

## Non-hyperbolic iterated function systems: semifractals and the chaos game

by

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**Abstract.** We consider iterated function systems (IFSs) on compact metric spaces and introduce the concept of *target sets*. Such sets have very rich dynamical properties and play a role similar to that played by semifractals introduced by Lasota and Myjak for regular IFSs. We study sufficient conditions which guarantee that the closure of the target set is a local attractor for the IFS. As a corollary, we establish necessary and sufficient conditions for the IFS to have a global attractor. We provide an example of a non-regular IFS whose target set is non-empty, showing that our approach gives rise to a new class of semifractals. Finally, we show that random orbits generated by IFSs draw target sets that are “stable”.

**1. Introduction.** Consider finitely many contractions  $f_1, \dots, f_k$  defined on a complete metric space  $X$  and a sequence of independent and identically distributed (i.i.d.) random variables  $\mathbb{X}_n$  with range  $\{1, \dots, k\}$ . It is well known that for every  $x$ , with probability 1, the *tail sequence of sets*  $\{f_{\mathbb{X}_{n-1}} \circ \dots \circ f_{\mathbb{X}_0}(x) : n \geq \ell\}_\ell$  converges to the unique compact set  $K$  satisfying

$$K = \bigcup_{i=1}^k f_i(K),$$

called the *Hutchinson attractor*. The existence and uniqueness of the latter was proved in [13].

In this paper, we consider a setting where  $f_1, \dots, f_k$  are just continuous maps defined on a compact metric space and we translate several results of “Hutchinson theory” to this non-contracting setting. The finite family of continuous maps  $f_1, \dots, f_k$  is called an *iterated function system* (IFS) and is

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denoted by  $\text{IFS}(f_1, \dots, f_k)$ . An IFS is called *hyperbolic* if all of its maps are (uniform) contractions, and *non-hyperbolic* otherwise.

A key ingredient in this study is the *Barnsley–Hutchinson operator* (BH-operator for short). Considering the hyperspace  $\mathcal{H}(X)$  of non-empty compact subsets of  $X$  endowed with the Hausdorff distance, the BH-operator associated to  $\text{IFS}(f_1, \dots, f_k)$  is defined by

$$(1.1) \quad \mathcal{B}: \mathcal{H}(X) \rightarrow \mathcal{H}(X), \quad \mathcal{B}(A) := \bigcup_{i=1}^k f_i(A).$$

An important part of Hutchinson theory is the study of the dynamics of the BH-operator and its fixed points. The *Hutchinson attractor* of a hyperbolic IFS is the unique fixed point of the BH-operator. Moreover, in this case, this attractor is global [2, 13].

A first important class of non-hyperbolic IFSs considered in this study is the one of IFSs *contracting on average* [3]. For this class, when studying IFSs from the probabilistic point of view, the results in [3] fully recover the results about stationary measures in [13]. On the other hand, from the topological point of view, the scenario may change substantially. The dynamics of the BH-operator of a contracting-on-average IFS is, in general, very different from the one in the hyperbolic case. For instance, it may have several compact fixed points, or none at all. Nevertheless, by [15], it always has one (not necessarily compact) fixed point with very rich dynamical properties, which is called its *semifractal* and, in a certain sense, generalizes the Hutchinson attractor. We observe that semifractals were introduced for *regular* IFSs (see Section 2.1 for definition), which is a class of IFSs much more general than those contracting on average. A semifractal is characterised by the following two properties:

- (SF1) it is the *minimum fixed* point of the BH-operator (i.e., it is contained in any other fixed point of the BH-operator) and
- (SF2) it attracts each of its compact subsets under the action of the BH-operator.

Another class of non-hyperbolic IFSs was considered by Edalat [12], also fully recovering aspects of Hutchinson theory. In what follows, we assume that  $X$  is compact. The  $\text{IFS}(f_1, \dots, f_k)$  is called *weakly hyperbolic* if it satisfies the following topological condition:

$$(1.2) \quad \lim_{n \rightarrow \infty} \text{diam}(f_{\omega_0} \circ \dots \circ f_{\omega_n}(X)) = 0$$

for every  $\omega = \omega_0 \omega_1 \omega_2 \dots \in \Sigma_k := \{1, \dots, k\}^{\mathbb{N}}$ . In [12] it is proved that the BH-operator of a weakly hyperbolic IFS has a global attracting fixed point. Every hyperbolic IFS is weakly hyperbolic. Conversely, by a recent result

in [1], after a change of metric, every weakly hyperbolic system can be made hyperbolic.

Bearing the discussion above in mind, we introduce the subset  $S_{\text{wh}} \subset \Sigma_k$  of *weakly hyperbolic sequences* defined by

$$(1.3) \quad S_{\text{wh}} := \left\{ \omega \in \Sigma_k : \lim_{n \rightarrow \infty} \text{diam}(f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X)) = 0 \right\}.$$

Note that if the IFS is weakly hyperbolic then  $S_{\text{wh}} = \Sigma_k$ . In this paper, we will recover parts of Hutchinson theory in “truly” non-hyperbolic settings where we only require that  $S_{\text{wh}} \neq \emptyset$ . In this analysis, a key object is the so-called *target set* defined as follows. Associated to  $S_{\text{wh}}$  we consider the *coding map*  $\pi: S_{\text{wh}} \rightarrow X$  that projects  $S_{\text{wh}}$  into the phase space  $X$ , defined by

$$(1.4) \quad \pi: S_{\text{wh}} \rightarrow X, \quad \pi(\omega) := \lim_{n \rightarrow \infty} f_{\omega_0} \circ \cdots \circ f_{\omega_n}(p),$$

where  $p$  is any point of  $X$ . By definition of  $S_{\text{wh}}$ , this limit always exists and is independent of  $p \in X$ . The set of weakly hyperbolic sequences generates the *target set* defined by

$$(1.5) \quad A_{\text{tar}} := \pi(S_{\text{wh}}).$$

Our results show that the target set encodes crucial aspects of the dynamics of the BH-operator. Let us now briefly discuss these results; the precise statements can be found in Section 2.

In what follows we always assume that  $S_{\text{wh}} \neq \emptyset$ . Under this assumption, the closure of the target set satisfies **(SF1)** and **(SF2)** (Theorem 1). We also give an example of a non-regular IFS whose target set is non-empty. This shows that our approach gives rise to a new class of semifractals.

In general, the closure of the target set may fail to be a global attractor. In Theorem 2, we study necessary and sufficient conditions guaranteeing that the closure of the target set is a “local” attractor. As a consequence, we state a sufficient and necessary condition ensuring that the BH-operator has a global attractor, generalising Edalat’s result [12] mentioned above. Moreover, we show that if the BH-operator has a unique fixed point, then it is necessarily a global attractor (Theorem 3).

Finally, we investigate under which conditions *a random orbit generated by an IFS draws the target*. For a given IFS  $(f_1, \dots, f_k)$ , consider a sequence of i.i.d. random variables  $\mathbb{X}_n$  with range in  $\{1, \dots, k\}$  and for  $i \in \{1, \dots, k\}$  let  $p_i := \text{prob}(\mathbb{X}_n = i) > 0$ . The procedure of choosing  $i$  at random with probability  $p_i$  and moving from  $x$  to  $f_i(x)$  is an algorithm called the (*probabilistic*) *chaos game*. In this way, for each  $x \in M$ , we get a Markov chain  $\mathbb{Z}_n := f_{\mathbb{X}_{n-1}} \circ \cdots \circ f_{\mathbb{X}_0}(x)$  moving through the set  $X$ . For hyperbolic IFSs, a consequence of standard ergodic theorems (see for instance [8]) is that the tail of the Markov chain  $\{\mathbb{Z}_n : n \geq \ell\}_\ell$  converges in the Hausdorff distance to the Hutchinson attractor with probability 1.

A standard way of taking an i.i.d. sequence with range in  $\{1, \dots, k\}$  is the following. Consider the space  $\Sigma_k$  as a probability space endowed with the product  $\sigma$ -algebra and the product measure  $\mathbb{P} = \nu^{\mathbb{N}}$ , where  $\nu$  is the probability measure in  $\{1, \dots, k\}$  given by  $\nu(i) = p_i$ . Then the natural projections  $\mathbb{X}_n(\omega) = \omega_n$  is a sequence of i.i.d. random variables. An advantage of considering  $\Sigma_k$  as the sample space of the random variables  $\mathbb{X}_n$  is that we can use the structure of  $\Sigma_k$  to improve the understanding and results about the chaos game, and also to get more elementary proofs. The results in [4] and Theorem 4 illustrate well these assertions. Let us explain this a bit more precisely.

We show in Theorem 4 that if the closure of the target set is *stable* (see Section 2.3 for the definition) then the subset  $\mathcal{D}$  of *disjunctive sequences* of  $\Sigma_k$  (see Section 2.3 for details) is such that for every  $x \in X$  the sequence of tail sets satisfies

$$\lim_{\ell \rightarrow \infty} \{f_\omega^n(x) : n \geq \ell\}_\ell = \text{closure}(A_{\text{tar}}), \quad \text{where} \quad f_\omega^n(x) := f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0}(x),$$

for every  $\omega \in \mathcal{D}$ . We observe that it follows from its definition that the set  $\mathcal{D}$  does not depend on  $x$  and satisfies  $\mathbb{P}(\mathcal{D}) = 1$ .

The procedure of choosing  $\omega$  in a prescribed set and considering the deterministic orbit  $f_\omega^n(x)$  is called the *deterministic chaos game*. Notice that in this study there is no stationary process associated, hence it is not possible to use ergodic theorems (as in the classical theory) or Markov operators (as in [16, 17]) to prove that deterministic orbits draw the attractor. It is worth mentioning that, remarkably, even in the probabilistic chaos game it is possible to avoid such probabilistic techniques: see for instance [5, 6, 7]. Here Theorem 4 contributes to a series of works about the deterministic chaos game [4, 7, 18]; the novelties of our result are discussed in Section 2.3.

**Organisation of the paper.** In Section 2 we precisely state the main results of this paper. The theorems are proved in Section 3. In Section 4, we present some examples illustrating our results and hypotheses.

**Standing hypotheses.** We always consider a compact metric space  $(X, d)$ , an IFS  $(f_1, \dots, f_k)$  of continuous maps defined on  $X$ , and the corresponding BH-operator  $\mathcal{B}$ ; these hypotheses are omitted in the statement of our theorems.

Convergence of compact sets is always considered with respect to the *Hausdorff distance*  $d_H$  defined as follows. Given a point  $x \in X$  and a set  $A \subset X$ , the distance between  $x$  and  $A$  is defined by

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

Recall that  $\mathcal{H}(X)$  is the set of non-empty compact subsets of  $X$ . The *Hausdorff distance* between  $A, B \in \mathcal{H}(X)$  is defined by

(1.6)

$$d_H(A, B) := \max\{h_s(A, B), h_s(B, A)\}, \quad \text{where } h_s(A, B) := \sup_{a \in A} d(a, B).$$

We observe that  $(\mathcal{H}(X), d_H)$  is a compact metric space [2].

## 2. Main results

**2.1. Target sets and semifractals.** Our first result shows that the closure of the target set has the properties **(SF1)** and **(SF2)** characterising the semifractals of regular IFSs introduced by Lasota and Myjak [15]. An IFS  $(f_1, \dots, f_k) = \mathfrak{F}$  is said to be *regular* if there are numbers  $1 \leq i_1 < i_2 < \dots < i_\ell \leq k$  such that IFS  $(f_{i_1}, \dots, f_{i_\ell}) = \mathfrak{F}'$  has a global attractor. The global attractor of  $\mathfrak{F}'$  is called a *nucleus* of  $\mathfrak{F}$  (an IFS may have several nuclei). It is proved in [15] that the BH-operator of any regular IFS  $\mathfrak{F}$  has a fixed point, called a *semifractal*, satisfying **(SF1)** and **(SF2)**. In the next theorem, replacing the regularity of the IFS by the condition  $S_{\text{wh}} \neq \emptyset$ , we recover these properties. Recall the definition of the target set  $A_{\text{tar}}$  in (1.5).

**THEOREM 1.** *Consider an IFS with  $S_{\text{wh}} \neq \emptyset$ . Then the following hold:*

- (1) *The set  $\overline{A_{\text{tar}}}$  is the minimum fixed point of  $\mathcal{B}$ .*
- (2) *For every compact set  $K \subset \overline{A_{\text{tar}}}$ ,*

$$\lim_{n \rightarrow \infty} d_H(\mathcal{B}^n(K), \overline{A_{\text{tar}}}) = 0.$$

**REMARK 2.1.** The regularity of the IFS does not imply that  $S_{\text{wh}} \neq \emptyset$  or vice versa. Example 4.1 provides an IFS with  $S_{\text{wh}} \neq \emptyset$  which is not regular. Hence, our approach provides a new class of semifractals. Since minimum fixed points are unique, it follows that for regular IFSs with  $S_{\text{wh}} \neq \emptyset$  the semifractal (as defined by Lasota and Myjak) is the closure of the target set.

**REMARK 2.2.** Zorn's lemma provides a *minimal fixed point* for the BH-operator (i.e., a fixed point that properly contains no other fixed point). However, as mentioned above, minimum fixed points may not exist. Note that, by definition, a minimum fixed point is minimal, but the converse is not true in general. Above we have seen two different contexts leading to minimum fixed points: regularity of the IFS and the existence of a weakly hyperbolic sequence. It is natural to ask for a unifying condition guaranteeing the existence of a minimum fixed point of the BH-operator.

**2.2. Attractors of the Barnsley–Hutchinson operator.** To state our next results precisely we need the following definition.

**DEFINITION 2.3** (Strict and Conley attractors). A compact set  $A \subset X$  is a *strict attractor* of the BH-operator if there is an open neighbourhood  $U$

of  $A$  such that

$$\lim_{n \rightarrow \infty} \mathcal{B}^n(K) = A \quad \text{for every compact set } K \subset U.$$

The *basin of attraction* of  $A$  is the largest open neighbourhood of  $A$  for which the above property holds. A strict attractor whose basin of attraction is the whole space is a *global attractor*.

A compact set  $S \subset X$  is a *Conley attractor* of the BH-operator if there exists an open neighbourhood  $U$  of  $S$  such that

$$\lim_{n \rightarrow \infty} \mathcal{B}^n(\overline{U}) = S.$$

REMARK 2.4. The continuity of the BH-operator implies that Conley and strict attractors are fixed points of  $\mathcal{B}$ . Note also that strict attractors are Conley attractors but the converse is not true in general.

The next theorem shows that the target set plays a key role in the study of strict attractors of the BH-operator for IFSs with  $S_{\text{wh}} \neq \emptyset$ .

THEOREM 2. *Consider an IFS with  $S_{\text{wh}} \neq \emptyset$ . Then the following hold:*

- (1) *The BH-operator has at most one strict attractor. Moreover, if such a strict attractor exists then it is equal to  $\overline{A_{\text{tar}}}$ .*
- (2)  *$\overline{A_{\text{tar}}}$  is a Conley attractor if and only if it is a strict attractor.*

Vince [20] obtained a result similar to Theorem 2 for a completely different class of non-hyperbolic IFSs. He shows that an IFS consisting of Möbius maps (Möbius IFS) has at most one strict attractor and that a compact set is a Conley attractor if and only if it is a strict attractor. Note that for Möbius IFSs the set  $S_{\text{wh}}$  is empty.

The next theorem generalises Edalat's [12] results in two ways: (i) it applies to IFSs which are not weakly hyperbolic, (ii) it provides not only a sufficient condition (as in [12]) but also a necessary condition for the existence of a global attractor. To state the theorem consider the set

$$X^* := \bigcap_{n \geq 0} \mathcal{B}^n(X),$$

which is a fixed point of  $\mathcal{B}$  (see Lemma 3.1). Note that  $X^*$  is the *maximum fixed point* of  $\mathcal{B}$  (i.e., every fixed point of  $\mathcal{B}$  is contained in  $X^*$ ).

THEOREM 3. *Consider an IFS with  $S_{\text{wh}} \neq \emptyset$ . The following three assertions are equivalent:*

- (1)  $\overline{A_{\text{tar}}} = X^*$ .
- (2) *the BH-operator has a unique fixed point.*
- (3)  $X^*$  *is a global attractor of the BH-operator.*

Hutchinson theory, both in the hyperbolic and in the non-hyperbolic version, has a topological and a probabilistic aspect. Despite their different

nature, these two points of view “talk to each other” through the coding map, which plays a key role in both approaches. Let us illustrate this “talk”. Starting with the probabilistic side, it is well known (see Letac [19] <sup>(1)</sup>) that if  $\mathbb{P}(S_{\text{wh}}) = 1$ , then the distribution (“probabilistic image”) of the coding map is an attracting <sup>(2)</sup> stationary measure of the Markov chain  $\omega \mapsto \mathbb{Z}_n(\omega) := f_\omega^n(x)$ , for every  $x$ . In other words, if  $S_{\text{wh}}$  is large in the probabilistic sense, one of the main results of the hyperbolic Hutchinson theory (uniqueness of stationary measures) is recovered. Considering now the topological side, item (1) of Theorem 3 shows that if the target set (i.e. the topological image of the coding map) is topologically big then the BH-operator has a global attractor. Note that when  $\mathbb{P}(S_{\text{wh}}) = 1$ , it is clear that  $\overline{A_{\text{tar}}}$  is the support of the unique stationary measure  $m$ , which is given by  $m(A) = \mathbb{P}(\pi^{-1}(A))$ . This discussion leads to the following question:

QUESTION 2.5. *Theorem 3 states that the target set is “big” if and only if the BH-operator has a global attractor. On the other hand, Letac’s result [19] only provides a sufficient condition: if  $S_{\text{wh}}$  is “big” then there is an attracting stationary measure. Is the converse true?*

REMARK 2.6. The name *target set* for  $A_{\text{tar}}$  is justified by the following characterisation:

$$(2.1) \quad A_{\text{tar}} = \left\{ x \in X : \text{there is } \omega \in \Sigma_k \text{ with } \{x\} = \bigcap_n f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X) \right\}.$$

This follows by observing that  $\{f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X)\}$  is a sequence of nested compact sets. Observe that for weakly hyperbolic IFSs, in [12] it is proved that

$$(2.2) \quad X^* = \bigcup_{\omega \in \Sigma_k} I_\omega, \quad \text{where} \quad I_\omega := \bigcap_n f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X).$$

In particular,  $A_{\text{tar}} = X^*$  for weakly hyperbolic IFSs.

It is worth mentioning that there are non-weakly hyperbolic IFSs still satisfying  $A_{\text{tar}} = \overline{A_{\text{tar}}} = X^*$  (see Example 4.3). However, since there are cases where  $A_{\text{tar}} \subsetneq \overline{A_{\text{tar}}} = X^*$  (Example 4.4), taking the closure of the target set in item (1) of Theorem 3 is necessary.

**2.3. The chaos game.** We now see how random iterations draw “stable” target sets. For that we need some preliminary definitions. Recall that the shift on  $\Sigma_k$  is the map  $\sigma : \Sigma_k \rightarrow \Sigma_k$  defined by  $(\sigma(\omega))_n = \omega_{n+1}$ . A sequence of  $\Sigma_k$  whose orbit is dense in  $\Sigma_k$ , with respect to the shift map, is called *disjunctive*. We denote by  $\mathcal{D}$  the set of disjunctive sequences. As a consequence

<sup>(1)</sup> Note that the terminology in [19] is different. We translated the results to our terminology.

<sup>(2)</sup> Under the action of the Markov operator associated with the Markov chain.

of Birkhoff's theorem,  $\mu(\mathcal{D}) = 1$  for every ergodic measure  $\mu$  (with respect to  $\sigma$ ) with full support.

A fixed point  $K$  of the BH-operator is *stable* if for every open neighbourhood  $V$  of  $K$  there is an open neighbourhood  $V_0$  of  $K$  such that

$$(2.3) \quad \mathcal{B}^n(V_0) \subset V \quad \text{for every } n \geq 0.$$

For instance,  $\overline{A_{\text{tar}}}$  is stable if it is a Conley attractor (but this condition is not necessary, see Example 4.2) or if all the maps of the IFS are *non-expanding* (i.e., Lipschitz with constant  $\leq 1$ ) <sup>(3)</sup>.

**THEOREM 4.** *Consider an IFS such that  $\overline{A_{\text{tar}}}$  is a stable fixed point of the BH-operator. Then for every  $x \in X$  and every disjunctive sequence  $\omega \in \Sigma_k$  we have*

$$\lim_{\ell \rightarrow \infty} \overline{\{f_\omega^n(x) : n \geq \ell\}} = \overline{A_{\text{tar}}}.$$

We now compare Theorem 4 with some previous results in [4, 7, 18].

In [4] the authors consider an IFS  $(f_1, \dots, f_k)$  having a strict attractor  $K$  that is *strongly fibred*, meaning that for every  $x \in K$  and every neighbourhood  $V$  of  $x$ , there is a finite word  $c_1 \dots c_\ell$  such that  $f_{c_\ell} \circ \dots \circ f_{c_1}(K) \subset V$ . Under this condition, it is proved that for every  $x$  in the basin of attraction of  $K$  we have

$$(2.4) \quad \lim_{\ell \rightarrow \infty} \{f_\omega^n(x) : n \geq \ell\} = K$$

for every disjunctive sequence  $\omega$ . We observe that the closure of the target set is a strongly fibred fixed point but, in general, it is not a strict attractor, therefore the results in [4] cannot be applied since being stable is a weaker condition than being a strict attractor. Even in the case when  $\overline{A_{\text{tar}}}$  is a strict attractor, Theorem 4 provides further information: the deterministic chaos game can start at *any point*  $x$  and not only in the basin of attraction of the set  $\overline{A_{\text{tar}}}$ .

The deterministic chaos game for *well-fibred quasi-attractors* is studied in [7]. We refrain from giving the precise definition of a well-fibred quasi-attractor. For our goals it is sufficient to say that every strong fibred fixed point of BH-operator is also a well-fibred quasi-attractor. In [7] it is proved that given any well fibred quasi-attractor  $K$ , for every disjunctive sequence  $\omega$ , the tail sequence  $\{f_\omega^n(x) : n \geq \ell\}_\ell$  converges to  $K$  for every point  $x$  in the set

$$\mathcal{B}^*(K) := \left\{ x \in X : \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \mathcal{B}^n(\{x\})} = K \right\}.$$

Since the closure of the target set is strongly fibred, it is also well-fibred and hence the result in [7] can be applied. In this case, Theorem 4 also provides

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<sup>(3)</sup> In the literature, such systems are also called *non-expansive*.



additional information since, even if  $A_{\text{tar}}$  is stable, the set  $\mathcal{B}^*(K)$  may not be the whole space: see Example 4.2.

Finally, [18] studies the deterministic chaos game for strict attractors of non-expanding IFSs. As observed above, in this case the target set is stable (but not necessarily a strict attractor—see Example 4.2) and hence Theorem 4 applies to cases which are not covered by [18].

### 3. Proofs of the theorems

**3.1. Preliminary results.** We start with two auxiliary results. The first one is an elementary lemma about fixed points of the BH-operator and the second one is an important lemma that will be used in the proofs of Theorems 1 and 2.

LEMMA 3.1. *Let  $A \in \mathcal{H}(X)$  be such that  $\mathcal{B}(A) \subset A$ . Then the set*

$$(3.1) \quad A^* := \bigcap_{n \geq 0} \mathcal{B}^n(A)$$

*is a fixed point of  $\mathcal{B}$ . In particular,  $X^*$  is a fixed point of  $\mathcal{B}$ .*

*Proof.* We need the following well-known result whose proof is included for completeness.

CLAIM 3.2. *Let  $(L_n)$  be a sequence of nested compact sets, i.e., with  $L_{n+1} \subset L_n$ . Then*

$$\lim_{n \rightarrow \infty} d_H(L_n, L) = 0, \quad \text{where } L := \bigcap_{n \geq 0} L_n.$$

*Proof.* If the claim is false, there are  $\epsilon > 0$  and a subsequence  $n_\ell \rightarrow \infty$  such that  $d_H(L_{n_\ell}, L) \geq \epsilon$  for all  $\ell$ . Since  $L \subset L_n$  for every  $n$ , for each  $\ell$  there is a point  $p_\ell \in L_{n_\ell}$  such that  $d(p_\ell, L) \geq \epsilon$ . By compactness, taking a subsequence if necessary, we can assume that  $p_\ell \rightarrow p$ . As  $(L_n)$  is nested it follows that  $p \in L$ , contradicting  $d(p_\ell, L) \geq \epsilon$  for all  $\ell$ . ■

Applying Claim 3.2 to the sequence  $L_n = \mathcal{B}^n(A)$  and using the continuity of  $\mathcal{B}$ , we get the conclusion of Lemma 3.1 ■

LEMMA 3.3. *For an IFS with  $S_{\text{wh}} \neq \emptyset$  and every compact set  $K$ ,*

$$\lim_{n \rightarrow \infty} h_s(\overline{A_{\text{tar}}}, \mathcal{B}^n(K)) = 0.$$

*Proof.* Towards a contradiction, assume that there are  $K \in \mathcal{H}(X)$  and a sequence  $(n_\ell)$  such that  $h_s(\overline{A_{\text{tar}}}, \mathcal{B}^{n_\ell}(K)) \geq \epsilon$  for every  $\ell$ . Note that for each  $\ell$  there is  $p_{n_\ell} \in \overline{A_{\text{tar}}}$  with  $d(p_{n_\ell}, \mathcal{B}^{n_\ell}(K)) \geq \epsilon$ . By compactness, we can assume that  $p_{n_\ell} \rightarrow p^* \in \overline{A_{\text{tar}}}$  and  $\mathcal{B}^{n_\ell}(K) \rightarrow \widehat{K}$  for some compact set  $\widehat{K}$ . We need the following elementary claim whose proof we present for completeness:

CLAIM 3.4. *For sequences  $(A_n)$  in  $\mathcal{H}(X)$  and  $(p_n)$  in  $X$ , if  $A_n \rightarrow A$  in the Hausdorff distance and  $p_n \rightarrow p$ , then*

$$d(p, A) = \lim_{n \rightarrow \infty} d(p_n, A_n).$$

*Proof.* We use the following “triangular” inequality: given a point  $q$  and compact sets  $A$  and  $B$  we have

$$d(q, A) \leq d(q, B) + d_H(A, B).$$

Let  $(A_n)$  and  $(p_n)$  be as in the claim. Applying twice the “triangular” inequality above we get

$$d(p, A) \leq d(p, p_n) + d(p_n, A) \leq d(p, p_n) + d(p_n, A_n) + d_H(A_n, A).$$

By hypothesis,  $d(p, p_n) \rightarrow 0$  and  $d_H(A_n, A) \rightarrow 0$ . We conclude that

$$(3.2) \quad d(p, A) \leq \liminf_n d(p_n, A_n).$$

Applying again the “triangular” inequality twice, we get

$$d(p_n, A_n) \leq d(p_n, p) + d(p, A_n) \leq d(p_n, p) + d(p, A) + d_H(A, A_n).$$

Arguing as above, we deduce that

$$(3.3) \quad \limsup_n d(p_n, A_n) \leq d(p, A).$$

Now (3.2) and (3.3) imply the claim. ■

It follows from Claim 3.4 that

$$(3.4) \quad d(p^*, \widehat{K}) \geq \epsilon.$$

We now derive a contradiction. By construction, there is  $\ell_0$  such that

$$(3.5) \quad h_s(B^{n_\ell}(K), \widehat{K}) < \epsilon/3 \quad \text{for all } \ell \geq \ell_0.$$

Take  $q \in B_{\epsilon/3}(p^*) \cap A_{\text{tar}}$ . By the definition of  $A_{\text{tar}}$  in (2.1), there is a sequence  $\omega = \omega_0 \omega_1 \dots \in S_{\text{wh}}$  such that

$$\bigcap_{n \geq 0} f_{\omega_0} \circ \dots \circ f_{\omega_n}(X) = \{q\}.$$

Therefore there is  $m_0$  such that

$$f_{\omega_0} \circ \dots \circ f_{\omega_{m-1}}(K) \subset f_{\omega_0} \circ \dots \circ f_{\omega_{m-1}}(X) \subset B_{\epsilon/3}(p^*) \quad \text{for every } m \geq m_0.$$

Since  $f_{\omega_0} \circ \dots \circ f_{\omega_{m-1}}(K) \subset \mathcal{B}^m(K)$ , we have

$$\mathcal{B}^{n_\ell}(K) \cap B_{\epsilon/3}(p^*) \neq \emptyset \quad \text{for every } \ell \text{ large enough.}$$

Taking now any  $z \in \mathcal{B}^{n_\ell}(K) \cap B_{\epsilon/3}(p^*)$  (with large  $\ell$ ) and using (3.5) we get

$$d(p^*, \widehat{K}) \leq d(p^*, z) + d(z, \widehat{K}) \leq \epsilon/3 + \epsilon/3 < \epsilon,$$

contradicting (3.4). This ends the proof of Lemma 3.3. ■

**3.2. Proof of Theorem 1.** (1) follows from two assertions that we will prove below:

- If  $K$  is a non-empty compact fixed point of  $\mathcal{B}$ , then  $\overline{A_{\text{tar}}} \subset K$ .
- $\overline{A_{\text{tar}}}$  is a fixed point of  $\mathcal{B}$ .

To prove the first assertion, take any point  $p \in A_{\text{tar}}$ . By (2.1) there is a sequence  $\omega$  such that

$$(3.6) \quad \{p\} = \bigcap_{n \geq 0} f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X) \supset \bigcap_{n \geq 0} f_{\omega_0} \circ \cdots \circ f_{\omega_n}(K).$$

Since the last intersection is non-empty and contained in  $K$ , it follows that  $p \in K$ . As this holds for every  $p \in A_{\text{tar}}$ , we see that  $A_{\text{tar}} \subset K$ , so clearly  $\overline{A_{\text{tar}}} \subset K$ .

It remains to prove the second assertion,  $\mathcal{B}(\overline{A_{\text{tar}}}) = \overline{A_{\text{tar}}}$ . Note that for  $p$  as in (3.6) and every  $i = 1, \dots, k$ ,

$$(3.7) \quad \{f_i(p)\} = \bigcap_{n \geq 0} f_i \circ f_{\omega_0} \circ \cdots \circ f_{\omega_n}(X).$$

This implies that  $\mathcal{B}(A_{\text{tar}}) \subset A_{\text{tar}}$ . Hence, by continuity of the maps  $f_i$ ,  $\mathcal{B}(\overline{A_{\text{tar}}}) \subset \overline{A_{\text{tar}}}$ . By definition of  $(\overline{A_{\text{tar}}})^*$  in (3.1), it is clear that  $(\overline{A_{\text{tar}}})^* \subset \overline{A_{\text{tar}}}$ . By Lemma 3.1,  $(\overline{A_{\text{tar}}})^*$  is a fixed point of  $\mathcal{B}$ . Since, by the first assertion,  $\overline{A_{\text{tar}}}$  is contained in every fixed point of  $\mathcal{B}$ , it follows that  $\overline{A_{\text{tar}}} \subset (\overline{A_{\text{tar}}})^*$  and hence  $\overline{A_{\text{tar}}} = (\overline{A_{\text{tar}}})^*$ .

To prove item (2) of the theorem, consider any non-empty compact subset  $K \subset \overline{A_{\text{tar}}}$ . It follows from (3.6) that  $\mathcal{B}(K) \subset \overline{A_{\text{tar}}}$  and hence  $\mathcal{B}^n(K) \subset \overline{A_{\text{tar}}}$  for every  $n \geq 0$ . In particular, recalling the definition of  $h_s$  in (1.6), it follows that

$$\lim_{n \rightarrow \infty} h_s(\mathcal{B}^n(K), \overline{A_{\text{tar}}}) = 0.$$

On the other hand, Lemma 3.3 yields  $\lim_{n \rightarrow \infty} h_s(\overline{A_{\text{tar}}}, \mathcal{B}^n(K)) = 0$  and hence

$$\lim_{n \rightarrow \infty} d_H(\mathcal{B}^n(K), \overline{A_{\text{tar}}}) = 0.$$

**3.3. Proof of Theorem 2.** If there are no strict attractors we are done. Otherwise, assume that there is a strict attractor  $K$ . Since  $K$  is a fixed point of  $\mathcal{B}$ , Theorem 1(1) implies that  $\overline{A_{\text{tar}}} \subset K$ . By definition of a strict attractor, the set  $K$  attracts every compact set in a neighbourhood of it, and thus  $\overline{A_{\text{tar}}}$  is attracted by  $K$ . Therefore  $\overline{A_{\text{tar}}} = K$ , proving (1).

Since every strict attractor is a Conley attractor, to prove (2) it is enough to see that if  $\overline{A_{\text{tar}}}$  is a Conley attractor then it is a strict attractor. Thus, we now assume that  $\overline{A_{\text{tar}}}$  is a Conley attractor and hence it has an open neighbourhood  $U$  such that  $\lim_{n \rightarrow \infty} \mathcal{B}^n(\overline{U}) = \overline{A_{\text{tar}}}$ . It is enough to check that  $\lim_{n \rightarrow \infty} \mathcal{B}^n(K) = \overline{A_{\text{tar}}}$  for every  $K \in \mathcal{H}(\overline{U})$ . In other words, we need

to see that for every  $\epsilon > 0$  there is  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ ,

$$(3.8) \quad d_H(\overline{A_{\text{tar}}}, \mathcal{B}^n(K)) = \max\{h_s(\overline{A_{\text{tar}}}, \mathcal{B}^n(K)), h_s(\mathcal{B}^n(K), \overline{A_{\text{tar}}})\} \leq \epsilon.$$

Since  $\lim_{n \rightarrow \infty} \mathcal{B}^n(\overline{U}) = \overline{A_{\text{tar}}}$ , there is  $n_0$  such that for every  $n \geq n_0$ ,

$$h_s(\mathcal{B}^n(\overline{U}), \overline{A_{\text{tar}}}) \leq \epsilon,$$

which implies that for every  $n \geq n_0$ ,

$$h_s(\mathcal{B}^n(K), \overline{A_{\text{tar}}}) \leq h_s(\mathcal{B}^n(\overline{U}), \overline{A_{\text{tar}}}) \leq \epsilon.$$

Hence, to prove (3.8) it remains to see that  $h_s(\overline{A_{\text{tar}}}, \mathcal{B}^n(K)) \leq \epsilon$  for every  $n$  sufficiently large. This is exactly the content of Lemma 3.3. ■

**SCHOLIUM 3.5.** *If  $U$  is a neighbourhood of  $\overline{A_{\text{tar}}}$  and  $\lim_{n \rightarrow \infty} \mathcal{B}^n(\overline{U}) = \overline{A_{\text{tar}}}$ , then every compact subset  $K$  of  $\overline{U}$  also satisfies  $\lim_{n \rightarrow \infty} \mathcal{B}^n(K) = \overline{A_{\text{tar}}}$ .*

**3.4. Proof of Theorem 3.** Note that by hypothesis the set  $A_{\text{tar}}$  is non-empty.

The implication (1) $\Rightarrow$ (2) follows from Theorem 1(1), which in this case means that “the minimum fixed point  $\overline{A_{\text{tar}}}$  and the maximum fixed point  $X^*$  are the same”. The converse (1) $\Leftarrow$ (2) follows from the fact that  $\overline{A_{\text{tar}}}$  and  $X^*$  are fixed points of  $\mathcal{B}$ .

The implication (3) $\Rightarrow$ (2) is immediate.

To prove (1) $\Rightarrow$ (3) note that, by Claim 3.2,  $X^* = \lim_{n \rightarrow \infty} \mathcal{B}^n(X)$  and thus  $X^*$  is a Conley attractor. Since, by hypothesis,  $\overline{A_{\text{tar}}} = X^*$ , we see that  $\overline{A_{\text{tar}}}$  is a Conley attractor and hence it follows from Scholium 3.5 that  $\overline{A_{\text{tar}}}$  is a strict attractor whose basin is the whole space. ■

**3.5. Proof of Theorem 4.** By hypothesis,  $\overline{A_{\text{tar}}}$  is a stable fixed point of the BH-operator. Fix  $x$  and  $\omega$ , and consider the nested sequences of sets

$$Y_\ell(x, \omega) := \{f_\omega^n(x) : n \geq \ell\}.$$

In view of Claim 3.2, it is enough to see that

$$(3.9) \quad \overline{A_{\text{tar}}} = \bigcap_{\ell \geq 0} \overline{\{f_\omega^n(x) : n \geq \ell\}} = \bigcap_{\ell \geq 0} \overline{Y_\ell(x, \omega)}.$$

To simplify notation, write  $Y_\ell = Y_\ell(x, \omega)$ .

*Proof of “ $\subset$ ”.* Fix  $p \in A_{\text{tar}}$  and  $\ell \geq 0$ . We need to see that for every neighbourhood  $V$  of  $p$ ,

$$(3.10) \quad V \cap Y_\ell \neq \emptyset.$$

By definition of  $A_{\text{tar}}$ , there is an arbitrarily long finite sequence  $c_0 \dots c_r$  such that

$$(3.11) \quad f_{c_r} \circ \dots \circ f_{c_0}(X) \subset V.$$

We can assume that  $r \geq \ell$ . Since  $\omega$  has dense orbit, there is  $m_1$  such that

$$\omega_{m_1} = c_0, \quad \omega_{m_1+1} = c_1, \quad \dots, \quad \omega_{m_1+r} = c_r.$$

Therefore, (3.11) yields

$$f_\omega^{m_1+r+1}(x) \in V.$$

Since  $m_1 + r \geq \ell$  we have  $f_\omega^{m_1+r+1}(x) \in Y_\ell$  and hence  $V \cap Y_\ell \neq \emptyset$ .

*Proof of “ $\supset$ ”.* Take any neighbourhood  $V$  of  $\overline{A_{\text{tar}}}$ . Since  $\overline{A_{\text{tar}}}$  is stable it has a neighbourhood  $V_0 \subset V$  such that  $\mathcal{B}^n(V_0) \subset V$  for every  $n \geq 0$ . By definition of  $A_{\text{tar}}$ , there is a sequence  $c_0 \dots c_r$  with  $f_{c_r} \circ \dots \circ f_{c_0}(X) \subset V_0$ . As above, since  $\omega$  is disjunctive, there is  $m_1$  with  $\omega_{m_1} = c_0, \omega_{m_1+1} = c_1, \dots, \omega_{m_1+r} = c_r$ . Taking  $n_0 = m_1 + r$  implies that  $f_\omega^{n_0+1}(x) \in V_0$ . From  $\mathcal{B}^n(V_0) \subset V$  for every  $n \geq 0$  it follows that  $Y_{n_0} \subset V$ , and thus  $\overline{Y_{n_0}} \subset \overline{V}$ . As the sequence  $(\overline{Y_\ell})$  is nested, we have  $\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{V}$ . Since this holds for every neighbourhood  $V$  of  $\overline{A_{\text{tar}}}$ , we conclude that

$$\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{A_{\text{tar}}},$$

proving “ $\supset$ ”. ■

**4. Examples.** In this section, we present some examples that add a bit to our understanding of the target set and also illustrate the hypotheses in the theorems. The first example, Example 4.1, is the most important one; it exhibits a non-regular IFS whose target is non-empty. Before going into the details, let us make a few general observations.

Requiring the target set to be non-empty is equivalent to assuming that  $S_{\text{wh}} \neq \emptyset$ . To construct an IFS  $(f_1, \dots, f_k)$  defined on a compact metric space  $X$  with  $S_{\text{wh}} \neq \emptyset$ , a natural (and also the simplest) procedure is asking some map to be a contraction, or more generally, asking that for some  $\ell$  in  $\{1, \dots, k\}$ , there is  $p$  such that

$$(4.1) \quad \bigcap_{i \geq 1} f_\ell^i(X) = \{p\}.$$

However, since the single map  $f_\ell$  has a fixed point (the point  $p$ ) that is a global attractor, such a hypothesis leads to a regular IFS (recall the definition in Section 2.1). We observe that this is precisely the scenario that one finds when studying contracting-on-average IFSs on compact metric spaces. In Example 4.1, we show how to construct IFSs with  $S_{\text{wh}} \neq \emptyset$  without assuming that one of the maps has a fixed point that is a global attractor as in (4.1). The idea is to use Theorem 4.1 in [11] claiming that for IFSs on compact subsets of  $\mathbb{R}^m$  satisfying a simple topological condition we have  $\mathbb{P}(S_{\text{wh}}) = 1$  for every product measure  $\mathbb{P} = \nu^{\mathbb{N}}$ , where  $\nu$  is any probability measure on  $\{1, \dots, k\}$  with  $\nu(\{i\}) > 0$  for every  $i$ . In particular,  $S_{\text{wh}} \neq \emptyset$  for the class of

IFSs in [11, Theorem 4.1]. In Example 4.1 the IFS is defined on  $[0, 1]$ , but using the same idea we can also produce examples in higher dimensions.

In Example 4.2, we construct an IFS satisfying the hypothesis in Theorem 4 such that the closure of the target set is not a Conley attractor.

We now discuss Examples 4.3 and 4.4 together. First recall Remark 2.6 stating that for weakly hyperbolic IFSs the target set is equal to the maximum fixed point, i.e.,  $A_{\text{tar}} = X^*$ . Example 4.3 shows that this equality can also happen for non-weakly hyperbolic IFSs. This is an example of a “genuinely” non-hyperbolic IFS whose BH-operator has a global attractor fixed point (recall Theorem 3). However, Example 4.4 shows that the equality  $A_{\text{tar}} = X^*$  is not necessary for the BH-operator to have a global attractor fixed point. In fact, Example 4.4 gives an IFS where  $\overline{A_{\text{tar}}}$  coincides with the maximum fixed point but  $A_{\text{tar}} \subsetneq \overline{A_{\text{tar}}}$ . In other words, in Theorem 3 it is indispensable to consider the closure of the target set.

In Examples 4.4 and 4.3, we consider IFSs consisting of two maps on the interval  $[0, 1]$ . We observe that to construct IFSs with the target set having the properties in these examples is much easier if we allow the IFS to have more than two maps. However, IFSs consisting of two maps whose target sets have the properties described above can be found in the constructions of two well known class of attractors (of a dynamical system), *bony attractors* and *porcupine-like horseshoes* [9, 14]. Hence, we also take the opportunity in this section to describe the target set of the IFSs used in the construction of these attractors.

We close this section with a quite simple Example 4.5 showing that, in general, IFSs may not have a minimum fixed point.

EXAMPLE 4.1 (A non-regular IFS with  $S_{\text{wh}} \neq \emptyset$ ). Consider an IFS  $(f_1, f_2)$  such that  $f_1$  and  $f_2$  are continuous injective maps on  $[0, 1]$  satisfying:

- (1)  $f_1$  has exactly two fixed points 0 and 1, where 0 is a repeller and 1 is an attractor;
- (2)  $f_2$  has exactly three fixed points  $p_1 < p_2 < p_3$ , where  $p_1$  and  $p_3$  are attractors and  $p_2$  is a repeller;
- (3)  $f_2([0, 1]) = [\alpha, \beta] \subset (0, 1)$  and  $p_1 < f_1(p_1) < \beta$ ;
- (4) there is  $c \in (\alpha, p_1)$  such that  $f_1(c) = f_2(c)$  and  $f_1(x) \neq f_2(x)$  for all  $x \neq c$ .

These conditions are depicted in Figure 1. We claim that:

- (i) IFS  $(f_1, f_2)$  is non-regular;
- (ii)  $S_{\text{wh}} \neq \emptyset$ .

Recall first the definition of a regular IFS in Section 2.1. To prove (i), first note that neither IFS  $(f_1)$  nor IFS  $(f_2)$  has a global attractor. Thus, it remains to see that IFS  $(f_1, f_2)$  has no global attractor. We prove this by showing that

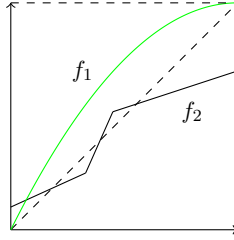


Fig. 1. A non-regular IFS with  $S_{\text{wh}} \neq \emptyset$

the BH-operator  $\mathcal{B}$  associated with  $\text{IFS}(f_1, f_2)$  has at least two different fixed points (which prevents regularity). Indeed,  $[0, 1]$  and  $[p_1, 1]$  are fixed points of  $\mathcal{B}$ : It is clear that  $[0, 1]$  is a fixed point, and to see that  $[p_1, 1]$  is one, note that  $p_1 < f_1(p_1) < \beta$ , hence

$$\mathcal{B}([p_1, 1]) = f_2([p_1, 1]) \cup f_1([p_1, 1]) = [p_1, \beta] \cup [f_1(p_1), 1] = [p_1, 1].$$

This implies that  $\text{IFS}(f_1, f_2)$  is non-regular.

To prove (ii), note that 1 is an attracting fixed point of  $f_1$  and  $f_2([0, 1]) \subset (0, 1)$ . Hence we get the *splitting property*  $f_1^n \circ f_2([0, 1]) \cap f_2([0, 1]) = \emptyset$  for  $n$  sufficiently large. Now Theorem 4.1 in [11] implies that  $S_{\text{wh}} \neq \emptyset$ .

EXAMPLE 4.2 (An IFS without strict attractors and such that  $\overline{A_{\text{tar}}}$  is stable). Consider maps  $f_1, f_2: [0, 2] \rightarrow [0, 2]$  defined by (see Figure 2)

$$f_1(x) = \frac{1}{3}x, \quad f_2(x) = \begin{cases} \frac{1}{3}x + \frac{2}{3} & \text{for } x \in [0, 1] \\ x & \text{for } x \in [1, 2]. \end{cases}$$

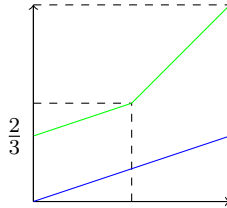


Fig. 2. The set  $\overline{A_{\text{tar}}}$  is Lyapunov stable and is not a Conley attractor

We claim that:

- (i)  $S_{\text{wh}} \neq \emptyset$ ;
- (ii)  $\overline{A_{\text{tar}}}$  is not a Conley attractor;
- (iii)  $\overline{A_{\text{tar}}}$  is stable.

Item (i) follows by observing that  $f_1$  is a contraction.

To prove (ii), consider the auxiliary  $\text{IFS}(g_1, g_2)$  where  $g_1 = f_1|_{[0,1]}$  and  $g_2 = f_2|_{[0,1]}$ . Let  $\mathcal{C}$  be the standard ternary Cantor set in  $[0, 1]$ . We claim that  $\overline{A_{\text{tar}}} = \mathcal{C}$ . Note that  $\mathcal{C}$  is the Hutchinson attractor of  $\text{IFS}(g_1, g_2)$  (see,

for instance, [13, Section 3.3, Example 1]). In particular,  $\mathcal{C}$  is the unique fixed point of the BH-operator  $\mathcal{B}$  of IFS( $f_1, f_2$ ) contained in  $[0, 1]$ . Since  $[0, 1]$  is  $\mathcal{B}$ -invariant, by Lemma 3.1 and minimality (item (1) in Theorem 1) we have  $\overline{A_{\text{tar}}} \subset [0, 1]^* \subset [0, 1]$  and therefore  $\overline{A_{\text{tar}}} = \mathcal{C}$ , proving the claim. To see that  $\mathcal{C}$  is not a Conley attractor, observe that every open neighbourhood of  $\mathcal{C}$  necessarily contains an interval of the form  $[1, \delta)$ . Since  $f_2(x) = x$  for all  $x \in [1, \delta)$ , the assertion follows.

To see (iii), just note that  $f_1, f_2$  are non-expanding maps.

In the next example, we consider the underlying IFS of the bony attractor in [14].

EXAMPLE 4.3 (A non-weakly hyperbolic IFS on  $[0, 1]$  with  $A_{\text{tar}} = [0, 1]$ ). Consider the IFS( $f_1, f_2$ ) whose maps  $f_1$  and  $f_2$  are defined on  $[0, 1]$  as follows (see Figure 3):

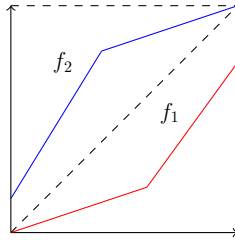


Fig. 3. The underlying IFS of a bony attractor

- (1)  $f_1$  is the piecewise-linear map with “vertices”  $(0, 0)$ ,  $(0.6, 0.2)$ , and  $(1, 0.8)$ ;
- (2)  $f_2$  is the piecewise-linear map with “vertices”  $(0, 0.15)$ ,  $(0.4, 0.8)$ , and  $(1, 1)$ .

As observed in [14], these maps have the following properties:

- (3)  $f_1 \circ f_2$  has a repelling fixed point;
- (4)  $f_1^3, f_1^2 \circ f_2, f_2^2 \circ f_1$ , and  $f_2^5$  are uniform contractions and the union of their images is  $[0, 1]$ .

We claim that:

- (i) the IFS( $f_1, f_2$ ) is non-weakly hyperbolic;
- (ii)  $A_{\text{tar}} = [0, 1]$ .

The fact that  $f_1 \circ f_2$  has a repelling fixed point implies (i): the periodic sequence  $\overline{12}$  does not belong to  $S_{\text{wh}}$ .

To prove (ii), consider the finite set of words corresponding to the compositions in item (4),

$$W = \{111, 112, 221, 22222\},$$



and let  $E_W$  be the subset of  $\Sigma_2$  consisting of sequences  $\omega$  that are concatenations of words of  $W$ . It follows from the classical Hutchinson theory for hyperbolic IFSs that  $E_W \subset S_{\text{wh}}$  and  $\pi(E_W) = [0, 1]$ , where  $\pi$  is the coding map (this follows from (4)). Thus  $A_{\text{tar}} = \pi(S_{\text{wh}}) = [0, 1]$ .

In the next example we consider the underlying IFS of the porcupine-like horseshoes of [9] and translate the construction of [10, p. 12] to our context.

EXAMPLE 4.4 ( $A_{\text{tar}} \subsetneq \overline{A_{\text{tar}}}$  and  $\overline{A_{\text{tar}}}$  is the whole space). Consider the IFS( $f_1, f_2$ ) whose maps  $f_1$  and  $f_2$  are defined on  $[0, 1]$  as follows (see Figure 4):

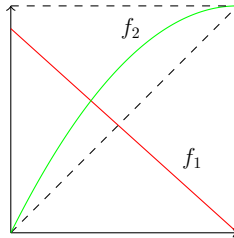


Fig. 4. The underlying IFS of a porcupine-like horseshoe

- (1)  $f_1(x) = \lambda(1 - x)$ ,  $\lambda \in (0, 1)$ ;
- (2)  $f_2$  is an injective continuous map with exactly two fixed points, the repelling fixed point 0 and the attracting fixed point 1;
- (3)  $f_2$  is a uniform contraction on  $[f_2^{-1}(\lambda), 1]$ .

We claim that:

- (i)  $\overline{A_{\text{tar}}} = [0, 1]$ ;
- (ii)  $A_{\text{tar}} \subsetneq \overline{A_{\text{tar}}}$ .

To prove (i), we start by observing that  $\lambda \in \overline{A_{\text{tar}}}$ . To see why, take any open neighbourhood  $V \subset (0, 1)$  of  $\lambda$  and note that  $f_1^{-1}(V)$  is a neighbourhood of 0. Consider the fixed point  $p = \frac{\lambda}{1+\lambda} \in (0, 1)$  of  $f_1$  and note  $\bar{1} \in S_{\text{wh}}$  and hence  $p \in A_{\text{tar}}$ . Since  $f_2^n(p) \rightarrow 1$  as  $n \rightarrow \infty$  and  $f_1(1) = 0$ , there is  $\ell$  such that  $f_1 \circ f_2^\ell(p) \in f_1^{-1}(V)$ . Hence  $f_1^2 \circ f_2^\ell(p) \in V$ . The invariance of  $A_{\text{tar}}$  implies that  $A_{\text{tar}} \cap V \neq \emptyset$ . Since this holds for every neighbourhood  $V$  of  $\lambda$  we get  $\lambda \in \overline{A_{\text{tar}}}$ .

We now prove that  $A_{\text{tar}}$  is dense in  $[0, 1]$ , i.e.  $J \cap A_{\text{tar}} \neq \emptyset$  for every open interval  $J \subset (0, 1)$ . If  $\lambda \in J$  we are done. Otherwise  $\lambda \notin J$  and either  $J \subset (\lambda, 1] =: I_2$  or  $J \subset [0, \lambda) =: I_1$ . We claim that

$$(4.2) \quad \lambda \in f_{\omega_m}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J) \quad \text{for some } \omega_0 \dots \omega_m.$$

Since  $\lambda \in \overline{A_{\text{tar}}} \cap f_{\omega_m}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J)$ , the invariance of  $A_{\text{tar}}$  implies that  $J \cap A_{\text{tar}} \neq \emptyset$ . Thus it remains to prove (4.2). For that let  $\omega_0 = i$  if  $J \subset I_i$  and

define recursively  $\omega_{\ell+1} = i$  if  $f_{\omega_\ell}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J) \subset I_i$ . Note that if at some step  $f_{\omega_\ell}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J)$  is neither in  $I_1$  nor in  $I_2$  then  $\lambda \in f_{\omega_\ell}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J)$  and we are done. As  $f_2^{-1}$  is a uniform expansion on  $I_2$  and  $f_1^{-1}$  is a uniform expansion on  $I_1$ , the size of the interval  $f_{\omega_\ell}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J)$  increases exponentially and the recursion stops after finitely many steps: there is  $m$  such that  $\lambda \in f_{\omega_m}^{-1} \circ \dots \circ f_{\omega_0}^{-1}(J)$ , proving (4.2).

To prove (ii), it is enough to see that  $1 \notin A_{\text{tar}}$ . This follows by noting that  $\bar{2} \notin S_{\text{wh}}$  and  $1 \notin f_{\omega_0} \circ \dots \circ f_{\omega_n}([0, 1])$  for any finite sequence  $\omega_0 \dots \omega_n$  such that  $\omega_i = 1$  for some  $i$ .

**EXAMPLE 4.5** (An IFS without a minimum fixed point). Consider the IFS  $(f_1, f_2)$  where  $f_1, f_2: [0, 1] \rightarrow [0, 1]$  are defined by  $f_1(x) = x$  and  $f_2(x) = 1 - x$ . For each  $x \in [0, 1]$ , the set  $\{x, 1 - x\}$  is a fixed point of  $\mathcal{B}$  which is also minimal. Obviously, there is no fixed point contained in all fixed points and hence there is no minimum fixed point. Note that, in this example, we have  $S_{\text{wh}} = \emptyset$ .

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