[X, \mathcal{B}P] \cong [X,P]$ for a CW complex X. Although Raptis proved that $[X,\mathcal{B}P] \cong [\mathrm{Cov}_P(X)]$ using this fact, it cannot be applied to non-Hausdorff spaces, in particular to finite spaces X of this study.

3.2. The classification theorem for simplicial complexes. This subsection is concerned with what the order complex $\mathcal{K}(P)$ of a finite space P classifies. We focus on contiguity classes of simplicial maps $\mathrm{sd}^k(K) \to \mathcal{K}(P)$ between the kth iterated barycentric subdivision of a simplicial complex K and $\mathcal{K}(P)$.

Remark 3.9. Let K be a simplicial complex and let P be a poset. A relation matrix of $\mathcal{F}(K)$ and P^{op} can be regarded as an open cover $P^{\mathrm{op}} \to \mathcal{O}(\mathcal{F}(K))$ of the finite space $\mathcal{F}(K)$ well-indexed by P. Moreover, the finite space $\mathcal{O}(\mathcal{F}(K))$ coincides with the finite space $\mathrm{Sub}(K)$ of subcomplexes of K with the inclusion order. Therefore, a principal relation matrix of $\mathcal{F}(K)$ and P^{op} can be identified with a cover of subcomplexes $P^{\mathrm{op}} \to \mathrm{Sub}(K)$ of K well-indexed by P.

For a simplicial map $f\colon K \to \mathcal{K}(P)$, a relation matrix $f^*\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$ is defined as follows:

$$f^*(\sigma,p) = \begin{cases} 1 & \text{if } \min f(\sigma) \geq p \text{ in } P, \\ 0 & \text{otherwise.} \end{cases}$$

It should be noted that σ is a subset of the vertices of K and $f(\sigma)$ becomes a totally ordered subset of P. This relation matrix is principal because $P_\sigma = \{p \in P \mid p \le \min f(\sigma)\}$ has the unique maximal element $p_\sigma = \min f(\sigma)$ in P.

Moreover, for a principal relation matrix $R\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$, we obtain a map $\widetilde{R}\colon \mathcal{F}(K) \to P$ sending σ to the unique maximal element p_σ in $R(\sigma)$ (recall Definition 3.1). It induces a simplicial map on the order complexes

$$R^* = \mathcal{K}(\widetilde{R})\colon \mathrm{sd}(K) \to \mathcal{K}(P).$$

We notice that this situation is slightly different from that in the previous subsection on finite spaces. We should take the barycentric subdivision into account.

Proposition 3.10. *For a simplicial map $f\colon K \to \mathcal{K}(P)$, there is a simplicial approximation $\lambda\colon \mathrm{sd}(K) \to K$ to the identity on $|K|$ with $f \circ \lambda = f^{**}\colon \mathrm{sd}(K) \to \mathcal{K}(P)$.*

Proof. For a simplex σ of K (a vertex of $\mathrm{sd}(K)$), the image $f(\sigma)$ is totally ordered in P. Numbers can be assigned to the vertices of a simplex $\sigma = \{v_0, v_1, \ldots, v_m\}$ with $f(v_0) \le f(v_1) \le \cdots \le f(v_m)$, and a vertex $v_0 = \lambda(\sigma)$ can be chosen so that $f(v_0)$ is the minimal element in $f(\sigma)$. This gives rise to a simplicial map $\lambda\colon \mathrm{sd}(K) \to K$. Indeed, for an element $S = \sigma_0 < \sigma_1 < \cdots < \sigma_n$ of $\mathrm{sd}(K)$ (a decreasing sequence of simplices $\sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_n$ in K), the vertices $\lambda(\sigma_i)$ span a simplex of K as a face of σ_0. We can easily verify that this is a simplicial approximation to the identity on $|K|$ and $f \circ \lambda = f^{**}$. ∎

Proposition 3.11. *For a principal relation matrix $R\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$, we have*

$$R^{**} = R \circ \left(\tau_{\mathcal{F}(K)} \times \mathrm{id}_{P^{\mathrm{op}}}\right) : \mathcal{F}(\mathrm{sd}(K)) \times P^{\mathrm{op}} \to \{0,1\},$$

where $\mathcal{F}(\mathrm{sd}(K))$ and $\mathrm{sd}(\mathcal{F}(K))$ are identified.

Proof. For an element $S = \sigma_0 < \sigma_1 < \cdots < \sigma_n$ of $\mathrm{sd}(\mathcal{F}(K)) = \mathcal{F}(\mathrm{sd}(K))$ expressed as a sequence of simplices of K, we have $R^{**}(S,p) = 1$ if and only if

$$p \le \min R^*(S) = \min\{p_{\sigma_i} \in P\} = \min\{p_{\sigma_i} \in P \mid p_{\sigma_0} \le \cdots \le p_{\sigma_n}\} = p_{\sigma_0}.$$

This inequality implies that $R(\sigma_0, p) = 1$. Recall the associated map

$$\tau_{\mathcal{F}(K)}\colon \mathrm{sd}(\mathcal{F}(K)) \to \mathcal{F}(K)$$

to the barycentric subdivision in Definition 2.6. We have

$$R^{**}(S,p) = R(\sigma_0, p) = \mu \circ \left(\tau_{\mathcal{F}(K)} \times \mathrm{id}_{P^{\mathrm{op}}}\right)(S,p)$$

for any $(S,p) \in \mathrm{sd}(\mathcal{F}(K)) \times P^{\mathrm{op}}$. ∎

Proposition 3.12. *For two contiguous simplicial maps $f, g\colon K \to \mathcal{K}(P)$, the induced relation matrices $f^*, g^*\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$ are homotopic.*

Proof. For any simplex σ of K, the union $f(\sigma) \cup g(\sigma)$ is totally ordered in P by the contiguity relation. A relation matrix $R\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$ is defined by

$$R(\sigma, p) = \begin{cases} 1 & \text{if } \min(f(\sigma) \cup g(\sigma)) \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

We have $f^* \geq R \leq g^*$ and then $f^* \simeq g^*$. ■

PROPOSITION 3.13. *For two homotopic principal relation matrices $R, T\colon \mathcal{F}(K) \times P^{\mathrm{op}} \to \{0,1\}$, the induced simplicial maps R^*, T^* are contiguous.*

Proof. Without loss of generality, we may assume that $R \leq T$. The induced maps $\widetilde{R}, \widetilde{T}\colon \mathcal{F}(K) \to P$ satisfy the inequality $\widetilde{R} \leq \widetilde{T}$. By [BM12, Proposition 4.11], the maps $R^* = \mathcal{K}(\widetilde{R})$ and $T^* = \mathcal{K}(\widetilde{T})$ are contiguous. ■

REMARK 3.14. For the natural map $\tau_P\colon \mathrm{sd}(P) \to P$, the induced map $\mathcal{K}(\tau_P)$ on the order complexes is a simplicial approximation to the identity on the classifying space $\mathcal{B}P$.

Moreover, for a simplicial complex K and a simplicial approximation

$$\lambda\colon \mathrm{sd}(K) \to K$$

to the identity on $|K|$, the induced map

$$\mathcal{F}(\lambda)\colon \mathrm{sd}(\mathcal{F}(K)) = \mathcal{F}(\mathrm{sd}(K)) \to \mathcal{F}(K)$$

is homotopic to $\tau_{\mathcal{F}(K)}$. Indeed, $\tau_{\mathcal{F}(K)}(S) = \sigma_0$ for an element $S = \sigma_0 \leq \cdots \leq \sigma_n$ of $\mathcal{F}(\mathrm{sd}(K))$. Furthermore,

$$\mathcal{F}(\lambda)(S) = \{\lambda(\sigma_0), \ldots, \lambda(\sigma_n)\}$$

is a face of σ_0 and $\mathcal{F}(\lambda) \leq \tau_{\mathcal{F}(K)}$.

DEFINITION 3.15. For finite spaces X and P, recall the set $\mathrm{RM}(X, P)$ of principal relation matrices of X and P^{op}, and the set $\mathrm{RM}[X, P]$ of homotopy classes. The maps $\tau_i = \tau_{\mathrm{sd}^i(X)}\colon \mathrm{sd}^i(X) \to \mathrm{sd}^{i-1}(X)$ induce the following sequence:

$$\mathrm{RM}[X, P] \xrightarrow{\tau_1^*} \mathrm{RM}[\mathrm{sd}(X), P] \xrightarrow{\tau_2^*} \mathrm{RM}[\mathrm{sd}^2(X), P] \to \cdots .$$

Let $\mathrm{RM}[X, P]_{\mathrm{sd}}$ denote the colimit of this sequence.

Moreover, for simplicial complexes K and L, let $\mathrm{Map}(K, L)$ denote the set of simplicial maps from K to L, and $[K, L]$ the set of contiguity classes. Simplicial approximations $\lambda_i\colon \mathrm{sd}^i(K) \to \mathrm{sd}^{i-1}(K)$ to the identity on $|K|$ induce the following sequence:

$$[K, L] \xrightarrow{\lambda_1^*} [\mathrm{sd}(K), L] \xrightarrow{\lambda_2^*} [\mathrm{sd}^2(K), L] \to \cdots .$$

We should note that the induced map λ_i^* does not depend on the choice of the approximation λ_i, for each i. Let $[K, L]_{\mathrm{sd}}$ denote the colimit of this sequence.

THEOREM 3.16. *For a simplicial complex K and a finite space P, there is a natural isomorphism between $[K, \mathcal{K}P]_{\mathrm{sd}}$ and $\mathrm{RM}[\mathcal{F}(K), P]_{\mathrm{sd}}$.*

Proof. A map $\varphi\colon [K, \mathcal{K}P]_{\mathrm{sd}} \to \mathrm{RM}[\mathcal{F}(K), P]_{\mathrm{sd}}$ is given by $\varphi[f] = [f^*]$. This does not depend on the choice of the representative and is well-defined by Remark 3.14 and Proposition 3.12. Similarly, the map

$$\psi\colon \mathrm{RM}[\mathcal{F}(K), P]_{\mathrm{sd}} \to [K, \mathcal{K}P]_{\mathrm{sd}}$$

given by $\varphi[\mu] = [\mu^*]$ is well-defined by Remark 3.14 and Proposition 3.13. We can verify that the maps φ and ψ are inverses to each other by Propositions 3.10 and 3.11. ∎

REMARK 3.17. Theorem 3.16 presents an alternative description of Raptis's classification theorem [Rap10, Proposition 5.4] for finite simplicial complexes and finite spaces. The simplicial approximation theorem yields an isomorphism between $[|K|, \mathcal{B}P]$ and $[K, \mathcal{K}(P)]_{\mathrm{sd}}$ ([Spa66, Theorem 3.5.8]) for a simplicial complex K and a finite space P. Hence, there are natural bijections

$$\mathrm{RM}[\mathcal{F}(K), P]_{\mathrm{sd}} \cong [K, \mathcal{K}P]_{\mathrm{sd}} \cong [|K|, \mathcal{B}P] \cong [\mathrm{Cov}_P(|K|)].$$

Here the last bijection comes from [Rap10]. Note that $\mathrm{RM}(\mathcal{F}(\mathrm{sd}^k(K)), P)$ is a subset of maps $P^{\mathrm{op}} \to \mathcal{O}(\mathcal{F}(\mathrm{sd}^k(K)))$ for $k \geq 0$ and $\mathrm{Cov}_P(|K|)$ is a subset of maps $P^{\mathrm{op}} \to \mathcal{O}(|K|)$. The above isomorphism

$$\Phi\colon \mathrm{RM}[\mathcal{F}(K), P]_{\mathrm{sd}} \to [\mathrm{Cov}_P(|K|)]$$

is induced by McCord's map $|K| = |\mathrm{sd}^{k+1}(K)| \to \mathcal{F}(\mathrm{sd}^k(K))$. Here, McCord's map determines a map $\mathrm{Sub}(\mathrm{sd}^k(K)) = \mathcal{O}(\mathcal{F}(\mathrm{sd}^k(K))) \to \mathcal{O}(|K|)$ sending a subcomplex L in $\mathrm{sd}^k(K)$ to the open star $\mathrm{st}(L) = \bigcup_{v \in L} \mathrm{st}(v)$ in $|K|$. This fact suggests that any open cover of $|K|$ well-indexed by P can be expressed up to homotopy as a well-indexed cover of open stars of subcomplexes in the iterated barycentric subdivision $\mathrm{sd}^k(K)$ for some $k \geq 0$.

4. Möbius numbers of filter and ideal relations.

Filter and ideal relations of finite posets play an important role in the calculation of Euler characteristic and Möbius number [Wal81, Section 3]. The *Euler characteristic* $\chi(P)$ of a finite poset P is defined as the alternate sum of numbers of n-chains of P:

$$\chi(P) = \sum_{n \geq 0} (-1)^n C_n(P),$$

where $C_n(P)$ is the number of totally ordered subsets of P with $n+1$ elements. This agrees with the standard topological Euler characteristic of the classifying space of P:

$$\chi(P) = \chi(\mathcal{B}P) = \sum_{n \geq 0} (-1)^n \operatorname{rank}(\mathrm{H}_n(\mathcal{B}P)).$$

The *Möbius number* of a finite poset P is a similar invariant to Euler characteristic defined by $\mu(P) = \chi(P) - 1$. Note that $\chi(P^{\mathrm{op}}) = \chi(P)$ and $\mu(P^{\mathrm{op}}) = \mu(P)$. These concepts are closely related to the *Möbius function* $\mu\colon P \times P \to \mathbb{Z}$ defined as follows:

$$\mu(x,y) = \begin{cases} \mu(\{p \in P \mid x < p < y) & \text{if } x < y, \\ 1 & \text{if } x = y, \\ 0 & \text{if } x \not\leq y. \end{cases}$$

The Euler characteristic of a finite poset P can be calculated as the sum

$$\chi(P) = \sum_{p,q \in P} \mu(p,q),$$

and the Möbius number $\mu(P)$ is the value $\mu(\hat{0}, \hat{1})$ of the Möbius function on the extended poset $\hat{P} = P \amalg \{\hat{0}, \hat{1}\}$, where $\hat{0}$ and $\hat{1}$ are the unique minimal and maximal element respectively. See [Rot64] for the details on the Möbius function and Euler characteristic for finite posets.

For a filter relation $R \subset P \times Q^{\mathrm{op}}$, the *R-join* $P +_R Q^{\mathrm{op}}$ is given as a poset whose underlying set is the disjoint union $P \amalg Q$. The order $x \leq y$ on $P +_R Q^{\mathrm{op}}$ is defined by

(1) $x \leq y$ in P^{op},
(2) $x \leq y$ in Q^{op}, or
(3) $x \in P$, $y \in Q$, and $(x, y) \in R$.

Although [Bac77] and [Wal81] employed ideal relations, we consider filter relations in this study. Walker calculated the Möbius number of R-join by directly counting chains.

THEOREM 4.1 ([Wal81, Theorem 3.1]). *Let R be a filter relation of two finite posets P and Q, and denote $P_{>p} = \{x \in P \mid x > p\}$ and $Q_{<q} = \{x \in Q \mid x < q\}$ for $p \in P$ and $q \in Q$. We have*

$$\mu(P +_R Q^{\mathrm{op}}) = \begin{cases} \mu(Q) + \sum_{p \in P} \mu(P_{>p})\mu(R(p)), \\ \mu(P) + \sum_{q \in Q} \mu(Q_{<q})\mu(R^{-1}(q)), \end{cases}$$

where $R(p) = \{q \in Q^{\mathrm{op}} \mid (p,q) \in R\}$ and $R^{-1}(q) = \{p \in P \mid (p,q) \in R\}$.

We will calculate the Möbius number $\mu(R)$ of a filter relation R as a dual statement of the above. This immediately follows from Leinster's product formula for Euler characteristic of Grothendieck construction [Lei08]. Let A be a poset regarded as a small category and let $F\colon A \to \mathbf{Posets}$ be a functor from A to the category of posets. The *Grothendieck construction* (poset limit) $\mathrm{Gr}(F)$ is a poset with the underlying set $\{(a, x) \in A \times Fa\}$ and $(a, x) \leq (b, y)$ given by $a \leq b$ in A and $F(a \leq b)(x) \leq y$.

THEOREM 4.2. *If F is a functor from a finite poset A to the category of finite posets, then*

$$\mu(\mathrm{Gr}(F)) = \mu(A) - \sum_{a \in A} \mu(A_{>a})\mu(Fa).$$

Proof. Recall a weighting $w\colon A \to \mathbb{Z}$ on A defined by $w(a) = \sum_{b \in B} \mu(a, b)$. By Hall's Theorem ([Rot64, Proposition 6] and [Lei08, Corollary 1.5]),

$$w(a) = \sum_{n \geq 0} (-1)^n |\{n\text{-chains in } A \text{ starting from } a\}|.$$

Note that an n-chain in A starting from a corresponds to an $(n-1)$-chain in $A_{>a}$. Hence, $w(a) = -\mu(A_{>a})$ for every $a \in A$. Our desired formula straightforwardly follows from [Lei08, Proposition 2.8]:

$$\begin{aligned}
\mu(\mathrm{Gr}(F)) &= \chi(\mathrm{Gr}(F)) - 1 = \sum_{a \in A} w(a)\chi(Fa) - 1 \\
&= \sum_{a \in A} w(a)\left(\mu(Fa) + 1\right) - 1 = \sum_{a \in A} w(a)\mu(Fa) + \chi(A) - 1 \\
&= \mu(A) - \sum_{a \in A} \mu(A_{>a})\mu(Fa). \ \blacksquare
\end{aligned}$$

Given a filter relation $R \subset P \times Q^{\mathrm{op}}$, we have two canonical functors $F_R\colon P \to \mathbf{Posets}$ and $F^R\colon Q^{\mathrm{op}} \to \mathbf{Posets}$ given by $F_R(p) = R(p)$ and $F^R(q) = R^{-1}(q)$, respectively. It should be noted that both Grothendieck constructions $\mathrm{Gr}(F_R)$ and $\mathrm{Gr}(F^R)$ agree with the original relation filter R. Applying Theorem 4.2 to F_R and F^R, we obtain the following formulas.

COROLLARY 4.3. *If R is a filter relation of two finite posets P and Q, then*

$$\mu(R) = \begin{cases} \mu(Q) - \sum_{q \in Q} \mu(Q_{<q})\mu(R^{-1}(q)), \\ \mu(P) - \sum_{p \in P} \mu(P_{>p})\mu(R(p)). \end{cases}$$

COROLLARY 4.4. *If R is a filter relation of two finite posets P and Q, then*

$$\mu(R) + \mu(P +_R Q^{\mathrm{op}}) = \mu(P) + \mu(Q).$$

Given a map between finite posets $f\colon P \to Q$, the associated filter relation $f^* \subset P \times Q^{\mathrm{op}}$ is homotopy equivalent to P. Indeed, we have the projection $f^* \to P$ and its homotopy inverse $P \to f^*$ sending p to $(p, f(p))$. Hence, $\mu(f^*) = \mu(P)$. Corollary 4.3 derives the same formula as [Bac77, Theorem 5.5] and [Wal81, Theorem 3.1].

COROLLARY 4.5 ([Bac77], [Wal81]). *If $f\colon P \to Q$ is a map on finite posets P and Q, then*

$$\mu(P) = \mu(f^*) = \mu(Q) - \sum_{q \in Q} \mu(Q_{>q})\mu(f^{-1}Q_{\leq q}).$$

5. LS category for maps on simplicial complexes and finite spaces. The Lusternik–Schnirelmann (LS) category $\mathrm{cat}(X)$ of a space X is a fundamental homotopy invariant defined by the minimal number (minus 1) of contractible open sets within X covering X [LS34]. More generally, we can consider the LS category for maps.

DEFINITION 5.1. For a continuous map $f\colon X \to Y$ between arbitrary spaces X and Y, an open set U of X is called *categorical* of f if the restriction $f|_U\colon U \to Y$ is null-homotopic. The *LS category* $\mathrm{cat}(f)$ is the smallest non-negative integer n such that there exist $n+1$ categorical open sets of f covering X. Particularly, the LS category $\mathrm{cat}(X)$ of a space X agrees with $\mathrm{cat}(\mathrm{id}_X)$ for the identity id_X on X.

The LS category of maps is a homotopy invariant, i.e., if two maps f, g are homotopic, then $\mathrm{cat}(f) = \mathrm{cat}(g)$ holds. A simplicial analog of the LS category based on the contiguity relation has recently been presented for simplicial complexes [FMV15] and simplicial maps [SS17].

DEFINITION 5.2. For a simplicial map $f\colon K \to L$ between simplicial complexes K and L, a subcomplex U of K is called *categorical* of f if the restriction $f|_U\colon U \to Y$ is contiguous to the constant simplicial map onto a vertex of L. The *simplicial LS category* $\mathrm{scat}(f)$ is the smallest non-negative integer n such that there exist $n+1$ categorical subcomplexes of f covering K. In particular, the simplicial LS category $\mathrm{scat}(K)$ of a simplicial complex K is defined as $\mathrm{cat}(\mathrm{id}_K)$ for the identity id_K on K.

The simplicial LS category of simplicial maps is invariant under the contiguity relation, i.e., if two simplicial maps f, g are contiguous, then $\mathrm{scat}(f) = \mathrm{scat}(g)$. Moreover, $\mathrm{scat}(K) = 0$ if and only if K is strongly collapsible (see [BM12]). For a simplicial map $f\colon K \to L$, we have $\mathrm{cat}(|f|) \le \mathrm{scat}(f)$. This inequality may be strict owing to the smaller number of subcomplexes of K than that of open sets of $|K|$. This can be resolved by taking the iterated barycentric subdivision into account. In [SS17, Section 4.1], the inequality $\mathrm{scat}(\mathrm{sd}(f)) \le \mathrm{scat}(f)$ was proved for the induced simplicial map $\mathrm{sd}(f) = \mathcal{K}(\mathcal{F}(f))\colon \mathrm{sd}(K) \to \mathrm{sd}(L)$. This yields the following decreasing sequence:

$$\mathrm{scat}(f) \ge \mathrm{scat}(\mathrm{sd}(f)) \ge \mathrm{scat}(\mathrm{sd}^2(f)) \ge \cdots .$$

This sequence might be expected to converge to $\mathrm{cat}(|f|)$; however, this is *not* true in general.

EXAMPLE 5.3. Let K be a collapsible simplicial complex which is not strongly collapsible (see, e.g., [BM12, Example 2.13]). The iterated barycentric subdivision $\mathrm{sd}^k(K)$ is not strongly collapsible for all $k \ge 0$ [BM12, Theorem 4.15]. Hence,

$$\mathrm{scat}(\mathrm{sd}^k(\mathrm{id}_K)) = \mathrm{scat}(\mathrm{id}_{\mathrm{sd}^k(K)}) = \mathrm{scat}(\mathrm{sd}^k(K)) \geq 1$$

for any n, whereas $\mathrm{cat}(|\mathrm{id}_K|) = \mathrm{cat}(\mathrm{id}_{|K|}) = \mathrm{cat}(|K|) = 0$.

Now we consider a slightly different idea. For a simplicial map $f\colon K \to L$, take a simplicial approximation $\lambda_k\colon \mathrm{sd}^k(K) \to K$ to the identity on $|K|$ and consider the simplicial LS category $\mathrm{scat}(f \circ \lambda_k)$. It should be noted that this does not depend on the choice of the approximation λ_k. The value $\mathrm{scat}(f \circ \lambda_k)$ becomes smaller as k becomes larger:

$$\mathrm{scat}(f) \geq \mathrm{scat}(f \circ \lambda_1) \geq \mathrm{scat}(f \circ \lambda_2) \geq \cdots .$$

The next theorem shows that the above sequence converges to $\mathrm{cat}(|f|)$. The proof is essentially based on González's simplicial approach to topological complexity [Gon18], and the author's categorical approach to LS category [Tan18].

THEOREM 5.4. *For a simplicial map* $f\colon K \to L$, *we have* $\mathrm{scat}(f \circ \lambda_k) = \mathrm{cat}(|f|)$ *for sufficiently large* $k \geq 0$.

Proof. We have $\mathrm{scat}(f \circ \lambda_k) \geq \mathrm{cat}(|f|)$ for all $k \geq 0$, since $|f \circ \lambda_k| \simeq |f|$ for any simplicial approximation λ_k to the identity on $|K|$. Let us show the converse inequality for large k. We assume that $\mathrm{cat}(|f|) = n$ with categorical open sets $U_0, \ldots, U_n$ of $|f|\colon |K| \to |L|$ covering $|K|$. For large $\ell \geq 0$, the realization $|\sigma|$ of each simplex σ in $\mathrm{sd}^\ell(K)$ is included in some U_j by the Lebesgue covering lemma. The subcomplexes $K_j = \{\sigma \in \mathrm{sd}^\ell(K) \mid |\sigma| \subset U_j\}$ cover $\mathrm{sd}^\ell(K)$, and the realization $|K_j|$ is categorical of $|f|$ for each j. By the simplicial approximation theorem and [Spa66, Corollary 3.4.4], there is a simplicial approximation $g\colon \mathrm{sd}^k(K) \to L$ of $|f|$ for sufficiently large $k \geq \ell$, and the restriction $g|_{\mathrm{sd}^{k-\ell}(K_j)}$ is a simplicial approximation to $|f||_{|K_j|}$ for each j. This implies that $g|_{\mathrm{sd}^{k-\ell}(K_j)}$ is contiguous to the constant map onto a vertex, thus $\mathrm{scat}(g) \leq n$. For a simplicial approximation $\lambda_k\colon \mathrm{sd}^k(K) \to K$ to the identity on $|K|$, the composition $f \circ \lambda_k\colon \mathrm{sd}^k(K) \to L$ is also a simplicial approximation of $|f|$. The universality of simplicial approximation shows that g and $f \circ \lambda_k$ are contiguous and

$$\mathrm{scat}(f \circ \lambda_k) = \mathrm{scat}(g) \leq n. \blacksquare$$

Let $f\colon P \to Q$ be a map between finite spaces and let $\tau_P^i\colon \mathrm{sd}^i(P) \to P$ be the composition $\tau_P^i = \tau_{\mathrm{sd}^{i-1}(P)} \circ \cdots \circ \tau_P$. We have the following decreasing sequence:

$$\mathrm{cat}(f) \geq \mathrm{cat}(f \circ \tau_P) \geq \mathrm{cat}(f \circ \tau_P^2) \geq \cdots .$$

As a consequence of Theorem 5.4, this sequence converges to $\mathrm{cat}(\mathcal{B}f)$:

COROLLARY 5.5. *For a map* $f\colon P \to Q$ *between finite spaces,* $\mathrm{cat}(f \circ \tau_P^k) = \mathrm{cat}(\mathcal{B}f)$ *for sufficiently large* $k \geq 0$.

Proof. We have

$$\mathrm{cat}(f\circ\tau_P^k)\geq \mathrm{cat}(\mathcal{B}(f\circ\tau_P^k))=\mathrm{cat}(\mathcal{B}f)\qquad\text{for any } k\geq 0.$$

Moreover, Theorem 5.4 implies the equality

$$\mathrm{cat}(\mathcal{B}f)=\mathrm{cat}(|\mathcal{K}f|)=\mathrm{scat}(\mathcal{K}f\circ\lambda_k)$$

for a simplicial approximation $\lambda_k\colon\ \mathrm{sd}^k(\mathcal{K}(P))\to\mathcal{K}(P)$ to the identity on $\mathcal{B}P$ and large $k\geq 0$. By [SS17, Proposition 32],

$$\mathrm{scat}(\mathcal{K}f\circ\lambda_k)\geq \mathrm{cat}(\mathcal{F}(\mathcal{K}f\circ\lambda_k))=\mathrm{cat}(\mathrm{sd}(f)\circ\mathcal{F}(\lambda_k))=\mathrm{cat}(f\circ\tau_P^{k+1}).$$

This implies that $\mathrm{cat}(\mathcal{B}f)\geq\mathrm{cat}(f\circ\tau_P^{k+1})$, thus $\mathrm{cat}(\mathcal{B}f)=\mathrm{cat}(f\circ\tau_P^{k+1})$. ∎

Theorem 5.4 and the simplicial approximation theorem suggest that for every continuous map $\varphi\colon\ |K|\to|L|$, there exists a simplicial map $f\colon\ \mathrm{sd}^k(K)\to L$ with $\mathrm{cat}(\varphi)=\mathrm{scat}(f)$. Moreover, the proof of Corollary 5.5 suggests that there exists a map $g\colon\ X\to P$ on finite spaces with $\mathrm{cat}(\varphi)=\mathrm{cat}(g)$.

5.1. Calculation of LS category using relation matrices. As seen in the previous subsection, the LS category of a map on finite regular cell complexes coincides with the LS category of some map $f\colon\ X\to P$ on finite spaces. Because X has only finitely many open sets, it is important for computing $\mathrm{cat}(f)$ to discriminate categorical open sets of f. A map $f\colon\ X\to P$ on finite spaces corresponds to a principal relation matrix of X and P^{op}, as seen in Section 3. We will describe null-homotopic maps on finite spaces in terms of the associated matrices.

Definition 5.6. Let $R\colon\ X\times P^{\mathrm{op}}\to\{0,1\}$ be a principal relation matrix of finite spaces X and P^{op}. A point $(x,p)\in X\times P^{\mathrm{op}}$ is called *reversible* for R if the following hold:

(1) If $R(x,p)=1$, then the relation matrix R' defined by

$$R'(y,q)=\begin{cases}0 & \text{if } y\leq x \text{ and } q\geq p,\\ R(y,q) & \text{otherwise}\end{cases}$$

is principal.

(2) If $R(x,p)=0$, then the relation matrix R' defined by

$$R'(y,q)=\begin{cases}1 & \text{if } y\geq x \text{ and } q\leq p,\\ R(y,q) & \text{otherwise}\end{cases}$$

is principal.

The notation $R\approx_{(x,p)}R'$ is used in this case, and the equivalence relation generated by $\approx_{(x,p)}$ is denoted by $\approx$. In other words, for two distinguished principal relation matrices R and R' of X and P^{op}, the relation $R\approx R'$ holds if and only if $R=R'$ or there is a finite sequence of principal relation matrices

$R = T_0, T_1, \dots, T_n = R'$ with reversible points $(x_1, p_1), \dots, (x_{n-1}, p_{n-1})$ satisfying either $T_{i-1} \approx_{(x_i,p_i)} T_i$ or $T_i \approx_{(x_i,p_i)} T_{i-1}$, for each i.

For two principal relation matrices R and R' of finite spaces X and P^{op}, it is obvious that $R \approx R'$ implies $R \simeq R'$ in $\mathrm{RM}(X, P)$. The converse is also true, as we will see below.

LEMMA 5.7. *Let $f, g\colon X \to P$ be maps between finite spaces. These maps are homotopic if and only if $f^* \approx g^*$ as principal relation matrices.*

Proof. Suppose that f, g are homotopic. By Lemma 2.1, without loss of generality we may assume that $f = g$ except at a point $x \in X$ and $f(x) < g(x)$. The point $(x, g(x))$ is reversible for f^* and $f^* \approx_{(x,g(x))} g^*$. Conversely, if $f^* \approx_{(x,p)} g^*$, then Definition 5.6 ensures that $f^* \simeq g^*$ in $\mathrm{RM}(X, P)$. This implies that $f \simeq g$ by Proposition 3.7. ∎

The next corollary immediately follows from Lemma 5.7 and Proposition 3.7.

COROLLARY 5.8. *Two principal relation matrices R and R' are homotopic $R \simeq R'$ if and only if $R \approx R'$.*

Lemma 2.1 states that the relation $\simeq_{(x,p)}$ generates the homotopy relation on $\mathrm{Map}(X \times P^{\mathrm{op}}, \{0, 1\})$. On the other hand, $\approx_{(x,p)}$ generates the homotopy relation on $\mathrm{RM}(X, P)$.

DEFINITION 5.9. Let X and P be finite spaces. For a point $p \in P$, the *constant relation matrix* on p is defined by

$$c_p(x, q) = \begin{cases} 1 & \text{if } q \le p \text{ in } P, \\ 0 & \text{otherwise.} \end{cases}$$

Note that a constant relation matrix is not a constant map $X \times P^{\mathrm{op}} \to \{0, 1\}$. However, for the constant map $f_p\colon X \to P$ onto $p \in P$, we have $f_p^* = c_p$.

COROLLARY 5.10. *A map $f\colon X \to P$ between finite spaces is null-homotopic if and only if the associated relation matrix f^* satisfies $f^* \approx c_p$ for some $p \in P$.*

For a principal relation matrix $R\colon X \times P^{\mathrm{op}} \to \{0, 1\}$, we can choose total orders on X and P^{op} compatible with the partial orders (linear extensions) respectively, that is, we can write $X = \{x_1, \dots, x_m\}$ and $P = \{p_1, \dots, p_n\}$ so that $x_k \le x_\ell$ in X implies $k \le \ell$ and $p_i \le p_j$ in P implies $i \ge j$. We will describe R as a $\{0, 1\}$-matrix with size $m \times n$ in the following examples.

EXAMPLE 5.11. The finite space S^n consists of $2n + 2$ points $\{x_0, y_0, \dots, x_n, y_n\}$ with the order $w_i < z_{i+1}$ for any i and $w, z \in \{x, y\}$. The inclusion $S^1 \hookrightarrow S^2$ determines the relation matrix R described in Table 1.

Table 1. Matrix associated with the inclusion $S^1 \hookrightarrow S^2$

R	x_0	y_0	x_1	y_1	x_2	y_2
x_0	1	0	0	0	0	0
y_0	0	1	0	0	0	0
x_1	1	1	1	0	0	0
y_1	1	1	0	1	0	0

This associated relation matrix resembles the board game “Reversi”, as switching a reversible point influences other components according to the partial order on $S^1 \times (S^2)^{\mathrm{op}}$. For example, the point (x_0, y_2) is reversible, and switching it yields the principal relation matrix R' on the left-hand side of Table 2. Next, switching the reversible point (y_0, y_2), we obtain a vertical striped matrix on the right-hand side of Table 2. Constant relation matrices are always formed of vertical stripes. Indeed, this matrix represents the constant relation matrix c_{y_2}, and we have $R \approx_{(x_0,y_2)} R' \approx_{(y_0,y_2)} c_{y_2}$. Corollary 5.10 implies that the inclusion $S^1 \hookrightarrow S^2$ is null-homotopic.

Table 2. Switching reversible points

R'	x_0	y_0	x_1	y_1	x_2	y_2
x_0	1	1	1	1	0	1
y_0	0	1	0	0	0	0
x_1	1	1	1	1	0	1
y_1	1	1	1	1	0	1

c_{y_2}	x_0	y_0	x_1	y_1	x_2	y_2
x_0	1	1	1	1	0	1
y_0	1	1	1	1	0	1
x_1	1	1	1	1	0	1
y_1	1	1	1	1	0	1

Example 5.12. For the identity $\mathrm{id}\colon S^1 \to S^1$, the associated (adjacent) matrix has no reversible point (see the left-hand side of Table 3). Although there is no principal matrix R with $\mathrm{id}^* \approx_{(x,y)} R$, we should consider the possibility of the converse relation $R \approx_{(x,y)} \mathrm{id}^*$ for some reversible point (x, y). If such an R with a reversible point (x, y) exists, then $|R(x, y) - \mathrm{id}^*(x, y)| = 1$. However, for any point $(x, y) \in S^1 \times S^1$, the map R can never be a principal matrix. For example, if $(x, y) = (x_0, x_0)$, then R is represented as a non-principal matrix on the right-hand side of Table 3. This fact also follows from Stong’s result on the minimality of the finite space [Sto66]. Hence, $\mathrm{cat}(S^1) \neq 0$.

Table 3. Computation of $\mathrm{cat}(S^1)$ using the associated matrix

id^*	x_0	y_0	x_1	y_1
x_0	1	0	0	0
y_0	0	1	0	0
x_1	1	1	1	0
y_1	1	1	0	1

R	x_0	y_0	x_1	y_1
x_0	0	0	0	0
y_0	0	1	0	0
x_1	1	1	1	0
y_1	1	1	0	1

We next consider the open sets $U = \{x_0, x_1, y_1\}$ and $V = \{y_0, x_1, y_1\}$ of S^1, as well as the associated matrices $\mu_U\colon U \times (S^1)^{\mathrm{op}} \to \{0, 1\}$, $\mu_V\colon V \times (S^1)^{\mathrm{op}} \to$

$\{0,1\}$ with the inclusions. We can verify that $\mu_U \approx_{(y_1,y_0)} \nu \approx_{(x_1,y_0)} c_{x_0}$ and $\mu_V \approx_{(y_1,x_0)} \rho \approx_{(x_1,x_0)} c_{y_0}$, where ν and ρ are the matrices described in Table 4. Thus, $\mathrm{cat}(S^1) = 1$.

Table 4. Matrices ν and ρ

ν	x_0	y_0	x_1	y_1
x_0	1	0	0	0
x_1	1	1	1	0
y_1	1	0	0	0

ρ	x_0	y_0	x_1	y_1
y_0	0	1	0	0
x_1	1	1	1	0
y_1	0	1	0	0

Conclusion remarks. In this paper, we introduced a combinatorial method for calculating the LS category of a continuous map $\varphi\colon |K| \to |L|$ on the realizations of simplicial complexes K and L. The first step of the calculation is to approximate φ with a simplicial map or a map on finite spaces using the simplicial approximation theorem. We need to consider $\mathrm{sd}^k(K)$ for sufficiently large $k \geq 0$ and search for categorical subcomplexes covering $\mathrm{sd}^k(K)$. However, the number of simplices and subcomplexes of $\mathrm{sd}^k(K)$ increases exponentially with k. This may present a difficulty in exhaustively searching for subcomplexes for large k. It is thus hoped that the proposed method may develop into effective heuristic algorithms.

Acknowledgements. I would like to thank the referees for helpful suggestions and comments. This work was supported by JSPS KAKENHI Grant Number JP17K14183.

References

[AS13] S. Aaronson and N. A. Scoville, *Lusternik–Schnirelmann category for simplicial complexes*, Illinois J. Math. 57 (2013), 743–753.

[Bac77] K. Baclawski, *Galois connections and the Leray spectral sequence*, Adv. Math. 25 (1977), 191–215.

[BM08] J. A. Barmak and E. G. Minian, *Simple homotopy types and finite spaces*, Adv. Math. 218 (2008), 87–104.

[BM12] J. A. Barmak and E. G. Minian, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. 47 (2012), 301–328.

[CL+03] O. Cornea, G. Lupton, J. Oprea and D. Tanré, *Lusternik–Schnirelmann Category*, Math. Surveys Monogr. 103, Amer. Math. Soc., Providence, RI, 2003.

[FMV15] D. Fernández-Ternero, E. Macías-Virgós and J. A. Vilches, *Lusternik–Schnirelmann category of simplicial complexes and finite spaces*, Topology Appl. 194 (2015), 37–50.

[Gon18] J. González, *Simplicial complexity: piecewise linear motion planning in robotics*, New York J. Math. 24 (2018), 279–292.

[HV93] K. A. Hardie and J. J. C. Vermeulen, *Homotopy theory of finite and locally finite T_0 spaces*, Exposition. Math. 11 (1993), 331–341.

[Lei08] T. Leinster, *The Euler characteristic of a category*, Doc. Math. 13 (2008), 21–49.

[LS34] L. Lusternik et L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationnels*, Hermann, Paris, 1934.

[McC66] M. C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, Duke Math. J. 33 (1966), 465–474.

[Moe95] I. Moerdijk, *Classifying Spaces and Classifying Topoi*, Lecture Notes in Math. 1616, Springer, Berlin, 1995.

[MW07] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) 165 (2007), 843–941.

[Rap10] G. Raptis, *Homotopy theory of posets*, Homology Homotopy Appl. 12 (2010), 211–230.

[Rot64] G.-C. Rota, *On the foundations of combinatorial theory, I. Theory of Möbius functions*, Z. Wahrsch. Verw. Gebiete 2 (1964), 340–368.

[Seg78] G. Segal, *Classifying spaces related to foliations*, Topology 17 (1978), 367–382.

[Spa66] E. Spanier, *Algebraic Topology*, Springer, New York, 1966.

[SS17] N. A. Scoville and W. Swei, *On the Lusternik–Schnirelmann category of a simplicial map*, Topology Appl. 216 (2017), 116–128.

[Sto66] R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. 123 (1966), 325–340.

[Tan18] K. Tanaka, *Lusternik–Schnirelmann category for categories and classifying spaces*, Topology Appl. 239 (2018), 65–80.

[Wal81] J. W. Walker, *Homotopy type and Euler characteristic of partially ordered sets*, Eur. J. Combin. 2 (1981), 373–384.

[Wei05] M. Weiss, *What does the classifying space of a category classify?*, Homology Homotopy Appl. 7 (2005), 185–195.

Kohei Tanaka
Institute of Social Sciences
School of Humanities and Social Sciences
Academic Assembly
Shinshu University
3-1-1 Asahi, Matsumoto
Nagano 390-8621, Japan
E-mail: tanaka@shinshu-u.ac.jp